Issues in the Origin and Development of *Hisab al-Khata'ayn* (Calculation by Double False Position)

Questions Sur l'Origine et le Développement de *Hisab al-Khata'ayn* (Calcul par Double Fausse Position)

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Abstract/ Résumé

The method of double false position, a numerical algorithm for evaluating linearly related quantities, was known in Asia prior to the Islamic era and the rise of algebra. It was discussed extensively in ancient Chinese texts (where it was known by such names as *ying bu tsu shu*, "rule of too much and not enough") and later in Arabic works by such practitioners as Abu Kamil, Qusta ibn Luqa, and Ibn al-Banna'. The question how *hisab al-khata'ayn* entered the corpus of mathematical literature in the Arab world, and whether it might have been borrowed from China, from India, or elsewhere, remains controversial. Among the issues bearing on the question are the diverse terminologies, justifications, and applications associated with the method in different traditions. An analysis of these traditions finds no support for the theory that *hisab al-khata'ayn* was borrowed from written sources in other cultures, but does suggest that the technique might have been used by nonscientists well before it appeared in Arab mathematical literature.

La méthode de double fausse position, un algorithme numérique pour l'évaluation des quantités qui sont linéairement reliées, a été connue en Asie avant l'ère islamique et le triomphe de l'algèbre. Elle a été discutée en détail en textes chinois antiques (aux noms tels que *ying bu tsu shu*, "règle d'excès et deficit") et plus tard dans les œuvres arabes par des praticiens tels que Abu Kamil, Qusta ibn Luqa, et Ibn al-Banna'. Il reste une question controversée comment *hisab al-khata'ayn* est arrivé au *corpus* de la littérature mathématique du monde arabe et s'il pourrait avoir été emprunté à la Chine, l'Inde, ou ailleurs. Parmi les questions importantes dans cette discussion est la variabilité dans la terminologie, la justification, et les applications liées à la méthode dans diverses traditions. Une analyse de ces traditions ne trouve aucun soutien de la théorie que *hisab al-khata'ayn* a été emprunté aux sources écrites dans d'autres cultures, mais suggère que la technique pourrait avoir été employée par des praticiens autres que des scientifiques bien avant qu'elle soit apparue en littérature mathématique arabe.

A merchant wishes to purchase 100 handkerchiefs from a producer. Those made of silk cost 5 *dirhams* each, and those of cotton cost 3 *dirhams* each. For a total of 428 *dirhams*, how many of each type can he purchase? The merchant makes two guesses, 60 and 70 silk handkerchiefs, respectively, and he notes how close each guess takes him to the total of 428 *dirhams* available:

$$60(5) + 40(3) = 420 \textit{ dirhams} \text{ (8 too few)}$$
$$70(5) + 30(3) = 440 \textit{ dirhams} \text{ (12 too many).}$$

Since these results bound his target of 428 *dirhams*, the correct number of silk handkerchiefs lies between his guesses of 60 and 70. The answer is a weighted average of these, weighting the first guess more heavily since it came closer to the target:

$$60\left(\frac{12}{12+8}\right)+70\left(\frac{8}{12+8}\right)=\frac{60(12)+70(8)}{12+8}=64 \text{ silk handkerchiefs.}$$

Notice that the guesses and the resulting errors are "cross-multiplied", i.e., each guess is multiplied by the error associated with the other guess. The resulting algorithm can be symbolized as

$$x=\frac{x_1e_2 \pm x_2e_1}{e_2 \pm e_1}, \qquad \text{(Equation 1)}$$

where e_1 and e_2 denote the two errors, and subtraction is used when the errors are of the same type (i.e., both excesses or both deficits).

This method of solving problems by interpolating between, or extrapolating from, two guesses, or suppositions, was used for centuries as a rote arithmetical procedure that could be pulled from the mathematical tool-kit whenever needed. In Arabic the algorithm is known as *hisab al-khata'ayn*, "reckoning from two falsehoods". Leonardo Fibonacci of Pisa rendered this in Latin as *regulis elchatayn*, although in Europe such terms as *regula falsi positionis*, *regula duorum falsorum* and "rule of double false position" became the most common.

As we can see in the above example where the merchant in effect solved the linear equation $5x + 3(100 - x) = 428$, double false position in its simplest form requires at most a memorization of the calculation steps and some skill in arithmetic, but no algebraic machinery. Yet its domain of application encompasses the full panoply of questions that call for the evaluation of linear relationships, and even of nonlinear ones in cases where an approximation will suffice.

A number of writers have missed this point. They have characterized double false position as nothing more than an antiquated way to solve simple equations of the form $ax + b = c$. "To the student of today, having a good symbolism at his disposal, it seems impossible that the world should ever have been troubled by" such an equation, claimed D. E. Smith (1925: 437). Frank Swetz wrote, "While to us the solution of such problems would be considered trivial, for centuries before the adoption of a manipulative symbolism, their literal presentation posed a high degree of difficulty" (1994: 334). Shen Kangshen and his colleagues commented that if one could transpose *b* to the other side of the equation by subtraction, one could solve the problem by the method of simple false position, which requires only one guess instead of two[2]; but, they wrote, "In antiquity people did not know how to move terms from one side of an equation to the other" (Shen et al. 1999: 351). All of these comments overlook the fact that the practitioners were often working with problems in which such equations were not evident. In some cases, they wouldn't even have known what an "equation" is.

To begin to better appreciate the actual contexts in which double false position was applied, let's look at the *Liber Augmenti et Diminutionis* (*Book of Increase and Decrease*), a medieval Latin text largely devoted to explaining this technique. This book, probably from the 12th Century, is the translation of some now-lost earlier work attributed to one "Abraham" and generally believed to have been written in Arabic or Hebrew.[3]

Chapter 4 of the *Liber Augmenti et Diminutionis* is based on problems involving a man who steals apples from an orchard. The first of these reads:

> A certain man went into an orchard and picked some apples. Now the orchard had three gates each guarded by a watchman. So the man gave the first watchman half of what he picked plus two apples more. He gave the second watchman half of what was left plus two more apples. The third got half of what remained and two apples more. The man was left with one apple. How many did he pick? (Hughes 2001: 124)

Even a "good symbolism" and a "literal presentation" wouldn't allow this problem to be dispatched easily, as Smith and Swetz assumed:

$$[(x - \frac{x}{2} - 2) - \frac{1}{2}(x - \frac{x}{2} - 2) - 2] - \frac{1}{2}[(x - \frac{x}{2} - 2) - \frac{1}{2}(x - \frac{x}{2} - 2) - 2] - 2 = 1\,.$$

Similarly, the ability to move terms from one side of an equation to the other, thought pivotal by Shen and his colleagues, wouldn't be of much help here. Only with great difficulty could the above equation be placed in a compact form, corresponding to $ax + b = c$ in modern notation. Much more easily, the author of *Liber* shows how to use double false position to solve the problem. He supposes that the man picked 100 apples, and by placing this directly into the terms of the problem he quickly finds that this would have left him with 9 apples, an excess of 8 beyond what was stated. Similarly, a guess of 200 apples leads to an excess of 20½. The author then turns these results into the correct answer as follows:

$$\frac{100(20\frac{1}{2}) - 200(8)}{20\frac{1}{2} - 8} = 36 \text{ apples (Libri 1838: 337).}$$

It's also possible to choose the two suppositions in such a way that fractional results are avoided.

In Chapter 5 of the *Liber Augmenti et Diminutionis*, the double false technique is used to solve a series of problems stated with more than one unknown and more than one constraint.[4] The first of these reads:

> Two men meet, each with so much money. The first said to the second, "Give me a drachma and I will have as much as you have left." The second said to the first, "Rather, give me four of your drachmas and I will have twice as much as you." How much did each have originally? (Hughes 2001: 126)

Almost every problem in the *Liber Augmenti et Diminutionis* is first solved by double false position and then by one or more other methods, be they algebraic (the operations of *al-jabr wa'l-muqabala*; *regula infusa*; the method of partial residue; substitution of an auxiliary variable) or arithmetic (single false position; inversion or "working backward"). We see from this book that double false position*, in itself a nonalgebraic technique, could be used as an alternative to algebraic methods in the solution of intricate problems*. It freed practitioners from the need to introduce cumbersome equations or to reduce problems to compact form, which could be an especially onerous task if fractions were involved.

In antiquity, double false position was being applied to many other nonroutine problems. In modern notation, these might take the form $ax + b = cx + d$; even if none of the four coefficients were spelled out in the problem, still, given two suppositions and their corresponding excesses or deficiencies, the exact solution could be found. In other cases, the problems involved piecewise linear or even nonlinear relationships. Further details are provided below.

"It Belongs to the Geometrical Art"

A famous discussion of double false position was included in the treatise *Talkhis a'mal al-hisab* (*Summary of Operations of Computation*) by Abu al-Abbas Ahmad ibn Muhammad, known as Ibn al-Banna' (1256-1321). Ibn al-Banna', who spent most of his life in Marrakech, called this technique "the method of scales", in reference to a balance diagram that he used as a mnemonic device for the cross-multiplication of numbers involved in this algorithm. The same diagram and terminology had been used earlier by Abu Zakariyya' Muhammad ibn 'Abdullah, known as al-Hassar, based in Sebta or elsewhere in Morocco, in his 12th-Century arithmetic *Kitab al-bayan wa'l-tadhkar* (*The Book of Proof and Recall*) (Suter 1901: 29-31; Djebbar 1995). Similar diagrams were used later in Europe.

Ibn al-Banna' makes an interesting remark in opening his discussion: *Wa ammâ al-kiffât fahâ min al-sinâ'at al-handasiyyah* ("As for the [method of] scales, it belongs to the geometrical art."). Al-Qalasadi (15th C.), in a later commentary, showed astonishment that Ibn al-Banna' had branded this technique as part of geometry. He remarked that it would be more accurate to say the technique is based on the arithmetical theory of proportions. He went on to hazard the guess that Ibn al-Banna' must have been preoccupied with questions of geometry at the time, which he said would also explain his having fallen into the habit of drawing diagrams such as these scales! Seconding al-Qalasadi's astonishment, Franz Woepcke wrote that "there is absolutely nothing geometrical in the rule of double false position as it was practiced by the Arab arithmeticians." As a result, he goes so far as to insist that the term *al-handasiyyah,* in the remark made by Ibn al-Banna', must be construed to mean "Hindu" rather than "geometrical" (Woepcke 1863: 510-513).

Considering whether the rule of double false position can be derived from geometrical considerations, or else from arithmetical or other reasoning, is one way to approach the issue of how this technique first came into the hands of Arab authors. On the latter question, at least three possibilities must be considered:

- One or more of them independently discovered the technique.
- They borrowed the technique ready-made from one or more older treatises written in another language.
- They recorded and systematized a technique that was already being used by non-mathematicians.

Discussion of *hisab al-khata'ayn* appeared in Arabic mathematical treatises in the late 9th Century CE and possibly earlier. The oldest Arabic writing that we have on the technique is that of Qusta ibn Luqa (died c. 912), a Christian Arab mathematician from Baalbek, on the coast of Lebanon. Below, I discuss his treatise *Maqala li-Qusta ibn Luqa fi al-burhan 'ala 'amal hisab al-khata'ayn* and show that geometry figures prominently there, contradicting Woepcke's claim. Lost to us is the *Kitab al-khata'ayn* of Abu Kamil Shuja' ibn Aslam of Egypt (died c. 930).[5]

Qusta begins[6] by noting that *hisab al-khata'ayn* can be used to resolve "all problems of calculation in which no roots appear", or what we would call linear problems. In problems of this type, he points out, whenever two values of the unknown are supposed, their difference will be proportional to the difference of the resulting errors. He then provides an arithmetical demonstration of the algorithm based on the problem $\frac{1}{2}x + \frac{1}{4}x = 10$. Using the suppositions x = 4 and 8, he obtains deficiencies 10 – 3 = 7 and 10 – 6 = 4, respectively; thus, increasing the supposition by 4 decreases the error by 3. Based on this ratio 4/3, he argues that to wholly eliminate the deficiency, 4, we must further increase the supposition, 8, by a proportional amount to arrive at the correct answer:

$$8 + \frac{4}{3}(4) = 13\frac{1}{3}.$$

He then shows that a cross-multiplication produces the same result:

$$8 + \frac{4}{3}(4) = 8 + \frac{8-4}{7-4}(4)$$

$$= \frac{8(7-4) + (8-4)4}{7-4}$$

$$= \frac{8(7) - (4)4}{7-4}.$$

In modern symbols, his demonstration amounts to:

$$x = x_2 + \frac{x_2 - x_1}{e_1 - e_2} e_2 \quad \text{(Equation 2)}$$

$$x = \frac{x_2 (e_1 - e_2) + (x_2 - x_1) e_2}{e_1 - e_2}$$

$$x = \frac{x_2 e_1 - x_1 e_2}{e_1 - e_2}. \qquad \text{(Equation 3)}$$

Qusta does not mention what might be obvious, that his arithmetical reasoning is valid also for affine functions. This can be seen by rewriting his equation $\frac{1}{2}x + \frac{1}{4}x = 10$ in a form such as $\frac{1}{2}x + \frac{1}{4}x + 5 = 15$. For the latter, the suppositions 4 and 8 still yield deficiencies 7 and 4. Since for affine functions the difference of the suppositions remains proportional to the difference of the corresponding errors, Qusta's reasoning and his calculations remain valid for this case, with no change needed whatsoever.

The bulk of Qusta's treatise is devoted to a geometric proof of the validity of the technique. He represents the linearity of the problem as a right triangle whose height and hypotenuse increase proportionally with its base. His diagram shows three instances of this right triangle; their bases represent the two suppositions x_1 and x_2 and the correct answer x, while the discrepancies among their heights represent the errors e_1 and e_2 and the difference $e_2 - e_1$. Stated in terms of this notation, Qusta is able to interpret x_1e_2, x_2e_1, and $x(e_2 - e_1)$ as rectangles. The difference between the first two is a gnomon whose area he shows (relying on Euclid I: 43) is equal to that of the third:

$$x_1e_2 - x_2e_1 = x(e_2 - e_1),$$

from which the result follows,

$$x = \frac{x_1e_2 - x_2e_1}{e_2 - e_1}.$$

Because he can treat only positive lengths and areas, Qusta must consider successively the three different cases (two excesses, two deficiencies, or one of each), but all three portions of his proof involve the same reasoning based on proportionality, which was also the basis for his arithmetical demonstration.

The way that Qusta organizes his treatise suggests, I think, the respective roles that he assigned to two ways of looking at the algorithm, one arithmetical and the other geometrical. He begins with a specific numerical example to show how the method is used; then he examines the same numerical problem more closely to demonstrate, *via* arithmetical reasoning, why the method should give the correct answer. Finally, he devotes the bulk of his attention to a detailed geometric proof in the Euclidean style. The arithmetical reasoning is already rather convincing, even though based on one example alone; the geometric proof is entirely general and wholly convincing.

Qusta seems to view double false position as a tool for use in the realm of practical arithmetic— the realm where it perhaps first arose— and for assistance in solving simple algebra problems. But in his mind its justification, and the resulting confidence in its generalized use, is less than satisfactory until and unless the technique can be firmly established by geometric proof. This should certainly not surprise us, for it is only another example of “the care that characterized mathematicians of the Islamic countries in basing on Greek mathematical theories the rules utilized in applied mathematics” (Youschkevitch 1976: 45; my translation).

Nearly three centuries later al-Samaw’al ben Yayha (b. 1130), a young physician and mathematician from Baghdad, gave a virtual carbon-copy of Qusta’s geometric proof, but he followed this with an algebraic proof, which was just as general and similarly subdivided into the three cases.[7] The only traces of geometry remaining in the algebraic proof are the use of line-segment notation to represent quantities; the accompanying diagrams are merely mnemonics to keep these symbols in mind, as opposed to spatial figures enclosing areas.

As for Fibonacci, he began his discussion of double false position with an arithmetical justification— based, as with Qusta ibn Luqa, on a specific numerical problem and on proportionality arguments, and yielding in succession the two methods represented by Equations (2) and (3) above, respectively. These latter he considered two different forms of the same *regulis elchatayn*, noting that the second was known by the special name *regula augmenti et diminucionis*. He followed this with a three-case algebraic proof much the same as that of al-Samaw’al (Sigler 2002: 447-452). There was no geometric demonstration.

Returning finally to Ibn al-Banna’, we can surmise that even if he viewed double false position as an arithmetical technique, he considered that it was based on a geometric foundation, and in this sense “belonged” to geometry. He did not bother to re-iterate the geometric proof that had been given by others before him going back at least as far as Qusta ibn Luqa. Also relevant here was the increasing independence of Arab and Islamic mathematics from Greek tradition, and the growing power of the methods at its disposal, notably the symbolic algebra that arose in the Maghreb and Andalusia beginning in the 13th Century. These developments eventually rendered the geometric demonstration of *hisab al-khata’ayn* superfluous. Ultimately, geometry was banished from discussions of this technique, to the point that al-Qalasadi, as we have seen, was astonished to even hear that the two had once been linked.

The Possibility of a Borrowing

Is there evidence that *hisab al-khata’ayn* came to the Arab world from older traditions in other cultures?

The method of *simple* false position was used in ancient Egypt and Mesopotamia, but no evidence for the use of *double* false position has been found in either of these cultures (on Mesopotamia, e.g., see Høyrup 2002: 103). The method of double false position, which can be applied to affine problems of the form $ax + b = c$, is not equivalent in any straightforward way to simple false position, which is effective only for problems of the form $ax = b$. Some have conjectured that the rule of double false position might have been derived algebraically from two applications of simple false position (see, for example, Easton 1967: 57). However, this derivation would have been daunting in the absence of the algebraic symbols that came much later. It is far easier to derive double false position directly from the same underlying rule of proportionality as is single false position, and this is exactly what Qusta ibn Luqa did in his treatise.

Neither have Greek and Byzantine traditions supplied us with any evidence of their having used double false position in the form that we have been considering, even though they studied problems that would have lent themselves to this technique (e.g., see Youschkevitch 1976: 46).

Some writers (e.g., Høyrup 2002: 103) have seen Heron’s approximation of $\sqrt[3]{100}$ in *Metrica* III.20 (1st Century CE) as an early use of double false position. His approximation began with the observation that since $64 < 100 < 125$, the cube root must lie between 4 and 5. Noting the respective errors, 36 and 25, he then made a calculation without explanation:

$$4 + \frac{5(36)}{5(36) + 4(25)} = 4\frac{9}{14} \quad \text{(Heath 1960: 341).}$$

As has been pointed out, if double false position (using Equation 1) is carried out not for the constraint $x^3 = 100$ but for the constraint $x^2 = 100/x$, and with suppositions 4 and 5, then the end result is the same as Heron's:

$$\frac{4(5)+5(9)}{5+9} = 4\frac{9}{14} \ .$$

However, comparing this calculation with Heron's, it seems far-fetched to suppose— and in any case is impossible to infer with confidence— that he arrived at his own approximation procedure by the same reasoning. The most that can be said is that he seems to have developed a cube-root approximation scheme that gives the same result as does a certain application of double false position in the case where the suppositional values differ by 1. But how would he have extended his technique to other situations, in particular to cases where the two suppositional values differ by a number other than 1? His approximation of the cube root, while sound, is not evidence that he was using the *generalized* linear interpolation scheme represented by Equation (1).

"Too much and not enough" in ancient China

Generalized methods that bear a decided resemblance to *hisab al-khata'ayn* were used in China before c. 100 BCE, when the seminal text *Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*) was first compiled by one or more authors.

Of the nine chapters in this book, Chapter 7 is devoted mainly to problems of the type involving "too much" (*ying*) or "not enough" (*bu tsu*). For example, problems involving both an excess (*tiao*) and a deficiency (*nu*) are solved by a procedure called *ying bu tsu shu* ("the rule of too much and not enough"). The procedure and the name are adjusted for the other cases.

Based largely on hearing about or examining this chapter, a number of writers have concluded that the method of double false position must have come to the Arab world from China. For example, Karine Chemla argued that "the stability of the way of expressing these rules and of the way of applying them makes it difficult to believe in independent discoveries", and on this basis she assumed "the transmission to the West" of *ying bu tsu shu* first to the Arab world and from there to Europe (Chemla 1997: 97, 108-109). Shen and his colleagues advanced the "conjecture that the Rule of Double False Position originated in China and was probably introduced to Europe by the Arabian mathematicians *via* the Silk Road" (Shen et al. 1999: 354). Liu Dun agreed, claiming that the method "spread from China into central Asia and then to Europe between the 9th and 13th centuries" (Liu 2002: 156).

Unfortunately, speculation along these lines has been fueled in part by linguistic misunderstandings. Some writers (e.g., Li and Du 1987: 40; Shen et al. 1999: 354; Joseph 2000: 172) were under the mistaken impression that the term *al-khata'ayn* refers to *khitai*, a medieval Arabic name for China, corresponding to "Cathay" in English. As we have seen, the phrase *al-khata'ayn* means "two falsehoods". A number of other writers (e.g., Youschkevitch 1976: 166 n. 19; Shen et al. 1999: 354; Hughes 2001: 109) have claimed that a striking similarity exists between the phrases *ying bu tsu* and *augmenti et diminutionis*. The phrase *ying bu tsu*, which means "too much and not enough", refers to the respective errors (e_1 and e_2 in Equations 2 and 3 above) that result from the two suppositions; it was used only in the case of one excess and one deficit, while other phrases were used in the other cases. By contrast, the phrase *augmenti et diminutionis*, which means "increase and decrease", refers not to the errors but to the suppositions themselves (x_1 and x_2 in Equations 2 and 3 above), either one of which must be increased or decreased to obtain the correct answer. This explains why *augmenti et diminutionis* was used as a generic label for all three types of problem, not just the case of one excess and one deficit.

The theory of a transmission from China to the Arab world has also been energized by an observed similarity in the outward appearance of the respective techniques, especially the symbolic formulae (such as

Equation 1 above) that modern writers use to summarize them. But a closer examination of Chapter 7 in the *Jiuzhang Suanshu* shows that not only its terminology but its whole approach to double false position, including the nature of the heuristic and the type of applications considered, is rather different from that used in the Arab world.

To show this, I want to make a detailed analysis of the first problem from that chapter. Problem 1 is especially important because it exemplifies the joint purchase, a whole class of Chinese problems that cannot be solved by the version of double false position that was practiced by medieval Arabs:

> Now an item is purchased jointly; everyone contributes 8 [coins], the excess is 3; everyone contributes 7, the deficit is 4. Tell: The number of people, the item price, what is each? Answer: 7 people, item price 53. (Shen et al. 1999: 358)

The original text provided just the final answers and a formulaic procedure for solving such problems on a standard Chinese counting board, with rods to represent numbers. There was no attempt to justify the steps of the procedure. However, in a later edition from about 260 CE, Liu Hui provided some explanatory comments. These are invaluable to us because they reflect how the Chinese viewed these problems. I have distilled Liu's annotations for Problem 1 (Shen et al. 1999: 359-360) down to the following chain of deductions. We are given:

8 coins per person → 1 item and 3 more coins
7 coins per person and 4 more coins → 1 item

Thus, Liu reasoned,

4(8) coins per person → 4 items and 4(3) more coins
3(7) coins per person and 3(4) more coins → 3 items

Therefore,

4(8) + 3(7) coins per person → 4 + 3 items

$$\frac{4(8)+3(7)}{4+3} \text{ coins per person} \rightarrow 1 \text{ item}$$

So, each person must contribute 53/7 the value of one coin.

Before we finish solving this problem, let's pause to take note of the character of Liu's reasoning here. His approach is based on the notion of balancing excess against deficit and thereby eliminating or canceling these quantities from consideration, as if on a balance sheet. There is no appeal to proportionality arguments, which, as we saw above, formed the unifying theme in the early Arabic explanations of *hisab al-khata'ayn* whether these explanations were stated in an arithmetical or geometric form. Needham points out that Chapter 7's emphasis on balancing excess and deficit reflects the Confucian doctrine of balancing *yin* and *yang* to achieve harmony (Needham 1959: 119).

Second, the mathematical operations that Liu invokes in this problem (multiplying both sides of a balance-expression by a constant; adding one balance-expression to another; canceling terms common to both sides) remind us of those that are used in the very next chapter of the *Jiuzhang Suanshu*, which instructs how to manipulate a *fangcheng* (rectangular array of numbers) in order to solve a system of linear equations. For that matter, they are also reminiscent of some operations used in *al-jabr wa'l-muqabala*. Yet the Chapter 7 operations are fundamentally arithmetical, not algebraic, inasmuch as they are carried out on numbers only, not on expressions involving unknown quantities (such as *shai'*, *mal*, or *jidhr*). Liu's term for the cross-multiplication 4(8) + 3(7) is *qi*, or "homogenization" of the suppositions 8 and 7, and his term for the equalization 4(3) = 3(4) is *tong*, or "uniformization" of the excess and deficit. But these are arithmetical operations *par excellence*. In fact, earlier Liu uses this exact same terminology to explain the procedure for solving arithmetic problems like 8/3 + 7/4.

Thus, the widespread impression (e.g., Swetz 1994: 334; Chemla 1997: 97, 108; Shen et al. 1999: 358) that Chapter 7 has a precociously "algebraic" character would appear to be mistaken. It is all too easy for modern observers to be misled by the routine use of algebraic notation to represent problems and techniques that employed no such symbolism in the original. In fact, however, if algebra had been a method of Chapter 7, then these problems would have been solved in a much more direct way. Considering the number of people involved in the above joint purchase as an unknown quantity or thing (x in modern notation), then the total price of the item could immediately have been written as $8x - 3 = 7x + 4$, an equation that could then be solved with a single *jabr* and a single *muqabala* (to invoke the later Arabic terms) yielding $x = 7$ people, and thus $7x + 4 = 53$ coins. *Vice versa*, considering the item price as an unknown quantity (x), then the number of people could immediately have been written as $(x + 3)/8 = (x - 4)/7$, an equation that could then be solved to yield $x = 53$ coins, and thus $(x - 4)/7 = 7$ people. But instead of naming the thing (*shai'*) desired and finding it directly in this way, which is a fundamental characteristic of the algebraic approach, the approach used in Chapter 7 is exactly *not* to name it and to find it very *indirectly*.

Note, thirdly, that the given values 8 and 7 in this problem *are not possible values for either of the unknowns asked for*, namely the number of people and the total price. Instead, they are the "coefficients" of those unknowns (see previous paragraph). They *are* possible values of the unknown contribution per person, which explains why the correct value for that quantity can be determined, as we saw earlier in Liu's reasoning, from the standard formula for double false position (Equation 1): the 8 and 7 are thought of as suppositions and the +3 and –4 as the resulting errors, the goal being to determine the contribution per person that corresponds to an excess of 0 coins beyond the purchase price.

On the other hand, finishing the problem by determining the number of people and the item price cannot be a routine double-false calculation. To use the same formula to determine the price, for example, we would have to reverse inputs and outputs, treating the +3 and –4 as suppositions, and the 8 and 7 as the resulting errors. The goal would be to determine the (necessarily negative) surplus of coins that corresponds to a contribution of 0 coins per person. The same formula would then lead to

$$\frac{3(7) - 8(-4)}{7 - 8} = -53,$$

meaning that if every person contributed nothing, there would be a deficit of 53 coins (thus, the item price must be 53).

However, signed numbers are not used in this chapter of the *Jiuzhang Suanshu*. That is why an "alternative rule" had to be given for finishing the calculation. Naturally this rule, like the one given earlier, took different forms depending on the situation (two excesses, two deficiencies, etc.). For the type of problem we're considering, the alternative rule that is stated for calculating the item price is to treat both excess and deficiency as positive numbers and to "subtract the smaller from the greater" (Shen et al. 1999: 359) to find the divisor, or in modern symbols:

$$x = \frac{x_1 e_2 + x_2 e_1}{|e_2 - e_1|} = \frac{3(7) + 8(4)}{|7 - 8|} = 53.$$

Once the item price of 53 is known, it can be combined with the contribution of 53/7 per person to determine that the number of purchasers is 53 ÷ (53/7) = 7 persons.

We see that the joint-purchase problems are fundamentally different from those appearing in the Arabic literature. In the latter, a rule relating input to output values is supplied as an intrinsic part of each problem, while the input values used as suppositions are chosen at will by each problem solver. It is just the opposite in a joint-purchase problem: there, the rule relating input to output values is unknown, while the values used as suppositions are directly supplied as an intrinsic part of each problem. Because these input values are not suppositions about the unknowns requested, finding the latter requires further work, an "alternative rule" beyond the double-false procedure. As indicated above, it is impossible to resolve joint-purchase problems using only the standard *al-khata'ayn* algorithm (Equation 1) unless negative numbers are accepted.

Significantly, of the 19 problems solved with the aid of double false position in the *Jiuzhang Suanshu*, the first 8 are of the joint-purchase variety. On the other hand, as far as I can determine, joint-purchase problems solved with the aid of double false position never appeared in medieval Arabic works. In Europe they didn't appear until 1430, and these were solved with procedures different from those used in the *Jiuzhang Suanshu* (Chemla 1997: 109-110; Sesiano 1985: 133).

Problems 11, 12, and 19 are not joint-purchase examples, but they are especially striking because they involve piecewise-linear functions. Here is Problem 19:

> Now a good horse and an inferior horse set out from Chang'an to Qi. Qi is 3000 *li* from Chang'an. The good horse travels 193 *li* on the first day and daily increases by 13 *li*; the inferior horse travels 97 *li* on the first day and daily decreases by ½ *li*. The good horse reaches Qi first [and] turns back to meet the inferior horse. Tell: how many days [till they] meet and how far has each traveled? Answer: $15\frac{135}{191}$ days. (Shen et al. 1999: 378)

Since the Chinese thought of the horses' speeds as constant on any given day, with the stated increases and decreases occurring overnight, the cumulative distance traveled by each horse is a piecewise-linear continuous function of the number of days elapsed. As Shen and his colleagues show, the positions bounding neighboring linear "pieces" describe a quadratic polynomial. To apply double false position to solve such a problem, care must be taken (and was taken in this text) to use a pair of suppositional values that lie in the same linear "piece" as the correct answer.

We can say that in general, Chapter 7 is characterized by applications of double false position— such as to joint-purchase and piecewise-linear problems— that are qualitatively different from those studied by Arab mathematicians, even requiring a different form of the algorithm. The divergence in algorithm and applications, alongside the reliance (noted above) on balance and elimination, rather than ratio and proportion, as heuristic, and the absence of any attempt at a general mathematical justification, sets the Chinese discussion quite apart from those that appeared later in the Arab world. Chemla's impression, mentioned above, that between Chinese and Arab traditions there was a "stability of the way of expressing these rules and of the way of applying them", appears to be unfounded. In the *Jiuzhang Suanshu*, we do not find any basis for the notion that double false position was transmitted from east to west.[8]

Iterative Approximation in India

In the full title and in the first few sentences of the *Liber Augmenti et Diminutionis*, which we considered earlier, it's stated cryptically that the material was compiled by one Abraham based on works about number-guessing written by Indian savants. Prompted by this remark, some historians (e.g., Smith 1925: 437) have supposed that the rule of double false position came to the Arab world from India.

But only in recent years has evidence for the early use of double false position in India come to light. Kim Plofker, an expert in early Indian mathematics, has identified a few instances in Sanskrit astronomical treatises in which approximate solutions to nonlinear problems were found by iterative linear interpolation (Plofker 2002).

These problems were all of the "pursuit" type: given the positions of two bodies moving at two different speeds, determine when they reach a specified configuration. The earliest passage cited by Plofker, from the astronomical work *Paitamaha-siddhanta* (5^{th}-6^{th} Century CE), tells how to estimate the time of occurrence of a *mahapata*, a particular astrologically-ominous configuration of the sun and moon. The same technique was used later by Brahmagupta (7^{th} C.) to compute the time of occurrence not only of a *mahapata* but also of a certain type of planetary conjunction. Both of these calculations reappeared in a work by the Indian author Lalla (8^{th} C.).

In a *mahapata*, the sun and moon must stand on opposite sides of, and be almost exactly equidistant from, a solstice or an equinox, and at the same time the magnitudes of their declinations from the solar ecliptic must be exactly equal. The *pata*, or difference between these declinations, is a known trigonometric function of their longitudes, which themselves vary with time in a known way. Thus, the heart of calculating the time of occurrence of the *mahapata* is to approximate when the *pata* will reach zero.

The *Paitamaha-siddhanta* instructs that to calculate this moment, first one should calculate the time when the longitudes of the sun and moon will be exactly equidistant from the solstice or equinox (as selected). Then, one should select a second "desired" (i.e., arbitrary) time. The *pata* is then calculated for each of these times. The text puts it this way:

> At these two times of the first and second [approximations] there exists a future or a past *pata*; their sum or difference is the divisor of the desired time [interval] multiplied by the first [approximation]. Thereby, repeatedly making a second [approximation], the time is corrected by a process of iteration. (Plofker 2002: 181)

Plofker is confident in the meaning of this passage. She interprets it in the following modern form, relying on signed numbers to cover both past and future events, the symbol t_0 to signify the time of longitudinal symmetry, t_1 to signify the arbitrarily chosen time, and f to signify the *pata* function:

$$t_2 = t_0 - \frac{(t_1 - t_0)\, f(t_0)}{f(t_1) - f(t_0)} \quad \text{(Equation 4)}$$

In subsequent iterations, each new approximation t_{k+1} would be calculated in this same way from the previous one t_k and the initial one t_0. Equation 4 is equivalent to one form of double false position (see Equation 2).

Although Plofker feels this clearly demonstrates that an iterated form of double false position was known in India, she is cautious to point out that there is still no evidence for the use of the non-iterated form of double false position (in either variant, Equation 2 or 3), and that the iterated form "remained a method specific to these few 'astronomical problems of pursuit.'" She goes on to add, "Moreover, the evidence for its adoption in any mathematical tradition other than the Indian during this time appears so far to be slight."

In a celebrated incident in 771 CE, during the reign of the caliph Abu Ja'far al-Mansur, another astronomical work called the *Surya-siddhanta* was brought to Baghdad by an Indian scholar. This work, soon translated into Arabic, had an impact on subsequent developments in trigonometry. Perhaps the type of iterative interpolation that we have just described could also have been transmitted to Islamic lands in such a manner early on, but if so there is no known evidence for its arrival or subsequent use by astronomers and mathematicians there (Plofker 2002: 184). Instead of using two-point iterative methods, these latter preferred to adapt the fixed-point iterative methods that had been used by their Babylonian and Greek predecessors to approximate square roots. For example, it was a fixed-point iteration that Ghiyath al-Din Jamshid al-Kashi (15^{th} C.) used to make his famous estimate of sin(1º).

Abu al-Raihan al-Biruni, the great scientist born in 973 in what is now Uzbekistan, journeyed in the 1020s to newly conquered lands in India, staying for a few years to study Sanskrit texts. In his subsequent work *Ta'rikh al-hind*, he mentioned the Indians' use of iterative approximation in *mahapata* calculations, but seemingly with more of an ethnological than a technical or astrological interest (Plofker 2002: 182), and this was long after *hisab al-khata'ayn* had already made its appearance. By contrast, al-Biruni devoted an entire treatise to the Indian generalizations of the "Rule of Three" or "Rule of Proportion" (Youschkevitch 1976: 145).

We cannot rule out the possibility that in the early medieval period, one or more of the scientists in the Islamic world might have acquired the technique of double false position ready-made from an older treatise from India or even China, disseminating this knowledge to others. However, the evidence we have examined shows that this is not likely, due to the differences in approach, applications, and terminology. Muslim scientists were quick to credit those from whom they borrowed, so even their silence is telling in this regard. The transmission from the Middle East to Europe later in the Middle Ages is clear, with Fibonacci attributing his technique to the Arabs, adopting an Arabic name for it, and utilizing an approach and justification similar to theirs. Between those two regions there were definite lines of exchange and the translation of some written materials. By contrast, there is nothing like such evidence for a transmission of double false position from China or India to the Middle East in the early Middle Ages.

It seems far more likely that the appearance of double false position in these three lands is a case of convergence, in which a technique is discovered and rediscovered within different cultures, each facing its own problems and following its own logical path toward an analogous common solution. In cautioning against the tendency to jump to conclusions favoring transmission between cultures, Needham pointed out that the history of science is replete with examples of convergence, because "when presented with the same rather simple problems, people in different parts of the world solved them in the same way" (Needham 1988: 227). Indeed, we saw above that the technique of double false position is a relatively simple one whose logic can be discovered in a natural way by any of a number of different approaches.

"At once an oil merchant and an arithmetic teacher"

When he was still a young man in Bukhara, central Asia, around 1000 CE, the great scientist Abu 'Ali ibn Sina was sent by his father to an oil merchant to learn place-value arithmetic using Indian numerals. "This fact," observed Guillaume Libri in a helpful remark, "from which certain authors have believed it possible to conclude that such arithmetic only arrived quite late among the Arabs, only proves that in Bukhara, Ibn Sina's homeland, one could be at once an oil merchant and an arithmetic teacher. And from that we might deduce that this science was more widely known then in the Orient than it is today among us." (Libri 1838: 378-379; translation mine)

The exposition of the merchant's handkerchief problem that I invented to open this article, as well as the reasoning we examined from Liu Hui for the joint-purchase problems from China, make clear that double false position is a technique that even a nonscientist might *discover* in a natural way when faced with any of a number of practical problems. It's even more evident that such a person could readily *call upon* this technique to solve problems as needed, requiring nothing more than some basic arithmetical skills. Mnemonic devices, similar to the scales diagrams recorded by al-Hassar and Ibn al-Banna and the verses of Ibn al-Yasamin (Abdeljaouad 2004: 5) and Robert Recorde (Smith 1925: 439), would have facilitated the routine use of double false position by those to whom more advanced mathematics and technical writings were inaccessible.

I suggest, then, that the tradition of *mu'amalat* reckoning carried out by traders and other practical men might have been the primary setting for the use, even for the discovery, of double false position. As we have seen, the early treatments of *hisab al-khata'ayn* by trained mathematicians such as Qusta ibn Luqa typically included careful arithmetic and/or geometric demonstrations of its validity. It's quite possible, however, that none of these scholars was the first person in the Arab world to use the rule. Instead, they might have recorded and systematized a technique that had long been used by nonscientists, giving it a more rigorous and generalized foundation in theoretical mathematics. Jens Høyrup has argued that algebra, too, in the form found in Muhammad ibn Musa al-Khuwarizmi's seminal *Hisab al-jabr wa'l-muqabala* (c. 825 CE), bore a similar relation to pre-existing commercial, surveying, and other practices. As he reminds us, "The integration of practical mathematics (as carried by the sub-scientific traditions) with theoretical mathematics (as inherited from the Greeks) was a specific accomplishment of the early Islamic culture." (Høyrup 1998: 161)

Further light is shed on the nature and origin of *hisab al-khata'ayn* by considering how its use persisted despite the rise of algebra. Algebra, which was applied to many of the same types of problems, is actually the older tradition in terms of surviving Arabic manuscripts. To account for the widespread use of double false position, we have to consider not just the chronology within which it was first introduced or set down on paper, but also the practical contexts in which it was employed and its power, ease, and convenience in solving various types of problems. In a wide range of applications, *hisab al-khata'ayn* was a simple and effective method, and it required less advanced knowledge of mathematics than did algebraic techniques.

In this light, it's interesting to contrast *Hisab al-jabr wa'l-muqabala* with the *Liber Augmenti et Diminutionis*. In the former, al-Khuwarizmi never uses or even mentions *hisab al-khata'ayn,* even though most of the legacy-division and other linear problems that he solves algebraically are mathematically indistinguishable from those found in *Liber*. By contrast, as noted above, the problems in *Liber* are solved primarily with double false position and secondarily with other methods, including some algebraic ones. Hughes suggests (2001:107) that this is for pedagogical reasons, to show readers that problems can be solved in more than one way. I would add that the relative scarcity in *Liber* of solutions that rely on *al-jabr wa'l-muqabala*, the most sophisticated of the techniques[9], suggests that this text might have been intended for a readership of practical people who didn't necessarily want or need the most advanced mathematical techniques. It would have served well as a training manual for merchants, jurists, surveyors, builders, and the like. Supporting this observation is the complete absence in *Liber* of any generalized presentation, of any mathematical justification, or of any treatment of quadratic equations, all major contrasts to al-Khuwarizmi's work.

Within certain lines of work such as commerce, surveying, and the Islamic division of legacies, the status of double false position might have verged on "common knowledge". We know that it was one of the common techniques used by jurists in the medieval Maghreb for the calculations entailed by legacy division (Djebbar 2002: 224-225; Laabid 2004: 10). These jurist divisors used arithmetical and other non-algebraic methods in a tradition that ran parallel to, and remained distinct from, the algebraic tradition found in the mathematical literature. The persistent use of double false position by these divisors helped prolong the life of this technique for centuries. In addition to Islamic jurists, the method was probably used by a number of medieval Arab and Asian merchants; it may well have circulated among them by some combination of trading activity and occasional rediscovery.

Even if algebraic methods were often more powerful than these arithmetic ones, for a long time algebra did not prevail among such strata. In Europe, double false position was used commonly by nonscientists until the 19th Century, long after it had disappeared from mathematical textbooks.

We who work in the mathematical sciences and who study the history of their development should guard against the tendency to think that all scientific techniques spring from the work of scientists. In *hisab al-khata'ayn* we might have an example of an algorithm that arose within the stock of "common knowledge" of certain groups of nonscientists and found its way from there into Arabic treatises, rather than the other way around.

Endnotes

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2. Simple false position, which was used in ancient Egypt, is an application of the "Rule of Three" or "Rule of Proportion" to solve problems of the form $ax = b$. If x_1 is any supposition and $ax_1 = b_1$, then by ratio and

proportion $\frac{x}{x_1} = \frac{b}{b_1}$; thus, by the Rule of Three, $x = \frac{bx_1}{b_1}$. This last formula provides an algorithm for calculating the correct answer.

3. The complete Latin text is included in Libri (1838: 304-371). Hughes (2001) gives a summary of existing manuscripts and of claims as to the book's authorship, along with English renderings of the enunciations of the problems and a commentary on the methods used in their solution.
4. Similar problems, but with more unknowns and constraints, were solved *via* double false position by such practitioners as Fibonacci, in the thirteenth chapter of his 1202 *Liber Abaci* (Sigler 2002: 455-487), and by Christophore Clavius, in the thirteenth chapter of his 1583 *Epitome Arithmeticae Practicae* (Chabert 1999: 107-111).
5. A number of other lost works from before 1000 have been identified having titles like *Kitab al-jam‘ wa'l-tafriq* (*The Book of Aggregation and Separation*). By analogy to *The Book of Increase and Decrease*, some scholars have assumed that these are also treatises on the method of double false position. A measure of caution is called for, however, since the phrase *al-jam‘ wa'l-tafriq* was ubiquitous in this period and was used to refer to many different things (for more on this point, see Djebbar 2002: 216-220).
6. Suter (1908) includes a full German translation of this treatise. Chabert et al. (1999: 99-100) provide a translation of the first portion of Qusta's geometric proof only.
7. His discussion and proofs appear on pp. 151-163 of the Arabic text in Ahmad and Rashed (1972), and the editors' commentary about these appear on pp. 66-70 of their French introduction. Elsewhere, Rashed argues that al-Samaw'al made use of double false position in an unsuccessful effort to develop an iterative method for approximating n^{th} roots (Rashed 1994: 112-114).
8. For discussion of later treatments of double false position in China, see Chemla 1997 and Liu 2002. For its use in an earlier work, see Cullen 2004: 81-88.
9. By contrast, the simpler algebraic technique of *regula infusa*, whose invention has been attributed to a professional legacy-divisor from the Islamic law tradition, is used extensively in *Liber* but is nowhere to be found in al-Khuwarizmi's algebra (Hughes 2001: 120-123).

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