Free Molecule Aerodynamics

Alexander Barrie

December 5, 2004

Abstract

Flow that has a very low density cannot be modeled using a continuum assumption. In this case, each particle must be modeled individually. If the density is sufficiently low, collisions between particles can be ignored. With no collisions, particles have no effect on each other and each can be modeled as an independent contributor to the flow. These contributions can then be summed up to get pressure distribution and lift on a body.

1 Introduction

When the density of a flow is sufficiently low, it may be considered collisionless. The cutoff point is determined by a parameter called the Knudsen Number, Kn. Flows that have a large Kn and are therefore collisionless can be modeled as a summation of contributions from individual particles impacting the surface of the body. Because each particle is treated as an individual entity, there are essentially no assumptions about the flow as a whole. There are several assumptions about the individual particles, however. In order to use the analytic pressure and lift equations derived within this paper, it is assumed that all particles are similar in size, shape, initial velocity, and temperature. This does not have to be the case if an iterative solution is used. The particles are also assumed to have no charge, and be in a uniform environment with no electric or magnetic fields. It is more difficult to reach an analytic solution if these things are included. This model also assumes that all of the particles will be reflected by a body in the flow, not absorbed.[1, 2, 3, 4]

The primary area where the flow density is low enough for a collisionless assumption is in the very high atmosphere and in space. These are also the areas, however, where one must be careful about whether these assumptions can be made. Magnetic and electric fields, as well as particle charges must be carefully measured before a collisionless approach can be used.[1]

2 Determining Flow Type

There are three primary regimes for modeling flows. The most common regime for aerodymacists is continuum flow. This refers to flows that are dense enough that the flow can be treated as a continuous fluid. Particles in this regime are close enough to each other to transmit pressure waves such as sounds and shockwaves. [3, 2, 1]

The opposite extreme is referred to as collisionless flow, or free molecule aerodynamics. In this flow, the molecules are never close enough to interact. They travel independently and can have differing properties from molecule to molecule. Collisionless flow is rarely seen on Earth. The primary interest of this flow is for the upper atmosphere and in space. Satellites in low Earth orbits can use this type of flow to model the atmospheric drag, for example.[2, 1]

The third regime is a transition between the two extremes. This is classified as a flow that is not dense enough to be treated as a continuum, but the particles can interact with one another. Specifically, the molecules run into each other and transfer energy through collisions. This type of flow is also useful in the upper atmosphere such as the areas where space planes may soon be traveling.[2, 1]

2.1 Knudsen Number

The Knudsen Number, Kn, is the parameter used to determine flow types. It is defined as[1]

$$K_n = \frac{\lambda_{mfp}}{L_b} \tag{1}$$

λ_{mfp} is the mean free path of the flow. This is the average distance a particle will travel before it encounters another particle. L_b is a characteristic length of the object in the flow, typically a diameter, or side length. The mean free path can be found by[4]

$$\lambda_{mfp} = \frac{1}{n\sigma} \tag{2}$$

where n is the number density and σ is the collision cross sectional area of the particle type in question.

If Kn is much larger than one then a particle can move a large distance compared to the body in the flow before interacting another particle. This would make it a collisionless flow. If Kn is on the order of 1 then it will not get very far compared to its own size before striking another particle. This makes it a transitional flow. If Kn is much less than 1 then the particles can barely move at all and the flow can be considered a continuum.[1]

2.2 Collisionless Flow Example

Take, for example, a $10m$ spacecraft in the LEO environment. The number density of oxygen plasma particles is about $1 \times 10^{11} m^{-3}$.[1] The cross section is simply[4]

$$\sigma = \pi(2r)^2 = 3.52 \times 10^{-20} m^2 \tag{3}$$

where r is the collision particle radius. The factor of 2 is added since if the center of two particles are within 2 radii of each other, they will collide.[4] The mean free path is then calculated by[4]

$$\lambda_{mfp} = \frac{1}{n\sigma} = 2.8 \times 10^8 m \tag{4}$$

Dividing this by the $10m$ spacecraft length, the Knudsen number turns out to be

$$K_n = \frac{\lambda_{mfp}}{L_b} = 2.8 \times 10^7 \tag{5}$$

which is much larger than 1. Therefore, the flow can be treated as collisionless for aerodynamic evaluations. Of course, in a LEO environment there are several other factors that are not taken into account in a collisionless flow example such as charges of particles, magnetic and electric fields, temperature cycling, etc.[1, 4]

3 Collisionless Flow

3.1 Parameters Used

Several parameters are used in the proceeding calculations. They are explained in this section.

$$\beta = (2RT)^{-\frac{1}{2}} = \left(\frac{m}{2kT}\right)^{\frac{1}{2}} \tag{6}$$

$1/\beta$ is defined as the average velocity that a particle will be moving in an equilibrium gas due to its thermal energy. R is the gas constant, T is temperature, m is mass, and k is the Boltzmann constant.[1]

$$\begin{aligned} u' &= u - \vec{c_0}\cos\theta \\ v' &= v - \vec{c_0}\cos\theta \\ w' &= w \\ \vec{c'} &= \vec{c} - \vec{c_0} \end{aligned} \tag{7}$$

These $'$ terms refer to the thermal velocity components. $\vec{c_0}$ is the overall macroscopic velocity, $\vec{c}$ is the molecular velocity, and u, v, and w are the three directional components. Theta is simply the polar angle coordinate around the body.[2, 1]

$$s = u\beta \tag{8}$$

The speed ratio, s, includes all of the flow parameters. Equations can then be put solely in terms of s and α. It is similar to the Reynold's number in continuum flow.[3]

$$f_0 = \left(\frac{\beta^3}{\pi^{\frac{3}{2}}}\right) \exp(-\beta^2 \vec{c'}^2) \tag{9}$$

Equation 9 is a distribution function that accurately models equilibrium gas flows. It can be derived from the Boltzmann distribution function as shown by Bird.[2]

3.2 Particle Flux

The particle flux to a surface is the number of particles that reach the surface in a given amount of time. This quantity is the essential element in calculating aerodynamic properties. Each particle delivers momentum and energy to the body so the total number of particles, and their individual properties are needed for determining the flow parameters.

The flux of a quantity Q is expressed as[2]

$$N_i \doteq n\overline{Q\vec{c_0}} \tag{10}$$

where n is the number density. If the free stream velocity is aligned with the x-axis then the inward flux is[2]

$$\dot{N}_i = n\overline{Qu} = n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} Qufdudvdw \tag{11}$$

where f is a distribution function. Using f_0 from Equation 9 as the distribution function, and substituting in Equation 7 leads to

$$\dot{N}_i = \frac{n\beta^3}{\pi^{\frac{3}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\vec{c_0}\cos\theta}^{\infty} Q(u' + \vec{c_0}\cos\theta) \exp[-\beta^2(u'^2 + v'^2 + w'^2)]du'dv'dw' \tag{12}$$

This equation can be used to calculate the flux of a parameter Q in a collisionless flow that is aligned with the x axis.

3.3 Reflected Particles

In standard continuum flow, particles will approach a body and increased pressures will force the particles to flow along streamlines around the body.[3] In collisionless flow, however, the particles do not get close enough to exert pressures on each other. The particles will therefore continue until they impact the body. Once they hit the surface they will be reflected back into the flow.[2]

There are two ways in which a particle may be reflected. The first type is called specular reflection. This is where the normal velocity component is reversed and the parallel velocity component is kept constant. It is like light being reflected on a mirror. The second method of reflection is called diffuse reflection. In this case the reflected velocity is independent of the original velocity direction and is instead chosen by a distribution function.[2]

The fraction of specular reflection, ε, is then defined to allow for different reflection models. A value of 1 assumes that all particles are reflected specularly. A value of 0 assumes all particles reflected diffusely. Values in between allow for a mix of both models. Determining ε for a given flow can be very difficult. It is determined experimentally in most cases.[2]

3.4 Pressure Coefficient

The contributions to the pressure from incident and reflected particles must be measured separately. Equation 12 can be integrated with $Q = mu = m(u' + \vec{c_0}\cos\theta)$ to get the incident portion of the pressure.[2]

$$\frac{\beta^2 p_i}{\rho} = \frac{s\cos\theta\exp(-s^2\cos^2\theta) + \sqrt{pi}[1 + \mathrm{erf}(s\cos\theta)] \times (\frac{1}{2} + s^2\cos\theta)}{2\sqrt{\pi}} \tag{13}$$

For the reflected portion that is reemitted specularly, the reflected pressure is simply equal to the incident pressure.[2]

$$p_r = p_i \tag{14}$$

For the diffusely reflected particles, they are absorbed and reemitted with velocity corresponding to the equilibrium distribution function. For that reason, p_r is given by the flux equation for a stationary gas.[2]

$$p_r = \frac{n_r m}{4\beta_r^2} \tag{15}$$

In this case n_r is the number density of a fictitious gas that would represent the flow inside the surface as it effuses across the surface. Given that the flux out must equal the flux in, equation 12 can be used to calculate this number density as[2]

$$n_r = n_\infty\sqrt{\frac{T_\infty}{T_r}}\left(\exp(-s^2\sin^2\alpha) + \sqrt{\pi}s\sin\alpha\,(1 + \mathrm{erf}(s\sin\alpha))\right) \tag{16}$$

The standard C_p equation can then be implemented, with a little bit of rearranging.[3, 2]

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{\frac{p}{p_\infty} - 1}{\frac{1}{2}\gamma(M)^2} = \frac{\frac{p}{p_\infty} - 1}{s^2} \tag{17}$$

Combining the incident and reflected pressure terms, $\frac{p}{p_\infty}$ can be calculated. When the incident and reflected terms are combined, the end result is[2]

$$\begin{aligned} \frac{p}{p_\infty} = (2\beta_\infty^2 \frac{p}{\rho_\infty}) &= [(1+\varepsilon)\pi^{-\frac{1}{2}} \sin\alpha + \frac{1}{2}(1-\varepsilon)(\frac{T_r}{T_\infty})^{\frac{1}{2}}] \\ &\times \exp(-s^2 \sin^2\alpha) + [(1+\varepsilon)(\frac{1}{2} + s^2 \sin^2\alpha) \\ &+ \frac{1}{2}(1-\varepsilon)(\frac{T_r}{T_\infty})^{-\frac{1}{2}} \pi^{\frac{1}{2}} s \sin\alpha][1 + \mathrm{erf}(s \sin\alpha)] \end{aligned} \tag{18}$$

with ε allowing for both diffuse and specular reflections to occur.

It is undesirable to have negative pressure coefficients, so for collisionless flow the pressure coefficient is usually modified to be[2]

$$C_p' = \frac{\frac{p}{p_\infty}}{s^2} \tag{19}$$

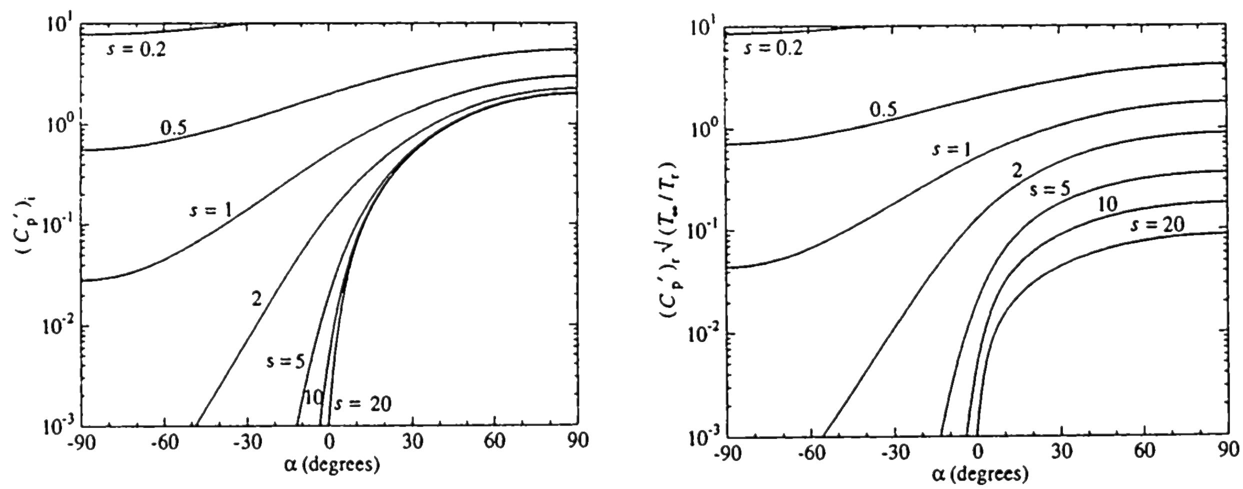

Figure 1: $(C_p')_i and \sqrt{T_\infty/T_r}(C_p)_r'$ vs. angle of attack for a variety of flow conditions[2]

Pressure coefficients for a variety of speed ratios are shown in Figure 1. The incident and reflected values are shown separately to get a better idea of how each contributes.

The reflection in this case is a diffuse reflection. The temperature ration is added as a means of normalization.

3.5 Lift Coefficient

Once the pressure distribution is found, the lift can be calculated. Figure 2 shows the lift coefficient for a flat plate at varying angles of attack and types of reflection. In this case $s = 10$ and $T_r = T_\infty$. The method of reflection plays a large role in the lift generated. The best case is an entirely specular reflection model, which can generate several times the lift generated in the entirely diffuse model.

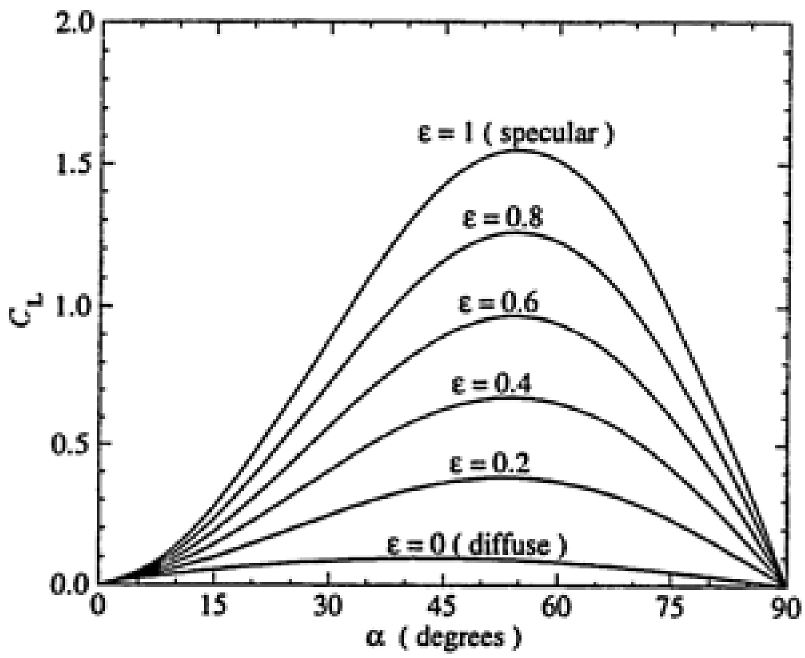

Figure 2: C_L vs. α for a flat plate using a variety of reflection types[2]

4 Conclusion

With a sufficiently high Knudsen Number, flows may be treated as collisionless. In this case, all inter-particle interactions are ignored. This allows for the individual particle effects on a body due to impacts to be summed up and resolved into a pressure coefficient. The pressure may then be evaluated and the lift coefficient can be calculated.

Lift coefficients for a flat plate were calculated. These calculations assumed neutral particles in a neutral, uniform environment with no magnetic or electric fields.

References

[1] Daniel Hastings and Henry Garret. *Spacecraft Environment Interactions.* Cambridge University Press, 2004.

[2] G. Bird. *Molecular Gas Dynamics and the Direct Simulation of Gas Flows.* Clarendon Press, 1994.

[3] K. Karamcheti. *Principles of Ideal Fluid Aerodynamics.* Robert E. Kreiger Publishing Company, 1966.

[4] B. Carroll and D. Ostlie. *An Introduction to Modern Astrophysics.* Addison-Wesley, 1996.