UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIV

1997

REPRESENTATION OF GENERALISED CREATION AND ANNIHILATION OPERATORS IN FOCK SPACE

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Dedicated to the memory of Professor Włodzimierz Mlak

1. Introduction. In [2] Bargmann defines a Hilbert space B_n of all complex holomorphic functions on $\mathbf{C}^n$, square integrable with respect to the Gaussian measure. Similarly in [3] he defines a Hilbert space B_∞ of all complex holomorphic functions on l^2 such that its basis is an amalgamation of basis of B_n over all integers $n \geq 1$. One would like to have an appropriate Gauss measure μ on l^2 such, that the space B_∞ can be regarded as a space of all complex holomorphic functions on l^2 and square integrable with respect to this measure. But such measure doesn't exist ([25]).

The Bargmann spaces B_n and Toeplitz operators on these spaces appeared to be very useful to describing certain physical observables [2, 4, 5, 18; and also 15, 16, 19]. The Bargmann space B_∞ was suggested by Bargmann [3] in 1961 as a convenient functional model to realize ideas of Fock [10], Dirac [9], Friedrichs [11], Cook [7] and Segal I.E. [22; see also 23, 24].

In [3; pages 202–203] Bargmann defined the generalised creation and annihilation operators in direction $a \in l^2$ and pointed out that these operators are the same as those introduced by Friedrichs [11; pages 38–41]. The detailed account Bargmann intended to publish later. Because it has not happened and because of importance of the above outlined Bargmann model, we present here a complate proof of this theorem (Section 4).

In Section 3 unitary isomorphisms between $\operatorname{Exp} H$ and B_∞ is completely described.

Section 2 is devoted to describing the Bargmann spaces B_n, B_∞ and generalised creation and annihilation operators.

2. The Bargmann space of infinite order. In this section we recall Bargmann's definition of the Hilbert space B_∞ and some properties of this space which are useful in the sequel.

We denote by τ the set of all sequences of nonnegative integers with finite number of nonzero entries. In the sequel the set $\mathbf{Z}_+^n$ (resp. $\mathbf{C}^n$) will be interpreted as a subset of the set τ (resp. l^2). Throughout this paper we shall use the following notation:

$$\|z\|^2 := \sum_{i=1}^{\infty} |z_i|^2, \; z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}, \; \alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!,$$

$$\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n, \ldots), \; |\alpha| = \sum_{i=1}^{\infty} \alpha_i, \; e_\alpha(z) = \frac{z^\alpha}{\sqrt{\alpha!}},$$

for $z := (z_1, \ldots, z_n, z_{n+1}, \ldots) \in l^2$ and $\alpha := (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \tau$.

We now recall Bargmann's definition of the Hilbert space B_∞ ([3; p. 200]), called the Bargmann's space of an infinite order. Denote by $l^2(\tau)$ the set of all square-summable sequences indexed by the set τ. For any sequence $(f) := (f_\alpha)_{\alpha \in \tau} \in l^2(\tau)$ we define the function $f \colon l^2 \to \mathbf{C}$ as follows:

$$f(z) := \sum_{\alpha \in \tau} f_\alpha e_\alpha(z), \; z \in l^2 \tag{2.1}$$

Then f is well defined ([26; pages. 1013–1014]).
Let $B_\infty := \{f \colon (f) \in l^2(\tau)\}$. Then the function $l^2(\tau) \ni (f) \to f \in B_\infty$ is a linear bijection ([26; p. 1014]). Let now $< f, g > := < (f), (g) >_{l^2(\tau)}$. Then the map $l^2(\tau) \ni (f) \to f \in B_\infty$ is an isometric and in consequence $(B_\infty, < \cdot, \cdot >)$ is a Hilbert space. The set $\{e_\alpha \colon \alpha \in \tau\}$ is an orthonormal basis of B_∞ ([26; p. 1014]). All functions from B_∞ are entire as functions from l^2 into $\mathbf{C}$ ([14; chap. 3, sec. 3], [2; p. 1018], see also for definition [6], [8]). B_∞ is a Hilbert space with reproducing kernel ([1; pages 342-346], [26; p. 1014])

$$K(w, z) := \exp < w, z > \text{ for } w, z \in l^2 \tag{2.2}$$

and the reproducing formula is fulfilled:

$$f(z) = < f, K(\cdot, z) >, \; f \in B_\infty. \tag{2.3}$$

The following property is very important for B_∞:

$$(2.4)\qquad \begin{cases} \text{If } f \in B_\infty \text{ then } f(z) = \sum_{\alpha\in\tau} f_\alpha e_\alpha(z),\ z \in l^2. \\ \text{Conversely if } f(z) = \sum\limits_{\alpha\in\tau} f_\alpha e_\alpha(z),\ z \in l^2 \text{ and } \sum\limits_{\alpha\in\tau} |f_\alpha|^2 < \infty, \\ \text{then } f \in B_\infty \text{ ([2; p. 1018]).} \end{cases}$$

Note that:
a) there are entire functions on l^2 such that $f \notin B_\infty$, e.g.

$$f(z) := \sum_{j=0}^{\infty} (j!)^{\frac{-1}{4}} (z_j)^j.$$

b) in general the product $f \cdot g$ of two functions $f, g \in B_\infty$ need not to be in B_∞ e.g. the function

$$f(z) = \sum_{m\in N} \sum_{n_1,\cdots,n_m \in N} \left[m \cdot 2^{\frac{m}{2}} \cdot (n_1+1) \cdot \ldots \cdot (n_m+1)\right]^{-1} \cdot e_{(0,\ldots,0,n_1,\ldots,n_m,0,0,\ldots)}$$

is in $f \in B_\infty$ and $f^2 \notin B_\infty$.
c) in general, the inequality $\|f \cdot g\| \leq \|f\| \cdot \|g\|$ in B_∞ is not true e.g.
if $f(z) := z^\alpha$ and $g(z) = z^\beta$ with $\alpha = (2,1,0,0,\ldots)$ and $\beta = (1,2,0,0,\ldots)$, then $\|f\| = \|g\| = \sqrt{2}$ and $\|f \cdot g\| = 3 > \|f\| \cdot \|g\| = 2$.
For others properties of the Hilbert space B_∞ we refer to [25], [26], [27].

Finally we recall definitions and fundamental properties of generalised creation and annihilation operators ([26; pages 1021-1022]). We define generalised creation and annihilation operators in B_∞ in direction $a \in l^2$ as follows:

$$(2.5)\qquad D(A_a^+) := \{f \in B_infty : <\cdot, a> f(\cdot) \in B_\infty\}$$

$$(2.6)\qquad D(A_a^-) := \{f \in B_\infty :\ (z \to \frac{d}{d\lambda} f(z+\lambda a)|_{\lambda=0}) \in B_\infty\}$$

$$(2.7)\qquad (A_a^+ f)z := <z, a> f(z),\ f \in D(A_a^+),\ z \in l^2$$

$$(2.8)\qquad (A_a^- f)z := \frac{d}{d\lambda} f(z+\lambda a)|_{\lambda=0},\ f \in D(A_a^-),\ z \in l^2.$$

We can also describe domains of the creation and the annihilation operators:

(2.9) $$f \in D(A_a^+) \text{ if and only if } \sum_{\beta\in\tau} |\sum_{i\in\mathbf{N}} f_{\beta-\delta_i}\overline{a}_i\sqrt{\beta_i}|^2 < \infty$$

where $\delta_i = (0, \dots, 0, 1, 0, 0, \dots) \in \tau$ (1 is on "i"- position) and $f_{\beta-\delta_i} := 0$ if $\beta_i = 0$.

If $f \in D(A_a^+)$, then $A_a^+ f = \sum_{\beta\in\tau} (\sum_{i\in\mathbf{N}} f_{\beta-\delta_i}\overline{a}_i\sqrt{\beta_i})e_\beta$.

(2.10) $$f \in D(A_a^-) \text{ if and only if } \sum_{\beta\in\tau} |\sum_{i\in\mathbf{N}} f_{\beta+\delta_i} a_i\sqrt{\beta_i + 1}|^2 < \infty.$$

If $f \in D(A_a^-)$, then $A_a^- f = \sum_{\beta\in\tau} (\sum_{i\in\mathbf{N}} f_{\beta+\delta_i} a_i\sqrt{\beta_i + 1})e_\beta$.

In addition we have:

(2.11) $$K(\cdot, w) \in D(A_a^+) \cap D(A_a^-)$$

(2.12) $$A_a^- K(\cdot, w) =< a, w > K(\cdot, w) \text{ for } w \in l^2.$$

For others properties of generalised creation and annihilation operators we refer the readers to [26].

3. Fock representation of B_∞. In this section we describe unitary isomorphism between Bargmann space B_∞ and an exponential Hilbert space.

First we recall some definitions [12]. Let H be a separable Hilbert space with basis $\{b_i\}_{i\in\mathbf{N}}$. Let $H^{\otimes^n}$ (resp. $f^{\otimes^n}$) be the tensor product of n-copies of H (resp. $f \in H$) and let u_δ be a unique unitary operator such that

$$u_\delta := H^{\otimes^n} \ni f_1 \otimes \dots \otimes f_n \to f_{\delta(1)} \otimes \dots \otimes f_{\delta(n)} \in H^{\otimes^n}.$$

where δ is a permutation of n-variables. Let Q_q stand for the Hilbert subspace of B_∞ spanned by the vectors $\{e_\alpha \colon \alpha \in \tau \text{ and } |\alpha| = q\}$, where $q \in \mathbf{Z}_+$. We know that $B_\infty = \bigoplus_{q\in\mathbf{Z}_+} Q_q$ ([26; p. 1015]).

Let $H^{\otimes^o} := \mathbf{C}$ and let $P_n := \frac{1}{n!}\sum_\delta u_\delta \colon H^{\otimes^n} \to H^{\odot^n}$ be the orthogonal projection (the summation is over all n-permutations). We define $\operatorname{Exp} H := \bigoplus_{n=0}^\infty H^{\odot^n}$, where $H^{\odot^n} := P_n(H^{\otimes^n})$, and $\operatorname{Exp} f := \sum_{n=0}^\infty \frac{f^{\otimes^n}}{\sqrt{n!}}$ for each $f \in H$. Then $< \operatorname{Exp} f, \operatorname{Exp} g >= \exp < f, g >$, the set $\{\operatorname{Exp} f : f \in H\}$ is linearly dense in $\operatorname{Exp} H$ and the set

$$\left\{\left(\frac{n!}{G!}\right)^{\frac{1}{2}} P_n(b_{\gamma_1} \otimes \dots \otimes b_{\gamma_n}) : \gamma_1 \leq \dots \leq \gamma_n\right\}$$
$$= \left\{\left(\frac{n!}{\alpha!}\right)^{\frac{1}{2}} P_n(b_{\beta_1}^{\otimes^{\alpha_1}} \otimes \dots \otimes b_{\beta_s}^{\otimes^{\alpha_s}}) : s \in \{1, \dots, n\}, \alpha, \beta \in \mathbf{N}^s,\right.$$
$$\left. |\alpha| = n \text{ and } \beta_1 < \dots < \beta_s\right\}$$

is an orthonormal basis of the Hilbert space $H^{\odot^n}$, where $G = G(\gamma) \in \tau$ and

$$G_k := \operatorname{card}\{s \in \{1, \dots, n\} : \gamma_s = k\}$$

(the map $\{\gamma \in \mathbf{N}^n : \gamma_1 \leq \ldots \leq \gamma_n\} \ni \gamma \to G(\gamma) \in \tau$ is an injection). It is clear that $|G| = n$.

Now we define a unitary isomorphism between B_∞ and Exp H.

Let $J_q\Big(\Big(\frac{q!}{G!}\Big)^{\frac{1}{2}} P_q(b_{\gamma_1} \otimes \ldots \otimes b_{\gamma_q})\Big) := e_G$ for all $q \in N$. Since J_q transforms the orthonormal basis of $H^{\odot^q}$ to the orthonormal basis of Q_q, it can be extend to the unitary isomorphism between $H^{\odot^q}$ and Q_q. Let $J_o := \mathrm{Id}_{\mathbf{C}}$. Then $J := \bigoplus_{q \in \mathbf{Z}_+} J_q$ is a unitary isomorphism from $\operatorname{Exp} H$ into $B_\infty = \bigotimes_{q \in \mathbf{Z}_+} Q_q$.

Let in the sequel $\{\delta_i\}$ denote the canonical orthonormal basis of l^2 and let the function u stands for antylinear, isometric isomorphism from H onto l^2 such that $U(\alpha b_i) = \overline{\alpha}\delta_i$ for all $\alpha \in \mathbf{C}$, and $i \geq 1$. We now show that the isomorphism J can be describe as follows:

PROPOSITION 1.

$$J(\operatorname{Exp}\ f) = K(\cdot, Uf), \tag{3.1}$$

$$(J\Phi)z = < \Phi, \operatorname{Exp}(U^{-1}z) >_{\operatorname{Exp} H} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} < \Phi_n, (U^{-1}z)^{\otimes^n} >_{H^{\otimes^n}} \tag{3.2}$$

for each $\Phi \in \operatorname{Exp} H$, *where* $\Phi = \bigoplus_{n=0}^{\infty} \Phi_n$, $\Phi_n \in H^{\odot^n}$, $z \in l^2$.

PROOF. Since $< K(\cdot, Uf), K(\cdot, Ug) > = < \operatorname{Exp} f, \operatorname{Exp} g >_{\operatorname{Exp} H}$ and the set $\{\operatorname{Exp} f \colon f \in H\}$ is linearly dense in $\operatorname{Exp} H$, then there exists a unique isomorphism F from B_∞ to $\operatorname{Exp} H$ such that $F(\operatorname{Exp} f) = K(\cdot, Uf)$. More precise we can write:

$$\begin{aligned} F(\operatorname{Exp} f)z &= K(z, Uf) = \exp(< z, Uf >_{l^2}) = \exp(< f, U^{-1}z >_H) \\ &= < \operatorname{Exp} f, \operatorname{Exp}(U^{-1}z) >_{\operatorname{Exp} H} . \end{aligned}$$

But the functions $F(\cdot)z$ and $< \cdot, \operatorname{Exp} U^{-1}z >$ are linear and continuous, and the set $\{\operatorname{Exp} f : f \in H\}$ is lenearly dense in $\operatorname{Exp} H$. So the isomorphism F obtain the following from:

$$F(\Phi)z = < \Phi, \operatorname{Exp}(U^{-1}z) >_{\operatorname{Exp} H} \quad \text{for each } \Phi \in \operatorname{Exp} H.$$

If we now show that $F = J$, then our proof will be finished. We show this equation on vectors from the basis of $\operatorname{Exp} H$:

$$F\Big(\Big(\frac{n!}{G!}\Big)^{\frac{1}{2}} P_n(b_{\gamma_1} \otimes . \otimes b_{\gamma_n})\Big) z$$
$$= \frac{1}{\sqrt{G!}} < P_n(b_{\gamma_1} \otimes . \otimes b_{\gamma_n}), (U^{-1}z)^{\otimes^n} >_{H^{\otimes n}}$$
$$= \frac{1}{\sqrt{G!}} < b_{\gamma_1} \otimes \ldots \otimes b_{\gamma_n}, (U^{-1}z)^{\otimes^n} >_{H^{\otimes n}}$$
$$= \frac{1}{\sqrt{G!}} \prod_{i=1}^{n} < b_{\gamma_i}, U^{-1}z >_H = \frac{1}{\sqrt{G!}} \prod_{i=1}^{n} z_{\gamma_i} = e_G(z).$$

Thus $F = J$. QED.

PROPOSITION 2.

$$J\Phi = \sum_{\alpha \in \tau} \sqrt{\frac{|\alpha|!}{\alpha!}} < \Phi_{|\alpha|}, b^{\otimes^\alpha} > e_\alpha \tag{3.3}$$

for all $\Phi = \bigoplus_{n=0}^{\infty} \Phi_n \in \operatorname{Exp} H$, *where* $b^{\otimes^\alpha} := \bigotimes_{i=1}^{\infty} b_i^{\otimes^{\alpha_i}}$

PROOF. It follows from the Proposition 1 that

$$(J\Phi)z = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} < \Phi_n, (U^{-1}z)^{\otimes^n} >_{H^{\otimes n}} \quad \text{for all } z \in l^2. \tag{3.4}$$

We note that the following analogue of multinomial Newton formula is true

$$(\sum_{i=1}^{\infty} x_i)^{\otimes^n} = \sum_{\substack{\alpha \in \tau \\ |\alpha| = n}} \frac{n!}{\alpha!} P_n(x^{\otimes^\alpha}),$$

where $x = (x_1, \ldots, x_m, \ldots)$ is a sequence of elements of some Hilbert space such, that the series $\sum_{i=1}^{\infty} x_i$ is convergent.

The proof of (3.3) follows from linearity, continuity of the tensor product and the projection P_n, and the following computations.

The condition (3.4) implies:

$$(U^{-1}z)^{\otimes^n} = (\sum_{i=1}^{\infty} \overline{z}_i b_i)^{\otimes^n} = \sum_{\substack{\alpha \in \tau \\ |\alpha| = n}} \frac{n!}{\alpha!} \overline{z}^\alpha P_n(b^{\otimes^\alpha})$$

for all $z = (z_1, \dots, z_m, \dots) \in l^2$. Hence

$$< \Phi_n, (U^{-1}z)^{\otimes^n} >_{H^{\otimes n}} = \sum_{\substack{\alpha \in \tau \\ |\alpha| = n}} \frac{n!}{\alpha!} z^\alpha < \Phi_n, P_n(b^{\otimes^\alpha}) >_{H^{\otimes n}}$$

$$= \sum_{\substack{\alpha \in \tau \\ |\alpha| = n}} \frac{n!}{\sqrt{\alpha!}} < \Phi_n, P_n(b^{\otimes^\alpha}) > e_\alpha(z).$$

So by (3.4) we have

$$(J\Phi)z = \sum_{n \in \mathbf{Z}_+} \frac{1}{\sqrt{n!}} \sum_{\substack{\alpha \in \tau \\ |\alpha| = n}} \frac{n!}{\sqrt{\alpha!}} < \Phi_{|\alpha|}, b^{\otimes^\alpha} > e_\alpha(z)$$

$$= \sum_{\alpha \in \tau} \sqrt{\frac{|\alpha|!}{\alpha!}} < \Phi_n, b^{\otimes^\alpha} > e_\alpha(z).$$

According to (2.4), the proof is completed. QED.

4. Representation of generalised creation and annihilation operators in Fock space. In this section we precisely describe a representation of generalised creation and annihilation operators in Fock space $\operatorname{Exp} H$ with help of the transformation J.
Bargmann in [3; pages 202–203] only stated these in the case $H = L^2(\mathbf{R}^3)$.
We first describe the above mentioned representation in general framework.

PROPOSITION 3. *Let $a \in l^2$. Then*

(4.1)
$$J(H^{\odot^n}) \subset D(A_a^{)}), \qquad \text{and}$$
$$J^{-1}A_a^+ J(\Phi_n) = \sqrt{n+1}P_{n+1}(\Phi_n \otimes U^{-1}a)$$

for all $\Phi \in H^{\odot^n}$, $n \geq 0$, and

(4.2)
$$J^{-1}A_a^- J(\operatorname{Exp} f) = < f, U^{-1}a >_H \operatorname{Exp} f$$

for each $f \in H$.[1]

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[1] $J(\operatorname{Exp} f) = K(\cdot, Uf) \in D(A_a^-)$. See the property (2.11).

PROOF. We first prove the condition (4.1). Let $\Phi_n \in H^{\odot^n}$. From the condition (3.2) we obtain:

$$\begin{aligned} A_a^+ J(\Phi_n) z &=< z, a >< \Phi_n, \operatorname{Exp}(U^{-1}z) >_{\operatorname{Exp} H} \\ &=< U^{-1}a, U^{-1}z >_H \frac{1}{\sqrt{n!}} < \Phi_n, (U^{-1}z)^{\otimes^n} >_{H^{\otimes^n}} \\ &= \frac{1}{\sqrt{n!}} < \Phi_n \otimes U^{-1}a, (U^{-1}z)^{\otimes^{n+1}} >_{H^{\otimes^{n+1}}} \\ &= \frac{1}{\sqrt{(n+1)!}} < \sqrt{n+1}\Phi_n \otimes U^{-1}a, (U^{-1}z)^{\otimes^{n+1}} >_{H^{\otimes^{n+1}}} \\ &=< P_{n+1}(\sqrt{n+1}\Phi_n \otimes U^{-1}a), \operatorname{Exp}(U^{-1}z) >_{\operatorname{Exp} H} \\ &= J(P_{n+1}(\sqrt{n+1}\Phi_n \otimes U^{-1}a))z \qquad \text{for all } a, z \in l^2. \end{aligned}$$

But $J(P_{n+1}(\sqrt{n+1}\Phi_n \otimes U^{-1}u)) \in B_\infty$. So $J(\Phi_n) \in D(A_a^+)$ and

$$J^{-1} A_a^+ J(\Phi_n) = \sqrt{n+1} P_{n+1}(\Phi_n \otimes U^{-1}a).$$

We now show the condition (4.2).
From Proposition 1 and the property (2.12) we obtain:

$$\begin{aligned} A_a^- J(\operatorname{Exp} f) = A_a^- K(\cdot, Uf) &=< a, Uf > K(\cdot, Uf) \\ &=< f, U^{-1}a >_H K(\cdot, Uf) \\ &= J(< f, U^{-1}a >_H \operatorname{Exp} f), \end{aligned}$$

This ends our proof. QED.

Proposition 3 shows us, that the generalised creation operator can be regarded as a weighted-operator shift

$$\operatorname{Exp} H \ni (\Phi_0, \Phi_1, \dots) \to (0, A_0^+\Phi_0, A_1^+\Phi_1, \dots) \in \operatorname{Exp} H$$

with the operator-weights $A_i^+ = \sqrt{i+1} P_{i+1}(\cdot \otimes U^{-1}a) \colon H^{\odot^i} \to H^{\odot^{i+1}}$.

Let now M is a σ-field on Ω and let m is a nonnegative measure on M. Then $L^2(\Omega, M, m)$ is a Hilbert space.

A measure m is called separable if there exists a countable subfamily $M' \subset M$ such that for any $\sigma \in M$ with $m(\sigma) < \infty$ and any $\epsilon > 0$ one can find a set $\gamma \in M'$ with $m(\gamma - \sigma) + m(\sigma - \gamma) < \epsilon$.

If m is a separable measure, then $L^2(\Omega, M, m)$ is a separable Hilbert space. On the other hand, if $L^2(\Omega, M, m)$ is separable and m is σ-finite, then m is a separable measure ([20; pages 82–83]).

For the Hilbert space $H = L^2(\Omega, M, m)$ we can rewrite Proposition 3 as follows:

PROPOSITION 4. *Let $H = L^2(\Omega, M, m)$ and m be a separable, nonnegative measure. Then:*

$$[(J^{-1}A_a^+J)\Phi_n](x_1, \dots, x_{n+1})$$

(4.3)
$$= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \Phi_n(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \cdot U^{-1}a(x_i)$$

$$a.e. \bigotimes^n \,]^2$$

for each $n \geq 1$ and $\Phi_n \in H^{\odot^n}$.

(4.4)
$$[(J^{-1}A_a^-J)\Phi_n](x_1, \dots, , x_{n-1})$$
$$= \sqrt{n} \int_\Omega \Phi_n(x_1, \dots, x_{n-1}, t)\overline{U^{-1}a(t)}m(dt) \quad a.e. \bigotimes^n m$$

for each $n \geq 1$ and

$$\Phi = \bigoplus_{n=0}^{\infty} \Phi_n \in \operatorname{Exp} H \quad \textit{such that} \quad J\Phi \in LIN\{\operatorname{Exp} f \colon f \in H\}$$

PROOF. The proof of (4.3) follows immediately from the condition (4.1), symmetry of Φ_i and the following computations:

$$P_{n+1}(\sqrt{n+1}\Phi_n \otimes U^{-1}a)(x_1, \dots, x_{n+1})$$
$$= \frac{1}{(n+1)!} \sum_{\sigma \in \beta(I,I)} (\sqrt{n+1}\Phi_n \otimes U^{-1}a)(x_{\sigma(1)}, \dots, x_{\sigma(n+1)})$$
$$= \frac{\sqrt{n+1}}{(n+1)!} \sum_{\sigma \in \beta(I,I)} \Phi_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \cdot U^{-1}a(x_{\sigma(n+1)})$$
$$= \frac{\sqrt{n+1}}{(n+1)!} \sum_{i=1}^{n+1} U^{-1}a(x_i) \sum_{\sigma \in \beta(I_{n+1}, I_i)} \Phi_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$
$$= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \Phi_n(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})U^{-1}a(x_i) \;\; a.e. \bigotimes^n m,$$

[2] $\bigotimes L^2(\Omega, M, m) \cong L^2(\Omega^n, \bigotimes^n M, \bigotimes^n m)$, where Ω^n denotes the Cartesian product of n-copies of the set Ω and $(\bigotimes^n M, \bigotimes^n m)$ denotes the tensor product of n-copies of the measure space (M, m) ([21, pages 51–52], [12]).

where $I := \{1, \dots, n+1\}, I_k := I \backslash \{k\}$ and $\beta(I_k, I_l)$ (resp. $\beta(I, I)$) is the set of all bijections from I_k to I_l (resp. from I onto I).
We now prove condition (4.4). It follows from condition (4.2):

$$P_{n-1}J^{-1}A_a^- J(\operatorname{Exp} f) = P_{n-1}(< f, U^{-1}a >_H \operatorname{Exp} f)$$
$$=< f, U^{-1}a >_H \frac{f^{\otimes^{n-1}}}{\sqrt{(n-1)!}} = \int_\Omega f(t)\overline{U^{-1}a(t)}m(dt) \cdot \frac{f^{\otimes^{n-1}}}{\sqrt{(n-1)!}}.$$

So we obtain

$$[P_{n-1}J^{-1}A_a^- J(\operatorname{Exp} f)](x_1, \dots, x_{n-1})$$
$$= \int_\Omega f(t)\overline{U^{-1}a(t)}m(dt) \cdot \frac{f^{\otimes^{n-1}}}{\sqrt{(n-1)!}}(x_1, \dots, x_{n-1})$$
$$= \frac{1}{\sqrt{(n-1)!}} \int_\Omega f^{\otimes^{n-1}}(x_1, \dots, x_{n-1}) f(t)\overline{U^{-1}a(t)}m(dt)$$
$$= \frac{1}{\sqrt{(n-1)!}} \int_\Omega f^{\otimes^{n}}(x_1, \dots, x_{n-1}, t)\overline{U^{-1}a(t)}m(dt)$$
$$= \sqrt{n} \int_\Omega \frac{f^{\otimes^{n}}}{\sqrt{n!}}(x_1, \dots, x_{n-1}, t)\overline{U^{-1}a(t)}m(dt)$$
$$= \sqrt{n} \int_\Omega (P_n \operatorname{Exp} f)(x_1, \dots, x_{n-1}, t)\overline{U^{-1}a(t)}m(dt) \quad a.e. \bigotimes^n m.$$

Now by linearity we obtain our statement. QED.

The condition (4.4) from Proposition 4 is true also for all $\Phi \in \operatorname{Exp} H$ such, that $J\Phi \in D(A_a^-)$, but only if we additionally assume that measure m is σ-finite. Then we can use the Fubini theorem ([13; p. 148]).

PROPOSITION 5. *Let $H = L^2(\Omega, M, m)$ and let m is separable and σ-finite. Then*

$$[(J^{-1}A_a^- J)\Phi_n](x_1, \dots, x_{n-1}) = \sqrt{n} \int_\Omega \Phi_n(x_1, \dots, x_{n-1}, t)\overline{U^{-1}a(t)}m(dt)$$

for $n \geq 1$ and

$$(J^{-1}A_a^- J)\Phi_0 = 0.$$

PROOF. We known from [26; p. 1025] that $(A_a^+)^* = A_a^-$. So obviously $J^{-1}A_a^-J = (J^{-1}A_a^+J)^*$. It follows from the above remarks, Fubini theorem and the conditions (4.1), (4.3), that for any $\Psi = \bigoplus \Psi_n$, $\Phi = \bigotimes \Phi_n \in \operatorname{Exp} H$ such, that $J\Psi \in D(A_a^-)$, $J\Phi \in D(A_a^+)$ we have:

$$
\begin{aligned}
&< P_{n-1}J^{-1}A_a^-J\Psi, \Phi >_{\operatorname{Exp} L^2(m)} \\
&=< \Psi, (J^{-1}A_a^+J)P_{n-1}\Phi >_{L^2(m)^{\otimes n}} \\
&=< \Psi_n, \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \Phi_{n-1}(\mathbf{x}_i)U^{-1}a(x_i) >_{L^2(m)^{\otimes n}} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{\Omega^n} \Psi_n(\mathbf{x})\overline{\Phi_{n-1}(\mathbf{x}_i)}\cdot\overline{U^{-1}a(x_i)}\bigotimes^n m\ (d\mathbf{x}) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{\Omega^{n-1}} \overline{\Phi_{n-1}(\mathbf{x}_i)}\Big(\int_{\Omega} \Psi_n(\mathbf{x})\overline{U^{-1}a(x_i)}(dx_i)\Big)\bigotimes^{n-1} m\ (d\mathbf{x}_i)
\end{aligned}
$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{x}_i = (x_1 \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $d\mathbf{x} = (dx_1, \dots, dx_n)$ and $d\mathbf{x}_i = (dx_1, \dots, dx_{i-1}, dx_{i+1}, \dots, dx_n)$. However the function

$$\Psi_{n-1}(\mathbf{x}_i) := \int_{\Omega} \Psi_n(\mathbf{x})\overline{U^{-1}a(x_i)}m(dx_i)$$

is symmetric and it belongs to $L^2(m)^{\odot^{n-1}}$.
So we can write:

$$
\begin{aligned}
&< (P_{n-1}J^{-1}A_a^-J)\Psi, \Phi >_{\operatorname{Exp} L^2(m)} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{\Omega^{n-1}} \\
&\Psi_{n-1}(\mathbf{x}_i)\overline{\Phi_{n-1}(\mathbf{x}_i)}m(dx_1)\dots m(dx_{i-1})m(dx_{i+1})\dots m(dx_n)
\end{aligned}
$$

It is obvious, that the number

$$
\begin{aligned}
N &:= \int_{\Omega^{n-1}} \Psi_{n-1}(\mathbf{x}_i)\overline{\Phi_{n-1}(\mathbf{x}_i)}m(dx_1)\dots m(dx_{i-1})m(dx_{i+1})\dots m(dx_n) \\
&= \int_{\Omega^{n-1}} [\Psi_{n-1}(y_1, \dots, y_{n-1})\overline{\Phi_{n-1}(y_1, \dots, y_{n-1})}m(dy_1)\dots m(dy_{n-1})
\end{aligned}
$$

is independent of "i". So finally we obtain:

$$< (P_{n-1}J^{-1}A_a^- J)\Psi, \Phi >_{\operatorname{Exp} L^2(m)}$$
$$= \frac{1}{\sqrt{n}} \cdot n \cdot N = \sqrt{n} \ < \Psi_{n-1}, \Phi_{n-1} >_{L^2(m)^{\odot^{n-1}}}$$
$$=< \sqrt{n} \int_\Omega \Psi_n(\cdot, t)\overline{U^{-1}a(t)}m(dt), \Phi >_{\operatorname{Exp} L^2(m)} .$$

It follows from the above, the property:
$A_a^-(Q_q \cap D(A_a^-)) = Q_{q-1} \cap R(A_a^-)$ ([26; conditions (2.10), (5.6), (5.8)]) and the density of $D(A_a^+)$ in B_∞ ([26; Proposition 4]) that our statement is true, where $R(A_a^-)$ denotes the image of the operator A_a^-. QED.

Proposition 5 shows us that for $H = L^2(\Omega, M, m)$ under appropriate assumptions about m, the generalised annihilation operator can be regarded as a weighted-operator shift

$$\operatorname{Exp} H \ni (\Phi_0, \Phi_1, \dots) \to (A_1^-\Phi_1, A_2^-\Phi_2, \dots) \in \operatorname{Exp} H$$

with the operator weights
$(A_i^-\Phi_i)z = \sqrt{i}\int_\Omega \Phi_i(z,t)\overline{U^{-1}a(t)}m(dt)$ for $i \geq 1$, where $A_i^- : H^{\odot^i} \to H^{\odot^{i-1}}$.
Such a definition for generalised annihilation and creation operators was proposed by Friedrichs ([11; pages 38–40]). If we take $a = \delta_i = (0, \dots, 0, 1, 0, \dots)$ then the generalised creation and annihilation operators in direction a are the following operator-weighted shifts with operator-weights

$$A_n^+\Phi_n(x_1, \dots, x_{i+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \Phi_n(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)b_i(x_i)$$

and respectively

$$A_n^-\Phi_n(x_1, \dots, x_{n-1}) = \sqrt{n}\int_\Omega \Phi_n(x_1, \dots, x_{n-1}, t)\overline{b_i(t)}m(dt),$$

and they represent the multiplication operator by n-th coordinate on B_∞ and the complex differentiation operator with respect to n-variable on B_∞, respectively.

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Received October 12, 1996