Subsets and Proper Subsets

1 Subsets $\subseteq$

A set A is a subset of a set B if every element in set A is also an element in set B.

Example 1 *Given that* $A = \{1, 5, 7, 11\}$ *and* $B = \{1, 3, 4, 5, 7, 9, 11, 13\}$, *is A is a subset of B?*

Since every element of set A is also an element of set B we say that A is a subset of B.

Or in symbolic form $A \subseteq B$.

An English interpretation of a subset would be that all of A is "inside" of B

Example 2 *Given that* $A = \{x \epsilon N | x < 10\}$ *and* $B = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$, *are either sets a subset of the other?*

Here neither all the elements of A are in B or visa versa. So what can we say? Well they both have a cardinal number of 9, so they are equivalent.

Notice that the symbol for subset looks like the symbol for "less than or equal" $\leq$. ($\subseteq$) A subset has to be smaller or equal to the other set.

2 Proper Subsets $\subset$

Now if $A \subseteq B$ and $A \neq B$ then we say that A is a Proper Subset of B. (every element of A is in B but they are not equal)

Example 3 *Given that* $A = \{a, e, i, o, u\}$ *(could this also be* $A = \{x \in letters | x \text{ is a vowel}\}$*???)*

and $B = \{x \in alphabet | x \text{ is any letter}\}$. *What can be said about the relationship between the two sets?*

Since every element is A is also an element of B, then $A \subseteq B$. *Because they are not equal, they become Proper Subsets* $A \subset B$. *Notice that if two sets are Proper Subsets that they are first Subsets.*

Example 4 *Given that* $A = \{x \in \mathbb{N} | \, x \text{ is a prime \# and } x < 13\}$ *and* $B = \{2, 3, 5, 7, 9\}$

1. *Is $A \subseteq B$? If so is $A \subset B$?*
2. *Is $B \subseteq A$? If so is $B \subset A$?*
   State your reasonings for each.

1. (a) i. There is an element in A that is not in B, namely 11, so $A \nsubseteq B$. To be proper subset you must first be a subset, so $A \not\subset B$
   ii. All the elements of B are also in A, thus $B \subseteq A$ and since $B \neq A$, we get also that $B \subset A$. (Both subset and proper-subset)

3 Subsets and the Empty Set:

Prove: The empty set is a Subset of ALL sets.

To prove this we are going to use what is called an indirect proof, where we assume the opposite of what we are trying to prove, come up with a contradiction, and then determine since what we assumed was true is false, that means that the opposite of what we assumed must be True.

Sounds confusing doesn't it.

Assume: The empty set is not a Subset of any set. $(\varnothing \nsubseteq A)$. Thus there must be an element in $\varnothing$ that is not in A. But there are no elements in $\varnothing$, so our assumption that $\varnothing \nsubseteq A$ must be wrong and its opposite true.

Thus: $\varnothing \subseteq A$ for any set A In English, "The empty set is a subset of all sets"

4 Number of Subsets

Making a table that lists out the number of subsets a given set has we find by increasing the number of elements that:

# elements	**Set**	**Subsets**	**#subsets**
0	$\{\}$	$\{\}$	1
1	$\{a\}$	$\{\}, \{a\}$	2
2	$\{a, b\}$	$\{\}, \{a\}, \{b\}, \{a, b\}$	4
3	$\{a, b, c\}$	$\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b\}$	8
4	$\{a, b, c, d\}$	Follow the pattern and the next would have:	16

There is a nice pattern here, to get to the next number of subsets we multiply the last answer by 2. Another way to achieve this would be to see that:

$$\begin{array}{cccccc} & 1 & 2 & 4 & 8 & 16 \\ = & 2^0 & 2^1 & 2^2 & 2^3 & 2^4 \end{array}$$ which is 2 raised to the power of the Cardinal Number of the set:

$$\#subsets = 2^{n(A)}$$

Example 5 *How many subsets does the set* $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ *have?*

Well its cardinal Number is 10 so: $2^{n(A)} = 2^{10} = 1024$ *subsets!*

On a calculator this can be found using ^ on most TI calculators or x^y or y^x buttons:

2 $\boxed{\wedge}$ 10 or 2 $\boxed{x^y}$ 10 or 2 $\boxed{y^x}$ 10

Example: True or False:

1. $\{1, 2, 3, 4\} \subseteq \{x \epsilon N | x < 5\}$

2. $\{1, 2, 3, 4\} \subset \{x \epsilon N | x < 5\}$

3. $\{\varnothing\} \subset \{\ \}$

4. $\{5\} \in \{1, 2, 3, 4, 5\}$

5. $6 \subset \{3, 6, 9, 12\}$

6. $\{\} \subset \{1\}$

Answers:

1. True, every element of the first set is also an element of the second set.

2. False, the sets are equal

3. False, there is an element in the first, namely ϕ, that is not in the second set.

4. False, the set of 5 {5} is not an element of the second set

5. False, 6 is not even a set, so it can't be a subset

6. True, the empty set is a subset of all sets and it is not equal to the second set.