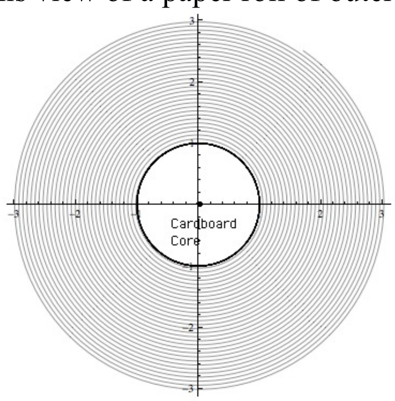
My wife, Joan, was looking at an ad in a grocery store flyer one morning. There was a special on rolls of paper towels available in two in two different size rolls, both of the about a foot wide and perforated into sheets 1 foot long.. The cost of the larger rolls was double that of the smaller roll. Joan wanted to know which size was the best buy.

On the basis of the pictures in the ad, I estimated that both types of rolls were wrapped around cardboard cylinders with an outer radius of about 1 inch, and that the outer layer of the larger roll was about 5 inches from the central axis of the cardboard core, while that of the smaller roll was about 3 inches from the central axis. Assuming that both rolls were wrapped with the same degree of tightness around their cardboard cores, say .01 inch in thickness for the paper and airspace between layers, I needed to compare how many squares of paper towel were on each of the two roll sizes in order to answer her question.

I will refer to these data and assumptions as **Joan's Paper Roll Problem** in the discussion below.

1. Modeling Joan's Paper Roll Problem

Because we are assuming that both rolls have the same width and the same tightness, we can model the paper rolls from a side view perpendicular to the central axis of the cardboard core of the rolls. Then the edge of the paper on the roll is an **Arithmetic spiral** winding out from the carbord core of radius 1 inch to the outside edge of the roll. The diagram below shows this view of a paper roll of outer radius 3 inches:



1. Calculus Solution of the Paper Roll Problem based on the Archimedean Spiral Model.

Since the width of the paper on both rolls is 1 ft., we can compute tha amount of paper on each roll by finding the length of the Archimedean spiral

(1) $$r = h(\theta) = A\,\theta$$

where θ is the angle in radians swept out as the spiral increases from the radius of 0 to the radius r of the roll. Each wind of the roll will increase r by T ft., the thickness of each sheet, while θ will increase by 2π radians, so $2\pi A = T$, so $A = \frac{T}{2\pi}$ ft. For the assumed thickness of T = (1/100) in. = 1/1200 ft. for each layer, $A = \frac{T}{2\pi} = \frac{1}{2400\pi}$ ft..

To reach the surface of the cardboard core, where r = 1/12 ft. the angle θ swept out must increase from 0 to the angle $\theta = \theta_1$ where $\frac{1}{12} ft. = \frac{1}{2400\pi}\theta_1$. Therefore, θ_1 = 200 π radians. Similarly, to reach the surface of a roll of radius R ft. in. roll, the angle θ swept out must increase from 0 to the angle $\theta = \theta_R$ so $R\ ft. = \frac{1}{2400\pi}\theta_R$, which implies that θ_R =2400 π R radians.

In particular, to reach the surface of the 5 in. roll, the angle θ swept out must increase from $\theta = 0\ radians$ to the angle $\theta = \theta_5\ radians$ so $\frac{5}{12} ft. = \frac{1}{2400\pi}\theta_5$, so θ_5 = 1000 π radians.

2. Solution of the Paper Roll Problem By Calculus.

The formula from calculus for the arc length of a curve r = h(θ) in polar coordinates between θ = c radians and θ = d radians is:

$$L(c,d) = \int_c^d \sqrt{h'(\theta)^2 + h(\theta)^2}\,d\theta$$

For the Archimedean spiral $h(\theta) = \frac{T}{2\pi}\theta,\ h'(\theta) = \frac{T}{2\pi}$, so this arc length is given by

$$(2) \qquad L(c,d = \frac{T}{2\pi}\int_c^d \sqrt{1+\theta^2}\,d\theta$$

This is an interesting integral to calculate! We will carry out the integration later but here is the indefinite integral given in one table of integrals:

$$\int \sqrt{1+\theta^2}\,d\theta = \frac{1}{2}\left[\theta\sqrt{1+\theta^2} + Ln(\sqrt{1+\theta^2} + \theta)\right] + c$$

We can express the solution (2) in terms of a function $f[\theta_d]$ of a single varible θ_d. That variable θ_d represents the angle in radians starting from 0 radians and wrapping counter clockwise around pole to the angle $\theta = \theta_d$ radians:

$$f(\theta_d) = \frac{T}{2\pi}\int_0^{\theta_d} \sqrt{1+\theta^2}\,d\theta = \frac{T}{4\pi}\left[\left[\theta\sqrt{1+\theta^2} + Ln(\sqrt{1+\theta^2} + \theta)\right]\right]_0^{\theta_d}$$

We use this result to obtain the following formula for the length L(c, d) of the arithmetic spiral:

$$L(c,d) = f[\theta_d] - f[\theta_c]$$

where $\frac{T}{4\pi} = \frac{.01}{4\pi}\text{ in.} = \frac{1}{400\pi}\text{ in.} = \frac{1}{4800\pi}\text{ ft.}$

We have already observed that $\theta_1 = 200\pi$ radians at the outer edge of the 1 in. cardboard core. Similarly, for the outer edge of a 3 in. roll, $\theta_3 = 600\pi$ radians , and for a 5 in. roll, $\theta_5 = 1000\pi$ radians. Therefore, the length in ft. of a roll of paper of radius 3 in. wrapped on cardboard core of radius 1 in. is

$$f[\theta_3] - f[\theta_1] = \frac{1}{4800\pi}\left[\theta_d\sqrt{1+\theta_d^{\,2}} + \mathrm{Ln}(\sqrt{1+\theta_d^{\,2}} + \theta_d)\right]_{200\pi}^{600\pi} \approx 209.44 \text{ ft.}$$

Similarly, the length of the 5 in. roll is

$$f[\theta_5] - f[\theta_1] = \frac{1}{4800\pi}\left[\theta_d\sqrt{1+\theta_d^{\,2}} + \mathrm{Ln}(\sqrt{1+\theta_d^{\,2}} + \theta_d)\right]_{200\pi}^{1000\pi} \approx 628.319 \text{ ft.}$$

3. Details of the Indefinite Integration of $\int\sqrt{1+\theta^2}\,d\theta$.

1) Let θ = tan r, dθ = $\sec^2$ r dr. Then $\int\sqrt{1+\theta^2}\,d\theta = \int \sec^3 r\,dr$.

2) Next, integrate-by-parts twice and simplify:

Let $u = \sec r,\ dv = \sec^2 r dr,$ *then* $du = \sec r \tan r\ dr$ *and* $v = \tan r$ *so*

$$\int \sec^3 r dr = \int \sec r \sec^2 r dr = \sec r \tan r - \int \tan^2 r \sec r dr =$$

$$\sec r \tan r - \int (\sec^2 r - 1) \sec r dr = \sec r \tan r + \int \sec r dr - \int \sec^3 r dr$$

$$\therefore 2\int \sec^3 r dr = \sec r \tan r + \int \sec r dr$$

3) From 2) we have

$$\int\sqrt{1+\theta^2}\,d\theta = \int \sec^3 r\ dr = \frac{1}{2}(\sec r \tan r + \int \sec r\ dr)$$

4) Integrate the sec r by multiplying by $1 = \dfrac{\sec r + \tan r}{\tan r + \sec r}$ to obtain an integral of the form $\int \dfrac{dw}{w}$ where w = tan r + sec r:

$$\int \sec r\ dr = \int \sec r \left[\frac{\sec r + \tan r}{\sec r + \tan r}\right] dr = \text{ (Multiply by 1)}$$

$$= \int \left[\frac{\sec^2 r + \sec r \tan r}{\tan r + \sec r}\right] dr) = \int \frac{dw}{w} = \text{ (where w = tan r + sec r)}$$

$$= \ln w + K = \ln(\tan r + \sec r) + K$$

5) By 4), we obtain:

$$\int\sqrt{1+\theta^2}\,d\theta = \int \sec^3 r\ dr = \frac{1}{2}[(\sec r \tan r + \int \sec r\ dr)] = \frac{1}{2}[(\sec r \tan r) + \ln(\tan r + \sec r)] + K$$

Use a right triangle with legs of length θ = sec r and 1 (different θ than that in the given integral !) to see that:

$$\int\sqrt{1+\theta^2}\,d\theta = \int \sec^3 r\ dr = \frac{1}{2}[(\theta\sqrt{1+\theta^2} + \ln(\sqrt{1+\theta^2} + \theta)] + K$$

Therefore, the total length $f[\theta_d]$ of a paper roll starting at c = 0 radians and wrapping until $\theta = \theta_d$ radians is:

$$f[\theta_d] = \frac{T}{2\pi}\int_0^{\theta_d}\sqrt{1+\theta^2}\,d\theta = \frac{T}{4\pi}\left[\left[\theta\sqrt{1+\theta^2} + \mathrm{Ln}(\sqrt{1+\theta^2} + \theta)\right]\right]_0^{\theta_d} =$$

$$= \frac{T}{4\pi}\left[\theta_d\sqrt{1+\theta_d^{\ 2}} + \mathrm{Ln}(\sqrt{1+\theta_d^{\ 2}} + \theta_d)\right]$$

4. Two Elementary Solutions. Dick Stanley and Patrick Callahan developed the following two simple and elegant approaches to a paper roll problem in which the successive layers are modeled as concentric paper cylinders. They developed two differentt solutions based on that model that they credited to Mickey and Minnie Mouse.

Mickey's approach: Find the number of concentric layers of paper on the roll and then divide by the length of the middle layer to find the total length of the paper on the roll.

Minnie's approach: Find the area of the paper at the end of the roll by subtracting the area of the core of the roll from the area of the entire roll end including the core. This area is equal to the length of the paper on the roll times its thickness.

Note that because, in both of these approaches, the paper roll consists of concentric cylinders of a fixed thickness rather than the Archimedean spiral used in Section 1, our calculus solutions of the Archimedean spiral model may not be the same as those obtained by the approaches by Mickey and Minnie. However, as long as the thickness of the cylindrical layers is very small in comparison to length of the roll, the differences in the computed paper roll lengths may turn out to be negligible. We'll see!

First, let's apply Mickey's appoach using assumed the paper thickness of T = .01 in. and a cardboard core radius of 1 in., the number of layers on the rolls of radius 3 in. and 5 in. respectively is given by the following calculations:

$$\frac{3\text{ in.}}{.01\text{ in}} - \frac{1\text{ in.}}{.01\text{ in}} - = 300 - 100 = 200\text{ layers}; \qquad \frac{5\text{ in.}}{.01\text{ in}} - \frac{1\text{ in.}}{.01\text{ in}} - = 500 - 100 = 400\text{ layers}.$$

The average radius of the layers in a 3 in. roll is 2 in., while the average radius of the layers in a 5 in. roll is 3 in. Consequently, the average length of a layer in a 3 in. roll is (2 in.))(2π) = 4π in. while the average length a of a layer in a 5 in. roll is (3 in)(2π) = 6π in.. It follows that:

i) the total length in feet of a 3 in. roll is (200) (4π) /12 ft $\approx$ 209.440 ft.

ii) the total length in feet of a 5 in. roll is (400) (6π) /12 ft $\approx$ 628.319 ft.

These values agree with the values s obtained with the calculus solution with the Archimedean spiral model to three decimal places.

Next, let's apply Minnie's idea: For the 3 in. roll with the 1 in. core, the area of the end of the roll minus the area of the 1 in. core is $3^2\pi - 1^2\pi = 8\pi$ square inches. For the 5 in. roll with the 1 in. core, the area of the end of the roll minus the area of the 1 in. core is $5^2\pi - 1^2\pi = 24\pi$ square inches.

From this, we can conclude that:

i) the length in feet of the 3 in.roll $\frac{8\pi}{.01} = \frac{800\pi}{12}$ = 209.440 ft.

ii) the total length in feet of a 5 in. roll is $\frac{24\pi}{.01} = \frac{2400\pi}{12} = 200\pi$ ft. $\approx$ 628.319 ft.

just as Mickey concluded.

Thus, both Mickey's and Minni's solutions of Joan's Paper Roll Problem are essentially the same and are very close to the values obtained by calculus in the previous sections.

We can generalize Mickey's solution to rolls whose outer layer have radius r inches that are wrapped around a cardboard core of radius 1 in. with the same tightness as the rolls in Joans's Paper Roll Problem so that the thickness of each layer is .01 in.

In the general case, the number of layers of paper in a roll of radius r inches is:

$$\frac{\text{r in.}}{\text{.01 in}} - \frac{\text{1 in.}}{\text{.01 in}} - = 100\text{ r} - 100 = 100\text{ (r} - 1\text{) layers}$$

and the average radius of the layers in the roll is (r + 1)/2 in.= (r + 1)/24 ft. so the total length L(r) of the paper on the roll in ft. is

$$L(r) = 100(r - 1)(2\pi)(\frac{r+1}{24}) = \frac{25\pi}{3}(r^2 - 1)$$

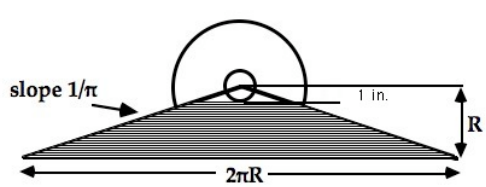
Therefore, the number of foot-square sheets of paper towel on a roll of outer radius r > 1/12 ft. is $\frac{25\pi}{3}(r^2-1)$. Note that this formula specilizes to the correct formula obtained above for rolls of radii 3 in. and 5 in.:

Next, let's apply Minnie's idea for a roll of outer radius r in. and core radius of 1 in: The area of the end of the roll in square inches minus the area of the 1 in. core is $\pi r^2 - \pi\, 1^2 = \pi(r^2-1)$ square inches. Thus, if L(r) is the length of the paper on the roll in inches, $\pi(r^2-1) = L(r)\frac{1}{100}$, so $L(r) = 100\,\pi(r^2-1)\,in. = \frac{25\pi}{3}(r^2-1)ft.$

Therefore, the number of foot-square sheets of paper towel on a roll of outer radius r > 1/12 ft. is $\frac{25\pi}{3}(r^2-1)$, which is the same as Mickey's solution.

An activity related to Mickey's and Minnie's solutions.

Dick Stanley and Patrick Callahan described the following activity that sheds light on the concentric cylinder model which is the basis of both Mickey's and Minnie's solution of the Paper Roll Problem. Suppose a paper roll of outer radius R with a cardboard core of radius 1 inch is resting on a horizontal table. Imagine cutting the paper roll along a line perpendicular to the carboard core with a razor blade until you reach that core. The layers of the roll would fall and form a trapezoid-shaped pile. The base of the pile would have length 2πR and the top of the pile would have length 2π. Consequently, the two edges of the pile would have slope 1/π as in the following diagram:



Note that in both Mickey's and Minnie's solution of Joan's Paper Roll Problem, the length L(r) in feet of the roll is the quadratic function of the outer radius r of the roll: $L(r) = \frac{25\pi}{3}(r^2 - 1)ft$.

The calculus solution of Joan's Paper Roll Problem given in Section 2 based on the arithmetic spiral model:

$$f[\theta_3] - f[\theta_1] = \frac{1}{4800\pi}\left[\theta_d\sqrt{1+\theta_d^{\,2}} + \mathrm{Ln}(\sqrt{1+\theta_d^{\,2}} + \theta_d)\right]_{200\pi}^{600\pi} \approx 209.44 \text{ ft.}$$

$$f[\theta_5] - f[\theta_1] = \frac{1}{4800\pi}\left[\theta_d\sqrt{1+\theta_d^{\,2}} + \mathrm{Ln}(\sqrt{1+\theta_d^{\,2}} + \theta_d)\right]_{200\pi}^{1000\pi} \approx 628.319 \text{ ft.}$$

which give the same values for the lengths in feet of the two roll sizes as the quadratic function $L(r) = \frac{25\pi}{3}(r^2 - 1)ft$.even though the indefinite integral does not appear to be quadratic.

In the next section, we will investigatee the mathematical connection between the arithmetic spiral and concentric cylinder models of Joan's Paper Roll Problem.

5. The Solution of the Archimedean Spiral Model of Joan's Paper Roll Problem is not quadratic but it is quadratic for all practical purposes.

We have shown in the preceding sections that the length in feet of a paper roll in the form of an Archimedean spiral of sheet thickness T = .01 in. = 1/1200 ft. that winds outward from a central angle of 0 radinas to a central angle of θ_d radians is given by the function f defined by:

$$f[\theta_d] = \frac{T}{2\pi}\int_0^{\theta_d}\sqrt{1+\theta^2}\,d\theta = \frac{T}{4\pi}\left[\left[\theta\sqrt{1+\theta^2} + \mathrm{Ln}(\sqrt{1+\theta^2} + \theta)\right]\right]_0^{\theta_d} =$$

$$= \frac{T}{4\pi}\left[\theta_d\sqrt{1+\theta_d^{\,2}} + \mathrm{Ln}(\sqrt{1+\theta_d^{\,2}} + \theta_d)\right]$$

For large positive values of $\theta = \theta_d$, the expression $\sqrt{1+\theta_d^{\,2}}$ is approximately equal to θ_d so, for large values θ_d, f[θ_d] is approximately equal to the function g defined by

g[θ_d] = q[θ_d] + nq[θ_d]

where:

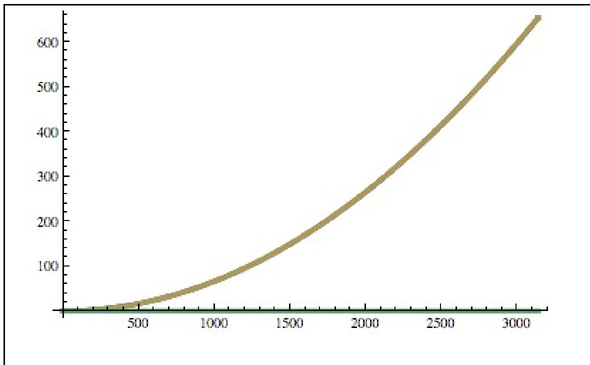
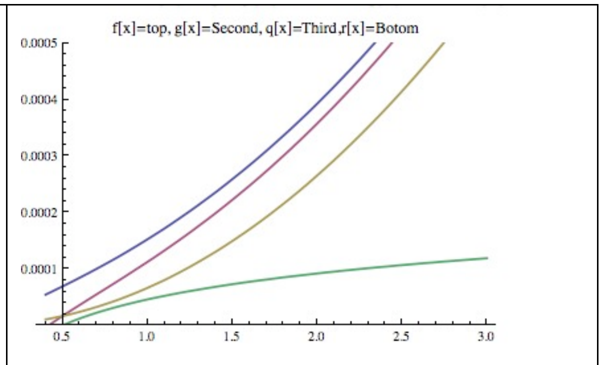
$$q[\theta_d] = \frac{T}{4\pi}\left[\theta_d^{\,2}\right] \qquad nq[\theta_d] = \frac{T}{4\pi}\mathrm{Ln}\left[2\,\theta_d\right]$$

We will call the function q and nq the **quadratic part of g** and the **non-quadratic part of g**. Note that q is a quadratic function that increases from its minimum value at 0 through positive values of θ_d.

Let's compare the values and graphs of the functions:

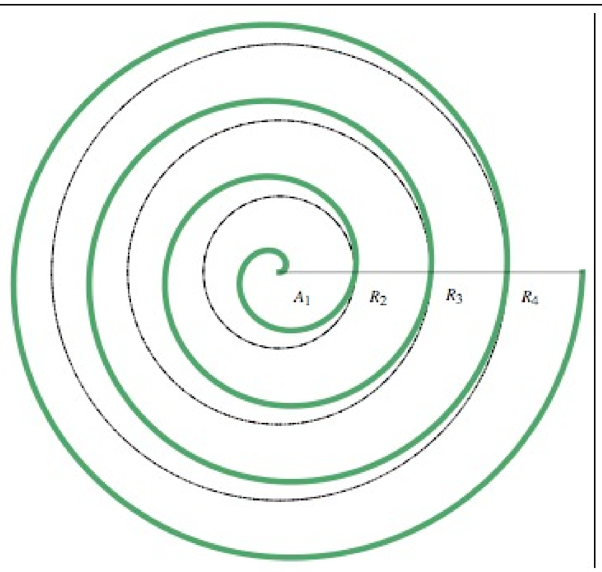
f = the solution by calculus Archimedean spiral model,
g = an approximation to the solution f for large values of the angle θ_d,
q = the quaratic part of g,
nq = the non-quadratic part of g.

When these four functions are graphed on the domain $0 < \theta_d < 3000\,\pi$ (below left) the functions f, g and q appear to coincide while the residual function r appears to be 0. However, the diagram (below right) show the graphs of the same four functions on the much smaller interval $.5 < \theta_d < \pi$. (The lower limit of .5π for θ_d was chosen to assure that the values of the residual part r(θ_d) are positive.)

	
$0 < \theta_d < 1000\,\pi$	$.5\pi < \theta_d < 3\,\pi \quad Range = (0, .815)$

3. A A very brief outline of a solution of the Paper Roll Problem based on remarkable results of Archimedes on the Archimedean Spiral.

In his book *On Spirals*, Archimedes (287 BC – 212 B.C.) used a beautiful exhaustion argument to compute the area A_1 enclosed by the first turn of the Archimedean spiral r = A θ as well as the areas A_n enclosed by the horizontal axis of nth turn of that spiral. His results are summarized in the diagram and table below:



$$r = \frac{T}{2\pi}\theta \;\Rightarrow\; A_1 = \frac{1}{3}\pi\left(2\pi\frac{T}{2\pi}\right)^2 = \frac{1}{3}\pi T^2$$

$$A_2 = \frac{7}{12}\pi\left(4\pi\frac{T}{2\pi}\right)^2 = \frac{7}{12}\pi(2T)^2 = \frac{7}{3}\pi T^2$$

$$R_2 = A_2 - A_1 = \left(\frac{7}{3} - \frac{1}{3}\right)\pi T^2 = 6\left(\frac{1}{3}\pi T^2\right) =$$

$$= 6A_1 = 2\pi T^2.$$

$$R_{n+1} = nR_2 = n\left(2\pi T^2\right) = nR_2 \;\; for\; n > 1.$$

For Minnie's solution of Joan's Paper Roll Problem, the cardboard core is of radius 1 in. and the outer radii of the two rolls are 3 in. and 5 in. respectively. The total length of the paper roll is the total area of the paper roll divided by the thickness. For a 3 in. radius roll with thickness T =.01 in. = 1/1200 ft., the result is

$$Length(3\ in.) = \frac{\left(2\pi T^2\right)\left(100 + 101 + 102 + \ldots\ldots + 298 + 299\right)}{T} = \left(2\pi T\right)\left(\frac{(299)(300)}{2} - \frac{(99)(100)}{2}\right) =$$

$$\pi T((299)(300) - (99)(100)) \approx 208.9159$$

$$Length(5\ in.) = \frac{\left(2\pi T^2\right)\left(100 + 101 + 102 + \ldots\ldots + 498 + 499\right)}{T} = \left(2\pi T\right)\left(\frac{(299)(300)}{2} - \frac{(99)(100)}{2}\right) =$$

$$\pi T((499)(500) - (99)(100)) \approx 627.2713$$