Gibbs Phenomenon for Wavelets

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Abstract

When a Fourier series is used to approximate a function with a jump discontinuity, an overshoot at the discontinuity occurs. This phenomenon was noticed by A. Michelson [Mi] and explained by J.W. Gibbs [Gi] in 1899. This phenomenon is known as the Gibbs effect.

In this paper, possible Gibbs effects will be looked at for wavelet expansions of functions at points with jump discontinuities. Certain conditions on the size of the wavelet kernel will be examined to determine if a Gibbs effect occurs and what magnitude it is. An if and only if condition for the existence of a Gibbs effect is presented, and this conditions is used to prove existence of Gibbs effects for some compactly supported wavelets. Since wavelets are not translation invariant, effects of a discontinuity will depend on its location. Also, computer estimates on the sizes of the overshoots and undershoots were computed for some compactly supported wavelets with small support.

1 Gibbs Phenomenon for Fourier Series

To illustrate what is happening in the Gibbs effect, let us examine the partial sums of a Fourier series. Let $g(x)$ be a periodic, piecewise smooth function with a jump discontinuity at x_0. For any fixed x_1, not equal to x_0, the partial sums of $g(x)$ at x_1 approach $g(x_1)$. That is, if s_n is the partial sum of g, then

$$\lim_{n\to\infty} s_n(x_1) = g(x_1)\,.$$

However, if x is allowed to approach the discontinuity as the partial sums are taken to the limit, an overshoot, or undershoot, may occur. That is,

$$\lim_{\substack{n\to\infty \\ x_n\to x_0^+}} s_n(x_n) \neq g(x_0^+)$$

and

$$\lim_{\substack{n\to\infty \\ x_n\to x_0^-}} s_n(x_n) \neq g(x_0^-)$$

are possible. This overshoot, or undershoot, is called the Gibbs phenomenon.

Proposition 1.1 *Let f be a function of bounded variation, 2π-periodic function. At each jump discontinuity x_0 of f, the Fourier series for f will overshoot (undershoot) $f(x_0^+)$ and undershoot (overshoot) $f(x_0^-)$ if $f(x_0^+)-f(x_0^-)$ is positive (negative). The overshoot and undershoot will be approximately 9% of the magnitude of the jump $\left|f(x_0^+)-f(x_0^-)\right|$.*

For further details for the Fourier series, see [Zy].

2 General wavelet structure and compactly supported wavelets

A general structure, called a multiresolution analysis, for wavelet bases in $L^2(\mathbf{R})$ was described by S. Mallat [Ma].

Let

$$... \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$$

be a family of closed subspaces in $L^2(\mathbf{R})$ where

$$\bigcap_{m\in\mathbf{Z}} V_m = \{0\}\,, \qquad \overline{\bigcup_{m\in\mathbf{Z}} V_m} = L^2(\mathbf{R}),$$

$$f \in V_m \Longleftrightarrow f(2\cdot) \in V_{m+1}\,.$$

and there is a $\phi \in V_0$ such that $\{\phi_{m,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis of V_m, where

$$\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n)\,. \tag{1}$$

Define W_m such that $V_{m+1} = V_m \oplus W_m$. Thus, $L^2(\mathbf{R}) = \sum \oplus W_m$. Then there exists a $\psi \in W_0$ such that $\{\psi_{m,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis of W_m, and $\{\psi_{m,n}\}_{m,n\in\mathbf{Z}}$ is a wavelet basis of $L^2(\mathbf{R})$,where

$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)\,. \tag{2}$$

The function ϕ is called the scaling function, and ψ is called the mother function.

Two special sums will be used in this paper. For $f \in L^2(\mathbf{R})$, the projection map of $L^2(\mathbf{R})$ onto V_m is

$$\Pi_m : L^2(\mathbf{R}) \to V_m$$

defined by

$$\Pi_m f(x) = \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x) = \sum_{n \in \mathbf{Z}} \langle f, \phi_{m,n} \rangle \, \phi_{m,n}(x) \,,$$

and $\Pi_m f(x)$ will be called a dyadic sum of f. Also, a general partial sum of f will be defined by a projection of $L^2(\mathbf{R})$ into V_{m+1}, namely,

$$\mathcal{S}_m^{l\sigma} : L^2(\mathbf{R}) \to V_{m+1}$$

is defined by

$$\mathcal{S}_m^{l\sigma} f(x) = \sum_{j=-\infty}^{m-1} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}(x) + \sum_{k=0}^{l} \left\langle f, \psi_{m,\sigma(k)} \right\rangle \psi_{m,\sigma(k)}(x) \,,$$

where $\{\sigma(k)\}_{k=0}^{l}$ is a set of $l+1$ distinct integers.

Compactly supported wavelets will be used to illustrate the Gibbs effects. The basic structure and properties needed for this paper are provided below. Further details can be found in I. Daubechies paper [Da].

Based on a decomposition and reconstruction algorithm of S. Mallat which utilizes wavelet's multiresolution analysis structure, I. Daubechies extracted the necessary conditions for constructing wavelets from a sequence of numbers $\{h(n)\}$ without reference to a multiresolution analysis.

Proposition 2.1 *Define* $m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}} h(n) e^{-in\xi}$*, where the* $h(n)$*'s satisfy*

$$\sum_{n \in \mathbf{Z}} |h(n)| \, |n|^{\epsilon} < \infty \qquad \text{for some } \epsilon > 0, \tag{3}$$

$$\sum_{n \in \mathbf{Z}} h(n) = \sqrt{2}, \tag{4}$$

and

$$\sum_{n \in \mathbf{Z}} h(n) h(n+2k) = \delta_{0,k} = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0. \end{cases} \tag{5}$$

Also, $m_0(\xi)$ *can be written in the following form:*
$m_0(\xi) = [\frac{1}{2}(1+e^{-i\xi})]^N F(\xi), N \in \mathbf{Z}_+$*, with* $F(\xi) = \sum_{n\in\mathbf{Z}} f(n)e^{-in\xi}$*, where*

$$\sum_{n\in\mathbf{Z}} |f(n)|\,|n|^{\epsilon} < \infty \qquad \text{for some } \epsilon > 0, \tag{6}$$

and

$$\sup_{\xi\in\mathbf{R}} |F(\xi)| < 2^{N-1}. \tag{7}$$

Then define $g(n) = (-1)^n h(-n+1)$*, and let* $\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$*,*

$$\phi(x) = \lim_{l\to\infty} \eta_l(x), \tag{8}$$

where

$$\eta_l(x) = \sqrt{2}\sum_{n\in\mathbf{Z}} h(n)\eta_{l-1}(2x-n) \qquad \text{and} \qquad \eta_0(x) = \chi_{[-\frac{1}{2},\frac{1}{2})}(x). \tag{9}$$

Also, define $\psi(x) = \sqrt{2}\sum_{n\in\mathbf{Z}}(-1)^n h(-n+1)\phi(2x-n)$*. Then, the set of* $\phi_{m,n}(x) = 2^{m/2}\phi(2^m x - n)$ *defines a multiresolution analysis, and the* $\{\psi_{m,n}(x)\}_{m,n\in\mathbf{Z}}$ *is the associated wavelet basis.*

Remark. Condition (3) guaranties that $\hat{\phi}$ is well defined; (4) makes $\int_{\mathbf{R}} \phi(x)\,dx = 1$; (5) gives the orthonormality to the $\{\phi_{m,n}\}$; and (6) and (7) ensure that ϕ is continuous.

Note that if $N = 1$ and $F(\xi) = 1$ in the above proposition, then $\phi(x) = \chi_{[0,1)}(x)$, and one gets the Haar system. In this case, the continuity condition (7) fails. I. Daubechies showed that $\phi(x)$ and $\psi(x)$ have compact support if and only if only a finite number of the $h(n)$'s in Proposition (2.1) are nonzero.

The following definition of I. Daubechies will be used in the following sections.

Definition 2.2 *Let* ${}_N\phi$ *and* ${}_N\psi$ *be the function defined by* $\{h(n)\}$ *satisfying the conditions of Theorem (2.1),* $h(n) = 0$ *for* $n < 0$ *and* $n > 2N-1$*,* $h(0) \neq 0$ *and* $h(2N-1) \neq 0$*.*

With these assumptions, the support size of ${}_N\phi$ can be determined.

Proposition 2.3 *The smallest interval which contains the support of ${}_N\phi$ is* $[0, 2N-1]$.

REMARK. In this paper, the fact that this is the smallest such interval is needed. A proof of this was not provided in [Da], so a proof of this will be provided here.

PROOF. From (8), it follows that the supp ${}_N\phi \subset [0, 2N-1]$. It is left to show that this interval is the smallest such interval.
Claim: There does not exist an $\epsilon > 0$ such that ${}_N\phi|_{[0,\epsilon]} \equiv 0$.
Assume there does exist such an ϵ, and let ϵ_0 be the largest such ϵ; a largest must exist since ${}_N\phi$ is continuous. By (8),

$$ {}_N\phi(x) = \sqrt{2} \sum_{n=0}^{2N-1} h(n)\, {}_N\phi(2x-n)\,, \tag{10} $$

and for $x \leq \min(\epsilon_0, 1/2 + \epsilon_0/2)$,

$$ 0 = {}_N\phi(x) = \sqrt{2} h(0) \phi(2x)\,. \tag{11} $$

Since $h(0) \neq 0$, (11) implies that $\phi(x) \equiv 0$ for $x \leq \min(2\epsilon_0, 1 + \epsilon_0)$. This violates the maximality of ϵ_0, hence there does not exist such an ϵ and the claim is true.
To show that there does not exist an $\epsilon > 0$ such that ${}_N\phi|_{[2N-1-\epsilon, 2N-1]} \equiv 0$, the same procedure can be carried out, using $x \geq 2N - 1 - \min(\epsilon_0, 1/2 + \epsilon_0/2)$ in (10) and using the fact that $h(2N-1) \neq 0$. $\square$

3 Gibbs phenomenon at the origin

To study the Gibbs effect of functions of bounded variation with a jump discontinuity at zero, it suffices to look at wavelet expansions of the function

$$ f(x) = \begin{cases} -1-x, & -1 \leq x < 0 \\ 1-x, & 0 < x \leq 1 \\ 0, & \text{else} \end{cases} \tag{12} $$

since other functions with a jump discontinuity at zero can be written in terms of f plus a function which is continuous at the origin.

3.1 A general formula for dyadic sums

We first restrict attention to the study of dyadic sum wavelet expansions of $f(x)$.

$$\Pi_m : L^p(\mathbf{R}) \longrightarrow V_m$$

$$\Pi_m f(x) = \int_{\mathbf{R}} f(y) K_m(x,y)\, dy \tag{13}$$

where $K_m(x,y) = \sum_{n\in\mathbf{Z}} \phi_{m,n}(x)\phi_{m,n}(y)$, and ϕ is the scaling function. Now,

$$\begin{aligned} \Pi_m f(x) &= \int_{-1}^{0} (-1-y) K_m(x,y)\, dy + \int_0^1 (1-y) K_m(x,y)\, dy \\ &= \int_0^\infty \chi_{[0,2^m]}(u)(1-2^{-m}u)\{K_0(2^m x, u) - K_0(2^m x, -u)\}\, du\,. \end{aligned}$$

Since what is of interest is the region about the origin as m tends to infinity, x will be set to $2^{-m}a$, where a is a fixed real number (see remark below). The above expression then becomes

$$\Pi_m f(2^{-m}a) = \int_0^\infty \chi_{[0,2^m]}(u)(1-2^{-m}u)\{K_0(a,u) - K_0(a,-u)\}\, du. \tag{14}$$

The absolute value of the argument of the integral is bounded by $|K_0(a,u)| + |K_0(a,-u)|$, which is an integrable function because of the rate of decay of wavelets. Also, the limit of this argument as m tends to infinity is $\chi_{[0,\infty)}(u)\{K_0(a,u) - K_0(a,-u)\}$. Thus, applying the Dominated Convergence Theorem to (14), one has

$$\begin{aligned} \lim_{m\to\infty} \Pi_m f(2^{-m}a) &= \int_0^\infty \{K_0(a,u) - K_0(a,-u)\}\, du \\ &= 2\int_0^\infty K_0(a,u)\, du - 1, \end{aligned}$$

since integrating the kernel over all reals is one.

Remark. If instead of choosing a as a fixed number, we had a sequence $2^{-m}a_m$, there would be two possibilities: If $a_m \to 0$, then we would end up with equation (15) with $a = 0$, and we would have the same expression as if we had chosen $a = 0$. If $a_m \to \infty$, since $2^{-m}a_m$ must tend to zero, a_m must tend to infinity slower that 2^m. Thus, because of the decay conditions of ϕ, the expression of equation (14) would tend to zero, and there would be no overshoot. This explains our choice of $x = 2^{-m}a$.

The following theorem has now been obtained.

Theorem 3.1 *For f defined in (12), $a \in \mathbf{R}$ and using the notation of (13)*

$$\lim_{m\to\infty} \Pi_m f(2^{-m}a) = 2\int_0^\infty K_0(a,u)\,du - 1. \tag{15}$$

Thus, studying a Gibbs phenomenon reduces to looking at the above integral of the wavelet kernel. Specifically, a Gibbs effect occurs near the origin if and only if

$$\int_0^\infty K_0(a,u)\,du > 1, \qquad \textit{for some } a > 0$$

and (or)

$$\int_0^\infty K_0(a,u)\,du < 0, \qquad \textit{for some } a < 0.$$

A similar result was proved independently by S.M. Gomes and E. Cortina [GoCo] for a more general class of expansions.

So far, results pertain to all wavelets. We will now look at wavelets which have compact support. It is easy to see from Theorem (3.1) that for the Haar system, where $\phi(x) = \chi_{[0,1)}(x)$, there is no Gibbs effect at the origin.

The existence of a Gibbs effect near the origin for wavelet expansions of f can be proved for certain compactly supported wavelets using the following result.

Theorem 3.2 *A Gibbs phenomenon for a dyadic wavelet expansion of $f(x)$ generated by the function ${}_N\phi$, defined in Definition (2.2), occurs at the right hand side of $f(x)$ if and only if there exists an $a > 0$ such that*

$$\begin{aligned} {}_N\phi(a+1)\int_0^1 {}_N\phi(t)\,dt \quad &+ \quad {}_N\phi(a+2)\int_0^2 {}_N\phi(t)\,dt + \ldots \\ &+ \quad {}_N\phi(a+(2N-2))\int_0^{2N-2} {}_N\phi(t)\,dt < 0. \end{aligned} \tag{16}$$

Proof. From Theorem (3.1), there exists a Gibbs effect at the right hand side of the origin if and only if there exists an $a > 0$ such that

$$\int_0^\infty K_0(a,u)\,du > 1 \qquad \left(= \int_{\mathbf{R}} K_0(a,u)\,du\right). \tag{17}$$

Since the support of ${}_N\phi$ is contained in $[0, 2N-1]$ and the integral of ${}_N\phi$ over the reals is one, (17) reduces to finding an $a > 0$ such that

$$\sum_{n=-\infty}^{2N-2} {}_N\phi(a+n) \int_n^{2N-1} {}_N\phi(t)\,dt > \sum_{n=-\infty}^{2N-2} {}_N\phi(a+n) \int_0^{2N-1} {}_N\phi(t)\,dt. \qquad (18)$$

Subtracting the appropriate terms of (18) yields (16). Thus, the theorem is proved. □

To prove the existence of a Gibbs effect for some compactly supported wavelets, the following technical lemmas will be needed.

Lemma 3.3 *For ${}_N\phi$, $N > 2$, $h(2N-2) + h(2N-1) \neq \sqrt{2}$.*

Proof. Assume the Lemma is false and $h(2N-2)+h(2N-1) = \sqrt{2}$. That would imply that $h^2(2N-2) + 2h(2N-2)h(2N-1) + h^2(2N-1) = 2$. Equation (5) implies that the sum of the first and last term is less than or equal to 1. Thus,

$$h(2N-2)h(2N-1) \geq \frac{1}{2},$$

which by the assumption of the proof can be rewritten as

$$\left[h(2N-1) - \frac{1}{\sqrt{2}}\right]^2 \leq 0.$$

This statement is only true if $h(2N-1) = 1/\sqrt{2}$. By the assumption, this implies that $h(2N-2) = 1/\sqrt{2}$ and then by equation (5), $h(n)$ would have to be zero for $N \leq 2N-3$. This last statement is false since $h(0) \neq 0$, from the construction requirements for these wavelets. Hence, the assumption in the proof is false and the Lemma is proved. □

Lemma 3.4 *For ${}_N\phi$, $N > 2$, there exists a positive integer $n < 2N-1$ such that $\int_0^n {}_N\phi(t)\,dt \neq 0$.*

Proof. Assume that the lemma is false; that is, assume that $\int_0^n {}_N\phi(t)\,dt = 0$ for all integers $n < 2N-1$. Then, since $\int_0^{2N-1} {}_N\phi(t)\,dt = 1$,

$$\int_{2N-2}^{2N-1} {}_N\phi(t)\,dt = 1 \qquad (19)$$

and

$$\int_{k-1}^{k} {}_N\phi(t)\,dt = 0 \qquad \text{for } k = 0, ..., 2N-2. \tag{20}$$

Using equations (19) and (20), and integrating equation (10) over $[2N-2, 2N-1]$, one obtains

$$\begin{aligned}
1 &= \int_{2N-2}^{2N-1} {}_N\phi(t)\,dt \\
&= \sqrt{2}\{h(2N-2)\int_{2N-2}^{2N-1} {}_N\phi(2t-(2N-2))\,dt \\
&\qquad +h(2N-1)\int_{2N-2}^{2N-1} {}_N\phi(2t-(2N-1))\,dt\} \\
&= \frac{\sqrt{2}}{2}\{h(2N-2)\int_{2N-2}^{2N-1} {}_N\phi(t)\,dt \\
&\qquad +h(2N-1)\int_{2N-3}^{2N-1} {}_N\phi(t)\,dt\} \\
&= \frac{\sqrt{2}}{2}\{h(2N-2)+h(2N-1)\}.
\end{aligned}$$

Thus, $h(2N-2) + h(2N-1) = \sqrt{2}$, which is false by Lemma (3.3). The assumption made in the proof is incorrect, and the lemma is true. □

The following result can now be proved.

Theorem 3.5 *If $h(2N-1) < 0$, then there exists a Gibbs phenomenon on the right hand side of the origin for the dyadic sum wavelet expansion of $f(x)$ generated by ${}_N\phi$.*

Proof. Letting n be the smallest integer to satisfy Lemma (3.4), equation (16) reduces to looking for an $a > 0$ such that

$$\begin{aligned}
{}_N\phi(a+n)\int_0^n {}_N\phi(t)\,dt + {}_N\phi(a+(n+1))\int_0^{n+1} {}_N\phi(t)\,dt + ... +& \\
{}_N\phi(a+(2N-2))\int_0^{2N-2} {}_N\phi(t)\,dt < 0.&
\end{aligned} \tag{21}$$

To simplify the above expression, a can be chosen such that $2N-2 \le a+n < 2N-1$. Then, equation (21) reduces to

$${}_N\phi(a+n)\int_0^n {}_N\phi(t)\,dt < 0. \tag{22}$$

By the assumption on n, $\int_0^n {}_N\phi(t)\,dt \neq 0$, and (22) can be verified if two numbers $x_1, x_2 \in [2N-2, 2N-1]$ can be found such that ${}_N\phi(x_1)$ and ${}_N\phi(x_2)$ have opposite signs.

Choose $x_1 \geq 2N - 1.5$ such that ${}_N\phi(x_1) \neq 0$. Then,

$$ {}_N\phi(x_1) = \sqrt{2}h(2N-1)\,{}_N\phi(2x_1 - (2N-1)). $$

Since $h(2N-1) < 0$, letting $x_2 = 2x_1 - (2N-1)$, the needed numbers x_1 and x_2 have been found. The theorem is now proved. □

The question now is for what values of N is $h(2N-1)$ negative? To answer this question, we need to look at I. Daubechies construction of her compactly supported wavelets, [Da]. In Section 4C of this paper, I. Daubechies defines a specific family of compactly supported wavelets. The coefficients, $h(n)$ of ${}_N\phi$ satisfy the following condition

$$ [1/2(1+e^{i\xi})]^N \sum_{n=0}^{N-1} q(n)e^{in\xi} = 2^{-1/2} \sum_{n=0}^{2N-1} h(n)e^{in\xi}, \tag{23} $$

where

$$ \left| \sum_{n=0}^{N-1} q(n)e^{in\xi} \right|^2 = \sum_{n=0}^{N-1} \left(\begin{array}{c} N-1+n \\ n \end{array} \right) \left[\frac{1}{2} - \frac{1}{4}(e^{i\xi} + e^{-i\xi}) \right]^n . \tag{24} $$

See [Da] for details.

In Equation (24), the highest exponential term on the left hand side is $q(N-1)q(0)e^{i(N-1)\xi}$, and on the right hand side, the coefficient on $e^{i(N-1)\xi}$ is negative when N is even. Thus, for N even, $q(0)$ and $q(N-1)$ have opposite signs.

In looking at the lowest and highest exponential terms in (23), we see that $h(0)$ and $h(2N-1)$ have the same signs as $q(0)$ and $q(N-1)$ respectively. Thus, when N is even, $h(0)$ and $h(2N-1)$ have opposite signs. Since the coefficients can be reversed without effecting the wavelet properties, we can choose $h(0)$ to be positive, as done in [Da]. Thus, $h(2N-1)$ is negative, and we can use Theorem (3.5) to get the following result.

Corollary 3.6 *If N is even, then there exists a Gibbs phenomenon on the right hand side of the origin for the dyadic sum wavelet expansion of $f(x)$ generated by ${}_N\phi$.*

REMARK. In [Da], the coefficients $h(n)$ were listed for the compactly supported wavelets $N = 2, 3, ..., 10$ and it can be seen that $h(0)$ and $h(2N-1)$ do not have opposite signs for the odd N wavelets listed.

REMARK. The proofs for these arguments have only worked for Gibbs effects on the right hand side of the discontinuity. To illustrate why this argument does not work for the left hand side, we can examine the wavelets generated by ${}_2\phi$ and we would need to show that $\int_0^\infty K_0(a,u)\,du < 0$, where $K_0(a,u) = \sum_{k\in\mathbf{Z}} {}_2\phi(a+k){}_2\phi(u+k)$. Since the support of ${}_2\phi$ is in $[0,3]$, our work simplifies to showing that ${}_2\phi(a+1)\int_1^3 {}_2\phi(t)\,dt + {}_2\phi(a+2)\int_2^3 {}_2\phi(t)\,dt < 0$. Using the argument of the above proof, we would restrict a between -2 and -1 and wish to show ${}_2\phi(a+2)\int_2^3 {}_2\phi(t)\,dt < 0$ by showing that the integral is nonzero and that ${}_2\phi(a_1+2)$ and ${}_2\phi(a_2+2)$ have opposite signs for some a_1 and a_2 in $(-2,-1)$. This can not be done with arguments used above. In fact, on the interval $(0,1)$, ${}_2\phi$ does not change sign; this can be seen in Daubechie's paper [Da] . In the numerical estimates of Gibbs effects which follow, Gibbs effects were observed on the left hand side, but the author has been unable to prove the existence.

Theorem (3.1) has been used to prove the existence of a Gibbs phenomenon. Now, sizes of Gibbs effects for some of I. Daubechies compactly supported wavelets will be approximated by values obtained in FORTRAN programs based on Theorem (3.1).

It is first necessary to determine where a possible Gibbs phenomenon could occur. To do this, Theorem (3.1) will be used to determine where a Gibbs effect could not occur for compactly supported wavelets.

Let

$$K_0(a,u) = \sum_{n\in\mathbf{Z}} {}_N\phi(a+n){}_N\phi(u+n).$$

For $a > 0$, when is

$$\int_0^\infty K_0(a,u)\,du \;\; = 1 \qquad \left(= \int_{\mathbf{R}} K_0(a,u)\,du \right) \tag{25}$$

true? Since the support of ${}_N\phi$ is contained in $[0, 2N-1]$,

$$\int_0^\infty K_0(a,u)\,du = \sum_{n=-\infty}^{2N-2} {}_N\phi(a+n)\int_n^{2N-1} {}_N\phi(t)\,dt. \tag{26}$$

Also,

$$\int_{\mathbf{R}} K_0(a,u)\,du = \sum_{n=-\infty}^{2N-2} {}_N\phi(a+n) \int_0^{2N-1} {}_N\phi(t)\,dt.$$

The two above sums are equal if ${}_N\phi(a+n) = 0$ for $n \geq 1$. This will at least be true if $a+n \geq 2N-1$, and thus, (25) is satisfied when $a \geq 2N-2$. Thus, there is no Gibbs effect, as defined in Theorem (3.1), for $a \geq 2N-2$ for the wavelet expansion generated by ${}_N\phi$.

Similarly, for $a < 0$, a Gibbs effect will not occur if

$$\int_0^{\infty} K_0(a,u)\,du = 0. \tag{27}$$

As seen from the sum of this integral, equation (26), equation (27) is true if ${}_N\phi(a+n) = 0$ for $1 \leq n \leq 2N-2$. This is true if $a+n \leq 0$, which implies that there is no Gibbs effect for $a \leq -(2N-2)$.

Thus, in searching for a Gibbs effect of dyadic wavelet expansions of f generated by ${}_N\phi$, one only needs to examine the region $\{2^{-m}a : a \in (-(2N-2), 2N-2)\}$ as m tends to infinity.

The next step is to use computer analysis to approximate the value of the integral $\int_0^\infty K_0(a,u)\,du$ for values of a in $[-(2N-2), 2N-2]$. From (8), ${}_N\phi(x)$ will be approximated by ${}_N\eta_l(x) = \sqrt{2}\sum_{n=0}^{2N-1} h(n)\, {}_N\eta_{l-1}(2x-n)$ for various values of l, where ${}_N\eta_0(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$. Several values of l were used until little change was noted in the output and computer time limited going any further.

Results from this computer analysis are approximate, but they do give a good idea of the size of the Gibbs effect. For any expansion by compactly supported wavelets, the Gibbs phenomenon on each side of the origin may differ because of the lack of symmetry of these wavelets. This is reflected in the results given in Table (1).

It can also be noted that the net Gibbs effect of both sides of the discontinuity seems to be decreasing with higher order wavelets. This agrees with work on periodic spline approximations done by F.B. Richards [Ri]. He examined higher order splines in approximating the function

$$g(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \end{cases} \tag{28}$$

in $L^2[-1,1]$. F.B. Richard numerically calculated the overshoot at $g(0^+)$ for splines of degree one through seven, and found that the overshoot was

$_N\phi$	$_N\phi \approx {}_N\eta_l$	left side of origin		right side of origin	
		a	$\lim_{m\to\infty} \Pi_m f(2^{-m}a)$	a	$\lim_{m\to\infty} \Pi_m f(2^{-m}a)$
$_2\phi$	$_2\eta_{11}$	-1.01	-1.04	.99	1.61
$_3\phi$	$_3\eta_8$	-1.0	-1.25	1.0	1.25
$_4\phi$	$_4\eta_7$	-0.9	-1.33	1.6	1.12
$_5\phi$	$_5\eta_7$	-0.8	-1.20	0.8	1.22

Table 1: Approximate maximum overshoot and undershoot for dyadic sum wavelet expansions $\lim_{m\to\infty} \Pi_m f(2^{-m}a)$ generated by $_N\phi$

larger than that of the Fourier series. He conjectured that this overshoot approaches the Fourier overshoot as the order of the splines goes to infinity. In a later paper with J. Foster [FoRi], this conjecture was proved.

3.2 A general formula for partial sums

For general wavelet expansions, again, $f(x)$ will be defined as in (12), and $\mathcal{S}_m^{l\sigma} f(x)$ is defined as its general partial sum,

$$\mathcal{S}_m^{l\sigma} f(x) = \Pi_m f(x) + \int_{\mathbf{R}} f(y) G_m^{l\sigma}(x,y)\, dy$$

where

$$G_m^{l\sigma}(x,y) = \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y). \tag{29}$$

It follows as in the dyadic sums case that

$$\begin{aligned}\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a) &= \left\{2\int_0^\infty K_0(a,u)\, du - 1\right\} \\ &\quad + \left\{\int_0^\infty G_0^{l\sigma}(a,u)\, du - \int_0^\infty G_0^{l\sigma}(a,-u)\, du\right\}.\end{aligned}$$

Since $\int_{\mathbf{R}} \psi = 0$, the following result is obtained.

Corollary 3.7 *For f defined in (12), $a \in \mathbf{R}$, and using the notation of (29),*

$$\begin{aligned}\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a) &= 2\int_0^\infty K_0(a,u)\, du - 1 + 2\int_0^\infty G_0^{l\sigma}(a,u)\, du \\ &= \lim_{m\to\infty} \Pi_m f(2^{-m}a) + \mathcal{G}^{l\sigma}(a)\end{aligned}$$

The term $\mathcal{G}^{l\sigma}(a)$ gives the value of the limit dependent on which additional terms are added to the dyadic sum. The $\mathcal{G}^{l\sigma}(a)$ term could shift the peak of the Gibbs effect, and could also change the size of it.

It is easy to see from Corollary (3.7) that there is no Gibbs phenomenon for partial sum Haar expansions of f.

To examine the addition of more terms for compactly supported wavelets, lets begin by examining our new term.

$$\mathcal{G}^{l\sigma}(a) = 2\sum_{k=0}^{l} {}_N\psi(a+\sigma(k)) \int_{\sigma(k)}^{\infty} {}_N\psi(t)\, dt.$$

Since the support of ${}_N\psi$ is $[-\frac{N}{2}, N]$, as shown in ([Da]), and $\int_{\mathbf{R}} {}_N\psi(t)\, dt = 0$, $\int_{\sigma(k)}^{\infty} {}_N\psi(t)\, dt \neq 0$ implies that $-\frac{N}{2} < \sigma(k) < N$, and $\mathcal{G}^{l\sigma}(a) \neq 0$ implies that $-\frac{N}{2} < a + \sigma(k) < N$. Thus, in looking for values of $\mathcal{G}^{l\sigma}(a)$, the only time that a nonzero value is obtained is when $-N - \frac{N}{2} < a < N + \frac{N}{2}$, and $-\frac{N}{2} < \sigma(k) < N$.

For the wavelets generated by ${}_2\phi$, $\sigma(k) = 0$ and 1 are the only values of concern. To illustrate that there is a change in the Gibbs effect, one can examine the case where $\sigma(k) = 0$ is the only term added. In this case, in looking at the value of $\mathcal{G}^{l\sigma}(a) = 2{}_2\psi(a) \int_0^{\infty} {}_2\psi(t)\, dt$, when $a = .99$, where the right hand Gibbs effect was observed in the numerical computations above, it can be seen from values in [Da] that this term is nonzero. Thus, either the size of the Gibbs effect is changed, or the value a for which the maximum jump occurs must moved.

To further look at this case, we shall investigate the numerical values obtained on the computer. From the above work, when $\sigma(k) = 0$, the values of $-1 < a < 2$ are of interest, and when $\sigma(k) = 1$, the interval $-2 < a < 1$ is what needs to be examined. Thus, the computer programs for this part will compute values for $\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$ over the region $-2 < a < 2$; this includes estimating the values for $\mathcal{G}^{l\sigma}(a)$, where $\sigma(k) = 0$ or -1 or both. Results are given in Table (2).

In this case, the data suggests that adding additional terms in the partial sum has lessened the Gibbs effect, but that the effect appears to occurs in the same location.

l	$\sigma(k)$	left side of origin		right side of origin	
		a	$\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$	a	$\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$
0		-1.01	-1.04	0.99	1.61
1	$\sigma(1) = 0$	-1.01	-1.04	0.99	1.30
1	$\sigma(1) = -1$	-1.01	-1.02	0.99	1.61
2	$\sigma(1) = 0$, $\sigma(2) = -1$	-1.01	-1.02	0.99	1.30

Table 2: Approximate maximum overshoot and undershoot for the general partial wavelet sums $\lim_{m\to\infty} \mathcal{S}_m^{l\sigma} f(2^{-m}a)$ generated by ${}_2\phi$ for various values of l and $\sigma(k)$ in $\mathcal{G}^{l\sigma}(a)$ (${}_2\phi$ is approximated by ${}_2\eta_{11}$.)

4 Gibbs phenomenon at a general point

Because of the translation and dilation procedure used to generate wavelets, wavelets are not translation invariant. With this fact, it is important to study Gibbs effects for wavelet expansions of functions with a discontinuity at a general point. We will work with the function

$$g(x) = f(x-b) = \begin{cases} (b+1) - x, & b < x \leq (1+b) \\ (b-1) - x, & b-1 \leq x < b \\ 0, & \text{else} \end{cases} \tag{30}$$

which has a discontinuity at the point b.

4.1 A general formula for dyadic sums

A dyadic sum expansion for the function g, defined in (30) will now be found.

$$\Pi_m : L^p(\mathbf{R}) \to V_m \tag{31}$$

$$\begin{aligned} \Pi_m g(x) &= \int_{\mathbf{R}} g(y) K_m(x,y)\, dy \\ &= \int_{b-1}^{b} [(b-1) - y] K_m(x,y)\, dy + \int_{b}^{b+1} [(b+1) - y] K_m(x,y)\, dy \\ &= \int_{2^m(b-1)}^{2^m b} [(b-1) - 2^{-m}t] K_0(2^m x, t)\, dt \\ &\quad + \int_{2^m b}^{2^m(b+1)} [(b+1) - 2^{-m}t] K_0(2^m x, t)\, dt\,. \end{aligned}$$

As m tends to infinity, points close to b are of interest, so we will let $x = 2^{-m}a + b$, where a is a fixed real number. With some changes of variables and combining, one gets

$$\begin{aligned}\Pi_m g \quad & (2^{-m}a + b) \\ = \quad & \int_0^{2^m} (1 - 2^{-m}u)\{K_0(a + 2^m b, u + 2^m b) - K_0(a + 2^m b, -u + 2^m b)\}\, du.\end{aligned} \tag{32}$$

As m approaches infinity, the $2^m b$ term in the argument of the kernels causes some difficulty. We want to remove the m dependence in the kernels. Note that

$$K_0(x, y) = \sum_{n\in\mathbf{Z}} \phi(x + n)\phi(y + n) = \sum_{n\in\mathbf{Z}} \phi(x + n' + n)\phi(y + n' + n),$$

where n' is any integer. Thus,

$$K_0(a + 2^m b, u + 2^m b) = K_0(a + 2^m b - [\![2^m b]\!], u + 2^m b - [\![2^m b]\!])\,.$$

where $[\![x]\!]$ is the greatest integer less that or equal to x. Since, as m varies, the value of $2^m b - [\![2^m b]\!]$ may vary, we will restrict m values to a set J such that this expression will be fixed for all m in J. If b is a rational number, there will be a finite number of such sets of m values, if b is irrational, there will be an infinite number of such sets and each will contain one value of m. The notation used is the following:

$$\mathbf{b_J} = 2^m b - [\![2^m b]\!], \qquad \text{for } m \in J. \tag{33}$$

Using the convention of (33), (32) becomes

$$\Pi_m g(2^{-m}a + b) = \int_0^{2^m} (1 - 2^{-m}u)\{K_0(a + b_J, u + b_J) - K_0(a + b_J, -u + b_J)\}\, du$$

for $m \in J$. Since the m dependence has been taken out of the above kernel's argument, the limit as m tends to infinity can now be taken, as was done in Section 3.1.

$$\begin{aligned}\lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a + b) \quad &= \quad \int_0^\infty \{K_0(a + b_J, u + b_J) - K_0(a + b_J, -u + b_J)\}\, du \\ &= \quad \int_{b_J}^\infty K_0(a + b_J, u)\, du - \int_{-\infty}^{b_J} K_0(a + b_J, u)\, du\end{aligned}$$

This yields a statement more general than that of Theorem (3.1).

J	left side of origin		right side of the origin	
	a	$\lim_{m\to\infty}\Pi_m g(2^{-m}a+b)$	a	$\lim_{m\to\infty}\Pi_m g(2^{-m}a+b)$
even	-0.34	-1.15	0.66	1.34
odd	-1.67	-1.002 *	1.33	1.33
* *Computer calculated number smaller, but digits insignificant*				

Table 3: Approximate maximum overshoot and undershoot for dyadic sum wavelet expansions $\lim_{m\to\infty}\Pi_m g(2^{-m}a+b)$ generated by ${}_2\phi$, where $b=\frac{1}{3}$ (${}_2\phi$ is approximated by ${}_2\eta_{11}$).

Theorem 4.1 *If g is defined as in (30), $a \in \mathbf{R}$, and using the notation of (31),*

$$\lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a+b) = 2\int_{b_J}^{\infty} K_0(a+b_J,u)\,du - 1\,,$$

where $2^m b - [\![2^m b]\!] = b_J$ for $m \in J$.

Remark. If $b = 2^k$ for some integer k, then Theorem (4.1) simplifies to the case $b=0$, which is Theorem (3.1).

Again, using Theorem (4.1) it is easy to show that there is no Gibbs phenomenon in this case.

The next examples we look at are the compactly supported wavelets. Again, as done previously, it can be shown that in looking for a Gibbs phenomenon, the region that needs to be checked is $-(2N-1) < a < 2N-1$, where a is the number from Theorem (4.1).

Computer computations were done for the wavelet expansions of g generated by ${}_2\phi$, where the point of discontinuity is $b = 1/3$. In this case, $b_J = 1/3$ when $J = \{m : m \text{ is even}\}$, and $b_J = 2/3$ when $J = \{m : m \text{ is odd }\}$. The function ${}_2\phi$ was approximated by ${}_2\eta_{11}$, and the Gibbs phenomenon was checked for $-3 < a < 3$. Result are given in Table (3)

4.2 A general formula for partial sums

The last type of expansion that will be looked at is general partial sums of a function with a jump discontinuity at any point. This can be done by looking at partial sum of g. We will write,

$$\mathcal{S}_m^{l\sigma} g(x) = \Pi_m g(x) + \int_{\mathbf{R}} g(y) G_m^{l\sigma}(x,y)\,dy, \tag{34}$$

where

$$G_m^{l\sigma}(x,y) = \sum_{k=0}^{l} \psi_{m,\sigma(k)}(x)\psi_{m,\sigma(k)}(y).$$

Similar to the work done for $\Pi_m g(2^m a+b)$, and using the notation of equation (33),

$$\begin{aligned}
&\lim_{\substack{m\in J\\ m\to\infty}} \int_{\mathbf{R}} g(y)\, G_m^{l\sigma}(2^{-m}a+b,y)\,dy \\
&\quad= \int_{b_J}^{\infty} G_0^{l\sigma}(a+b_J,u)\,du - \int_{-\infty}^{b_J} G_0^{l\sigma}(a+b_J,u)\,du \\
&\quad= 2\int_{b_J}^{\infty} G_0^{l\sigma}(a+b_J,u)\,du.
\end{aligned}$$

This gives us the following corollary.

Corollary 4.2 *For g defined in (30), $a, b \in \mathbf{R}$, and using the notation of equation (34),*

$$\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b) = \lim_{\substack{m\in J\\ m\to\infty}} \Pi_m g(2^{-m}a+b) + \mathcal{G}_{J,b}^{l,\sigma}(a),$$

where

$$\mathcal{G}_{J,b}^{l,\sigma}(a) \equiv 2\int_{b_J}^{\infty} G_0^{l\sigma}(a+b_J,u)\,du.$$

Again it can be verified that there is no Gibbs effect for the general partial Haar sum expansion of a function with a jump discontinuity at any point. This shows that in all cases concerning the Haar system, no Gibbs phenomenon occurs.

For the specific case of ${}_2\phi$, nonzero $\mathcal{G}_{J,b}^{l,\sigma}(a)$ terms may occur for values of $\sigma(k)$ equal to $-1, 0$ and 1 and $-3 < a < 3$. This area for a is the same area that was checked for a Gibbs phenomenon in the dyadic sum case. The computer program for this part estimated the $\mathcal{G}_{J,b}^{l,\sigma}(a)$ term for $-3 < a < 3$, $l = 0, 1, 2$ or 3, $\sigma(n)$ taking any one of the values $-1, 0$, and 1, and J being the set of odd or even values of m. As with the dyadic case, b will be chosen to be $1/3$. The results of the computer estimates are given in Tables (4) and (5).

This data seems to suggest that the size and the location of the maximum overshoot and undershoot may vary with the addition of extra terms in a general partial sum.

J	l	$\sigma(n)$	a	$\lim \mathcal{S}_m^{l\sigma} g*$	a	$\lim \mathcal{S}_m^{l\sigma} g*$
even	0		-0.34	-1.15	0.66	1.34
even	1	$\sigma(1) = -1$	-0.34	-1.15	0.66	1.32
even	1	$\sigma(1) = 0$	-0.34	-1.15	0.66	1.34
even	1	$\sigma(1) = 1$	-0.34	-1.19	0.66	1.34
even	2	$\sigma(1) = -1$ $\sigma(2) = 0$	-0.34	-1.15	0.66	1.33
even	2	$\sigma(1) = -1$ $\sigma(2) = 1$	-0.34	-1.19	0.66	1.32
even	2	$\sigma(1) = 0$ $\sigma(2) = 1$	-0.34	-1.18	0.66	1.34
even	3	$\sigma(1) = -1$ $\sigma(2) = 0$ $\sigma(3) = 1$	-0.34	-1.18	0.66	1.33
* *stands for* $\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$						

Table 4: Approximate maximum overshoot and undershoot for general partial wavelet sums $\lim_{\substack{m\in J\\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a+b)$ generated by ${}_2\phi$, where $J = \{m : m \text{ even }\}$ and $b = \frac{1}{3}$. (${}_2\phi$ is approximated by ${}_2\eta_{11}$).

J	l	$\sigma(n)$	a	$\lim \mathcal{S}_m^{l\sigma} g$*	a	$\lim \mathcal{S}_m^{l\sigma} g$*
odd	0		-1.67	-1.002**	1.33	1.33
odd	1	$\sigma(1) = -1$	-1.67	-1.002**	1.33	1.16
odd	1	$\sigma(1) = 0$	-1.67	-1.002**	1.33	1.33
odd	1	$\sigma(1) = 1$	-1.17	-1.004**	1.33	1.33
odd	2	$\sigma(1) = -1$ $\sigma(2) = 0$	-1.67	-1.002**	1.30	1.16
odd	2	$\sigma(1) = -1$ $\sigma(2) = 1$	-1.17	-1.004**	1.33	1.16
odd	2	$\sigma(1) = 0$ $\sigma(2) = 1$	-1.67	-1.001**	1.33	1.33
odd	3	$\sigma(1) = -1$ $\sigma(2) = 0$ $\sigma(3) = 1$	-1.67	-1.001**	1.33	1.16
* *stands for* $\lim_{\substack{m \in J \\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a + b)$						
** *last digits may be insignificant.*						

Table 5: Approximate maximum overshoot and undershoot for general partial wavelet sums $\lim_{\substack{m \in J \\ m\to\infty}} \mathcal{S}_m^{l\sigma} g(2^{-m}a + b)$ generated by ${}_2\phi$, where $J = \{m : m \text{ odd }\}$ and $b = \frac{1}{3}$. (${}_2\phi$ is approximated by ${}_2\eta_{11}$).

In Section 3.2, the addition of extra terms to the dyadic sum seemed to reduce the Gibbs effect. The results in this section seem to show that the addition of terms can also increase the Gibbs phenomenon. Another point of interest occurs when $J = \{m : m \text{ odd }\}$. The value for a for greatest undershoot appears to change. Both of these points appear to be true, but further analysis is needed here. The author hopes to get further details on the behavior of the Gibbs effect with additional terms in future work.

To conclude, this paper has given an if and only if condition for a Gibbs phenomenon for wavelets The existence of Gibbs effects has been demonstrated for some compactly supported wavelets, and size estimates for Gibbs effects for some compactly supported wavelets were found.

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References

[Da] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. on Pure and Appl. Math., **XLI** (1988), 909-996.

[FoRi] J. Foster and F.B. Richards, *Gibbs-Wilbraham Splines*, Constr. Approx., to appear.

[Gi] J.W. Gibbs, *letter to the editor*, Nature **59** (1899), 606.

[GoCo] S.M. Gomes and E. Cortina, *Some results on the convergence of sampling series based on convolution integrals*, Preprint 1992, to appear in SIAM Jr. Math. Anal.

[Ma] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbf{R})$, Trans. of the Am. Math. Soc., **315** (1989), no. 1, 69-87.

[Mi] A.A. Michelson, *letter to the editor*, Nature **58** (1898), 544-45.

[Ri] F.B. Richards, *A Gibbs phenomenon for spline functions*, J. Approx. Theory, **66** (1991), 334-351.

[Zy] A. Zygmund, *Trigonometric Series*, Cambridge Mathematical Library, 2nd ed. (1959).

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