Mathematics in Ancient India

2. Diophantine Equations: The Kuttaka

Amartya Kumar Dutta

Amartya Kumar Dutta is an Associate Professor of Mathematics at the Indian Statistical Institute, Kolkata. His research interest is in commutative algebra.

A great mathematical creation could be a profound and potent idea creating a fundamental and revolutionary impact by its very simplicity and elegance, or it could be a deep technical work of great complexity and ingenuity. In the history of mathematics, India seems to have the greatest contribution in introducing simplifying innovations. Three grand instances are: the invention of the decimal notation and creation of modern arithmetic; the invention of the sine and cosine functions leading to the creation of modern trigonometry; and creation of algebra. These master-strokes have influenced and transformed the very nature of mathematics. However, due to the very simplicity of these profound contributions, we do not realise how much admiration they deserve.

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As examples of great works of complexity, one can cite the higher geometry of Archimedes and Apollonius, the calculus of Archimedes, and the Indian achievements in the study of indeterminate equations in higher arithmetic (number theory) and in analytic trigonometry. However, the latter works – too far ahead of contemporary mathematical maturity – did not reach Europe during renaissance and had to be reinvented by some of the greatest minds of Europe during the 17th-18th centuries. Consequently the astonishing feats achieved in India in these two core areas of mathematics – algebraic number theory and analysis – often tend to get overlooked in accounts on the development of mathematics. Ironically, it is also due to the algebraic intricacies and abstractions involved in indeterminate or Diophantine analysis, that Indian progress in this area often gets neglected in popular writings involving mathematics culture and history. These unfortunate omissions tend to

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create an erroneous impression that the contributions of ancient India have been limited to commercial arithmetic or elementary mathematics.

In Part 1 of this series, we had made only a brief mention of Diophantine equations. In this part, we shall elaborate a little more on the exciting Indian achievements in this field and discuss Diophantine equations in the Sulbasutras and *Aryabhatiya*. We first make a few introductory remarks regarding Diophantine equations.

Equations with integer coefficients whose solutions are to be found in integers (or sometimes rational numbers) are called Diophantine equations.

Diophantine Equations: An Introduction

Equations with integer coefficients whose solutions are to be found in integers (or sometimes rational numbers) are called Diophantine equations in the honour of Diophantus of Alexandria (250 AD) – the adjective 'Diophantine' pertains not so much to the nature of the equation as to the nature of the admissible solutions of the equation. Problems in Diophantine equations are easy to state but usually hard to solve – the difficulty arises due to the stringent restriction of admitting only integer solutions. Often it is difficult even to ascertain whether an integer solution exists or not – an extreme example is the famous Diophantine equation $x^n + y^n = z^n$, for arbitrary $n(> 2)$.

Indians were the first to systematically investigate methods for determination of integral solutions of Diophantine equations. Diophantus had actually investigated solutions of equations in rational numbers (not integers) – rational solutions of equations are of considerable geometric interest. For homogeneous equations, the two problems are equivalent; but, in general, the problem of finding integer solutions to an equation is much more difficult than that of finding rational solutions. For instance, it is trivial to describe all rational solutions of a linear equation $ax - by = c$ (a, b, c integers); whereas to describe all integer solutions requires some effort.

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While Diophantus was interested in finding one rational solution, Indians investigated *all* integral solutions of Diophantine equations of first and second degree. By fifth century AD, the Indians had discovered a general method for the solution of the first degree Diophantine equation in two variables which we shall discuss in this issue.

No general method is as yet known for solving general quadratic or higher Diophantine equations. Among the quadratic equations, the most famous are the special equations of the form $x^2 - Dy^2 = 1$, known as the Pell equation, for which Indians had evolved a brilliant algorithm during the 7th-11th century AD which we shall discuss in a subsequent issue.

Systematic investigation of integral solutions began in Europe only in the 17th century when interest in number theory was rekindled with the publication of Bachet's translation of Diophantus with a commentary.

Diophantine Equations in Sulbasutras

The Indian interest in integral solutions of Diophantine equations can be traced back to the Sulbasutras. Certain brick constructions of the Vedic fire-altars provide interesting examples of specific simultaneous indeterminate equations. For instance, the *Garhapatyagni* altar is stipulated to have five layers of bricks, each layer containing 21 bricks, and forming an area of one square unit (*vyayam*).

Now, not all bricks can be of same length; for, to have stability in the structure, the cleavages (between two adjacent bricks) of two successive layers should not coincide. Suppose each layer has x square bricks of length $\frac{1}{m}$ unit each and y square bricks of length $\frac{1}{n}$ unit each, with $m > n$. Then the altar-specifications lead to the simultaneous Diophantine equations

$$x + y = 21; \quad \frac{x}{m^2} + \frac{y}{n^2} = 1$$

having precisely two sets of positive integral solutions for (x, y, m, n), namely $(16, 5, 6, 3)$ and $(9, 12, 6, 4)$. The Baudhayana Sulbasutra prescribes making three types of square bricks of lengths $\frac{1}{6}, \frac{1}{4}$ and $\frac{1}{3}$ units and then placing 9 bricks of length $\frac{1}{6}$ unit and 12 bricks of length $\frac{1}{4}$ unit in the first, third and fifth layers; and 16 bricks of length $\frac{1}{6}$ unit and 5 bricks of length $\frac{1}{3}$ unit in the second and fourth layers. That takes care of both the ritual as well as the engineering requirements. Mathematically speaking, Baudhayana has given all the positive integral solutions of the simultaneous equations described above.

More difficult Diophantine problems arise from the constructions of the *syena-cit* (falcon-shaped fire-altar). Two such examples, due to Baudhayana and Apastamba respectively, are:

$$x + y + z + u = 200; \quad \frac{x}{m} + \frac{y}{n} + \frac{z}{p} + \frac{u}{q} = 7\frac{1}{2}.$$

$$x + y + z + u + v = 200; \quad \frac{x}{m} + \frac{y}{n} + \frac{z}{p} + \frac{u}{q} + \frac{v}{r} = 7\frac{1}{2}.$$

The actual difficulties of construction are much more than what appears from mere algebraic considerations. The bricks have further to be arranged in such a way as to have the prescribed shape of the falcon-altar!

The Sulbasutras also give several examples of 'Pythagorean triples' (a triple of positive integers (a, b, c) satisfying the equation $x^2 + y^2 = z^2$ is called a Pythagorean triple). Further, Katyayana gave an ingenious rule to combine n squares of length a to get a new square – this rule involves the formula $na^2 = ((n+1)/2)^2a^2 - ((n-1)/2)^2a^2$. From this identity it is easy to deduce a general formula to describe all Pythagorean triples. In view of the abundant numerical examples and Katyayana's

Systematic methods for finding integer solutions of Diophantine equations can be found in Indian texts from the time of Aryabhata (499 AD).

rule, it seems that the Vedic mathematicians were aware of such a general formula (for more details, see [1]).

Kuttaka Algorithm of Aryabhata

Introduction

Systematic methods for finding integer solutions of Diophantine equations can be found in Indian texts from the time of Aryabhata (499 AD). The first explicit description of the general integral solution of the linear Diophantine equation $ay - bx = c$ occurs in his text *Aryabhatiya*. This algorithm is considered to be the most significant contribution of Aryabhata in pure mathematics. The technique was applied by Aryabhata to give integral solutions of simulataneous Diophantine equations of first degree – a problem with important applications in astronomy.

Aryabhata describes the algorithm in just two stanzas of *Aryabhatiya* (verses 32 and 33 of the section *Ganita*). His cryptic verses were elaborated by Bhaskara I (6th century AD) in his commentary *Aryabhatiyabhasya*. Bhaskara I illustrated Aryabhata's rule with several examples including 24 concrete problems from astronomy. Without the explanation of Bhaskara I, it would have been difficult to interpret Aryabhata's verses (for details, see [2]).

Bhaskara I aptly called the method '*kuttaka*' (pulversisation) – the significance of the terminology would be clear from the algorithm. The kuttaka was subsequently discussed, with variations and refinements, by several Indian mathematicians including Brahmagupta (628 AD), Mahavira (850), Aryabhata II (950), Sripati (1039), Bhaskara II (1150) and Narayana (1350). The idea in kuttaka was considered so important by the Indians that initially the whole subject of algebra used to be called *kuttaka-ganita*, or simply *kuttaka* – Brahmagupta (628 AD) used this term! The current Sanskrit term *bijaganita* appeared much later.

Formulation

If we take a, b, c to be positive integers, then any linear Diophantine equation in two variables is of the form $ay - bx = \pm c$ or $ay + bx = \pm c$. Indians actually evolved algorithms for determining all *positive* integral solutions of such equations – an even more subtle problem. For the concrete applications in astronomy, one needed to pick out precisely the positive integral solutions of equations of the type $ay - bx = \pm c$. For simplicity, we shall not always be too meticulous on this technical point – sometimes we shall talk about integers rather than positive integers.

Indians evolved algorithms for determining all positive integral solutions of linear Diophantine equations in two variables.

The equation $ay - bx = \pm c$ was visualised by ancient Indians in the form $y = \frac{bx \pm c}{a}$. The quantities a, b, c, x, y were called *hara* (divisor), *bhajya* (dividend), *ksepa* (interpolator), *gunaka* (multiplier) and *phala* (quotient), respectively. During the early stages (i.e., the time of Aryabhata–Bhaskara I) when negative numbers had not yet taken firm roots in Indian algebra, the equation was arranged in the form $y = \frac{bx+c}{a}$ or $x = \frac{ay+c}{b}$ so as to ensure that the interpolator c comes with a positive sign. It would be beyond the scope of this article to go into all the finer historical details on the kuttaka – a thorough discussion involves 55 pages in ([3], Vol II. p. 87-141). For brevity, we present the essence of the algorithm adopting modern style and notations but using the various ideas introduced by the ancient Indian stalwarts.

Now, it is easy to see that the equation $ay - bx = c$ will have integral solutions only if c is divisible by the GCD (greatest common divisor) of a and b – this observation was made, in some form or the other, by almost all ancient Indian writers.

Using the Indian methods we shall deduce the converse:

Laghubhaskariya of Bhaskara I discusses a problem pertaining to the revolutions of Saturn which would lead to the equation $146564y-1577917500x = 24$.

If c is divisible by the GCD of a and b, then the equation $ay - bx = c$ has infinitely many integral solutions.

From now onwards, we shall assume that a, b, c are all positive and focus on the equations $ay - bx = \pm c$.

Reductions

The quantities a, b, c were usually very large in the concrete problems arising out of astronomy. For instance, *Laghubhaskariya* of Bhaskara I discusses a problem pertaining to the revolutions of Saturn which would lead to the equation $146564y - 1577917500x = 24$. To simplify the laborious calculations, various devices were employed by the Indians. It is now a standard trick in modern mathematics to be on the lookout for such simplifications or 'reductions' by which efforts are made to transfer a problem to a equivalent but neater and possibly more tractable problem where the underlying features become more transparent. We now describe some of the reductions performed by the ancient Indians before applying the kuttaka.

First of all, cancelling $\gcd(a, b, c)$ (which is simply $\gcd(a, b)$ by earlier remark), one assumes that a and b are coprime. By this reduction, not only do the coefficients become smaller (thereby simplifying computations), but, more importantly, one is now better equipped to tackle the problem as one has the advantage of the additional property of a and b being coprime – a property which is potentially useful. Bhaskara I, Brahmagupta, Aryabhata II, Sripati and Bhaskara II, among others, explicitly stated that all the coefficients should be divided by $\gcd(a, b)$, so that the coefficients in the reduced equation become relatively prime, or to use ancient terminology, (mutually) *drdha* (firm), *niccheda* (having no divisor), *nirapavarta* (irreducible).

Secondly, following Bhaskara I, one can reduce the problem of finding all the positive integral solutions to that of finding one positive integral solution. Suppose that (u, v) is a positive integral solution of $ay - bx = \pm c$. From (u, v), one first finds the minimum positive integral solution. Dividing u and v by a and b respectively, we have $u = pa + r$ and $v = qb + s$ for some whole numbers p, q, r, s such that $r < a$ and $s < b$. If $p = q$, then (r, s) is clearly a solution of $ay - bx = \pm c$; in fact, it is the minimum positive integral solution (proof is easy). If $p \neq q$, then it can be seen that $p < q$ when we consider the equation $ay - bx = c$; and $p > q$ when we deal with $ay - bx = -c$. The minimum positive integral solution in the two cases are $(r, s + (q - p)b)$ and $(r + (p - q)a, s)$, respectively. This rule for arriving at the minimum solution has been explained very lucidly by Aryabhata II but it is already implicit in Aryabhata–Bhaskara I. Next, if (α, β) is a minimum positive integral solution of $ay - bx = \pm c$, then Bhaskara I and his successors described the general positive integral solution as $(\alpha + ta, \beta + tb)$, where t is a positive integer. This can be easily verified using that a, b are coprime. In fact, the general integral solution of $ay - bx = \pm c$ can be seen to be $(x, y) = (u + ta, v + tb)$ where t is any integer and (u, v) any integer solution.

Thirdly, it is clearly enough to solve an equation of the type $ay - bx = \pm 1$; for, if $av - bu = \pm 1$, then $a(cv) - b(cu) = \pm c$. Such equations were called *sthira-kuttaka* (constant pulveriser). This simplification too was made by some of the Indian mathematicians right from Bhaskara I. In problems of astronomy involving the equations $ay - bx = \pm c$, the conditions were often such that the coefficients a, b would be the same in several equations but the interpolator c would vary. In such situations, working first with the constant pulveriser and then modifying the solution according to the specific problem would have been convenient.

We mention another subtle reduction of Aryabhata II, applicable in case there is a common factor between a

and c or between b and c, which would further reduce the size of the coefficients of x and y. Let $g_1 = \gcd\ (a, c)$, $a_1 = \frac{a}{g_1}$, $g_2 = \gcd\ (b, \frac{c}{g_1})$ and $b_1 = \frac{b}{g_2}$. Then Aryabhata II, and his successors like Bhaskara II, observed that the problem of solving $ay - bx = \pm c$ reduces to the problem of solving $a_1Y - b_1X = \pm 1$: if (u, v) is an integral solution of the latter, then $(\frac{cu}{g_2}, \frac{cv}{g_1})$ is an integral solution of the original equation.

Now, without loss of generality, we assume $a > b$. This step was achieved through two different methods. The later method corresponds to the modern approach: if $b > a$, think of the equation as $bx - ay = \mp c$ rather than $ay - bx = \pm c$. But earlier writers like Bhaskara I, who already had to arrange the equation $ay - bx = c$ so as to have only positive coefficients, used the following device when $b > a$: Let $b = aq + b_1$, where $b_1 < a$. Then the original equation transforms into the equation $ay_1 - b_1x = c$ (where $y_1 = y - qx$), which is of the desired form. If (u, v) is a solution of $ay_1 - b_1x = c$, then $(u, v + qu)$ is a solution of $ay - bx = c$. The assumption $a > b$ is not very crucial and not all Indian authors insisted on it; but it would facilitate our discussions.

The Main Algorithm

We now describe the central heart of Aryabhata's algorithm made explicit by Bhaskara I. For greater clarity and brevity, we take advantage of modern language, especially the subscript notation.

By successive division, we have $a = a_1b + r_1$, $b = a_2r_1 + r_2$, $r_1 = a_3r_2 + r_3$ and so on. Let n denote the number of steps after which the process terminates. Since the GCD of a and b is 1, the final relation is $r_{n-2} = a_nr_{n-1} + 1$ $(1 < r_{n-1} < r_{n-2} < \quad < r_1 < b)$. Thus, $r_n = 1$ and $r_{n+1} = 0$. Given a and b, the quantities $a_1, \quad , a_n$ can be quickly determined by the method of repeated division for computing GCD of a and b.

Now, for solving $ay-bx=1$, define quantities x_{n+2}, x_{n+1}, x_n, by backward induction as follows: define x_{n+2} and x_{n+1} to be whole numbers satisfying the relation $r_{n-1}x_{n+2} - x_{n+1} = (-1)^n$. Thus, if n is odd, one can simply take $x_{n+2} = 0$ and $x_{n+1} = 1$; if n is even, one can take $x_{n+2} = 1$ and $x_{n+1} = r_{n-1} - 1$. Now define x_m $(n \geq m \geq 1)$ by $x_m = a_m x_{m+1} + x_{m+2}$. For quick computation of x_n, $, x_2, x_1$, the Indians constructed convenient tables called *valli* (see *Table* 1 for one such example). Then we will have $ax_2 - bx_1 = 1$ (as will be clear from subsequent discussions). Thus $(x, y) = (x_1, x_2)$ is a solution of the equation $ay - bx = 1$.

Hidden in the deceptively simple kuttaka algorithm lies a beautiful idea which has inspired a powerful technique in modern number theory.

Aryabhata–Bhaskara I observed that one need not continue the repeated division till the stage $r_n = 1$. One can terminate at an intermediate stage k if one can readily obtain (by inspection) a positive integral solution (x_k, x_{k+1}) of the equation $r_{k-1}X_{k+2} - r_k X_{k+1} = (-1)^k$. One can then define x_k, x_{k-1}, $, x_2\ (= y), x_1\ (= x)$ recursively, as before, and obtain a solution of the equation $aY - bX = 1$. The quantity x_{k+1} was called *mati.*

For solving $ay - bx = -1$, one defines the *mati* x_{k+1} $(k \leq n)$ and x_{k+2} to be numbers satisfying $r_{k-1}X_{k+2} - r_k X_{k+1} = (-1)^{k+1}$; the rest is as above. Or, one could take the approach of Bhaskara II : solve $ay - bx = 1$ and use the fact that if (α, β) is the minimum positive integral solution of $ay - bx = 1$, then $(a - \alpha, b - \beta)$ is the minimum positive integral solution of $ay - bx = -1$ Brahmagupta, Bhaskara II and Narayana gave similar rules for deriving integer solutions of $ay + bx = c$ from $ay - bx = c$.

Hidden in the deceptively simple kuttaka algorithm lies a beautiful idea which has inspired a powerful technique in modern number theory.

The Key Idea

Using modern approach, the underlying idea of Aryabhata for solving $ay - bx = \pm 1$ may be formulated as follows (to fix our ideas, let us take the RHS to be 1):

(1) Assume the existence of a positive integral solution (x, y) to the equation $aY - bX = 1$; then

(2) Transform this equation by successive steps into equations with smaller and smaller solutions eventually arriving at an equation $a'Y' - X' = \pm 1$ with an obvious solution $(x', y') = (a't \mp 1, t)$ for any t;

(3) Then work backwards from this obvious solution (x', y') of the reduced equation to determine the desired solution (x, y) of the original equation.

To elaborate this interesting idea, assume that (x, y) is a positive solution of the equation $aY - bX = 1$. Recall that we are assuming $0 < b < a$ and a, b are coprime. If $a - b = 1$, then $(1, 1)$ is a solution of $aY - bX = 1$. So we may further assume that $a - b > 1$. Then $x > y$. Note that if b had been 1, we would have been through. So why not try to 'break' the coefficients a, b into smaller ones? Recall the relation $a = a_1 b + r_1$ with $0 < r_1 < b$. The pair (b, r_1) is smaller than the pair (a, b) but both determine each other. Now try to transform the equation $aY - bX = 1$ to get an equation with coefficients b and r_1. The relation $ay - bx = 1$ leads to the relation $(a_1 y - x)b + r_1 y = 1$.

Put $x = a_1 y + z$. Then clearly $0 < z < y$ and $bz - r_1 y = -1$. Now (y, z) is a positive solution of the equation $bZ - r_1 Y = -1$ which is again a linear equation of the original form; but now the solution (y, z) is coordinate-wise smaller than the original (x, y) and, moreover, the coefficients (of the new equation) too have become smaller since $0 < b < a$ and $0 < r_1 < b$. The two pairs of solutions being linearly related; if one can determine the smaller pair (y, z), one can easily compute the original (x, y).

Denote x, y, z by x_1, x_2, x_3, respectively. Keep on proceeding as above introducing $x_4, x_5, \cdots$, etc.. As per our earlier notations, since $r_n = 1$, we eventually arrive at the easily solvable equation $r_{n-1}X_{n+2} - X_{n+1} = (-1)^n$. From this equation, we work backwards to arrive at the solution (x, y) of the original equation $aY - bX = 1$.

Thus, the main algorithm itself is an illustration of a non-trivial application of the modern 'reduction' principle: for, it transfers, through a sequence of steps, a somewhat involved problem to a problem with obvious solution! Since the process involves the breaking up of the original data (both solutions and coefficients) into smaller and smaller numbers by repeated division, the Indians appropriately described the algorithm as 'kuttaka' (pulverizer).

A crucial principle involved in this algorithm is that a decreasing sequence of positive integers must terminate after a finite stage – which is but a version of Fermat's celebrated principle of descent!

A crucial principle involved in this algorithm is that a decreasing sequence of positive integers must terminate after a finite stage.

To realise the significance of the kuttaka idea, we briefly discuss some relevant developments that took place in Europe more than 1000 years after Aryabhata.

Aryabhata's Kuttaka and Fermat's Infinite Descent

The integer solution of the linear Diophantine equation was described in Europe for the first time in 1621 AD by Bachet, 1122 years after Aryabhata. Bachet's solution – which is essentially same as the kuttaka – was rediscovered by Euler in 1735. In the preface to his book on the *Theory of Numbers* (1798), the great Legendre paid a tribute to Bachet making a special mention of his '*very ingenious method*' for solving the indeterminate equation of the first degree.

Bachet's rediscovery of the kuttaka triggered several fruitful ideas in number theory. The principle was ingeniously developed by Fermat, Brouncker and Wallis in their researches on number theory both for constructing solutions of equations and for showing that certain equations do not have integral solutions! The most profound example is Fermat's famous principle of 'infinite descent' (or, simply 'descent').

As a simple illustration of the descent principle, one can show that the equation $x^4 + y^4 = z^2$ has no non-trivial integral solution (in particular, Fermat's last theorem is true for $n = 4$). The basic idea is: Assume (x_0, y_0, z_0) is a positive integral solution; without loss of generality, x_0 and y_0 are coprime and x_0 is even. Then $({x_0}^2, {y_0}^2, z_0)$ must be of the form $(2mn, m^2 - n^2, m^2 + n^2)$ – a fact known from the ancient times (it occurs explicitly in Euclid, Diophantus and Brahmagupta). Now after some elementary algebraic manipulations (see [4]), one can construct another solution (x_1, y_1, z_1) with $0 < z_1 < z_0$. But then, repeating the process, one would have an infinite sequence of solutions (x_i, y_i, z_i) with $z_0 > z_1 > z_2 > \quad > 0$ which is absurd!

The resemblance with the kuttaka idea is unmistakable! As Andre Weil (1906-1999), one of the giants in 20th century mathematics, remarked about Aryabhata's method: '*In later Sanskrit texts this became known as the kuttaka (= 'pulverizer') method; a fitting name, recalling to our mind Fermat's 'infinite descent'.*'

In a subsequent issue, we shall refer to another cute improvization of the 'kuttaka' or 'descent' principle in connection with the so-called Pell's equation.

Fermat used the descent method to prove that the area of a Pythagorean triangle (a right-angled triangle with rational sides) can neither be a square nor twice a square. He was extremely (and justly) proud of his descent principle. In a letter written towards the end of his life, Fermat (1601-1665) stated that all his proofs of his discoveries in number-theory used the descent method! He predicted that "*this method will enable extraordinary developments to be made in the theory of numbers.*" This technique has indeed turned out to be a powerful tool of fundamental importance in modern number theory! It has been crucially used, directly or implicitly, in several deep theorems in the area, especially in the study of elliptic curves.

The kuttaka may be interpreted as a technique in the theory of continued fractions.

Kuttaka and Continued Fractions

Continued fractions is an useful topic in number theory (for basic concepts and results, see [4]). Incidentally, Ramanujan had a phenomenal mastery of continued fractions.

The kuttaka may be interpreted as a technique in the theory of continued fractions – in fact, Aryabhata's formulation $y = \frac{bx+c}{a}$ and method of solution strongly suggests that the discovery of the kuttaka algorithm was preceded by a discovery of the basic principles of continued fractions (see [5]).

Knowledge of continued fractions is even more apparent in some of the later Indian works. In the original kuttaka of Aryabhata, after obtaining the quotients $a_1, \ \ , a_n$, one computes quantitites $x_n, x_{n-1}, \ \ $ in the backward direction (i.e., one moves from below to top in the table). But in certain later Indian texts like the anonymous Karanapaddhati and the Yuktibhasa (1540) of Jyesthadeva (1500-1600 AD), a more direct approach is taken : having obtained the quotients $a_1, \ \ , a_n$, instead of the x_is, one constructs two sequences of numbers p_m and q_m successively using the following recurrence relations

$$p_0 = 1, p_1 = a_1; \quad p_m = a_m p_{m-1} + p_{m-2}$$

$$q_0 = 0, q_1 = 1; \quad q_m = a_m q_{m-1} + q_{m-2}$$

till one reaches p_n and q_n (where n is as before). Then $aq_n - bp_n = (-1)^{n+1}$, i.e., (p_n, q_n) is a solution of one of the equations $ay - bx = \pm 1$ (the solutions of the other equation can be derived using techniques discussed earlier).

This direct method can be best understood in the language and framework of continued fractions. One has been calculating the simple continued fraction expansion of $\frac{a}{b}$; the quantities $a_1, a_2, \cdots, a_n, a_{n+1} (= r_{n-1})$ are the 'quotients', i.e., $\frac{a}{b} = [a_1, \cdots, a_n, r_{n-1}]$, and $\frac{p_m}{q_m}$ are the successive convergents (which are actually characterised by the above recurrence relations). Recall the identity $p_m q_{m-1} - q_m p_{m-1} = (-1)^m$. Since $p_{n+1} = a$ and $q_{n+1} = b$, we have the relation $aq_n - bp_n = (-1)^{n+1}$.

It is remarkable that the Indians had discovered the basic principles underlying the theory of continued fractions; especially the recurrence relations for defining p_m and q_m and the crucial identity $p_m q_{m-1} - q_m p_{m-1} = (-1)^m$. The solution of the linear Diophantine equation in the notation of continued fraction was given by Saunderson in England in 1740 and Lagrange in France in 1767.

Kuttaka and Matrix Operations

The kuttaka algorithm of Aryabhata may also be interpreted as the solution of a system of linear equations by matrix operations.

The linear transformations $a = a_1 b + r_1$, $b = b$; $x = a_1 y + z$, $y = y$ may be expressed as

$$\begin{pmatrix} a & b \\ x & y \end{pmatrix} = \begin{pmatrix} b & r_1 \\ y & z \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} b & r_1 \\ y & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{a_1} =$$

Proceeding in this manner, eventually the LHS can be expressed as a product of invertible matrices with entries $0, 1$.

Simultaneous Linear Indeterminate Equations

The original problem of Aryabhata (*dvicchedagra*), which led to the kuttaka, was to find a number N, which when divided by given numbers $a_1, \quad , a_k$, leave respectively given remainders $r_1, \quad , r_k$; the a_i were termed variously as *ccheda*, *bhajak* or *bhagahara* (divisors) and the r_i were called *agra* or *sesa* (remainders). Let $N = a_1y_1 + r_1 = a_2y_2 + r_2 = \quad = a_ky_k + r_k$. Applying kuttaka first on the equation $a_1y_1 + r_1 = a_2y_2 + r_2$, one can get a minimum positive as well as the general solution for (y_1, y_2). The general solution is in terms of a single parameter t_1; both the expressions $a_1y_1 + r_1$ and $a_2y_2 + r_2$ take the form $b_1t_1 + s_1$ for some b_1, s_1. The process is repeated with $b_1t_1 + s_1 = a_3x_3 + r_3$, and so on.

Mahavira, Aryabhata II, Sripati and Bhaskara II described similar methods for solving simultaneous linear Diophantine equations – *samslista kuttaka* (conjunct pulveriser) – of the form $b_1y_1 = a_1x \pm c_1$, $b_2y_2 = a_2x \pm c_2$, $b_3y_3 = a_3x \pm c_3 \quad \cdot$.

The kuttaka technique was applied by the Indians to solve important problems arising out of astronomy and calendar-making – for instance, the determination of the time when a certain constellation of the planets would occur in the heavens, especially eclipses. To see the connection, note that if k events $E_1, \quad \cdot E_k$ occur regularly with periods $a_1, \quad , a_k$ with E_i happening at times r_i, $r_i + a_i, \quad \cdot$, then the k events occur simultaneously at time N where N is a number which, when divided by each a_i, $(1 \leq i \leq k)$, leaves remainder r_i.

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We skip concrete examples from astronomy as some background details would be needed. Instead, we quote four historical problems that can be solved using kuttaka. The first two are popular riddles.

Eg 1. (Bhaskara I; 6th century) Find the least natural number N which leaves the remainder 1 when divided by 2,3,4,5 or 6 but is exactly divisible by 7. [Ans : 301] This problem also occurs in Ibn-al-Haitam (1000 AD) and Leonardo Fibonacci (1202 AD).

Eg 2 (Brahmagupta 628 AD) Find the least natural number N which on division by 6,5,4,3, leave the remainders 5,4,3,2, respectively. [Ans : 59]

Eg 3 (Bhaskara I) Find the least number N which when divided by 8 leaves 5 (as remainder), divided by 9 leaves 4 and divided by 7 leaves 1. [Ans: 85]

Eg 4 (Mahavira 850 AD) Five heaps of fruits added with two fruits were divided (equally) between nine travellers; six heaps added with four were divided amongst eight; four heaps increased by one were divided amongst seven. Tell the number of fruits in each heap. [(Least) Ans: 194]

From the time of Brahmagupta (628 AD), Indians began attempting the harder problems of solving various Diophantine equations of the second degree. We shall discuss some of their outstanding achievements in the next part of the article.

Suggested Reading

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Address for Correspondence
Amartya Kumar Dutta
Indian Statistical Institute
203, BT Road
Kolkata 700 032, India.