Transformation Groups, Vol. 4, No. 4, 1999, pp. 329–353

THE SIZE OF A HYPERBOLIC COXETER SIMPLEX

N. W. JOHNSON

Department of Mathematics and Computer Science
Wheaton College
Norton, MA 02766, U.S.A.
njohnson@wheatonma.edu

R. KELLERHALS

Mathematisches Institut
Universität Göttingen
37073 Göttingen,
Germany
ruth@cfgauss.uni-math.gwdg.de

J. G. RATCLIFFE

Department of Mathematics
Vanderbilt University
Nashville, TN 37240, U.S.A.
ratclifj@math.vanderbilt.edu

S. T. TSCHANTZ

Department of Mathematics
Vanderbilt University
Nashville, TN 37240, U.S.A.
tschantz@math.vanderbilt.edu

Abstract. We determine the covolumes of all hyperbolic Coxeter simplex reflection groups. These groups exist up to dimension 9. The volume computations involve several different methods according to the parity of dimension, subgroup relations and arithmeticity properties.

Introduction

Let $X^n = S^n, E^n$, or H^n denote either spherical, Euclidean, or hyperbolic n-space. A *Coxeter simplex* is a n-dimensional simplex in X^n, all of whose dihedral angles are submultiples of π or zero. We allow a simplex in H^n to be unbounded with ideal vertices on the sphere at infinity ∂H^n. A *Coxeter simplex reflection group* is a group generated by the reflections in the sides of a Coxeter simplex in X^n. A Coxeter simplex reflection group is a discrete group of isometries of X^n, with fundamental domain its defining Coxeter simplex. Spherical Coxeter simplex reflection groups are finite, whereas Euclidean and hyperbolic Coxeter simplex reflection groups are infinite. Coxeter simplex reflection groups arise naturally in geometry as groups of symmetries of regular tessellations of X^n. The spherical and Euclidean Coxeter simplices were classified by H. S. M. Coxeter [C1]. The hyperbolic Coxeter simplices were classified by H. S. M. Coxeter and G. J. Whitrow [CW], F. Lannér [L], J.–L. Koszul [K], and M. Chein [C]. For each dimension $n \geq 3$, there are only finitely many hyperbolic Coxeter simplices, and such simplices exist only in dimensions $n = 2, 3, \ldots, 9$.

By the *size* of a non-Euclidean Coxeter simplex, we mean its n-dimensional volume in X^n. For a spherical Coxeter simplex, this is just the volume of S^n,

$$\mathrm{vol}_n(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

Received March 27, 1998. Accepted July 24, 1998.

divided by the order of the corresponding Coxeter group. For a hyperbolic Coxeter simplex, there is no such general formula, and a variety of methods must be employed to calculate the size of an individual simplex. We are interested in calculating the sizes of hyperbolic Coxeter simplices because they are the most elementary building blocks for hyperbolic manifolds (see [R, Example 1, p. 505; Example 2, p. 509]), and the volume of a hyperbolic manifold of finite volume is its most important topological invariant.

Extending some of our own separate and joint investigations ([K1], [K2], [K3], [K5], and [RT]), in the present paper, we determine the size of every hyperbolic Coxeter simplex of dimension $n \geq 3$. Exact expressions and accurate numerical values for the volumes of these simplices are presented here for the first time. See the volume tables at the end of the paper for the size of every hyperbolic Coxeter simplex of dimension $n \geq 3$.

2. Dissecting simplices into orthoschemes

Let $<,>$ denote either the standard bilinear form on Euclidean n-space E^n or Lorentzian $(n+1)$-space $E^{n,1}$. We consider S^n to be the unit n-sphere in E^{n+1} and H^n to be the upper hemisphere of unit imaginary radius in $E^{n,1}$, that is,

$$H^n = \{x \in E^{n,1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}.$$

Then the standard bilinear forms on E^{n+1} and $E^{n,1}$ may be applied to tangent vectors of S^n and H^n, respectively.

Let $P \subset X^n$ denote a convex polyhedron bounded by finitely many hyperplanes H_i with unit normal vectors e_i, $i \in I$. If H_i, H_j $(i, j \in I, i \neq j)$ bound adjacent sides of P, then the dihedral angle $\alpha_{ij} = \angle(H_i, H_j)$ between these sides is given by $\cos\alpha_{ij} = -< e_i, e_j >$. If P has many right dihedral angles, then P can be described more conveniently by its weighted graph or scheme $\Sigma(P)$: The nodes i correspond to the bounding hyperplanes H_i of P. If H_i, H_j $(i, j \in I, i \neq j)$ are not perpendicular, their nodes are joined by an edge; in this case, a positive weight is put on the edge which equals $-< e_i, e_j >$. If P is a simplex, then all the sides of P are adjacent and we will usually label the edges of $\Sigma(P)$ by the corresponding dihedral angles of P. An unlabeled edge will denote an angle of $\pi/3$.

For Coxeter simplices in X^n (all the dihedral angles are of the form π/k, $k \geq 2$), two nodes related by the weight $\cos(\pi/k)$ are connected by a simple line for $k = 3$, or by a line marked k for $k \geq 4$. Such a graph is termed a *Coxeter diagram.* A diagram is called *spherical, Euclidean,* or *hyperbolic* according whether it describes a spherical, Euclidean, or hyperbolic simplex, respectively. A Coxeter reflection group can also be characterized by its *Witt symbol* or its *Coxeter symbol.* The Witt symbol for a hyperbolic Coxeter diagram is an extension of the notation for spherical and Euclidean diagrams proposed by Witt [W] (see the volume table at the end of the paper). The Coxeter symbol is a bracketed expression encoding the form of the Coxeter diagram. For example, $[p, q, r]$ is associated to a linear Coxeter diagram with 3 edges of consecutive markings p, q, r. The Coxeter symbol $[3^{i,j,k}]$ denotes a group with a Y-shaped Coxeter diagram with strings of i, j, and k edges emanating from a common node. The symbol $[3^{[n]}]$ belongs to a cyclic Coxeter diagram with n edges.

The most elementary class of convex polytopes in X^n consists of orthoschemes. An *orthoscheme* $R \subset X^n$ is an n-simplex bounded by hyperplanes $H_0, \ldots, H_n$ subject to the orthogonality conditions $H_i \perp H_j$ for $|i-j|>1$. An orthoscheme $R \subset X^n$ has at most n non-right dihedral angles $\alpha_i = \angle(H_{i-1}, H_i)$ with $1 \leq i \leq n$. In a hyperbolic orthoscheme, one has $\alpha_i < \pi/2$. Moreover, R can be described by a linear scheme

$$\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{\text{---}}\circ\text{---}\cdots\text{---}\circ\overset{\alpha_n}{\text{---}}\circ \ .$$

If p_i denotes the vertex of R opposite to the hyperplane H_i, the edge path $p_0p_1, \ldots, p_{n-1}p_n$ is totally orthogonal, that is, $p_{i-1}p_i \perp p_ip_{i+1}$ for $1 \leq i \leq n-1$. For $R \subset H^n$, this implies that at most p_0, p_n may be points at infinity in which cases R is called *simply* or *doubly asymptotic.*

An arbitrary convex polytope $P \subset X^n$ can be represented by a finite number of orthoschemes by means of dissection and complementation, and therefore, orthoschemes generate the polytope groups $\mathcal{P}(X^n)$ in scissors congruence theory. More precisely, let $\mathcal{P}(H^n)$ be the group for bounded polytopes in H^n, and let $\mathcal{P}(\overline{H}^n)$ be the group for possibly unbounded polytopes in H^n. One has the following results (cf. [Sa, Prop. 3.7, p. 195], [D, Prop. 6.4, p. 142]):

Proposition. (i) *The image of* $\mathcal{P}(H^n)$ *in* $\mathcal{P}(\overline{H}^n)$ *is generated by the simply asymptotic orthoschemes for all* $n \geq 2$.

(ii) *The group* $\mathcal{P}(\overline{H}^{2n+1})$ *is generated by the doubly asymptotic orthoschemes for all* $n \geq 1$.

Examples. (1) Every compact orthoscheme $R \subset H^n$ can be represented by $n+1$ simply asymptotic orthoschemes Q_i, $i = 0, \ldots, n$, according to

$$R = Q_0 - Q_1 + \cdots + (-1)^n Q_n. \qquad (1)$$

To see this, denote by $p_0, \ldots, p_n$, the vertices of R. Let $\overrightarrow{p_ip_0}$ be the ray through the edge p_ip_0 starting at p_i. Denote by $q_0 \in \partial H^n$ the intersection point of $\overrightarrow{p_1p_0}$ with the boundary of H^n at infinity, and set $q_n := p_0$. Choose points q_{n+i} (indices modulo $n+1$) on $\overrightarrow{p_ip_0}$, such that the plane spanned by $q_0, \ldots, q_{n-1}$ is orthogonal to the line through $q_{n-1}q_n$ in q_{n-1}. Then, it is easy to check that the simplices $Q_i := q_0 \cdots q_i p_{i+1} \cdots p_n$, with $0 \leq i \leq n$, are simply asymptotic orthoschemes and yield Equation (1).

(2) Every simply asymptotic orthoscheme $Q \subset H^n$ can be represented by finitely many doubly asymptotic orthoschemes.

To show this, let $Q = p_0 \cdots p_n$ with $p_n \in \partial H^n$. Denote by q_0 the intersection point of the ray $\overrightarrow{p_0p_1}$ starting at p_0 with ∂H^n. Again, choose points q_{n+i} (indices modulo $n+1$) on $\overrightarrow{p_0p_i}$, such that the hyperplane generated by $q_0, \ldots, q_{n-1}$ is orthogonal to $\overrightarrow{p_0p_n}$ at q_{n-1}, and put $q_n := p_0$. Then, the simplices $T_i := q_0 \cdots q_i p_{i+1} \cdots p_n$, $0 \leq i \leq n$, are asymptotic orthoschemes. In fact, apart from $T_n = q_0 \cdots q_n$ which is simply asymptotic, $T_0, \ldots, T_{n-1}$ are doubly asymptotic. Moreover, there is the following relation (cf. [D, Theorem 2.6, (i), p.127]) $Q = -\sum_{i=0}^{m} T_i + \sum_{i=m+1}^{n} T_i$, where $m = m(Q)$, $0 \leq m \leq n-1$, depends on the dihedral angles of Q. For example, $m = 0$ if the double of the dihedral angle of Q, visible as the planar angle of Q at p_2 in the triangle $p_0p_1p_2$, is still acute.

Now, combining this with the cutting and pasting relation (1), we obtain, for a simply asymptotic orthoscheme Q, the relation $2Q = \sum_{i=0}^{n} (-1)^i Q_i - \sum_{i=0}^{m} T_i + \sum_{i=m+1}^{n} T_i \quad (0 \leq m \leq n-1)$.

On the right hand side, Q_n and T_n are the only simply asymptotic orthoschemes. Furthermore, Q, Q_n and T_n share the same (spherical) vertex figure at $p_0 = q_n$. This implies that $\Sigma(T_n) = \Sigma(Q_n)$, and that Q_n and T_n are isometric.

Hence, for n *odd*, we obtain the relation

$$2Q = \sum_{i=0}^{n-1} (-1)^i\, Q_i \;-\; \sum_{i=0}^{m} T_i \;+\; \sum_{i=m+1}^{n-1} T_i \quad (0 \leq m \leq n-1), \tag{2}$$

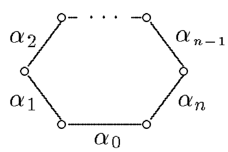
where all the summands on the right hand side are doubly asymptotic orthoschemes. In particular, the volume of a simply asymptotic n-orthoscheme Q is given by

$$\mathrm{vol}_n(Q) = \frac{1}{2}\left(\sum_{i=0}^{n-1} (-1)^i \,\mathrm{vol}_n(Q_i) \;-\; \sum_{i=0}^{m} \mathrm{vol}_n(T_i) \;+\; \sum_{i=m+1}^{n-1} \mathrm{vol}_n(T_i) \right),$$

where $0 \leq m \leq n-1$.

(3) A simplex $S \subset H^n$, $n \geq 2$, with cyclic diagram

$\Sigma(S)$:

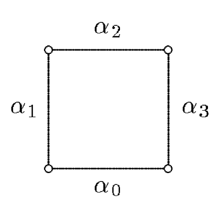


, $0 < \alpha_i < \pi/2$,

can be dissected into 2^{n-1} orthoschemes. To see this, denote by $H_0, \ldots, H_n$ the hyperplanes bounding S and forming the angles $\alpha_i = \angle(H_i, H_{i+1})$ (indices modulo $n+1$). Let p_i be the vertex of S opposite to H_i. Now, put hyperplanes h_i through p_0, for example, such that h_i is orthogonal to the edge $l_i := p_i p_{i+1}$ for $i = 1, \ldots, n-1$. Connecting the footpoints $h_i \cap S \notin l_i$ with the vertices $p_0, \ldots, p_n$ of S yields the desired dissection (look at the dissections induced in the vertex figures and use [D, Theorem 2.1, p. 125] for orthoschemes). Observe that, for $p_0 \in \partial H^n$, all dissecting orthoschemes are (at least) simply asymptotic.

As an illustration, consider the simplex $S \subset H^3$ with diagram

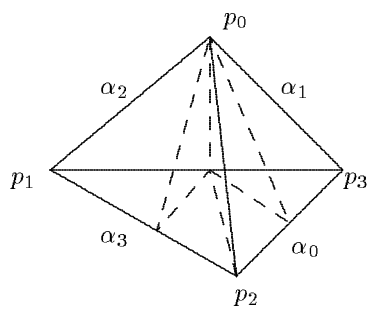
$\Sigma(S)$:



By the above construction, S is dissected into 4 orthoschemes

$$S = R_1 + R_2 + R_3 + R_4 \tag{3}$$

as follows:



.

For the dissecting orthoschemes, we obtain the following:

$$\begin{aligned} \Sigma(R_1) &: \circ\overset{\alpha_0}{—}\circ\overset{\alpha_1}{—}\circ\overset{\beta_2}{—}\circ \,, \\ \Sigma(R_2) &: \circ\overset{\alpha_0}{—}\circ\overset{\gamma_2}{—}\circ\overset{\gamma_3}{—}\circ \,, \\ \Sigma(R_3) &: \circ\overset{\gamma_1}{—}\circ\overset{\gamma_2'}{—}\circ\overset{\alpha_3}{—}\circ \,, \\ \Sigma(R_4) &: \circ\overset{\beta_1}{—}\circ\overset{\alpha_2}{—}\circ\overset{\alpha_3}{—}\circ \,, \end{aligned} \tag{4}$$

where $\gamma_2' := \frac{\pi}{2} - \gamma_2$, and $\tan\gamma_2 = \cos\alpha_0/\cos\alpha_3$. As for the remaining angles, let $\omega \in (0, \pi/2)$ be the auxiliary angle defined by

$$\cos\omega = \frac{\cos\alpha_0\cos\alpha_2 - \cos\alpha_1\cos\alpha_3}{\sqrt{\cos^2\alpha_0 + \cos^2\alpha_3}} \ .$$

Then, $\beta_2 + \gamma_3 = \omega, \beta_1 + \gamma_1 = \pi - \omega$. Finally, for

$$h := \frac{\sqrt{\cos^2\alpha_1 + \cos^2\alpha_2 - \cos^2\omega}}{\sin\omega} \ ,$$

one finds $\cot\gamma_1 = (\cos\alpha_3/\cos\alpha_0)\cdot h,\ \cot\gamma_3 = (\cos\alpha_0/\cos\alpha_3)\cdot h$.

For example, the Coxeter simplex $\widehat{AV}_3$ has a square diagram with $\alpha_0 = \pi/3$, $\alpha_1 = \pi/3$, $\alpha_2 = \pi/6$, and $\alpha_3 = \pi/3$. By the above construction, $\widehat{AV}_3$ is dissected into 4 orthoschemes $\widehat{AV}_3 = [3,3,6] + [3,4,4] + [4,4,3] + [3,6,3]$.

3. Volume formulae

There are different methods to deal with non-Euclidean simplicial volumes. We make use of Schläfli's volume differential formula. Let $\mathcal{S}^n_\kappa$, $n \geq 2$, denote the set of n-dimensional simplices of curvature $\kappa = \pm 1$. For $S \in \mathcal{S}^n_\kappa$, denote by S_j the sides of S forming the dihedral angles $\alpha_{jk} = \angle(S_j, S_k)$ sitting at the apecies $S_{jk} = S_j \cap S_k$ $(0 \leq j < k \leq n)$. Then, Schläfli's formula says that (cf. for example [G, p. 118–120])

$$d\mathrm{vol}_n(S) = \frac{\kappa}{n-1} \sum_{0 \leq j < k \leq n} \mathrm{vol}_{n-2}(S_{jk})\, d\alpha_{jk} \ , \quad \mathrm{vol}_0(\{\mathrm{pt}\}) := 1. \tag{5}$$

By (5), the volume of a non-Euclidean simplex is given by simple integrals. However, the integrands involved are very complicated analytical expressions in terms of the dihedral angles. For instance, for a 3-dimensional hyperbolic orthoscheme R with dihedral angles

$\alpha_1, \alpha_2, \alpha_3$, the associated edge lengths l_1, l_2, l_3, figuring as coefficients in (5), are given by

$$l_k = \frac{1}{2} \log \left| \frac{\cos(\theta - \overline{\alpha}_k)}{\cos(\theta + \overline{\alpha}_k)} \right| \quad \text{with} \quad \overline{\alpha}_k := \begin{cases} \alpha_2, & \text{for } k = 2; \\ \frac{\pi}{2} - \alpha_k, & \text{for } k = 1, 3, \end{cases}$$

where the additional angle is given by

$$\theta = \arctan \frac{\sqrt{\cos^2 \alpha_2 - \sin^2 \alpha_1 \sin^2 \alpha_3}}{\cos \alpha_1 \cos \alpha_3} \in [0, \pi/2].$$

Secondly, formula (5) indicates that volume computations can be treated for even and odd dimensions separately.

In the remaining part of this section, we collect the most important results about volumes of hyperbolic simplices of odd dimensions $n \geq 3$. For the even-dimensional volume problem, we refer to Section 4. In the special and important case of orthoschemes (see §2, Proposition), the integration of (5) could be performed for $n = 3$ and $n = 5$, while, for $n \geq 7$, there are no closed formulae available up to now.

For an orthoscheme $R \subset H^3$, the volume is given by (cf. [Lo] and [K1, Thm. II, p. 562])

$$\begin{aligned} \mathrm{vol}_3(R) = \frac{1}{4}\{ &\text{Л}(\alpha_1 + \theta) - \text{Л}(\alpha_1 - \theta) + \text{Л}(\tfrac{\pi}{2} + \alpha_2 - \theta) + \text{Л}(\tfrac{\pi}{2} - \alpha_2 - \theta) + \\ &\text{Л}(\alpha_3 + \theta) - \text{Л}(\alpha_3 - \theta) + 2\text{Л}(\tfrac{\pi}{2} - \theta)\}, \end{aligned} \tag{6}$$

where θ is defined as above and Л is the Lobachevsky function Л_2 defined in the appendix.

For example, for the orthoscheme $\overline{HV}_3 = [5, 3, 6]$, we have $\theta = \pi/6$ and Formula (6) yields the formula $\mathrm{vol}_3(\overline{HV}_3) = \frac{1}{2}\text{Л}(\frac{1}{3}\pi) + \frac{1}{4}\text{Л}(\frac{11}{30}\pi) - \frac{1}{4}\text{Л}(\frac{1}{30}\pi)$. The Coxeter simplex $\overline{HP}_3$ can be subdivided into two copies of $\overline{HV}_3$, and so the volume of $\overline{HP}_3$ is twice the volume of $\overline{HV}_3$ given by the above formula. Using Formula (6), R. Meyerhoff [M] computed numerical values of the volumes of all the hyperbolic Coxeter tetrahedra.

According to Coxeter [C2], the volume of an orthoscheme $R \subset H^3$ can also be expressed in terms of the Schläfli function $S(\alpha, \beta, \gamma)$ by the formula $\mathrm{vol}_3(R) = \frac{i}{4} S(\frac{\pi}{2} - \alpha_1, \alpha_2, \frac{\pi}{2} - \alpha_3)$. The Schläfli function is defined by the formula

$$S(\alpha, \beta, \gamma) = \sum_{n=1}^{\infty} \frac{(-r)^n}{n^2} (\cos 2n\alpha - \cos 2n\beta + \cos 2n\gamma - 1) - \alpha^2 + \beta^2 - \gamma^2,$$

where $r = (\sin\alpha \sin\gamma - d)/(\sin\alpha \sin\gamma + d)$ and $d = \sqrt{\cos^2\alpha \cos^2\gamma - \cos^2\beta}$.

For a doubly asymptotic orthoscheme $R \subset H^5$ with scheme

$$\Sigma(R) \;:\; \circ \overset{\alpha_1}{\text{—}} \circ \overset{\alpha_2}{\text{—}} \circ \overset{\alpha_3}{\text{—}} \circ \overset{\alpha_4}{\text{—}} \circ \overset{\alpha_5}{\text{—}} \circ \,,$$

put

$$\lambda = \tan\Theta = \frac{|\det \Sigma(R)|^{1/2}}{\cos\alpha_1 \cos\alpha_3 \cos\alpha_5}\,, \quad 0 \leq \Theta \leq \pi/2,$$

where $\det \Sigma(R) = \det\big((< e_i, e_j >)_{ij}\big)$ denotes the determinant of the Gram matrix of R. Furthermore, let $\alpha_0 \in [0, \pi/2]$ be the angle such that $\tan\alpha_0 = \cot\Theta \cdot \tan\alpha_3$. Then, (cf. [K5, Theorem 3])

$$\begin{aligned}\mathrm{vol}_5(R) = -\frac{1}{8}\{ I(\lambda^{-1}, 0; \alpha_1) + \frac{1}{2} I(\lambda, 0; \alpha_2) - I(\lambda^{-1}, 0; \alpha_0') + \frac{1}{2} I(\lambda, 0; \alpha_4) \\ + I(\lambda^{-1}, 0; \alpha_5) \} + \frac{1}{32}\{I_{\mathrm{alt}}(\lambda, \alpha_1; \alpha_2) + I_{\mathrm{alt}}(\lambda, \alpha_5; \alpha_2)\},\end{aligned} \tag{7}$$

where

$$I(a, b; x) := \int_{\pi/2}^{x} Л(y)\, d\arctan(a\tan(b + y)), \quad a, b \in \mathbb{R} \text{ fixed},$$

with $I(1, b; x) = -Л_3(x) - \frac{3}{16}\zeta(3)$, and $I_{\mathrm{alt}}(a, b; x)$ is defined by $I_{\mathrm{alt}}(a, b; x) := I_\delta(a, b; x) + I_\delta(a, -b; x)$ with $I_\delta(a, b; x) := I(a, -\frac{\pi}{2} - b; \frac{\pi}{2} + b + x) - I(a, -\frac{\pi}{2} - b; \frac{\pi}{2} + b + \frac{\pi}{2})$. We remark that the integral $I(a, b; x)$ is expressible in terms of (several dozens of) polylogarithms $\mathrm{Li}_k(z)$, $k \le 3$ (cf. [K5]). See the Appendix for the definitions of polylogarithms $\mathrm{Li}_k(z)$ and Lobachevsky functions $Л_k(z)$.

Especially, for a doubly asymptotic orthoscheme $R \subset H^5$ given by the diagram

$$\Sigma(R) \ : \quad \circ\overset{\alpha_1}{—}\circ\overset{\alpha_2}{—}\circ\overset{\alpha_3}{—}\circ\overset{\alpha_1}{—}\circ\overset{\alpha_2}{—}\circ \ ,$$

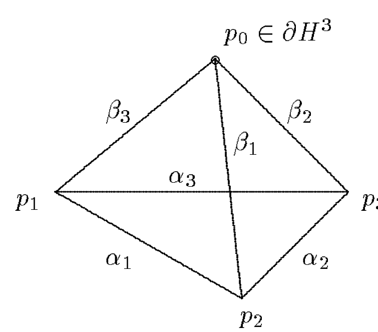
which means that $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1$, one deduces the comparatively simple result (cf. [K3, §3.1, Theorem])

$$\begin{aligned}\mathrm{vol}_5(R) = \frac{1}{4}\{Л_3(\alpha_1) + Л_3(\alpha_2) - \frac{1}{2}Л_3(\tfrac{\pi}{2} - \alpha_3)\} - \frac{1}{16}\{Л_3(\tfrac{\pi}{2} + \alpha_1 + \alpha_2) \\ + Л_3(\tfrac{\pi}{2} - \alpha_1 + \alpha_2)\} + \frac{3}{64}\zeta(3).\end{aligned} \tag{8}$$

For example, the doubly asymptotic Coxeter orthoscheme $\overline{X}_5 \subset H^5$ given by

$$\circ—\circ—\circ\overset{4}{—}\circ—\circ—\circ$$

has the size $\mathrm{vol}_5(\overline{X}_5) = 7\zeta(3)/9216$. By Equations (6) and (7), the problem of computing 3- and 5-dimensional polyhedral volume is solved by the Proposition of Section 2 by means of cutting into and pasting of orthoschemes. For example, for a simplex $S \subset H^3$ with at least one vertex at infinity where the dihedral angles $\beta_1, \beta_2, \beta_3$ add up to π, and whose remaining angles are $\alpha_1, \alpha_2, \alpha_3$ according to



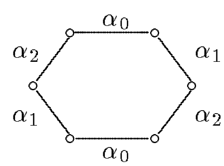
,

the volume is given by (cf. for example [G, (26), p. 130])

$$\begin{aligned}\mathrm{vol}_3(S) = \frac{1}{2}\sum_{i=1}^{3}\Big\{&\text{Л}(\beta_i) + \text{Л}\big(\tfrac{1}{2}(\pi+\alpha_i+\alpha_{i+1}-\beta_i)\big) + \text{Л}\big(\tfrac{1}{2}(\pi+\alpha_i-\alpha_{i+1}-\beta_i)\big) + \\ &\text{Л}\big(\tfrac{1}{2}(\pi-\alpha_i+\alpha_{i+1}+\beta_i)\big) + \text{Л}\big(\tfrac{1}{2}(\alpha_i+\alpha_{i+1}+\beta_i-\pi)\big)\Big\}.\end{aligned} \tag{9}$$

However, by using the dissection method and the results (6) and (7) to compute volume, it can be quite troublesome to determine the explicit shape of the cutting orthoschemes. For some combinatorially–metrically simple cases, the integration of (5) can directly be achieved. For instance, for a totally asymptotic simplex $Q \subset H^5$ with cyclic diagram

$\Sigma(Q)$:



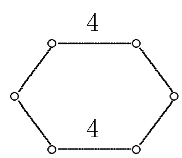
, $\cos^2\alpha_0 + \cos^2\alpha_1 + \cos^2\alpha_2 = 1,$

one obtains (cf. [K4, 3.4, Theorem 3])

$$\begin{aligned}\mathrm{vol}_5(Q) = &\frac{1}{2}\big\{\text{Л}_3(\alpha_0) + \text{Л}_3(\alpha_1) + \text{Л}_3(\alpha_2) - \text{Л}_3(\tfrac{\pi}{2}-\alpha_0) \\ &- \text{Л}_3(\tfrac{\pi}{2}-\alpha_1) - \text{Л}_3(\tfrac{\pi}{2}-\alpha_2)\big\} + \frac{7}{32}\zeta(3).\end{aligned} \tag{10}$$

Hence, for the Coxeter simplex $Q = \widehat{UR}_5$ with diagram

$\Sigma(\widehat{UR}_5)$:

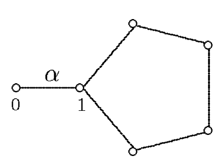


,

one obtains $\mathrm{vol}_5(\widehat{UR}_5) = 7\zeta(3)/288$.

Or, one contents oneself by representing volume as a single integral and by approximating it numerically. We demonstrate this in two examples. Consider the simply asymptotic simplex $T(\alpha) \subset H^5$ with diagram

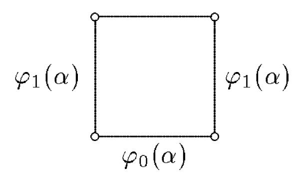
$\Sigma(T(\alpha))$:



.

For $\alpha \in [\pi/3, \pi/2)$, the simplex $T(\alpha)$ is hyperbolic. In the limiting case $\alpha = \pi/2$, the simplex $T(\pi/2)$ degenerates to a point-shaped simplex with volume equal to zero. By looking more closely to its shape, the face $F(\alpha) := T(\alpha) \cap H_0 \cap H_1$ associated to the angle α is of the form

$\Sigma(F(\alpha))$:



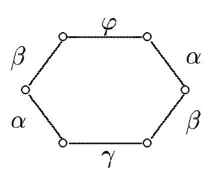
with dihedral angles given by the functions $\varphi_0(x) = 2\arctan\sqrt{(1-\cot^2 x)/2}$ and $\varphi_1(x) = \arctan\sqrt{2-\cot^2 x}$. The simplex $F(\alpha)$ can be cut into four orthoschemes according to (3) and (4) of Section 2. This yields, together with the formulae (5) and (6), the integral expression (for brevity, we discard the integration argument t)

$$\begin{aligned}\mathrm{vol}_5(T(\alpha)) &= -\frac{1}{4}\int_{\pi/2}^{\alpha} \mathrm{vol}_3(F(t))dt \\ &= -\frac{1}{4}\int_{\pi/2}^{\alpha} \Big\{\mathrm{vol}_3(\circ\!\!-\!\!-\!\!\circ\overset{\varphi_1}{-\!\!-}\circ\overset{\beta_2}{-\!\!-}\circ) + \mathrm{vol}_3(\circ\!\!-\!\!-\!\!\circ\overset{\gamma_2}{-\!\!-}\circ\overset{\gamma_3}{-\!\!-}\circ) + \\ &\qquad \mathrm{vol}_3(\circ\overset{\gamma_1}{-\!\!-}\circ\overset{\gamma_2'}{-\!\!-}\circ\overset{\varphi_1}{-\!\!-}\circ) + \mathrm{vol}_3(\circ\overset{\beta_1}{-\!\!-}\circ\overset{\varphi_0}{-\!\!-}\circ\overset{\varphi_1}{-\!\!-}\circ)\Big\}dt.\end{aligned}$$

By numerical evaluation, the volume of the Coxeter simplex $T(\pi/3) = \overline{P}_5$ is $\mathrm{vol}_5(\overline{P}_5) \simeq 0.00207405196$. One should compare this numerical value of the volume of $\overline{P}_5$ with the exact expression for the volume of $\overline{P}_5$ obtained by number-theoretical methods (see §5): $\mathrm{vol}_5(\overline{P}_5) = 5^{3/2}L(3,5)/4608$.

Analogously, we proceed in case of a doubly asymptotic simplex $Q(\alpha,\beta,\gamma;\varphi)$ in H^5 with scheme

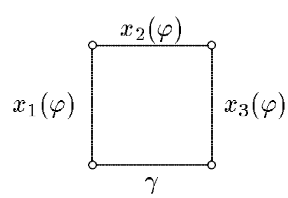
$\Sigma(Q(\alpha,\beta,\gamma;\varphi))$:



, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

For $\varphi = \gamma$, we are in the situation of (10). The apex simplex $F(\varphi)$ associated to the dihedral angle φ looks like

$\Sigma(F(\varphi))$:



with dihedral angles

$$x_1(y) = \arctan\frac{\sqrt{\sin^2\beta\sin^2 y - \cos^2\alpha}}{\cos\beta\sin y},$$

$$x_2(y) = \arctan\Big(\frac{\sqrt{\sin^2\alpha\sin^2\beta - \cos^2 y}}{\cos\alpha\cos\beta}\tan y\Big),$$
$$x_3(y) = \arctan\frac{\sqrt{\sin^2\alpha\sin^2 y - \cos^2\beta}}{\cos\alpha\sin y}.$$

Therefore, we can write (see (10))

$$\mathrm{vol}_5(Q(\alpha,\beta,\gamma;\varphi)) = -\frac{1}{4}\int_\gamma^\varphi \mathrm{vol}_3(F(t))dt + \frac{1}{2}\{Л_3(\alpha) + Л_3(\beta) + Л_3(\gamma) -$$
$$- Л_3(\tfrac{\pi}{2}-\alpha) - Л_3(\tfrac{\pi}{2}-\beta) - Л_3(\tfrac{\pi}{2}-\gamma)\} + \frac{7}{32}\zeta(3).$$

From this, the volume of the Coxeter simplex $Q(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{4};\frac{\pi}{3}) = \widehat{AU}_5$ can be estimated as $\mathrm{vol}_5(\widehat{AU}_5) \simeq 0.007573474422$. The Coxeter group corresponding to $\widehat{AU}_5$ is nonarithmetic, and so there is probably no simple, number-theoretical, exact expression for the volume of $\widehat{AU}_5$.

4. Volumes of even-dimensional Coxeter simplices

There is a qualitative difference between volume computations in even and in odd dimensions. Indeed, Schläfli's volume differential formula (see (5)) yields an inductive principle to derive non-Euclidean simplex volumes within the same parity of dimension. For even dimensions, the induction is based on the well-known area formula $\kappa\cdot\mathrm{vol}_2(S) = \alpha_1+\alpha_2+\alpha_3-\pi$ for a non-Euclidean triangle $S = S(\alpha_1,\alpha_2,\alpha_3)$ with angles $\alpha_1,\alpha_2,\alpha_3 \in [0,\pi/2]$ and curvature $\kappa = \pm 1$. Its simple, linear structure allows one to integrate the differential $d\,\mathrm{vol}_4$ along a suitable path in the space $\mathcal{S}^4_\kappa$ of 4-dimensional simplices of curvature κ. This process can be continued to higher even dimensions. It was L. Schläfli [S, No. 24] who observed a particular pattern how to build even-dimensional simplex volume up from certain lower- (and odd-) dimensional volumes. In this way, he could determine the content of all spherical Coxeter orthoschemes (see Remark (c) below). To describe Schläfli's reduction law, we make use of the normalized volume functions

$$\text{on } \mathcal{S}^n_{+1} \;:\quad f_n := c_n\,\mathrm{vol}_n,\quad f_0 := 1;$$
$$\text{on } \mathcal{S}^n_{-1} \;:\quad F_n := i^n\,c_n\,\mathrm{vol}_n,\quad F_0 := 1,$$

where

$$c_n = \frac{2^{n+1}}{\mathrm{vol}_n(S^n)} = \frac{2^n}{\pi^{\frac{n+1}{2}}}\,\Gamma\Big(\frac{n+1}{2}\Big).$$

Therefore, $f_n = 1$ precisely for a spherical simplex all of whose dihedral angles equal $\pi/2$. Moreover, for a linear spherical scheme Σ of order $n+1$ consisting of r disjoint components $\sigma_1,\ldots,\sigma_r$ of orders $n_1+1,\ldots,n_r+1 \geq 1$, the function f_n is multiplicative in the following sense (cf. [S, No. 23, p. 238]): $f_n(\Sigma) = f_{n_1}(\sigma_1)\cdots f_{n_r}(\sigma_r)$.

We reproduce the volume reduction law for the case of even-dimensional and eventually asymptotic hyperbolic simplices. For the proof we refer to [K2, §3] as it follows the same lines as the verification of the analoguous result for hyperbolic *orthoschemes*.

Theorem (Reduction formula for hyperbolic simplices). *Denote by* $S \subset \mathcal{S}^{2n}_{-1}$, $n \geq 1$, *a* $(2n)$*-dimensional hyperbolic simplex with scheme* Σ. *Then,*

$$F_{2n}(\Sigma) = \sum_{k=0}^{n} (-1)^k a_k \sum_{\sigma} f_{2n-(2k+1)}(\sigma), \qquad \sum f_{-1} := 1,$$

where σ *runs through all spherical subschemes of order* $2(n-k)$, *and where the coefficients* a_k *are the tangent numbers given by*

$$\tan x = \sum_{k=0}^{\infty} \frac{a_k}{(2k+1)!} x^{2k+1}.$$

Remarks. (a) The spherical subschemes σ of order $2(n-k)$, $0 \leq k \leq n$, of Σ represent iterated odd-dimensional vertex figures of S.

(b) The tangent numbers a_k are integers which are expressible in terms of the Bernoulli numbers B_k by

$$a_k = 2^{2k+1} \frac{2^{2k+2}-1}{k+1} B_{k+1} = 1, 2, 16, 16 \cdot 17, 256 \cdot 31, 512 \cdot 691, \ldots .$$

Here we use consecutive index notation for B_k so that $B_1 = 1/6, B_2 = 1/30, B_3 = 1/42, \ldots$.

(c) Based on the corresponding theorem for spherical orthoschemes (cf. [S, No. 26]), Schläfli obtained the following results (cf. [S, No. 28–30]):

$$f_1(I_2^p) = f_1(\circ\overset{p}{\text{———}}\circ) = \frac{2}{p}, \quad p \geq 2;$$

$$f_2(H_3) = f_2(\circ\text{———}\circ\overset{5}{\text{———}}\circ) = \frac{1}{15};$$

$$f_3(H_4) = f_3(\circ\text{———}\circ\text{———}\circ\overset{5}{\text{———}}\circ) = \frac{1}{900};$$

$$f_3(F_4) = f_3(\circ\text{———}\circ\overset{4}{\text{———}}\circ\text{———}\circ) = \frac{1}{72};$$

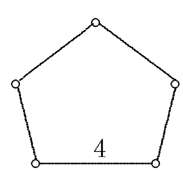
$$f_n(A_{n+1}) = f_n(\circ\text{———}\circ\text{———}\circ\text{———}\cdots\text{———}\circ\text{———}\circ) = \frac{2^{n+1}}{(n+2)!}, \quad n \geq 0;$$

$$f_n(B_{n+1}) = f_n(\circ\overset{4}{\text{———}}\circ\text{———}\circ\text{———}\cdots\text{———}\circ\text{———}\circ) = \frac{1}{(n+1)!}, \quad n \geq 1.$$

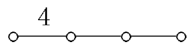
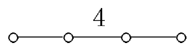
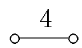
(d) The volumes of all hyperbolic Coxeter orthoschemes are listed in [K2, p. 206].

Examples. (1) Consider the Coxeter simplex $\widehat{AF}_4 \subset H^4$ with scheme

$\Sigma(\widehat{AF}_4)$:



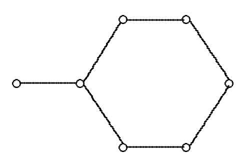
The scheme $\Sigma(\widehat{AF}_4)$ contains only spherical subschemes, and so describes a compact hyperbolic 4-simplex. More precisely, it contains the following list of subschemes with respective multiplicities:

order	subscheme	multiplicity
4		2
		2
		1
2		4
		1
		5

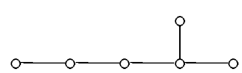
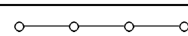
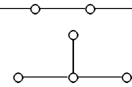
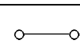
Therefore, by the above theorem, $F_4(\Sigma(\widehat{AF}_4)) = \sum f_3 - 2\sum f_1 + 16 = 11/360$, that is, $\mathrm{vol}_4(\widehat{AF}_4) = 11\pi^2/4320 \simeq 0.02513093713$.

(2) The Coxeter simplex $\overline{P}_6 \subset H^6$ with scheme

$\Sigma(\overline{P}_6)$:

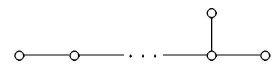


is simply asymptotic since its scheme contains the Euclidean cycle $\widetilde{A}_5$ of order 6. All other subschemes are spherical. Among these, the even order ones are tabulated in the following list:

order	subscheme	multiplicity
6		2
		2
		1
4		8
		1
		13
		5
		7
		1
2		7
		14

The spherical Coxeter simplices with diagrams

D_{n+1} :



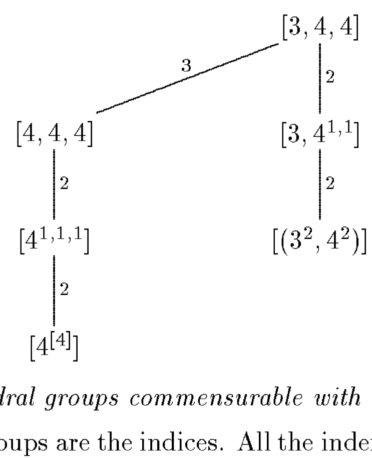
of order $n+1 \geq 4$ have volumes $f_n(D_{n+1}) = 2 \cdot f_n(B_{n+1}) = 2/(n+1)!$. This reflects actually a dissection property which can easily be seen by means of induction. Therefore, we obtain $F_6(\Sigma(\overline{P}_6)) = \sum f_5 - 2\sum f_3 + 16\sum f_1 - 272 = -13/11340$, whence $\mathrm{vol}_6(\overline{P}_6) = 13\pi^3/1360800 \simeq 0.00029620929$.

5. Volumes of arithmetic Coxeter simplices

A Coxeter simplex is said to be *arithmetic* if the corresponding Coxeter reflection group is arithmetic (cf. [G, p. 217 ff]). There are exactly 72 hyperbolic Coxeter simplices of dimension three and above. According to Vinberg [V], all but eight of them are arithmetic. The nonarithmetic simplices are $\widehat{BH}_3, \overline{HV}_3, \overline{HP}_3, \widehat{AV}_3, \widehat{BV}_3, \widehat{CR}_3, \widehat{HV}_3$, and $\widehat{AU}_5$. See the volume tables at the end of the paper for Coxeter diagrams of all the hyperbolic Coxeter simplices of dimension three and above. The volumes of the arithmetic hyperbolic Coxeter simplices can be computed using number-theoretical techniques. In this section, we shall describe the computation of the volumes of all the odd-dimensional arithmetic hyperbolic Coxeter simplices using number theory.

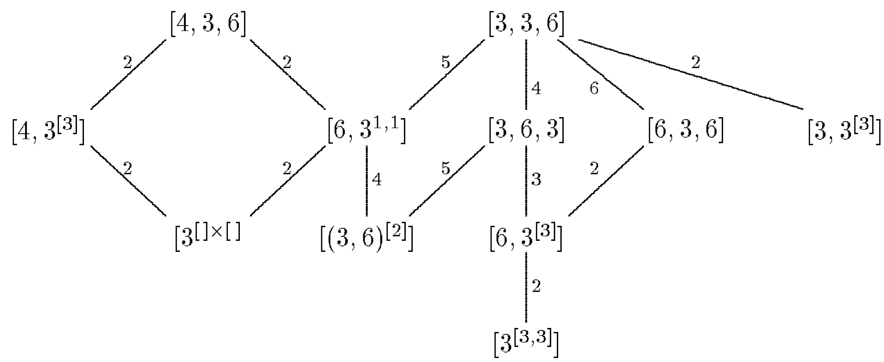
There are nine compact hyperbolic Coxeter tetrahedra. All except $\widehat{BH}_3$ are arithmetic. The volumes of the eight compact arithmetic tetrahedra were computed by Maclachlan and Reid [MR] in terms of zeta functions evaluated at 2, of certain number fields, using number-theoretical calculations of Borel [Bo] for the volumes of fundamental domains of arithmetic 3-dimensional hyperbolic groups. See Maclachlan and Reid [MR] for details.

There are 23 noncompact hyperbolic Coxeter tetrahedra. All but six of them are arithmetic. The arithmetic groups are commensurable with either the Picard group $\mathrm{PSL}(2,\mathbb{Z}[i])$ or the Bianchi group $\mathrm{PSL}(2,\mathbb{Z}[(1+\sqrt{-3})/2])$. In particular, the Coxeter group $[3,4,4]$ contains $\mathrm{PSL}(2,\mathbb{Z}[i])$ as a subgroup of index four and the Coxeter group $[3,3,6]$ contains $\mathrm{PSL}(2,\mathbb{Z}[(1+\sqrt{-3})/2])$ as a subgroup of index four. See §12–13 of Bianchi [B]. Commensurability diagrams are given below.



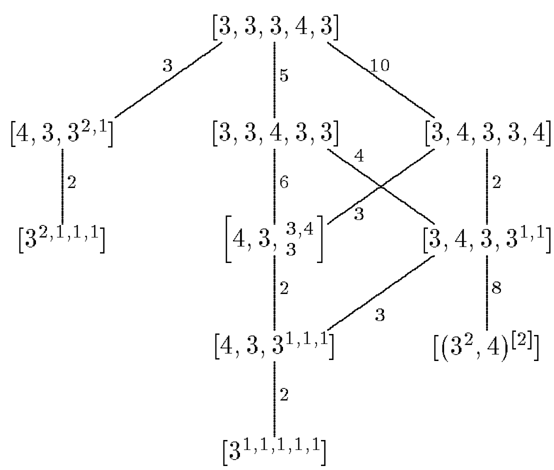
Coxeter tetrahedral groups commensurable with the Picard group

The numbers between groups are the indices. All the index two subgroup relations are due to a plane of symmetry in the Coxeter simplex of the subgroup according to Coxeter and Whitrow [CW, p. 431]. For a discussion of the remaining subgroup relations, see Johnson and Weiss [JW].



Coxeter tetrahedral groups commensurable with the Bianchi group

All the hyperbolic Coxeter simplices in dimensions above three are arithmetic except for the 5-simplex $\widehat{AU}_5$. There are 12 hyperbolic Coxeter 5-simplices. Ten of them are commensurable. The commensurability relationships were worked out by Johnson and Kellerhals (cf. [K3], for example). A commensurability diagram for the ten commensurable hyperbolic Coxeter 5-simplices follows. The numbers between the groups are the indices.



Commensurable, hyperbolic, Coxeter 5-simplex reflection groups

All the hyperbolic Coxeter simplices in dimensions five and above are noncompact. The volumes of all the noncompact arithmetic Coxeter simplices can be computed using Siegel's analytic theory of quadratic forms [Si1], [Si2].

Let f be a quadratic form in $n+1$ variables with integer coefficients that is equivalent over $\mathbb{R}$ to the *Lorentzian quadratic form* $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$. Let S be the matrix of the quadratic form f. The *group of units* of the form f is the group of all $(n+1) \times (n+1)$

matrices A, with integral entries, such that $A^tSA = S$. A unit of f is said to be *positive* or *negative* according as A leaves invariant each of the two connected components of $\{x \in \mathbb{R}^{n+1} : x^tSx < 0\}$ or interchanges them. The group of positive units of f corresponds to a discrete group Γ of isometries of hyperbolic n-space

$$H^n = \{x \in E^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}$$

under the equivalence of f with the Lorentzian quadratic form. Let q be a positive integer and let $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. Let $E_q(S)$ be the order of the group of units of f modulo q, that is,

$$E_q(S) = |\{A \in \mathrm{GL}(n+1, \mathbb{Z}_q) : A^tSA \equiv S \mod q\}|.$$

By Formula 82 in Siegel's paper [Si2], we have that

$$\mathrm{vol}_n(H^n/\Gamma) = 4|\det S|^{\frac{n+2}{2}} \prod_{k=1}^{n} \pi^{-\frac{k}{2}}\Gamma(\tfrac{k}{2}) \cdot \lim_{q\to\infty} \frac{2^{\omega(q)} q^{\frac{n(n+1)}{2}}}{E_q(S)}$$

where $\omega(q)$ is the number of distinct prime divisors of q.

We first consider the special case that f is the Lorentzian quadratic form. According to Vinberg [V], the orbit space H^n/Γ is a hyperbolic Coxeter simplex Δ^n for $n = 2, 3, \ldots, 9$. Ratcliffe and Tschantz [RT] evaluated the above limit and determined the volume of H^n/Γ explicitly as follows. Let B_k be the kth Bernoulli number with even index notation so that $B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,\ \ldots$. Let $\zeta(s)$ denote the Riemann zeta function and consider the Dirichlet L-function (cf. Appendix)

$$L(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots .$$

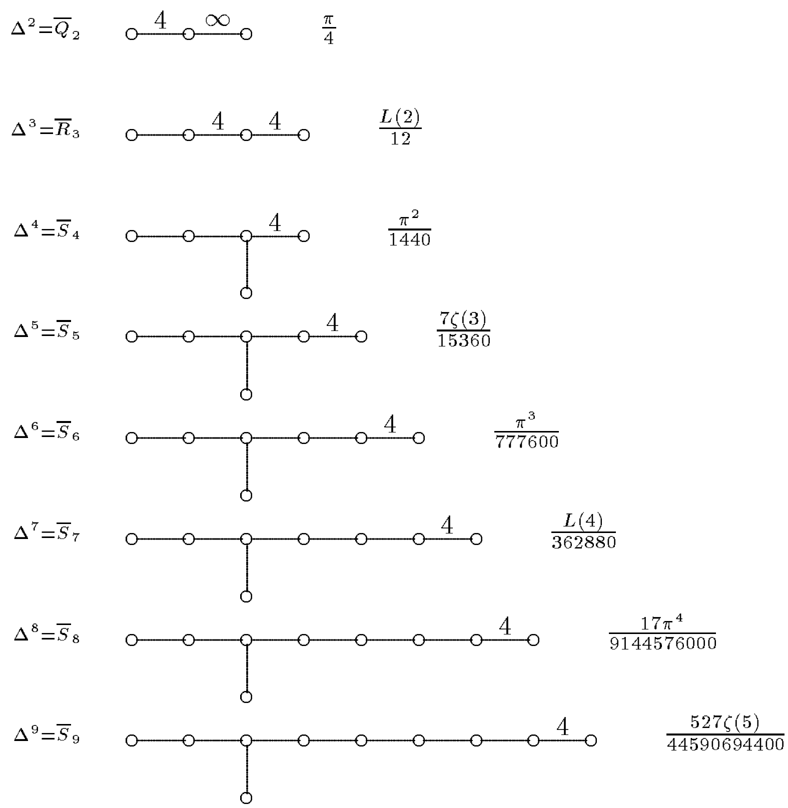
Theorem [RT, Thm 6, p. 66]. *The volume of H^n/Γ is given by*

$$\mathrm{vol}_n(H^n/\Gamma) = \begin{cases} (2^{\frac{n+1}{2}} - 1)(2^{\frac{n-1}{2}} \pm 1) \displaystyle\prod_{k=1}^{\frac{n-1}{2}} \frac{|B_{2k}|}{8k} \cdot \frac{\zeta(\frac{n+1}{2})}{2} & \text{if } n \equiv 1 \mod 4, \\ (2^{\frac{n}{2}} \pm 1) \displaystyle\prod_{k=1}^{\frac{n}{2}} |B_{2k}| \cdot \frac{\pi^{\frac{n}{2}}}{n!} & \text{if } n \text{ is even}, \\ \displaystyle\prod_{k=1}^{\frac{n-1}{2}} \frac{|B_{2k}|}{2k} \cdot L(\tfrac{n+1}{2}) & \text{if } n \equiv 3 \mod 4, \end{cases}$$

with the plus sign if $n \equiv 0, 1, 2 \mod 8$ and the minus sign if $n \equiv 4, 5, 6 \mod 8$.

The table below lists Coxeter diagrams and volumes of the simplices Δ^n for $n = 2, 3, \ldots, 9$.

We now assume that n is odd and $d = |\det S|$ is an odd prime number such that $(-1)^{(n-1)/2} d \equiv 1 \mod 4$.



Coxeter diagrams and volumes of the Coxeter simplices Δ^n

Consider the Dirichlet L-function

$$L(s,d) = \sum_{n=1}^{\infty} \left(\frac{n}{d}\right) n^{-s}$$

where (n/d) is a Legendre symbol. Using the same arguments as in [RT], Ratcliffe and Tschantz have proved that Siegel's formula for the volume of H^n/Γ yields that $\mathrm{vol}_n(H^n/\Gamma)$ is a rational multiple of $\sqrt{d}\,L(\frac{n+1}{2},d)$ which can be explicitly computed when $S = \mathrm{diag}(1,1,\ldots,1,-d)$ or $S = \mathrm{diag}(1,\ldots,1,d,-1)$. Let Γ^n_{-d} or Γ^n_d denote the group of positive units of the above diagonal forms, respectively.

In dimension three, we found that

$$\mathrm{vol}_3(\Gamma^3_{-3}) = 5\sqrt{3}\,L(2,3)/64,$$
$$\mathrm{vol}_3(\Gamma^3_{-7}) = 7^{3/2}L(2,7)/64.$$

The group Γ^3_{-3} is the Coxeter group of $\overline{BV}_3 = [4,3,6]$. The group Γ^3_{-7} contains a subgroup of index 2 which is contained in the Coxeter group of $\widehat{BB}_3$ as a subgroup of index 3. Thus we have

$$\mathrm{vol}_3(\overline{BV}_3) = 5\sqrt{3}\,L(2,3)/64,$$
$$\mathrm{vol}_3(\widehat{BB}_3) = 7^{3/2}L(2,7)/96.$$

Hyperbolic volumes in dimension three are often expressed in terms of the Lobachevsky function. The volumes of $\overline{BV}_3$ and $\overline{R}_3$, given above, agree with the volumes given in the volume tables, since $Л(\frac{1}{3}\pi) = \sqrt{3}\,L(2,3)/4$ and $Л(\frac{1}{4}\pi) = L(2)/2$.

The 5- and 7-dimensional hyperbolic Coxeter simplices $\overline{P}_5$, $\overline{T}_7$, and $\overline{P}_7$ have the property that twice their Gram matrix $(-\cos(\pi/m_{ij}))$ has integral entries. Now each of the corresponding Coxeter groups is a subgroup of finite index of the group of units of the quadratic form corresponding to twice its Gram matrix. See §4.1 of Chap.V of Bourbaki [Bou] for a discussion. The determinants of twice their Gram matrices are $-5, -3$, and -7, respectively. Consequently, the volumes of $\overline{P}_5$, $\overline{T}_7$, and $\overline{P}_7$ are rational multiples of $\sqrt{5}\,L(3,5)$, $\sqrt{3}\,L(4,3)$, and $\sqrt{7}\,L(4,7)$, respectively. By performing an accurate numerical integration of the volumes of $\overline{P}_5$, $\overline{T}_7$, and $\overline{P}_7$, using a Monte–Carlo method, and then performing a best rational fit, Ratcliffe and Tschantz determined, with a high degree of probability, that

$$\begin{aligned} \mathrm{vol}_5(\overline{P}_5) &= 5^{3/2}L(3,5)/4608, \\ \mathrm{vol}_7(\overline{T}_7) &= 3^{1/2}L(4,3)/860160, \\ \mathrm{vol}_7(\overline{P}_7) &= 7^{5/2}L(4,7)/3317760. \end{aligned}$$

We then searched for commensurability relationships between the Coxeter groups of $\overline{P}_5$, $\overline{T}_7$, and $\overline{P}_7$ and the unit groups Γ^5_{-5}, Γ^7_3, and Γ^7_{-7}, respectively. We first found that

$$\begin{aligned} \mathrm{vol}_5(H^5/\Gamma^5_{-5}) &= 3\cdot 5^{3/2}L(3,5)/2048, \\ \mathrm{vol}_7(H^7/\Gamma^7_3) &= 17\cdot 3^{1/2}L(4,3)/491520, \\ \mathrm{vol}_7(H^7/\Gamma^7_{-7}) &= 7^{5/2}L(4,7)/98304. \end{aligned}$$

The ratio of $\mathrm{vol}_5(H^5/\Gamma^5_5)$ with $5^{3/2}L(3,5)/4608$ is $27/4$. We verified that we had the correct volume of $\overline{P}_5$ by showing that its Coxeter group has a subgroup of index 27, which is a subgroup of Γ^5_{-5} of index 4. The ratio of $\mathrm{vol}_7(H^7/\Gamma^7_3)$ with $3^{1/2}L(4,3)/860160$ is $119/4$ and the ratio of $\mathrm{vol}_7(H^7/\Gamma^7_{-7})$ with $7^{5/2}L(4,7)/3317760$ equals $135/4$. Similarly to the above, we checked that we had the correct volumes of $\overline{T}_7$ and $\overline{P}_7$ by showing that the corresponding Coxeter group has a subgroup of index 119 and 135, respectively, which is a subgroup of Γ^7_3 of index 4 and a subgroup of Γ^7_{-7} of index 4, respectively. Details will appear in a forthcoming paper of Ratcliffe and Tschantz.

There are only four 7-dimensional hyperbolic Coxeter simplices, $\overline{T}_7, \overline{S}_7$, $\overline{Q}_7$, and $\overline{P}_7$. We have already determined the volumes of all these simplices except for $\overline{Q}_7$. It is obvious from the Coxeter diagrams of $\overline{S}_7$ and $\overline{Q}_7$ that $\overline{Q}_7$ can be subdivided into two copies of $\overline{S}_7$. Therefore the volume of $\overline{Q}_7$ is twice the volume of $\overline{S}_7$.

There are only three 9-dimensional hyperbolic Coxeter simplices, $\overline{T}_9, \overline{S}_9$, $\overline{Q}_9$. We have already determined the volume of $\overline{S}_9$. Again, it is easy to see that $\overline{Q}_9$ can be subdivided into two copies of $\overline{S}_9$. In order to find the volume of $\overline{T}_9$, Ratcliffe and Tschantz first performed an accurate numerical integration for the volume of $\overline{T}_9$. We then observed that the ratio of the volumes of $\overline{Q}_9$ and $\overline{T}_9$ is approximately 527. We next positioned the smaller simplex $\overline{T}_9$ in one corner of the larger simplex $\overline{Q}_9$. By reflecting the smaller simplex around inside $\overline{Q}_9$, we observed that 527 copies of $\overline{T}_9$ subdivide $\overline{Q}_9$. Therefore the volume of $\overline{T}_9$ is the volume of $\overline{Q}_9$ divided by 527, and so

$$\mathrm{vol}_9(\overline{T}_9) = \zeta(5)/22295347200.$$

Appendix. Polylogarithms and volume tables

The classical polylogarithms (cf. [L])

$$\mathrm{Li}_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n}, \quad z \in \mathbb{C},\ n \geq 1,$$

arise quite naturally as non-Euclidean volume functions, at least for lower orders n. Heuristically, this can be explained by their inductive behaviour

$$\mathrm{Li}_1(z) = -\log(1-z), \quad \mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(t)}{t} dt,$$

which should be compared with the volume differential formula in Section 3.

The polylogarithms $\mathrm{Li}_n(z)$ are closely related to certain Dirichlet L-series. For example,

$$\mathrm{Li}_n(1) = \zeta(n) = \sum_{r=1}^{\infty} \frac{1}{r^n} \qquad \text{(Riemann's zeta function)}$$

$$\mathrm{Im}\,\mathrm{Li}_n(e^{i\frac{\pi}{2}}) = L(n) = \sum_{r=1}^{\infty} \frac{\chi(r)}{r^n} = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + - \cdots ,$$

where χ denotes the Dirichlet character modulo 4.

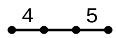
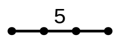
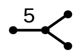
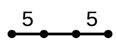
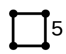
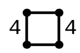
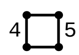
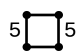
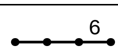
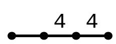
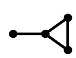
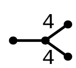
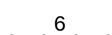
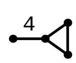
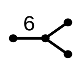
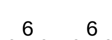
The Lobachevsky functions are derived from polylogarithms as

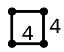
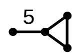
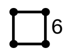
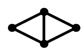
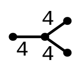
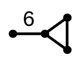
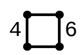
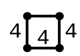
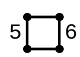
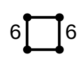
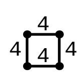
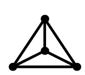
$$Л_{2m}(\alpha) := \frac{1}{2^{2m-1}} \mathrm{Im}\,\mathrm{Li}_{2m}(e^{2i\alpha}) = \frac{1}{2^{2m-1}} \sum_{r=1}^{\infty} \frac{\sin(2r\alpha)}{r^{2m}},$$

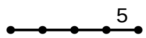
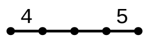
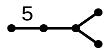
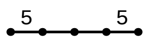
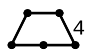
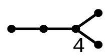
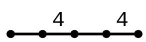
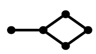
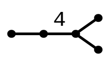
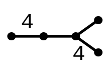
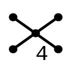
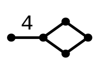
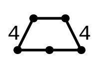
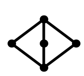
$$Л_{2m+1}(\alpha) := \frac{1}{2^{2m}} \mathrm{Re}\,\mathrm{Li}_{2m+1}(e^{2i\alpha}) = \frac{1}{2^{2m}} \sum_{r=1}^{\infty} \frac{\cos(2r\alpha)}{r^{2m+1}}.$$

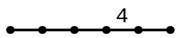
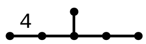
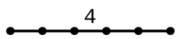
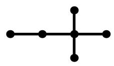
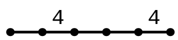
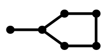
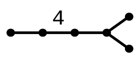
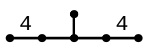
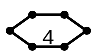
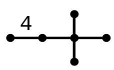
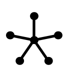
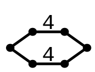
The functions $Л_k(\alpha)$ are π-periodic, even (odd) for k odd (even), and they satisfy the distribution law

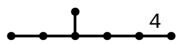
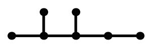
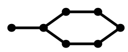
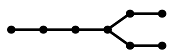
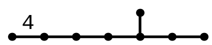
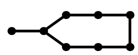
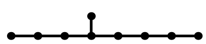
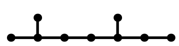
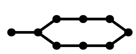
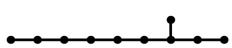
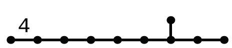
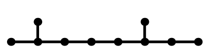
$$\frac{1}{m^{k-1}} Л_k(m\alpha) = \sum_{r=0}^{m-1} Л_k(\alpha + \frac{r\pi}{m}).$$

Coxeter Diagram	Notation	Witt Symbol	Volume Formula	Volume Value
	$[4,3,5]$	$\overline{\mathrm{BH}}_3$	$\frac{i}{4}S(\frac{1}{4}\pi, \frac{1}{3}\pi, \frac{3}{10}\pi)$	0.0358850633
	$[3,5,3]$	$\overline{\mathrm{J}}_3$	$\frac{i}{4}S(\frac{1}{6}\pi, \frac{1}{5}\pi, \frac{1}{6}\pi)$	0.0390502856
	$[5,3^{1,1}]$	$\overline{\mathrm{DH}}_3$	$\frac{i}{2}S(\frac{1}{4}\pi, \frac{1}{3}\pi, \frac{3}{10}\pi)$	0.0717701267
	$[(3^3,4)]$	$\widehat{\mathrm{AB}}_3$	See §2-3.	0.0857701820
	$[5,3,5]$	$\overline{\mathrm{K}}_3$	$\frac{i}{4}S(\frac{3}{10}\pi, \frac{1}{3}\pi, \frac{3}{10}\pi)$	0.0933255395
	$[(3^3,5)]$	$\widehat{\mathrm{AH}}_3$	See §2-3.	0.2052887885
	$[(3,4)^{[2]}]$	$\widehat{\mathrm{BB}}_3$	$7^{3/2}L(2,7)/96$	0.2222287320
	$[(3,4,3,5)]$	$\widehat{\mathrm{BH}}_3$	See §2-3.	0.3586534401
	$[(3,5)^{[2]}]$	$\widehat{\mathrm{HH}}_3$	See §2-3.	0.5021308905
	$[3,3,6]$	$\overline{\mathrm{V}}_3$	$\frac{1}{8}$ Л$(\frac{1}{3}\pi)$	0.0422892336
	$[3,4,4]$	$\overline{\mathrm{R}}_3$	$\frac{1}{6}$ Л$(\frac{1}{4}\pi)$	0.0763304662
	$[3,3^{[3]}]$	$\overline{\mathrm{P}}_3$	$\frac{1}{4}$ Л$(\frac{1}{3}\pi)$	0.0845784672
	$[4,3,6]$	$\overline{\mathrm{BV}}_3$	$\frac{5}{16}$ Л$(\frac{1}{3}\pi)$	0.1057230840
	$[3,4^{1,1}]$	$\overline{\mathrm{O}}_3$	$\frac{1}{3}$ Л$(\frac{1}{4}\pi)$	0.1526609324
	$[3,6,3]$	$\overline{\mathrm{Y}}_3$	$\frac{1}{2}$ Л$(\frac{1}{3}\pi)$	0.1691569344
	$[5,3,6]$	$\overline{\mathrm{HV}}_3$	See §3.	0.1715016613
	$[4,3^{[3]}]$	$\overline{\mathrm{BP}}_3$	$\frac{5}{8}$ Л$(\frac{1}{3}\pi)$	0.2114461680
	$[6,3^{1,1}]$	$\overline{\mathrm{DV}}_3$	$\frac{5}{8}$ Л$(\frac{1}{3}\pi)$	0.2114461680
	$[4,4,4]$	$\overline{\mathrm{N}}_3$	$\frac{1}{2}$ Л$(\frac{1}{4}\pi)$	0.2289913985
	$[6,3,6]$	$\overline{\mathrm{Z}}_3$	$\frac{3}{4}$ Л$(\frac{1}{3}\pi)$	0.2537354016

Coxeter Diagram	Notation	Witt Symbol	Volume Formula	Volume Value
	$[(3^2, 4^2)]$	$\widehat{\mathrm{BR}}_3$	$\frac{2}{3}\,\text{Л}(\frac{1}{4}\pi)$	0.3053218647
	$[5, 3^{[3]}]$	$\overline{\mathrm{HP}}_3$	See §3.	0.3430033226
	$[(3^3, 6)]$	$\widehat{\mathrm{AV}}_3$	See §2-3.	0.3641071004
	$[3^{[\,]\times[\,]}]$	$\overline{\mathrm{DP}}_3$	$\frac{5}{4}\,\text{Л}(\frac{1}{3}\pi)$	0.4228923360
	$[4^{1,1,1}]$	$\overline{\mathrm{M}}_3$	$\text{Л}(\frac{1}{4}\pi)$	0.4579827971
	$[6, 3^{[3]}]$	$\overline{\mathrm{VP}}_3$	$\frac{3}{2}\,\text{Л}(\frac{1}{3}\pi)$	0.5074708032
	$[(3, 4, 3, 6)]$	$\widehat{\mathrm{BV}}_3$	See §2-3.	0.5258402692
	$[(3, 4^3)]$	$\widehat{\mathrm{CR}}_3$	See §2-3.	0.5562821156
	$[(3, 5, 3, 6)]$	$\widehat{\mathrm{HV}}_3$	See §2-3.	0.6729858045
	$[(3, 6)^{[2]}]$	$\widehat{\mathrm{VV}}_3$	$\frac{5}{2}\,\text{Л}(\frac{1}{3}\pi)$	0.8457846720
	$[4^{[4]}]$	$\widehat{\mathrm{RR}}_3$	$2\,\text{Л}(\frac{1}{4}\pi)$	0.9159655942
	$[3^{[3,3]}]$	$\widehat{\mathrm{PP}}_3$	$3\,\text{Л}(\frac{1}{3}\pi)$	1.014916064

Coxeter Diagram	Notation	Witt Symbol	Volume Formula	Volume Value
	$[3,3,3,5]$	$\overline{\mathrm{H}}_4$	$\pi^2/10800$	0.00091385226
	$[4,3,3,5]$	$\overline{\mathrm{BH}}_4$	$17\pi^2/21600$	0.00776774420
	$[5,3,3^{1,1}]$	$\overline{\mathrm{DH}}_4$	$17\pi^2/10800$	0.01553548841
	$[5,3,3,5]$	$\overline{\mathrm{K}}_4$	$13\pi^2/5400$	0.02376015874
	$[(3^4,4)]$	$\widehat{\mathrm{AF}}_4$	$11\pi^2/4320$	0.02513093713
	$[4,3^{2,1}]$	$\overline{\mathrm{S}}_4$	$\pi^2/1440$	0.00685389195
	$[3,4,3,4]$	$\overline{\mathrm{R}}_4$	$\pi^2/864$	0.01142315324
	$[3,3^{[4]}]$	$\overline{\mathrm{P}}_4$	$\pi^2/720$	0.01370778389
	$[3,4,3^{1,1}]$	$\overline{\mathrm{O}}_4$	$\pi^2/432$	0.02284630648
	$[4,{}^{3,4}_{3}]$	$\overline{\mathrm{N}}_4$	$\pi^2/288$	0.03426945973
	$[4,3^{1,1,1}]$	$\overline{\mathrm{M}}_4$	$\pi^2/144$	0.06853891945
	$[4,3^{[4]}]$	$\overline{\mathrm{BP}}_4$	$\pi^2/144$	0.06853891945
	$[(3^2,4,3,4)]$	$\widehat{\mathrm{FR}}_4$	$\pi^2/108$	0.09138522594
	$[3^{[3]\times[\,]}]$	$\overline{\mathrm{DP}}_4$	$\pi^2/72$	0.13707783890

Coxeter Diagram	Notation	Witt Symbol	Volume Formula	Volume Value
	$[3,3,3,4,3]$	$\overline{\mathrm{U}}_5$	$7\zeta(3)/46080$	0.0001826041
	$[4,3,3^{2,1}]$	$\overline{\mathrm{S}}_5$	$7\zeta(3)/15360$	0.0005478123
	$[3,3,4,3,3]$	$\overline{\mathrm{X}}_5$	$7\zeta(3)/9216$	0.0009130206
	$[3^{2,1,1,1}]$	$\overline{\mathrm{Q}}_5$	$7\zeta(3)/7680$	0.0010956247
	$[3,4,3,3,4]$	$\overline{\mathrm{R}}_5$	$7\zeta(3)/4608$	0.0018260413
	$[3,3^{[5]}]$	$\overline{\mathrm{P}}_5$	$5^{3/2}L(3,5)/4608$	0.0020740519
	$[3,4,3,3^{1,1}]$	$\overline{\mathrm{O}}_5$	$7\zeta(3)/2304$	0.0036520826
	$[4,3,\genfrac{}{}{0pt}{}{3,4}{3}]$	$\overline{\mathrm{N}}_5$	$7\zeta(3)/1536$	0.0054781239
	$[(3^5,4)]$	$\widehat{\mathrm{AU}}_5$	See §3.	0.0075726186
	$[4,3,3^{1,1,1}]$	$\overline{\mathrm{M}}_5$	$7\zeta(3)/768$	0.0109562478
	$[3^{1,1,1,1,1}]$	$\overline{\mathrm{L}}_5$	$7\zeta(3)/384$	0.0219124956
	$[(3^2,4)^{[2]}]$	$\widehat{\mathrm{UR}}_5$	$7\zeta(3)/288$	0.0292166608

Coxeter Diagram	Notation	Witt Symbol	Volume Formula	Volume Value
	$[4, 3^2, 3^{2,1}]$	$\overline{S}_6$	$\dfrac{\pi^3}{777600}$	$0.3987432701 \times 10^{-4}$
	$[3^{1,1}, 3, 3^{2,1}]$	$\overline{Q}_6$	$\dfrac{\pi^3}{388800}$	$0.7974865401 \times 10^{-4}$
	$[3, 3^{[6]}]$	$\overline{P}_6$	$\dfrac{13\pi^3}{1360800}$	$2.9620928633 \times 10^{-4}$
	$[3^{3,2,2}]$	$\overline{T}_7$	$\dfrac{\sqrt{3}\, L(4,3)}{860160}$	$0.1892871372 \times 10^{-5}$
	$[4, 3^3, 3^{2,1}]$	$\overline{S}_7$	$\dfrac{L(4)}{362880}$	$0.2725266071 \times 10^{-5}$
	$[3^{1,1}, 3^2, 3^{2,1}]$	$\overline{Q}_7$	$\dfrac{L(4)}{181440}$	$0.5450532141 \times 10^{-5}$
	$[3, 3^{[7]}]$	$\overline{P}_7$	$\dfrac{7^{5/2} L(4,7)}{3317760}$	$4.1106779054 \times 10^{-5}$
	$[3^{4,3,1}]$	$\overline{T}_8$	$\dfrac{\pi^4}{4572288000}$	$0.0213042335 \times 10^{-6}$
	$[4, 3^4, 3^{2,1}]$	$\overline{S}_8$	$\dfrac{17\pi^4}{9144576000}$	$0.1810859845 \times 10^{-6}$
	$[3^{1,1}, 3^3, 3^{2,1}]$	$\overline{Q}_8$	$\dfrac{17\pi^4}{4572288000}$	$0.3621719690 \times 10^{-6}$
	$[3, 3^{[8]}]$	$\overline{P}_8$	$\dfrac{17\pi^4}{285768000}$	$5.7947515032 \times 10^{-6}$
	$[3^{6,2,1}]$	$\overline{T}_9$	$\dfrac{\zeta(5)}{22295347200}$	$0.0004650871 \times 10^{-7}$
	$[4, 3^5, 3^{2,1}]$	$\overline{S}_9$	$\dfrac{527\zeta(5)}{44590694400}$	$0.1225504411 \times 10^{-7}$
	$[3^{1,1}, 3^4, 3^{2,1}]$	$\overline{Q}_9$	$\dfrac{527\zeta(5)}{22295347200}$	$0.2451008823 \times 10^{-7}$

References

[B] L. Bianchi, *Sui gruppi di sostituzioni lineari con coefficienti appartenenti a corpi quadratici immaginari*, Math. Ann. **40** (1892), 332-412.

[Bo] A. Borel, *Commensurability classes and volumes of hyperbolic* 3*-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **8** (1981), 1–33.

[Bou] N. Bourbaki, *Éléments de mathématique. Fasc.* XXXIV*. Chapitres* IV*,* V *et* VI*: Groupes de Coxeter et systèmes de Tits. Groupes engendrés par des reflexions. Systèmes de racines*, Hermann, Paris, 1968. Russian translation: Н. Бурбаки, *Группы и алгебры Ли. Группы Кокстера и системы Титса. Группы, порожденные отражениями. Системы корней*, М., Мир, 1972.

[C] M. Chein, *Recherche des graphes des matrices de Coxeter hyperboliques d'ordre* ≤ 10, Rev. Française Informat. Rech. Opération **3** (1969), 3–16.

[C1] H. S. M. Coxeter, *Discrete groups generated by reflections*, Ann. Math. **35** (1934), 588–621.

[C2] H. S. M. Coxeter, *The functions of Schläfli and Lobatschevsky*, Quart. J. Math. (Oxford) **6** (1935), 13–29.

[CW] H. S. M. Coxeter, G. J. Whitrow, *World-structure and non-Euclidean honeycombs*, Proc. Royal Soc. London **A201** (1950), 417–437.

[D] H.E. Debrunner, *Dissecting orthoschemes into orthoschemes*, Geom. Dedicata **33** (1990), 123–152.

[G] Э.Б. Винберг (ред.), *Геометрия–2*, Итоги науки и техн.. Соврем. пробл. матем.. Фунд. направл., т. 29, ВИНИТИ, Москва, 1988. English translation: E. B. Vinberg (ed.), *Geometry*, II, Encyclopaedia of Math. Sciences, vol. 29, Springer–Verlag, Berlin–Heidelberg–New York, 1993.

[JW] N. W. Johnson, A. I. Weiss, *Quadratic integers and Coxeter groups*, Canad. J. Math. **51** (1999), to appear.

[K1] R. Kellerhals, *On the volume of hyperbolic polyhedra*, Math. Ann. **285** (1989), 541–569.

[K2] R. Kellerhals, *On Schläfli's reduction formula*, Math. Z. **206** (1991), 193–210.

[K3] R. Kellerhals, *On volumes of hyperbolic* 5*-orthoschemes and the trilogarithm*, Comment. Math. Helv. **67** (1992), 648–663.

[K4] R. Kellerhals, *On volumes of non-Euclidean polytopes*, in Polytopes: Abstract, Convex and Computational (T. Bisztriczky et al., ed.); vol. 440, Kluwer, Dordrecht, 1994, pp. 231–239.

[K5] R. Kellerhals, *Volumes in hyperbolic* 5*-space*, Geom. Funct. Anal. **5** (1995), 640–667.

[K] J.–L. Koszul, *Lectures on Hyperbolic Coxeter Groups, Notes by T. Ochiai*, Univ. of Notre Dame, Notre Dame, IN. (1968).

[L] F. Lannér, *On complexes with transitive groups of automorphisms*, Medd. Lunds Univ. Mat. Sem. **11** (1950), 1–71.

[Le] L. Lewin, *Polylogarithms and Associated Functions*, North Holland, 1981.

[Lo] Н. И. Лобачевский, *Применение воображаемой геометрии к некоторым интегралам*, Полн. собр. соч., М.–Л., том **3**, 1949, 181–294. German translation: N. I. Lobatschefskij, *Imaginäre Geometrie und ihre Anwendung auf einige Integrale*, Deutsche Übersetzung von H. Liebmann. Teubner, Leipzig, 1904.

[MR] C. Maclachlan, A. W. Reid, *The arithmetic structure of tetrahedral groups of hyperbolic isometries*, Mathematika **36** (1989), 221–240.

[M] R. Meyerhoff, *A lower bound for the volume of hyperbolic* 3*-orbifolds*, Duke Math. J. **57** (1988), 185–203.

[R] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds,* Graduate Texts in Math., vol. 149, Springer–Verlag, New York–Berlin–Heidelberg, 1994.

[RT] J. G. Ratcliffe, S. T. Tschantz, *Volumes of integral congruence hyperbolic manifolds*, J. Reine Angew. Math. **488** (1997), 55–78.

[Sa] C. H. Sah, *Scissors congruences,* I, *the Gauss–Bonnet map*, Math. Scand. **49** (1981), 181–210.

[S] L. Schläfli, *Die Theorie der vielfachen Kontinuität*, in Gesammelte Mathematische Abhandlungen, vol. 1, Birkhäuser, 1950.

[Si1] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. Math. **36** (1935), 527–606.

[Si2] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen,* II, Ann. Math. **37** (1936), 230–263.

[V] Э. Б. Винберг, *Дискретные группы, порожденные отражениями в пространствах Лобачевского*, Мат. Сборник **72** (1967), 471–488. English translation: E. B. Vinberg, *Discrete groups generated by reflections in Lobacevskii spaces,* Math. USSR–Sbornik **1** (1967), 429–444.

[W] E. Witt, *Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe*, Abh. Math. Sem. Univ. Hamburg **14** (1941), 289–322.