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Use of Circular Error Probability in Target Detection

By

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This paper describes an algorithm for calculating Circular Error Probability (CEP), which is frequently used with target location estimates. CEP relates measurement errors to the variation of a calculation that is computed from those measurements. A mathematical foundation is provided for the CEP estimate and the algorithm is shown to be extremely accurate. It differs from the true CEP estimate by less than one percent on average and has a maximum error of 1.5 percent. As such, it is a useful tool for assessing both the accuracy and sensitivity of any calculated quantity to errored input.

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INTRODUCTION

THE PROBLEM ADDRESSED BY CEP

This paper describes an algorithm for calculating Circular Error Probability (CEP). CEP relates measurement errors to the errors in a calculation that is computed from those measurements. As such, it is a useful tool for assessing the accuracy and sensitivity of any calculated quantity to errored inputs. CEP is frequently used with target location estimates. It is obtained under the assumption that the measurement variances are known.

THE INTERPRETATION OF CEP

The CEP associated with an estimated target location (x,y) is defined to be the radius of the smallest circle with center at (x,y) which has a 50% probability of containing the true target coordinates. Note that the value of CEP is a distance associated with the 50% probability and not a probability itself, as is suggested by the name. CEP provides a measure of the search area within which the true target location can be expected to be found. The better the accuracy of the (x,y) estimate, the smaller the value of CEP.

ORGANIZATION OF THE PAPER

In section 2 the relevant equations for CEP computation are developed. Computational steps are delineated along with required assumptions and a mathematical foundation is provided. A simplified algorithm that approximates CEP, and which is often quoted in literature, is discussed in this paper. The algorithm has a high degree of accuracy. A full derivation of the algorithm is found in appendix A. Also included in section 2 is a discussion of the Trace of the covariance error matrix. Under certain conditions the square root of the Trace is shown to be proportional to the CEP.

Section 3 is a summary of the algorithmic procedure for obtaining CEP from measurement data. It is the working guide to be used in CEP computation. Section 4 gives a brief description of a planar bistatic environment. This geometry is frequently used in emitter location. Section 5 gives examples of CEP calculations and their interpretation in a bistatic environment. Three commonly occurring cases are discussed. Section 6 is a summary of the paper.

SECTION 2
THE MATHEMATICS OF CEP

THE FUNDAMENTAL EQUATION

By definition the CEP is the radius of a circle about the (x,y) estimate which has a 50% probability of containing the target. The equation for any circle of radius CEP and center (x,y) is

$$(x'-x)^2 + (y'-y)^2 = CEP^2 \ .$$

Let $u = x'-x$ and $v = y'-y$, and rewrite the circle equation as

$$u^2 + v^2 = CEP^2 \ .$$

If we assume (u,v) are Gaussian variables with mean (0,0) and variances (σ_u^2, σ_v^2), their joint distribution is given by

$$p(u,v) = \frac{1}{2\pi\sigma_u\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{u}{\sigma_u}\right)^2 - \frac{1}{2}\left(\frac{v}{\sigma_v}\right)^2\right] \ , \qquad (1)$$

and the requirement that the CEP has a 50% probability of containing the target is expressed by the integral relation

$$\iint\limits_{u^2 + v^2 \le CEP^2} p(u,v)\,du\,dv = 0.5 \ . \qquad (2)$$

To obtain the correct value of CEP one must solve equation (2).

Evaluation of (2) involves numerical approximation since (except for some specific values of σ_u and σ_v) the integral has no known antiderivative. Nevertheless, an explicit formula for CEP can be obtained which approximates the solution to (2) with a high degree of accuracy. An approximating formula for CEP is described in the following subsection.

A SIMPLIFIED FORMULA FOR CEP

Whenever the following two conditions hold,

1. the error variables (dx,dy) are uncorrelated, and

2. the error probability densities are zero mean Gaussian distributions with variances $(\sigma_{dx}^2, \sigma_{dy}^2)$

then the following approximating formula for CEP may be used:

$$\text{CEP} = \begin{cases} \sigma_L(0.67+0.8w^2) & \text{if } 0< w <0.5 \\ 0.59\sigma_L(1+w) & \text{if } 0.5\leq w \leq 1 \end{cases} \quad (3)$$

where

$$\sigma_L = \text{Maximum}[\sigma_{dx}, \sigma_{dy}] \quad ,$$
$$\sigma_S = \text{Minimum}[\sigma_{dx}, \sigma_{dy}] \quad ,$$
$$w = \sigma_S/\sigma_L .$$

A derivation of (3) is found in appendix A.

Quality of the Approximation

Figure 1 shows the graph of (CEP/σ_L) versus σ_S/σ_L for both the "exact" value of CEP and the approximation. The "exact" value was computed from equation (2) to four decimal figure accuracy. The approximation was obtained from formula (3). A measure of the difference between the graphs is shown by figure 2.

In figure 2, deviations between the two graphs are displayed in terms of the percent error, defined by

$$\text{Percent Error} = 100 \left[\frac{(CEP)_{exact} - (CEP)_{approx}}{(CEP)_{exact}}\right] .$$

Figure 2 shows that the formula (3) approximation differs from the true CEP by a maximum of 1.5 percent, with the average error less than one percent. The approximation is thus seen to be an extremely accurate measure of true CEP.

DETERMINATION OF THE STANDARD DEVIATION

Assumptions

Use of the CEP formulas require knowledge of the standard deviations $(\sigma_{dx}, \sigma_{dy})$. In this section we show how to obtain $(\sigma_{dx}, \sigma_{dy})$ in terms of the measurement errors. The following assumptions are used:

- If m is a measurement used in the calculation of x or y, then dm (i.e., the error in m) is assumed to be a Gaussian random variable with mean zero and with known standard deviation σ_{dm}.

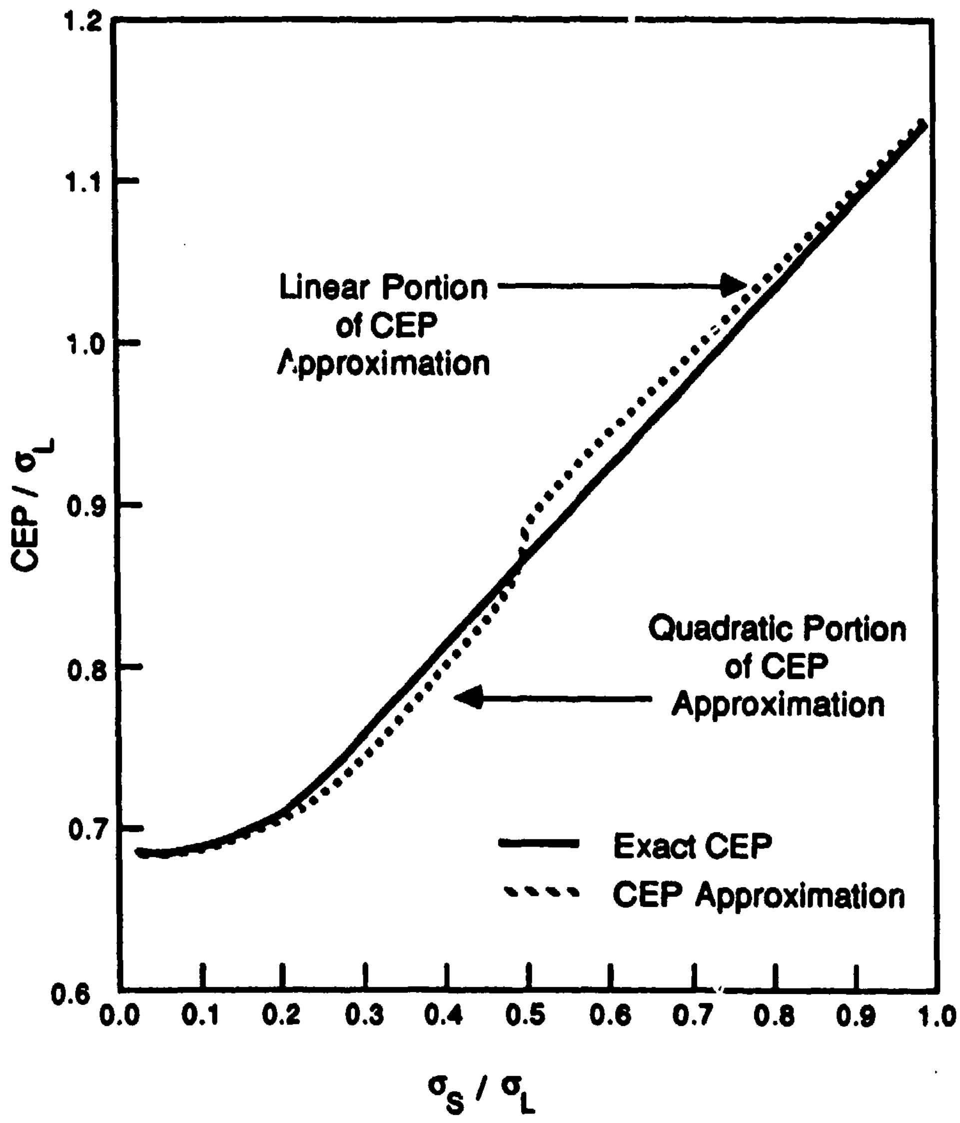

Figure 1. Comparison of CEP/σ_L and its Approximation

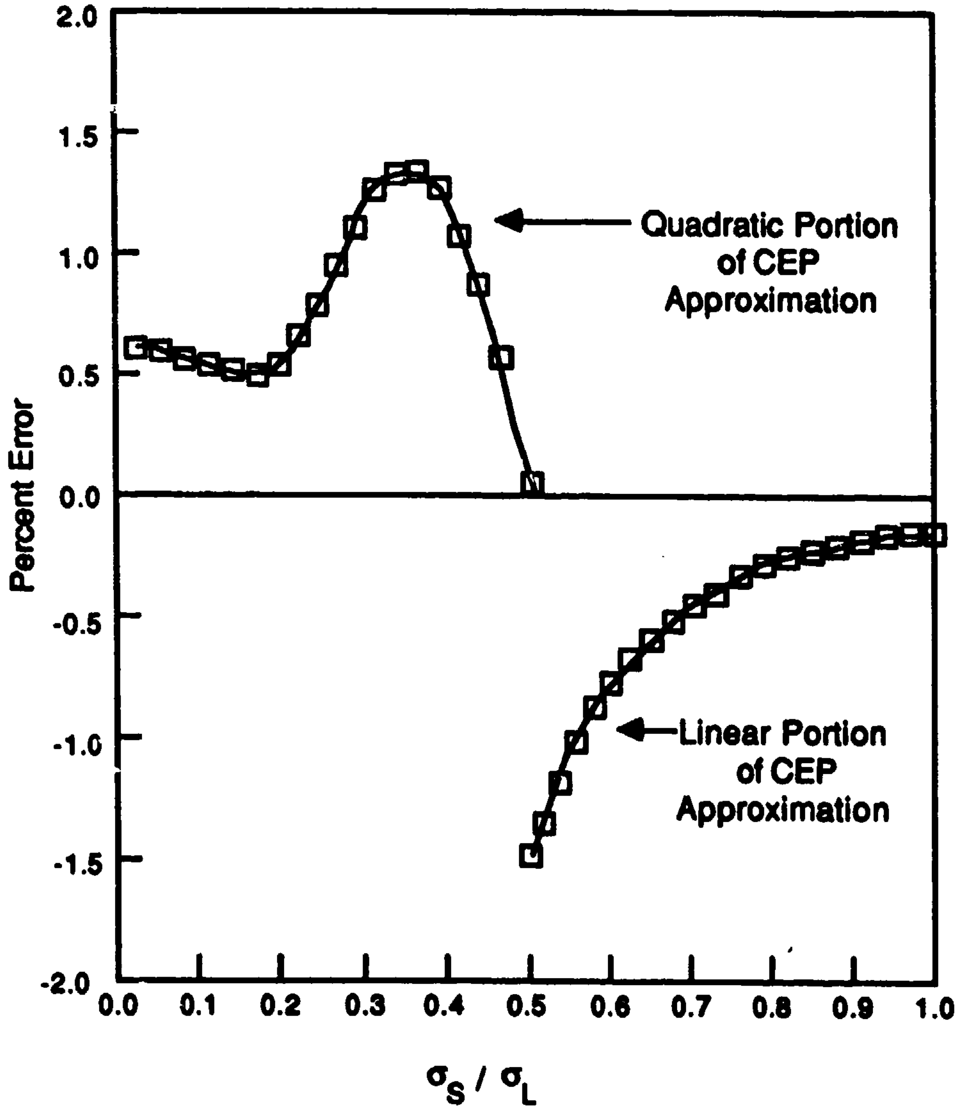

Figure 2. Percent Error in the CEP Approximation

- The standard deviation for each of the dm variables is sufficiently "small" so that second-order terms in the Taylor expansion of dm can be neglected.

- All of the dm variables are statistically independent.

For application of the CEP formulas, one must be sure that these assumptions are met.

An Expression for the (dx,dy) Errors

Our initial objective is to express the errors (dx,dy) in terms of the dm errors. Initially, we must have (x,y) expressed analytically in terms of measurements. If this can be done, then a good approximation to the (dx,dy) errors is provided by the derivative of (x,y). A required assumption for the approximation is that each of the dm errors are "small." For illustration we use the case where x and y are computed from three measurements (m_1, m_2, m_3) as we trace through the mathematical development. The extension to an arbitrary number of measurements follows a similar pattern.

The quantities (dx,dy) are estimated by taking the total derivative of x and y, namely,

$$dx = \frac{\partial x}{\partial m_1} dm_1 + \frac{\partial x}{\partial m_2} dm_2 + \frac{\partial x}{\partial m_3} dm_3 \qquad (4a)$$

$$dy = \frac{\partial y}{\partial m_1} dm_1 + \frac{\partial y}{\partial m_2} dm_2 + \frac{\partial y}{\partial m_3} dm_3 \quad . \qquad (4b)$$

Relationship (4) shows that errors dx and dy are expressed as a linear combination of the measurement errors. From statistical theory, the linearity provides an important observation, that is to say, if each of the error variables (dm_1, dm_2, dm_3) are assumed to be Gaussian with mean (0,0,0), then the variables (dx,dy) will also be Gaussian with mean (0,0). Thus, our assumption of Gaussian behavior for the measurement errors allows us to describe the errors (dx,dy) by means of a Gaussian analysis.

In matrix form, the (dx,dy) relations are expressed compactly as

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = M \begin{bmatrix} dm_1 \\ dm_2 \\ dm_3 \end{bmatrix} , \tag{5}$$

where M is the Jacobian of the transformation, expressed as

$$M = \begin{bmatrix} \frac{\partial x}{\partial m_1} & \frac{\partial x}{\partial m_2} & \frac{\partial x}{\partial m_3} \\ \frac{\partial y}{\partial m_1} & \frac{\partial y}{\partial m_2} & \frac{\partial y}{\partial m_3} \end{bmatrix} . \tag{6}$$

An Expression for the Covariance of (dx,dy)

The covariance of (dx,dy) is a 2x2 matrix from whose elements the standard deviations of dx and dy can be derived (a required next step in our CEP computation). In appendix B it is shown that the covariance of (dx,dy) is expressible in terms of the matrix M and the covariance of the (dm_1, dm_2, dm_3) variables by

$$\mathrm{Cov}(dx,dy) = M \; \mathrm{Cov}(dm1,dm2,dm3) \; M^* , \tag{7}$$

where M* is the transpose of matrix M. Multiplication of the expression in (7) gives

$$\mathrm{Cov}(dx,dy) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} , \tag{8}$$

where

$$a_{11} = \left(\frac{\partial x}{\partial m_1}\right)^2 dm_1 + \left(\frac{\partial x}{\partial m_2}\right)^2 dm_2 + \left(\frac{\partial x}{\partial m_3}\right)^2 dm_3 \quad (9a)$$

$$a_{12} = a_{21} = \left(\frac{\partial x}{\partial m_1}\right)\left(\frac{\partial y}{\partial m_1}\right) dm_1 + \left(\frac{\partial x}{\partial m_2}\right)\left(\frac{\partial y}{\partial m_2}\right) dm_2 + \left(\frac{\partial x}{\partial m_3}\right)\left(\frac{\partial y}{\partial m_3}\right) dm_3 \quad (9b)$$

$$a_{22} = \left(\frac{\partial y}{\partial m_1}\right)^2 dm_1 + \left(\frac{\partial y}{\partial m_2}\right)^2 dm_2 + \left(\frac{\partial y}{\partial m_3}\right)^2 dm_3 \quad . \quad (9c)$$

Extracting Standard Deviations from the Covariance Matrix

The standard deviations (σ_{dx}, σ_{dy}) can be extracted from the covariance matrix (8) using the following considerations.

If an appropriate coordinate system were chosen for the (dx,dy) joint distribution, and (dx,dy) were independent variables (as we are assuming), then the covariance of (dx,dy) would identify with the standard deviations (σ_{dx}, σ_{dy}) by

$$\text{Cov}(dx,dy) = \begin{bmatrix} \sigma_{dx}^2 & 0 \\ 0 & \sigma_{dy}^2 \end{bmatrix} \quad . \quad (10)$$

In general, however, the a_{12} term of the covariance matrix (8) will not be zero and a diagonalization of the matrix (equivalent to a rotation of the axes) is required to convert matrix (8) into the form of equation (10). After rotation, equation (8) will have the form

$$\text{Cov}(dx,dy) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad , \quad (11)$$

where λ_1, λ_2 are eigenvalues of (5). The eigenvalues are given by

$$\lambda_1 = \frac{\text{Trace}}{2} + \frac{\sqrt{\text{Trace}^2 - 4\text{Det}}}{2} \tag{12a}$$

$$\lambda_2 = \frac{\text{Trace}}{2} - \frac{\sqrt{\text{Trace}^2 - 4\text{Det}}}{2} \quad , \tag{12b}$$

where:

$$\text{Trace} = a_{11} + a_{22} \text{ and} \qquad \text{Det} = a_{11}a_{22} - a_{12}^2 \quad . \tag{12c}$$

Since eigenvalues of a matrix are invariant under rotation, the eigenvalues in (12) must equal the variances, namely,

$$(\lambda_1, \lambda_2) = (\sigma_{dx}^2, \sigma_{dy}^2) \quad .$$

As a caveat, note that when the eigenvalues are computed from (12), it is not clear (without further analysis) which eigenvalue associates with σ_{dx}^2 and which associates with σ_{dy}^2. However, this knowledge is not required for computation of the CEP. Only the ratio of smaller to larger eigenvalues is needed (see CEP formula (3)).

Significance of the Trace of the Covariance Matrix

Under certain conditions, the CEP can be shown to be proportional to the square root of the Trace. The behavior of the Trace thus provides both a qualitative and quantitative description of CEP behavior.

The necessary condition involves the radicand in (12) that is used to obtain the eigenvalues λ_1 and λ_2. For simplicity, we rewrite (12) in the form

$$\lambda_i = \frac{\text{Trace}\left[\,1 \pm \sqrt{1 - 4\text{Det}/\text{Trace}^2}\,\right]}{2}, \quad (i=1,2) \quad . \tag{13}$$

While the positiveness of the covariance matrix always guarantees that the radicand is positive, i.e.,

$$0 < 4\text{Det}/\text{Trace}^2 < 1 \quad ,$$

we consider the cases for which the Det is significantly smaller than the square of the Trace, i.e., the condition

$$0 < 4\text{Det}/\text{Trace}^2 \ll 1 \quad . \tag{14}$$

For this condition, we can neglect the second term in the radicand of (13) and replace it by zero as a first approximation. The two eigenvalues (equivalent to the two variances σ_{dx}^2 and σ_{dy}^2) take on the value λ_1=Trace and λ_2=0. Thus, the larger standard deviation becomes $\sigma_L=\sqrt{\text{Trace}}$ and the smaller is $\sigma_S=0$. Substituting these values into the CEP formula (3), we obtain w=0 and

$$\text{CEP} = 0.67\,\sqrt{\text{Trace}} \quad . \tag{15}$$

Therefore, under condition (14) the square root of the Trace can be used as a "quick look" estimate of the CEP. Some examples are discussed in section 5.

SECTION 3

A SUMMARY OF THE COMPUTATIONAL STEPS INVOLVED IN A CEP DETERMINATION OF TARGET ACCURACY

This section provides a working guide for computation of CEP. The reader is reminded that underlying assumptions must be met to insure both validity and accuracy of the CEP estimate. Chief among these are the following:

- The measurements used in the computation of target location have errors that can be treated as Gaussian random variables with zero mean and known standard deviation.
- The measurement errors are statistically independent.
- The standard deviation for each of the measurement errors is sufficiently small so that the second order terms in the Taylor expansion of the measurement error can be neglected.

Under these conditions the computational steps for obtaining CEP are summarized below.

Steps in the Computation of CEP

Step 1. Establish an analytic relationship between (x,y) and the measurement variables.

Step 2. Obtain partial derivatives of (x,y) with respect to the independent measurement variables being used in the (x,y) computation.

Step 3. Substitute the partial derivatives of (x,y) into equation (9) to obtain the matrix components a_{11}, a_{22}, a_{12}.

Step 4. Substitute the matrix components into formula (12c) to calculate the Trace and Determinant (Det). If relationship $4Det \ll Trace^2$ holds, then the product $0.67*\sqrt{Trace}$ provides an excellent estimate of the CEP. Alternatively, continue on and compute the eigenvalues λ_1 and λ_2 from formulas (12a) and (12b).

Step 5. Obtain the standard deviations σ_{dx} and σ_{dy} by taking the square root of λ_1 and λ_2.

Step 6. Substitute the values of σ_{dx} and σ_{dy} into formula (3) to obtain the CEP.

SECTION 4
BISTATIC TARGET LOCATION

In bistatic detection, two platforms are used to acquire information about a target T. The platforms may be either moving or stationary. Either one (or both) may be illuminating the target. In mathematical analysis, the two platforms and target are geometrically portrayed as a triangle on which simple equations can be used to describe distances and angles.

GEOMETRY

Figure 3 describes a planar view of the geometry used in bistatic calculations. Platform A is located at the origin (0,0). Platform B is located at $(R_i,0)$. The target T is located at (x,y). All these coordinates are measured relative to the local origin A.

In practice, the line directions and locations of the A,B,T triangle are also measured relative to an external reference frame. For example, if platform B was functioning as an information Collector, it would be natural to collect all directional information relative to its own heading. In this paper we assume that information is measured relative to some reference direction on platform B, and that this direction lies in the same plane as the triangle formed by A, B, and T.

Figure 3 displays the angles measured relative to platform B. ϕ_t defines the bearing angle (measured counterclockwise) from platform B direction to the target; ϕ_i defines the angle (measured counterclockwise) that the line connecting the two platforms makes with the platform B direction. Similarly, $\theta - \phi_i$ defines the bearing angle

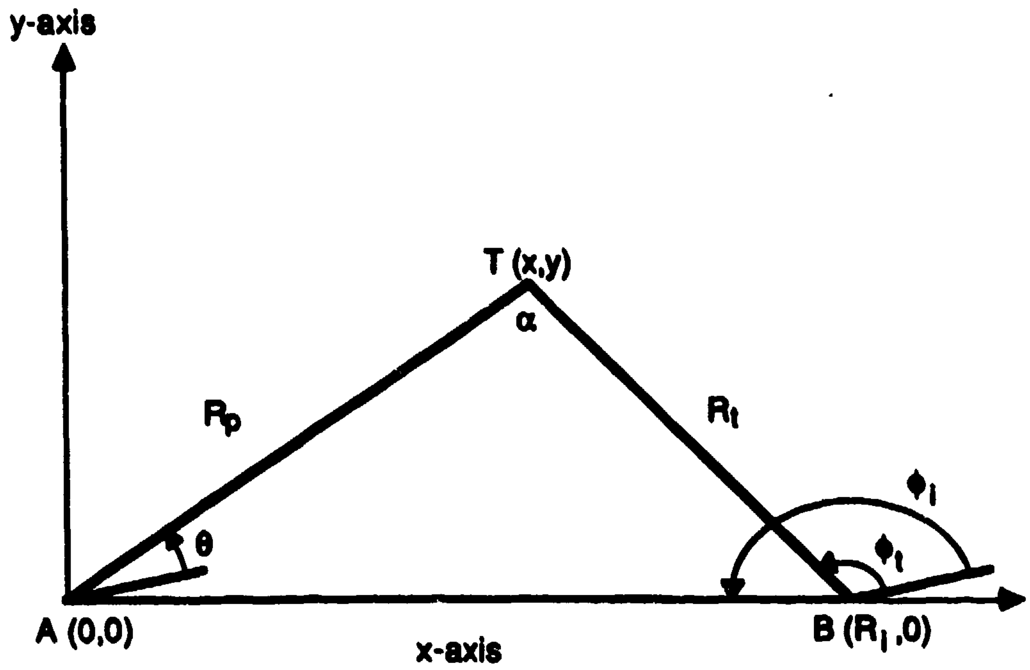

Figure 3. Bistatic Geometry

from A to the target T as seen from the platform B perspective. Θ is measured clockwise from the ϕ_i bearing line. From the geometry, the base angles of the triangle with vertices A,B,T are given by $\Pi + \Theta - \phi_i$ and $\phi_i - \phi_t$.

TYPICAL PLATFORM MEASUREMENTS

Various combinations of information can be used to acquire an estimate of target location. Some of the common measurements acquired are listed below.

R_i The measured distance between platform B and platform A. R_i has an associated measurement error, dR_i, whose average value is zero and whose standard deviation, σ_{dR_i}, is known.

ϕ_i The measured angle, relative to a platform B frame of reference, which defines the direction of the line from platform A to platform B. ϕ_i has an associated measurement error, $d\phi_i$, whose average value is zero and whose standard deviation, $\sigma_{d\phi_i}$, is known.

ϕ_t The measured angle, relative to a platform B frame of reference, which defines the bearing of the line from platform B to the target. ϕ_t has an associated measurement error, $d\phi_t$, whose average value is zero and whose standard deviation, $\sigma_{d\phi_t}$, is known.

θ The measured angle, relative to a platform B frame of reference, which defines the bearing of the line from platform A to the target. θ has an associated measurement error, $d\theta$, whose average value is zero and whose standard deviation, $\sigma_{d\theta}$, is known.

R_p The measured distance from platform A to the target. R_p has an associated measurement error, dR_p, whose average value is zero and whose standard deviation, σ_{dR_p}, is known.

R_t The measured distance from platform B to the target. R_t has an associated measurement error, dR_t, whose average value is zero and whose standard deviation, σ_{dR_t}, is known.

ΔT The Time-Difference-of-Arrival (TDOA) of a signal received by platform B from two different paths. If, for example, platform A were illuminating a target (e.g., in the case of a radar), or interrogating a target (e.g., in the case of a beacon system), the signal would be received directly by platform B (along the line connecting A and B) in time T_1, and received again at a later time T_2 (along the lines connecting A to T and T to B) from reflection off or response by the target. The time difference would be $\Delta T = T_2 - T_1$. If the signal is a response from the target, the time T_2 includes any fixed and/or random time delays introduced by the target transponder. Note that in the case of beacon systems, the "target" could be the Interrogator and A the transponder. ΔT has an associated measurement error, $d\Delta T$, whose average value is zero and whose standard deviation, $\sigma_{d\Delta T}$, is known.

SECTION 5
EXAMPLES

Assuming that the angle ϕ_i and the distance R_i between the platforms A and B have already been obtained (i.e., the base of the triangle with vertices A,B,T has been determined in the geometry of figure 3), then two additional measurements relating to target are required to compute the target location. We describe here the common cases where two angle bearings, or one angle bearing with TDOA measurement, are available.

CASE 1. TWO BEARING ANGLES TO TARGET ARE MEASURED

Bearing angles Θ and ϕ_t are acquired with known standard deviations $\sigma_{d\theta}$ and $\sigma_{d\phi_t}$, respectively. Geometrically, target location (x,y) will be the intersection of the relative bearing lines defined by $\Pi + \Theta - \phi_i$ and $\phi_i - \phi_t$ (see figure 4). Analytically, (x,y) is obtained by solving the simultaneous equations

$$\mathrm{Tan}(\Theta - \phi_i) = y/x \quad \text{and} \quad \mathrm{Tan}(\phi_t - \phi_i) = -y/(R_i - x) \ .$$

The partial derivatives of (x,y) with respect to the measurement variables are given in appendix C. The Trace and Det are obtained by substituting these derivatives into equation (13). Evaluation gives

$$\mathrm{Trace} = \frac{R_p{}^2 \sigma_{d\theta}{}^2 + R_t{}^2 \sigma_{d\phi_t}{}^2 + R_i{}^2 \sigma_{d\phi_i}{}^2 + \sin^2(\phi - \phi_i) \sigma_{dR_i}{}^2}{\sin^2(\alpha)}$$

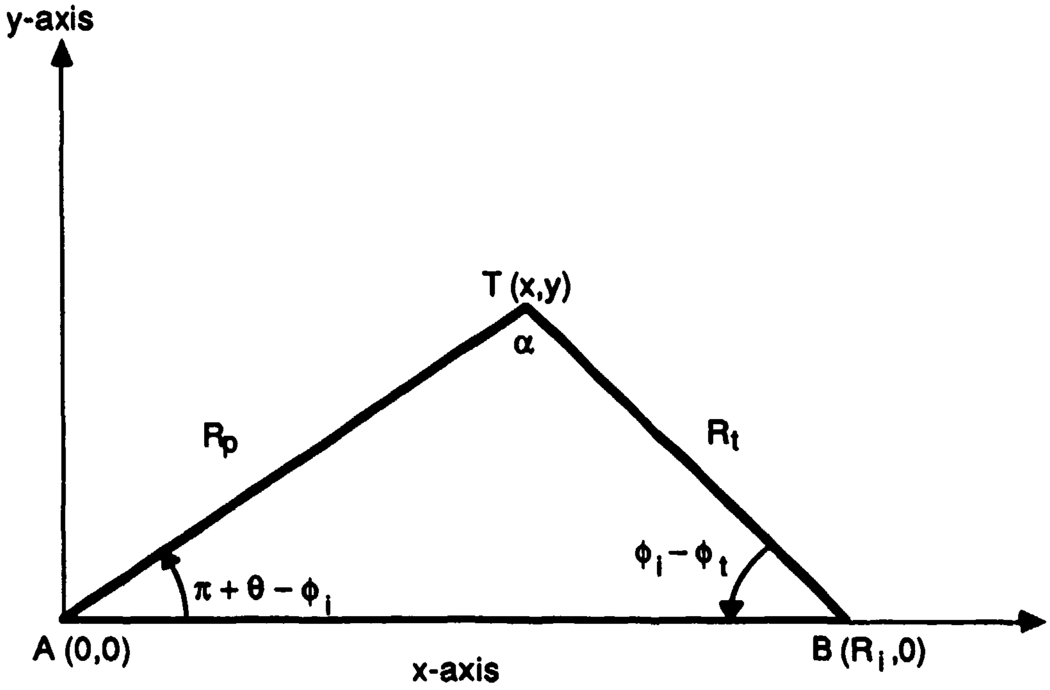

Figure 4. Case 1 Geometry: Target Location by the Intersection of Two Bearing Angles

and

$$\text{Det} = \frac{R_p^2 R_t^2(\sigma_{d\theta}^2 \sigma_{d\phi_t}^2 + \sigma_{d\theta}^2 \sigma_{d\phi_i}^2) + R_p^2 \sin^2(\phi_t - \phi_i)(\sigma_{d\theta}^2 + \sigma_{d\phi_i}^2)\sigma_{R_i}^2}{\sin^2(\alpha)}$$

where $\alpha = \phi_t - \theta$.

Figure 5 is a plot of CEP contours for the Case 1 geometry with $\phi_i = 0$ deg, $R_i = 150$ nmi and with standard deviations $\sigma_{d\theta} = 5$ deg, $\sigma_{d\phi_t} = 5$ deg, $\sigma_{d\phi_i} = 0$ deg, $\sigma_{dR_i} = 0$ deg. Platform locations A and B are on the axes. The graph is to be interpreted as follows: Select a point (x,y). Linearly extrapolate (as a first approximation) between curves to estimate the curve passing through (x,y). The contour number associated with that curve corresponds to the CEP for point (x,y).

As discussed earlier, the square root of the Trace may be used to qualitatively describe the CEP. For the geometry of figure 4, division by zero occurs in the Trace under three sets of conditions: 1) $\phi_t = 0$, $\theta = 0$, 2) $\phi_t = 180$, $\theta = 180$, and 3) $\phi_t = 90$, $\theta = 90$. Conditions 1 and 2 occur along the horizontal axis of figure 4. This axis, and a small band around it, is commonly called the "black hole" region. Along this line, the CEP is infinite. Targets near the horizontal axis are located very poorly using this measurement set. Condition 3 corresponds to targets being extremely far from platforms A and B. Such targets will also have extremely high CEP.

In general, the triangulation process of Case 1 for obtaining target location applies to any two bearing measurement systems (e.g., two ground radars, a ground radar and aircraft radar, etc.). An example of Case 1 application is the cooperative triangulation of two E-3 aircraft.

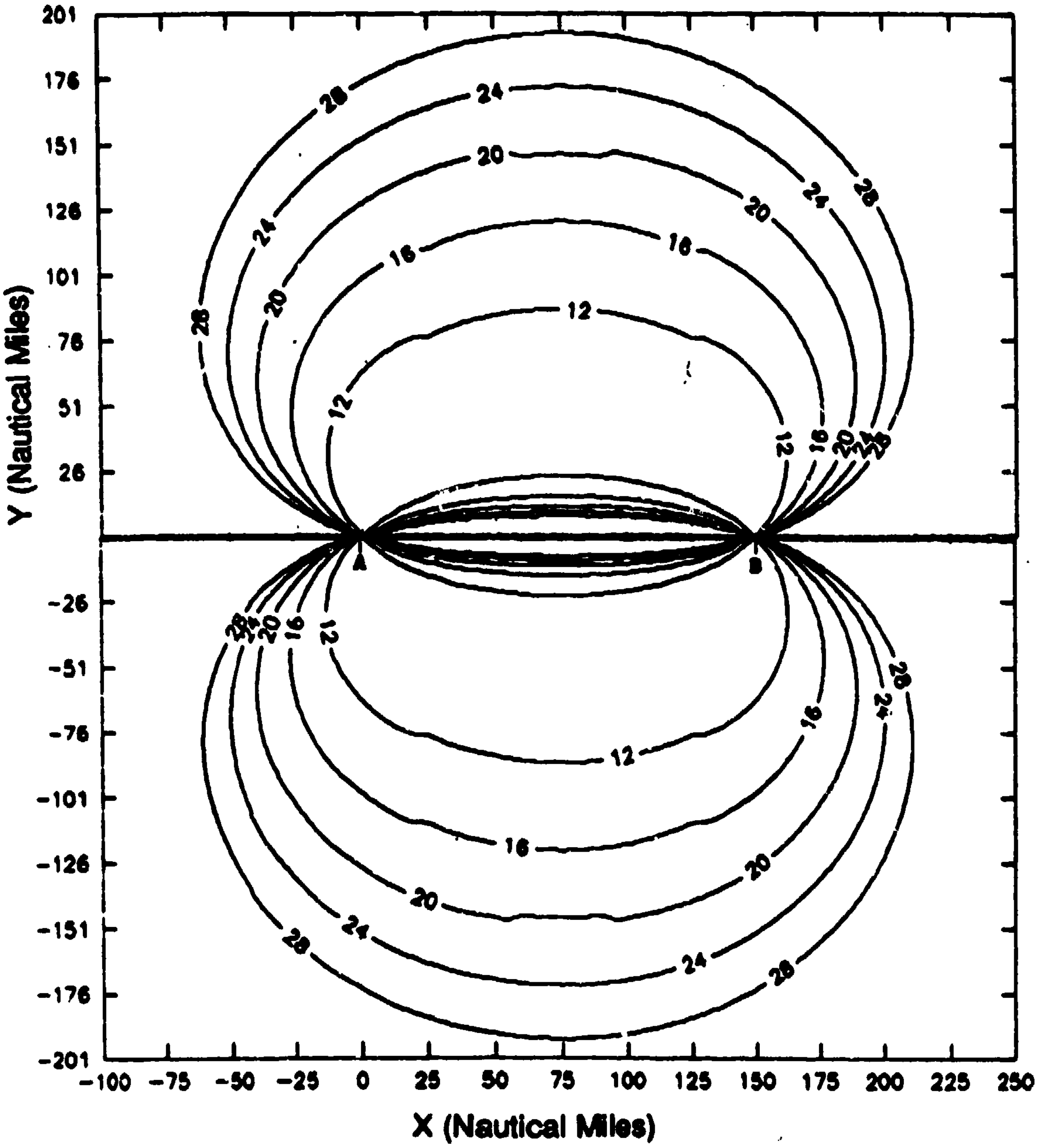

Figure 5. Case 1: CEP Contours for Two Bearing Angles

$\sigma_{d\theta}$ = 5 deg, $\sigma_{d\phi_t}$ = 5 deg, $\sigma_{d\phi_i}$ = 0 deg, R_i = 150 nmi

CASE 2. ONE BEARING ANGLE TO TARGET AND TDOA IS MEASURED FROM ILLUMINATION OF PLATFORM A BY THE TARGET (HYPERBOLA)

The geometry for this case is described in figures 6a and 6b. Bearing angle θ (or ϕ_t) is acquired with known standard deviation $\sigma_{d\theta}$ (or $\sigma_{d\phi_t}$). The time difference of arrival, ΔT, of the target signal from two different directions is also acquired at point B. The two directions traversed by the acquired signals are as follows: Direction 1 is the direct distance R_t from the target to point B. Direction 2 is the distance traveled by the signal (e.g., radar or beacon) from the target to the platform A plus the distance from the platform A to platform B (i.e., $R_p + R_i$). Time along direction 1 is received at T_1. Time along direction 2 is received at T_2. The time difference is $\Delta T = T_2 - T_1$. If the signal is a response from the target, the time T_2 includes any fixed and/or random time delays introduced by the target transponder. ΔT has an associated measurement error, $d\Delta T$, whose average value is zero and whose standard deviation, $\sigma_{d\Delta T}$, is known.

One can show from analytic geometry that the target location will lie on a hyperbola defined by ΔT and R_i (see figure 6b). The hyperbola has foci at $(-C_o\Delta T/2,0)$ and $(R_i+C_o\Delta T/2,0)$, and eccentricity equal to $R_i/C_o\Delta T$, where C_o is the speed of light. Geometrically, target location will be the intersection of the relative bearing line $\Pi + \theta - \phi_i$ (or $\phi_i - \phi_t$) with the hyperbola. Analytically, target location (x,y) is obtained by solving the simultaneous equations

$$\mathrm{Tan}(\theta-\phi_i) = y/x \qquad (\text{ or } \mathrm{Tan}(\phi_t-\phi_i) = -y/(R_i-x)\ )$$

and

$$R_p + R_i - R_t = C_o\Delta T.$$

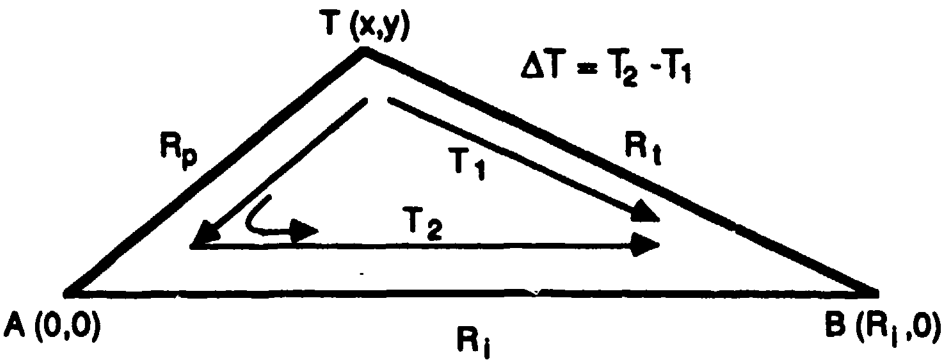

Figure 6a. Case 2: Signal Direction of the Time Measurements

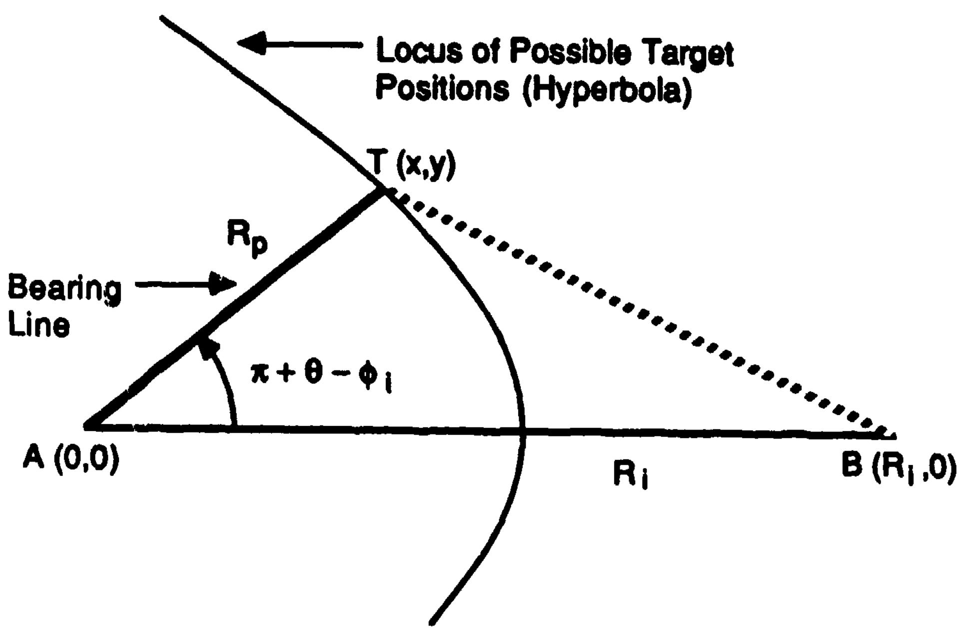

Figure 6b. Case 2: Target Location by the Intersection of a Bearing line with a Hyperbola

The partial derivatives of (x,y) with respect to the measurement variables are given in appendix C. The Trace and Det are obtained by substituting these derivatives into equation (13). Evaluation gives

$$\text{Trace} = \frac{2R_p^{\,2}(\sigma_{d\theta}^{\,2}+\sigma_{d\phi_i}^{\,2})(1-\cos\alpha) + C_o^{\,2}\sigma_{d\Delta T}^{\,2} + [1+\cos(\phi_t - \phi_i)]^2\sigma_{dR_i}^{\,2}}{(1-\cos\alpha)^2}$$

and

$$\text{Det} = \frac{R_p^2(\sigma_{d\theta}^2 + \sigma_{d\phi_i}^2)[C_o^{\,2}\sigma_{\Delta T}^2 + \{1+\cos(\phi_t - \phi_i)\}^2 \; \sigma_{R_i}^2]}{(1-\cos\alpha)^2} ,$$

where $\alpha = \phi_t - \theta$.

Figure 7 is a plot of CEP contours for the Case 2 geometry with [illegible] = 0 deg, R_i = 150 nmi and with standard deviations $\sigma_{d\theta}$ = 5 deg, $\sigma_{d\phi_i}$ = 0 deg, $\sigma_{d\Delta T} = 10^{-6}$ sec. Platform locations A and B are marked on the axes. Similar to Case 1, the graph is to be interpreted as follows: Select a point (x,y). Linearly extrapolate (as a first approximation) between curves to estimate the curve passing through (x,y). The contour number associated with that curve corresponds to the CEP for point (x,y).

As discussed earlier, the square root of the Trace may be used to qualitatively describe the CEP. In the above equation, division by zero occurs when $\cos(\alpha) = 1$. This condition occurs along the horizontal axis of figure 5 to the left of A (when $\theta \doteq 180$, $\phi_t = 180$), and to the right of B (when $\theta = 0$, $\phi_t = 0$). For this geometry, these external intervals on the horizontal axis (and a small band around them) are called "black hole" regions. Along these intervals, the CEP is infinite. Targets near these intervals have extremely poor

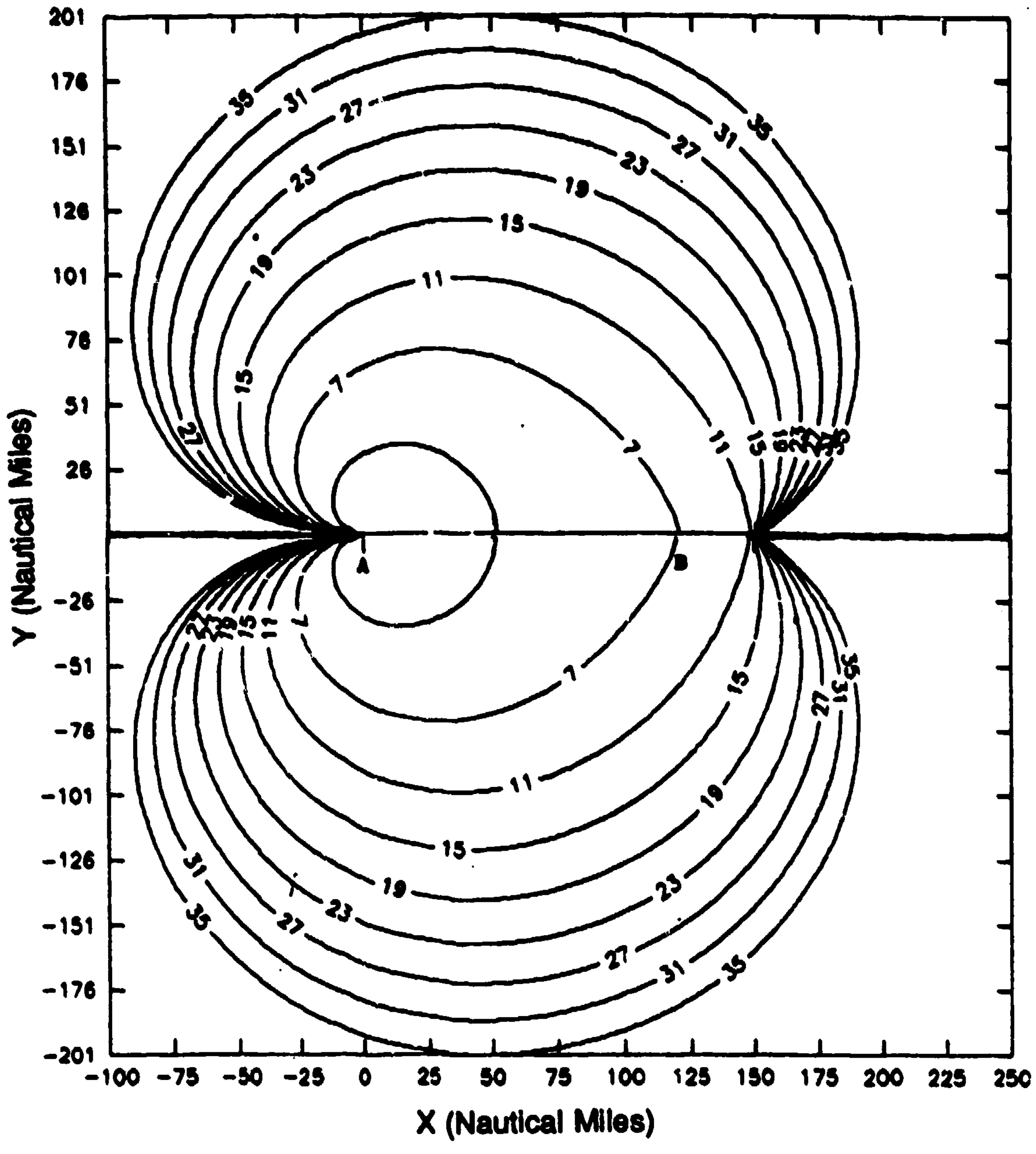

Figure 7. Case 2: CEP Contours for One Bearing Angle and a TDOA (Hyperbola)

$\sigma_{d\theta}$ = 5 deg, $\sigma_{d\Delta T}$ = 10^{-6} sec, $\sigma_{d\psi_i}$ = 0 deg, R_i = 150 nmi location measurement. In contrast, targets between A and B along the horizontal axis are able to be located with good accuracy.

Tactical Air Navigation System (TACAN) is an example of Case 2 application. Here the airborne target interrogates the ground station (i.e., the ground station at location A is the transponder).

CASE 3. ONE BEARING ANGLE TO TARGET AND TDOA IS MEASURED FROM ILLUMINATION OF THE TARGET BY PLATFORM A (ELLIPSE).

The geometry for this case is described in figures 8a and 8b. Bearing angle Θ (or ϕ_t) is acquired with known standard deviation $\sigma_{d\Theta}$ (or $\sigma_{d\phi_t}$). The time difference of arrival, ΔT, of the target signal from two different directions is also acquired at point B with standard deviation $\sigma_{d\Delta T}$. However, as opposed to Case 2, the directions traversed by the acquired signals are different. Direction 1 is the distance traveled by the signal from the platform A to the target plus the distance from the target to point B (i.e., $R_p + R_t$). Direction 2 is the direct distance R_i from the platform A to platform B.

The signal along direction 1 is received at T_1. The signal along direction 2 is received at T_2. The time difference is $\Delta T = T_2 - T_1$. If the signal is a response from the target, the time T_2 includes any fixed and/or random time delays introduced by the target transponder. ΔT has an associated measurement error, $d\Delta T$, whose average value is zero and whose standard deviation, $\sigma_{d\Delta T}$, is known.

For this case one can show from analytic geometry that the target location will lie on an ellipse defined by ΔT and R_i (see figure 8b). The ellipse has foci at $(R_i - C_o \Delta T/2, 0)$ and $(C_o \Delta T/2, 0)$, and with eccentricity equal to $-1 + C_o \Delta T/R_i$, where C_o is the speed of

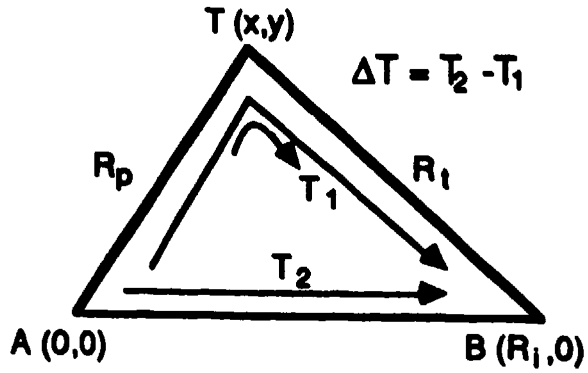

Figure 8a. Case 3: Signal Direction of the Time Measurements

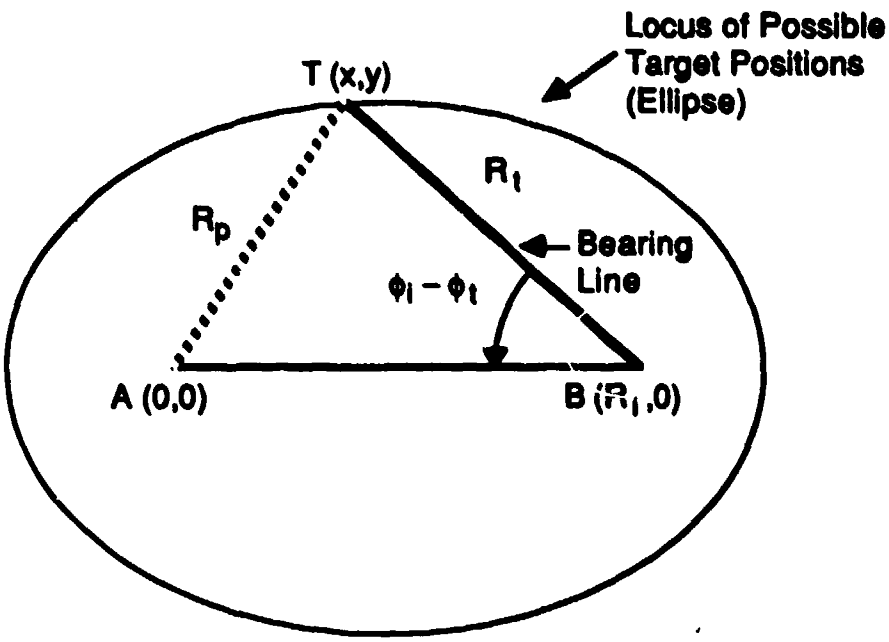

Figure 8b. Case 3: Target Location by the Intersection of a Bearing Line with an Ellipse

light. Geometrically, target location will be the intersection of the relative bearing line $\Pi + \Theta - \phi_i$ (or $\phi_i - \phi_t$) with the ellipse. Analytically, target location (x,y) is obtained by solving the simultaneous equations

$$\mathrm{Tan}(\Theta - \phi_i) = y/x \qquad (\text{ or } \mathrm{Tan}(\phi_t - \phi_i) = -y/(R_i - x)\ )$$

and

$$R_p + R_t - R_i = C_o \Delta T \quad .$$

The partial derivatives of (x,y) with respect to the measurement variables are given in appendix C. The Trace and Det are obtained by substituting these derivatives into equation (13). Evaluation gives

$$\mathrm{Trace} = \frac{2R_p^{\,2}(\sigma_{d\Theta}^{\,2} + \sigma_{d\phi_i}^{\,2})(1+\cos\alpha) + C_o^{\,2}\sigma_{d\Delta T}^{\,2} + [1+\cos(\phi_t - \phi_i)]^2 \sigma_{dR_i}^{\,2}}{(1+\cos\alpha)^2}$$

and

$$\mathrm{Det} = \frac{R_p^2(\sigma_{d\Theta}^2 + \sigma_{d\phi_i}^2)[C_o\sigma_{d\Delta T}^2 + \{1+\cos(\phi_t - \phi_i)\}^2 \sigma_{dR_i}^2]}{(1+\cos\alpha)^2} \quad ,$$

where $\alpha = \phi_t - \Theta$.

Figure 9 is a plot of CEP contours for the Case 3 geometry with ϕ_i = 0 deg, R_i = 150 nmi and with standard deviations $\sigma_{d\Theta}$ = 5 deg, $\sigma_{d\phi_i}$ = 0 deg, $\sigma_{d\Delta T}$ = 10^{-6} sec. Platform locations A and B are marked on the axes. Similar to Case 1, the graph is to be interpreted as follows: Select a point (x,y). Linearly extrapolate (as a first approximation) between curves to estimate the curve passing through (x,y). The contour number associated with that curve corresponds to the CEP for point (x,y).

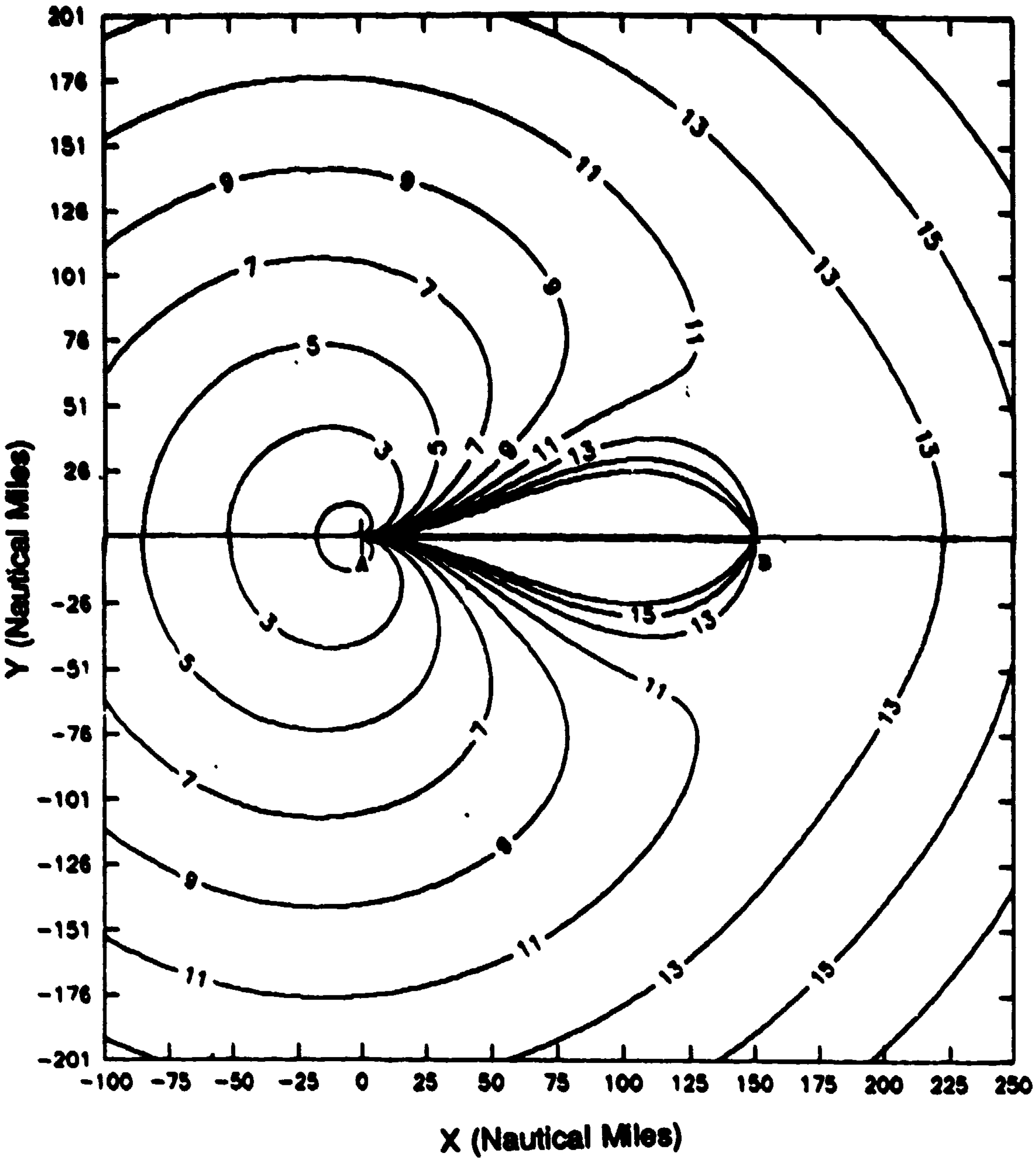

Figure 9. Case 3: CEP Contours for One Bearing Angle and a TDOA (Ellipse)

$\sigma_{d\theta}$ = 5 deg, $\sigma_{d\Delta T}$ = 10^{-6} sec, $\sigma_{d\phi_i}$ = 0 deg, R_i = 150 nmi

As discussed earlier, the square root of the Trace may be used to qualitatively describe the CEP. In the above equation, division by zero occurs when $\cos(\alpha) = -1$. This condition occurs along the horizontal axis of figure 8 on the interval between A and B (when $\theta = 0$, $\phi_t = 180$, or $\theta = 180$, $\phi_t = 0$). For Case 3 geometry, this interior interval on the horizontal axis, (and a small band around it) is called a "black hole" region. Along this interval the CEP is infinite. Targets near this interval have extremely poor location measurement. In contrast, targets outside of A and B along the horizontal axis are able to be located with good accuracy.

An example of Case 3 application is identification, friend or foe (IFF), interrogation of targets from a (location A) ground or airborne station.

SECTION 6
CONCLUSIONS

This paper describes an algorithm for calculating Circular Error Probability (CEP). CEP relates measurement errors to the variation of a calculation that is computed from those measurements. As such, it is a useful tool for assessing the accuracy and sensitivity of any calculated quantity to errored inputs. CEP is frequently used with target location estimates.

A mathematical foundation is provided for the CEP estimate, and the algorithm is shown to be extremely accurate. It differs from the true CEP estimate by less than one percent on average and has a maximum error of 1.5 percent.

Three common cases of CEP application are discussed. All deal with assessing the accuracy of a target location in a bistatic environment. Examples of CEP contours are generated for the special conditions R_i = 150 nmi, ϕ_i = 0 deg, σ_{dR_i} = 0 nm, $\sigma_{d\phi_i}$ = 0 deg. Regions of poor target detectability are indicated.

BIBLIOGRAPHY

Abramovitz, M., and F. A. Stegun, ed., Handbook of Mathematical Functions, National Bureau of Standards, Applied Math Series 55, Washington, D.C.: U.S. Government Printing Office, 1968.

Edwards, C. H., Jr., Advanced Calculus of Several Variables, New York, NY: Academic Press, 1973.

Skolnik, M. I., Introduction to Radar Systems, New York, NY: McGraw Hill Co., 1962.

Wegner, L. H., "On the Accuracy Analysis of Airborne Techniques for Passively Locating Electromagnetic Emitters," USAF Project Rand Report (R-722-PR, June 1971), Santa Monica, CA: Rand Corporation. [Note: Formulas for maximum and minimum variances (equations (26),(27), page 22 of this reference) are in error.]

APPENDIX A

DERIVATION OF THE SIMPLIFIED CEP FORMULA

CEP is defined as that value of R that satisfies the integral equation

$$\iint_{u^2+v^2 \leq R^2} p(u,v)\,du\,dv = 0.5 \qquad \text{(A1)}$$

where p(u,v) is the joint Gaussian distribution of variables (u,v) with mean (0,0) and variances $(\sigma_u{}^2, \sigma_v{}^2)$,

$$p(u,v) = \frac{1}{2\pi\sigma_u\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{u}{\sigma_u}\right)^2 - \frac{1}{2}\left(\frac{v}{\sigma_v}\right)^2\right] . \qquad \text{(A2)}$$

We make the following change of variables:

Let

$$\sigma_L = \text{Maximum}[\sigma_u, \sigma_v] ,$$
$$\sigma_s = \text{Minimum}[\sigma_u, \sigma_v] , \qquad \text{(A3)}$$
$$w = \sigma_s/\sigma_L, \quad 0 < w \leq 1 ,$$

and define

$$s = u/\sigma_L , \qquad \text{(A4)}$$
$$t = v/\sigma_L .$$

If, for the sake of discussion, we suppose that $\sigma_L = \sigma_u$ (the same argument will hold for $\sigma_L = \sigma_v$), then integral (A1) can be reexpressed as

$$F\left[\frac{CEP^2}{\sigma_L},w\right] = \frac{1}{2\pi w}\iint\limits_{s^2+t^2\leq\left[\frac{CEP}{\sigma_L}\right]^2} \exp\left[-\frac{s^2}{2}-\frac{t^2}{2w^2}\right]ds\,dt \tag{A5}$$

Symbolically, we write (A1) in the form

$$F\left[\frac{CEP^2}{\sigma_L{}^2},w\right] = 0.5 \tag{A6}$$

to emphasize the fact that integration of (A5) results in an implicit relation between $(CEP/\sigma_L)^2$ and w.

In general, it is not always possible to obtain an unambiguous solution from an implicit relation. (Remember that our object is to solve (A6) for CEP). However, if it can be shown that $(CEP^2/\sigma_L{}^2)$ is an increasing function of w, then the Inverse Mapping Theorem of Calculus argues that we may uniquely solve equation (A6) for $(CEP^2/\sigma_L{}^2)$ as a function of w.

We show that $(CEP^2/\sigma_L{}^2)$ is an increasing function of w by proving that its derivative with respect to w is positive. Differentiating (A6) we obtain

$$\frac{\partial F}{\partial w} + \frac{\partial F}{\partial(CEP^2/\sigma_L^2)}\cdot\frac{d(CEP^2/\sigma_L^2)}{dw} = 0\ . \tag{A7}$$

Solving for the derivative in (A7) gives

$$\frac{d(CEP^2/\sigma_L^2)}{dw} = -\frac{\partial F/\partial w}{\partial F/\partial(CEP^2/\sigma_L^2)}\ . \tag{A8}$$

The positiveness or negativeness of the partial derivatives on the right-hand side of (A8) can be obtained by the following arguments:

1. As (CEP^2/σ_L^2) increases, the circle area over which integration takes place increases. Therefore, the value of the integral F will also increase for each fixed w. We conclude that

$$\frac{\partial F}{\partial(CEP^2/\sigma_L^2)} > 0 \quad .$$

2. As w increases, the limits of integration remain constant but the integrand decreases. Therefore, the value of the integral (i.e., F) will decrease for each fixed (CEP^2/σ_L^2).

We conclude that

$$\frac{\partial F}{\partial w} < 0 \quad .$$

The above inequalities guarantee that the derivative in (A8) is positive. Thus (CEP^2/σ_L^2) is a monotonically increasing function of w.

The Inverse Mapping Theorem of Calculus can now be applied. It states that we can (in theory) uniquely solve equation (A6) for $(CEP^2/\sigma_L{}^2)$ as a function of w. Thus for some function h(w) we will have

$$\left[\frac{CEP}{\sigma_L}\right]^2 = h(w) \qquad \text{and} \qquad CEP = \sigma_L\sqrt{h(w)} \quad . \qquad (A9)$$

Equation (A9) gives an explicit representation for the form of the CEP solution as a function of w.

The function $\sqrt{h(w)}$ can be tabulated numerically from equation (A1) by selecting a value for w and then solving for the value of CEP/σ_L which will make the integral equal 0.5. Figure 5 shows the plot of the tabulated point pairs $(w, CEP/\sigma_L)$. One notes that on the interval $0 < w < 0.5$ the curve follows a quadratic pattern, while on the interval $0.5 \leq w \leq 1$ the curve is essentially linear.

The simplified approximation formula for CEP is nothing more than a reflection of this observation. It involves a quadratic fit to the curve for $\sqrt{h(w)}$ on the interval $0 < w < 0.5$ and a linear fit $h(w)$ on the interval $0.5 \leq w \leq 1$, namely,

$$\sqrt{h(w)} = \begin{cases} 0.67 + 0.8w^2 & \text{if } 0 < w < 0.5 \\ 0.59(1 + w) & \text{if } 0.5 \leq w \leq 1 \end{cases} \quad \text{(A10)}$$

Equation (A10) is an excellent approximation to CEP/σ_L. On average, the error to the true curve is less than 1% over the interval $0 < w \leq 1$.

APPENDIX B

THE COVARIANCE OF (dx,dy)

The covariance of a vector variable v with mean 0 is defined by

$$\mathrm{Cov}(v) = E(vv^*) \quad , \tag{B1}$$

where E(·) is the expected value operator and v* means the transpose of v. The operator E is linear and has the property for constants a,b and variables v_1, v_2 that

$$E(av_1 + bv_2) = aE(v_1) + bE(v_2) \quad . \tag{B2}$$

If v is of dimension N, the covariance of v will be an NxN matrix.

Let $v = (dx,dy)^*$ and $q = (dm_1, dm_2 . dm_3)^*$. From equation (5) of section 2 we are given the relation

$$v = Mq \quad , \tag{B3}$$

where M is a matrix of constants.

The following computations now give us the desired relationship:

$$vv^* = (Mq)(Mq)^* = Mqq^*M^*, \quad \text{(from property of transpose)}$$
$$E(vv^*) = E(Mqq^*M^*) = ME(qq^*)M^*, \quad \text{(from B2)}$$
$$\mathrm{Cov}(v) = ME(qq^*)M^* = M\mathrm{Cov}(q)M^*. \quad \text{(from B1, def. of covariance)}$$

APPENDIX C

CEP DERIVATIVE COMPUTATIONS FOR CASES 1, 2, 3

Case 1. $(\phi_i, R_i, \phi_t, \theta)$ are measured with error variances $(\sigma_{d\phi}, \sigma_{dR_i}, \sigma_{d\phi_t}, \sigma_{d\theta})$. Geometrically, target location (x,y) is obtained as the intersection of two bearing lines. From the triangle relationships in the geometry of figure 3, the equations from which target location can be obtained analytically are

$$\mathrm{Tan}(\theta-\phi_i) = y/x \quad \text{and} \quad \mathrm{Tan}(\phi_t-\phi_i) = -y/(R_i-x) \quad .$$

The partial derivatives of x and y are given by

$$\frac{\partial x}{\partial \phi_i} = \frac{R_t\cos(\theta-\phi_i)-R_p\cos(\phi_t-\phi_i)}{\sin(\alpha)} \; , \qquad \frac{\partial x}{\partial R_i} = \frac{\cos(\theta-\phi_i)\sin(\phi_t-\phi_i)}{\sin(\phi_t-\theta)} \; ,$$

$$\frac{\partial x}{\partial \phi_t} = \frac{R_t\cos(\theta-\phi_i)}{\sin(\phi_t-\theta)} \; , \qquad \frac{\partial x}{\partial \theta} = \frac{-R_t\sin(\theta-\phi_i)}{\sin(\phi_t-\theta)} \; ,$$

$$\frac{\partial y}{\partial \phi_i} = \frac{R_t\sin(\theta-\phi_i)-R_p\sin(\phi_t-\phi_i)}{\sin(\alpha)} \; , \qquad \frac{\partial y}{\partial R_i} = \frac{\cos(\theta-\phi_i)\sin(\phi_t-\phi_i)}{\sin(\phi_t-\theta)} \; ,$$

$$\frac{\partial y}{\partial \phi_t} = \frac{R_t\cos(\theta-\phi_i)}{\sin(\phi_t-\theta)} \; , \qquad \frac{\partial y}{\partial \theta} = \frac{-R_t\sin(\theta-\phi_i)}{\sin(\phi_t-\theta)} \; ,$$

where $\alpha = \phi_t - \theta$.

Case 2. $(\theta, \Delta T, R_i, \phi_i, \phi_t)$ are measured with error variances $(\sigma_{d\theta}, \sigma_{d\Delta T}, \sigma_{dR_i}, \sigma_{d\phi_i}, \sigma_{d\phi_t})$. The target location will lie on an hyperbola defined by ΔT and R_i. The hyperbola has foci at $(-C_o \Delta T/2, 0)$ and $(R_i + C_o \Delta T/2, 0)$, and eccentricity equal to $R_i / C_o \Delta T$, where C_o is the speed of light. Geometrically, target location will be the intersection of the relative bearing line $\Pi + \theta - \phi_i$ (or $\phi_i - \phi_t$) with the hyperbola. The equations from which the target location (x,y) can be obtained analytically are

$$\mathrm{Tan}(\theta - \phi_i) = y/x \qquad (\text{ or } \mathrm{Tan}(\phi_t - \phi_i) = -y/(R_i - x))$$

and

$$R_p + R_i - R_t = C_o \Delta T \quad .$$

The partial derivatives of x and y are given by

$$\frac{\partial x}{\partial \phi_i} = -\frac{R_p[\sin(\phi_t - \phi_i) - \sin(\theta - \phi_i)]}{1-\cos(\alpha)} \; , \qquad \frac{\partial x}{\partial R_i} = -\left[\frac{1+\cos(\phi_t - \phi_i)}{1-\cos(\alpha)}\right]\cos(\theta - \phi_i) \; ,$$

$$\frac{\partial x}{\partial \theta} = \frac{R_p[\sin(\phi_t - \phi_i) - \sin(\theta - \phi_i)]}{1-\cos(\alpha)} \; , \qquad \frac{\partial x}{\partial \Delta T} = \frac{C_o \cos(\theta - \phi_i)}{1-\cos(\alpha)} \; ,$$

$$\frac{\partial y}{\partial \phi_i} = -\frac{R_p[\cos(\phi_t - \phi_i) - \cos(\theta - \phi_i)]}{1-\cos(\alpha)} \; , \qquad \frac{\partial y}{\partial R_i} = -\left[\frac{1+\cos(\phi_t - \phi_i)}{1-\cos(\alpha)}\right]\sin(\theta - \phi_i) \; ,$$

$$\frac{\partial y}{\partial \theta} = -\frac{R_p[\cos(\phi_t - \phi_i) - \cos(\theta - \phi_i)]}{1-\cos(\alpha)} \; , \qquad \frac{\partial y}{\partial \Delta T} = \frac{C_o \sin(\theta - \phi_i)}{1-\cos(\alpha)} \; ,$$

where $\alpha = \phi_t - \theta$ and C_o is the speed of light.

Case 3. ($\phi_t, \Delta T, R_i, \phi_i$) are measured with error variances ($\sigma_{d\phi_t}$, $\sigma_{d\Delta T}, \sigma_{dR_i}, \sigma_{d\phi_i}$). The target location will lie on an ellipse defined by ΔT and R_i. The ellipse has foci at $(R_i - C_o \Delta T/2, 0)$ and $(C_o \Delta T/2, 0)$, and with eccentricity equal to $-1 + C_o \Delta T/R_i$, where C_o is the speed of light. Geometrically, target location will be the intersection of the relative bearing line $\Pi + \theta - \phi_i$ (or $\phi_i - \phi_t$) with the ellipse. The equations from which target location (x,y) can be obtained analytically are

$$\mathrm{Tan}(\theta - \phi_i) = y/x \qquad (\text{ or } \mathrm{Tan}(\phi_t - \phi_i) = -y/(R_i - x)\ )$$

and

$$R_p + R_t - R_i = C_o \Delta T \ .$$

The partial derivatives of x and y are given by

$$\frac{\partial x}{\partial \phi_i} = \frac{R_p[\sin(\phi_t - \phi_i) + \sin(\theta - \phi_i)]}{1 + \cos(\alpha)} , \qquad \frac{\partial x}{\partial R_i} = -\left[\frac{1 + \cos(\phi_t - \phi_i)}{1 + \cos(\alpha)}\right] \cos(\theta - \phi_i) ,$$

$$\frac{\partial x}{\partial \theta} = -\frac{R_p[\sin(\phi_t - \phi_i) + \sin(\theta - \phi_i)]}{1 + \cos(\alpha)} , \qquad \frac{\partial y}{\partial \Delta T} = \frac{C_o \cos(\theta - \phi_i)}{1 + \cos(\alpha)} ,$$

$$\frac{\partial y}{\partial \phi_i} = -\frac{R_p[\cos(\phi_t - \phi_i) + \cos(\theta - \phi_i)]}{1 + \cos(\alpha)} , \qquad \frac{\partial y}{\partial R_i} = -\left[\frac{1 + \cos(\phi_t - \phi_i)}{1 + \cos(\alpha)}\right] \sin(\theta - \phi_i) ,$$

$$\frac{\partial y}{\partial \theta} = -\frac{R_p[\cos(\phi_t - \phi_i) + \cos(\theta - \phi_i)]}{1 + \cos(\alpha)} , \qquad \frac{\partial y}{\partial \Delta T} = \frac{C_o \sin(\theta - \phi_i)}{1 + \cos(\alpha)} ,$$

where $\alpha = \phi_t - \theta$ and C_o is the speed of light.

APPENDIX D

TARGET LOCATION WHEN DISTANCE BETWEEN PLATFORMS IS NOT KNOWN

An interesting but more complex case occurs if one assumes that the distance R_i between the platforms is an unknown. In this case, two bearing angle measurements and a TDOA of Case 3 type (ellipse) will provide a solution not only for target location, but also for R_i.

Relationships between the sides and angles of triangle A,B,T in the geometry of figure 2 are given by

$$\frac{R_p}{\sin(\phi_t-\phi_i)} = \frac{R_t}{\sin(\theta-\phi_i)} = \frac{R_i}{\sin(\alpha)} \quad \text{(The Law of Sines)} \quad ,$$

$$y = -(R_i-x)\tan(\phi_t-\phi_i) \quad \text{and} \quad y = x\tan(\theta-\phi_i) \quad ,$$

$$R_p + R_t - R_i = C_o \Delta T \ .$$

The solution of these simultaneous equations for x,y and R_i is

$$R_i = \frac{C_o \Delta T \sin(\alpha)}{\sin(\theta-\phi_i) + \sin(\phi_t-\phi_i) + \sin(\alpha)}$$

$$x = \frac{C_o \Delta T \cos(\theta-\phi_i)\sin(\phi_t-\phi_i)}{\sin(\theta-\phi_i) + \sin(\phi_t-\phi_i) + \sin(\alpha)}$$

$$y = \frac{C_o \Delta T \sin(\theta-\phi_i)\sin(\phi_t-\phi_i)}{\sin(\theta-\phi_i) + \sin(\phi_t-\phi_i) + \sin(\alpha)}$$

where $\alpha = \phi_t - \theta$ and C_o is the speed of light.

The partial derivatives of x, y and R are given by

$$\frac{\partial x}{\partial \phi_t} = \frac{R_t \cos(\theta-\phi_i)[1+\cos(\phi_t-\phi_i)]}{D} ,$$

$$\frac{\partial x}{\partial \theta} = -\frac{R_p[1+\sin(\phi_t-\phi_i)\sin(\theta-\phi_i)+\cos(\phi_t-\phi_i)]}{D} ,$$

$$\frac{\partial x}{\partial \Delta T} = \frac{C_o \cos(\theta-\phi_i)\sin(\phi_t-\phi_i)}{D} ,$$

$$\frac{\partial y}{\partial \phi_t} = -\frac{R_t \sin(\theta-\phi_i)[1+\cos(\phi_t-\phi_i)]}{D} ,$$

$$\frac{\partial y}{\partial \theta} = -\frac{R_p \sin(\phi_t-\phi_i)[1-\cos(\theta-\phi_i)]}{D} ,$$

$$\frac{\partial y}{\partial \Delta T} = \frac{C_o \sin(\theta-\phi_i)\sin(\phi_t-\phi_i)}{D} ,$$

$$\frac{\partial R_i}{\partial \phi_t} = \frac{R_t[\sin(\alpha)\cos(\phi_t-\phi_i)+\sin(\phi_t-\phi_i)+\sin(\theta-\phi_i)]}{D\sin(\phi_t-\phi_i)} ,$$

$$\frac{\partial R_i}{\partial \theta} = -\frac{R_p[\sin(\alpha)\sin(\theta-\phi_i)+\cos(\theta-\phi_i)+\cos(\phi_t-\phi_i)]}{D\cos(\theta-\phi_i)} ,$$

$$\frac{\partial R_i}{\partial \Delta T} = \frac{C_o \sin(\alpha)}{D} ,$$

where $D = \sin(\phi_t-\phi_i) + \sin(\theta-\phi_i) - \sin(\alpha)$,

$\alpha = \phi_t - \theta$ and C_o is the speed of light.