DECLINATION LINES DETAILED

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INTRODUCTION

Declination lines on planar sundials can be drawn using the formulae giving the x and y co-ordinates from the sub-nodal point on the dial face and requiring the sun's altitude and azimuth to be calculated for each point.[1] This is both complicated and time consuming and does not reveal the relationship between the style height, sub-style line and declination lines.

Earlier work of mine describing the use of direction cosines in the delineation of planar sundials[2] led to the production of a formula giving the distance R of a declination line from the origin of the dial along an hour line in terms of style height SH, the hour angle h_0 and L the distance of the nodus from the origin along the style.

$$R = \frac{L\cos\delta}{\sin(E+\delta)} \tag{1}$$

where

$$\tan E = \frac{\cos h_0}{\tan SH} \tag{2}$$

E is the angle between the sun's direction at equinox and the hour line in the hour plane.[2]

Waugh[3] described a graphical method attributed to Leybourn[4] and Lennox-Boyd[5] offered a formula based on this construction in terms of SH and X, the hour line angle, which leads to the same value of E, which he refers to as t.

I have developed my formula, eliminating E, to allow declination lines to be drawn using polar co-ordinates knowing only the style height SH and the hour angle h_0. I have also developed Leybourn's method, producing a protractor to enable declination lines to be drawn simply for any style height and declination angle. Declination lines on planar dials are hyperbolae and I show that these can be plotted with cartesian co-ordinates using only the style height SH and the declination δ. This form also allows previously delineated dial faces to be simply checked for accuracy and their style height and nodus distance determined.

POLAR CO-ORDINATE FORM

The most important simplifying feature is the recognition that the position and shape of declination lines with respect to the origin and the sub-style line on any planar dial depend <u>only</u> on the style height of the dial and the distance L of the nodus from the origin along the sloping style. Inspection of any planar dial with declination lines shows that they are symmetrical about the sub-style line.

Fig. 1 illustrates this for a horizontal, a vertical south facing and a vertical declining dial all reading local apparent time (L.A.T.). They are for different latitudes but all have the same style height and the same pattern of declination lines about the sub-style line. They have different labels on their hour lines depending on their orientation and time zone, but the relationship between the origin, the intersection of the sub-style line with the declination lines and the shape of the curves depend only on the style height ***SH*.**

The position of every point on a declination line is given by the polar co-ordinates R and X_0, given by the formulae:-

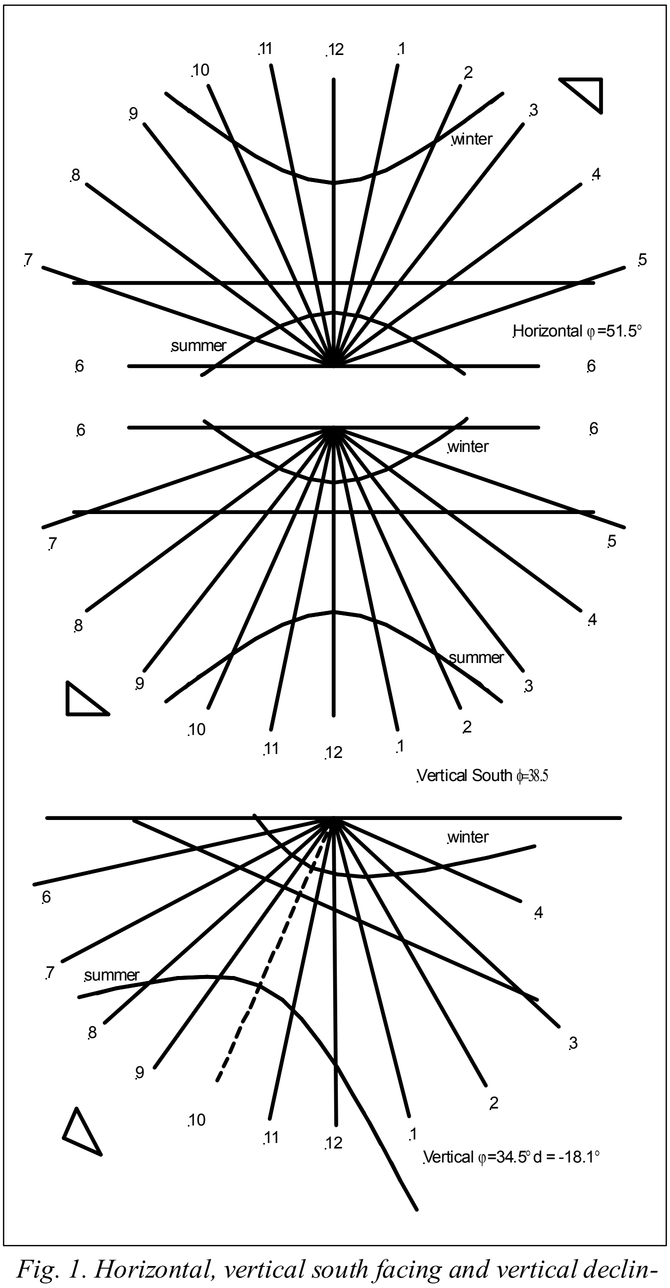

Fig. 1. Horizontal, vertical south facing and vertical declining dial faces all with the same style height.

$$\tan X_0 = \sin SH \times \tan h_0 \tag{3}$$

$$R = \frac{L\sqrt{(1-\cos^2 SH \times \sin^2 h_0}}{\cos SH \times \cos h_0 - \sin SH \times \tan \delta} \tag{4}$$

$$h_0 = (T_{24} - 12) \times 15°$$

where

X_0 = hour line angle (X_0=0 at noon LAT)
SH = style height
h_0 = hour angle (on sub-style line $h_0 = 0$)
δ = sun's declination
L = distance of nodus from origin along sloping style.
T_{24} = time in 24-hour clock notation.

The sign of SH is positive for a horizontal dial and negative for a vertical dial, as this gives the correct distances from the origin for the summer and winter solstice lines.

Declination lines can be drawn using the polar co-ordinates R and X_0 shown above.

The position and separation of the hour lines also only depends on the style height but they are labelled in accordance with the relevant longitude or equivalent longitude or standard time zone for dials that are not horizontal or south facing vertical.

HOUR LINE ANGLES

The hour angle h_0 used to determine the declination lines is the hour angle for a horizontal dial indicating local apparent time or a vertical dial indicating local apparent time at the location of equivalent latitude and longitude where the dial would be a south facing vertical dial.[3] To calculate the hour line angles X for a horizontal dial indicating time for a standard time zone or a vertical declining dial indicating local apparent time at its actual location or standard time at that location is quite straightforward. In this case we use the formula:

$$\tan X = \sin SH \times \tan h \tag{5}$$

a) Standard Time

The formula 5 above applies for all calculations, but the value of h must be chosen to fit the time indication required. If the location is θ degrees east of the standard meridian

$$h = h_0 + \theta$$

If the location is west of the standard meridian the value of θ is negative.

These corrections apply to horizontal and vertical dials.

b) Vertical Declining Dials

A vertical declining dial at latitude φ and longitude λ declining by angle d is the same as a south facing vertical dial at the equivalent latitude φ' and equivalent longitude λ' given by the formulae[6]

$$\cos\phi' = \cos d \times \cos\phi \quad \text{and} \quad \tan\lambda' = \frac{\tan d}{\sin\phi}$$

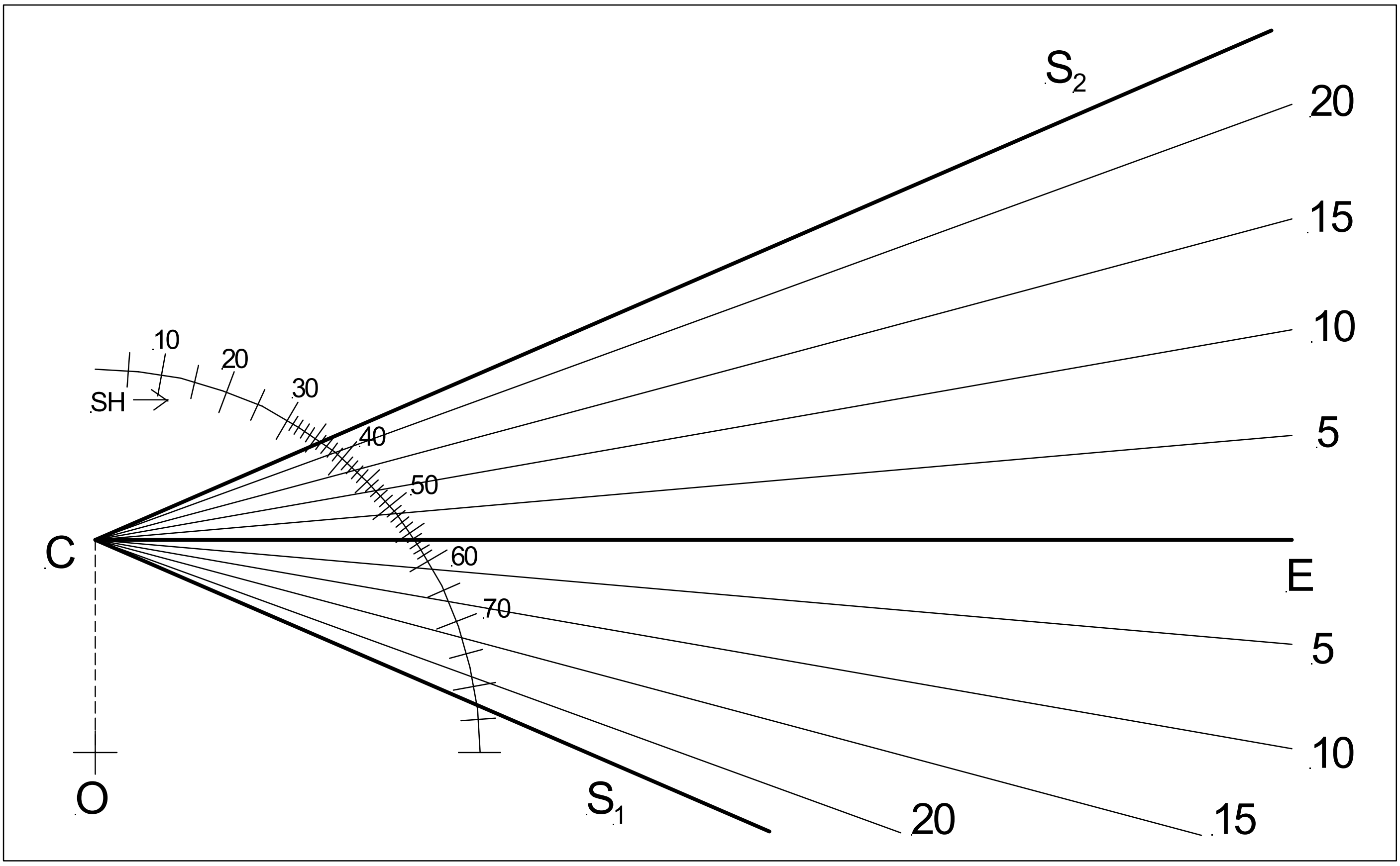

Fig. 2. Protractor for drawing declination lines based on Leybourn.[4]

If the declination d is west of south the value of λ' is positive, if d is east of south it is negative.

So in total we have

$$h = h_0 + \theta + \lambda'$$

And for a vertical dial

$$SH = 90\text{-}\varphi'$$

These formulae allow the hour line angles to be correctly labelled for any type of planar dial. It is for this reason that the vertical declining dial face in Fig. 1 has differently labelled hour lines from the vertical and are valid for different latitudes.

GRAPHICAL METHOD

The graphical solution is simple to perform if a little less accurate. The diagram in Fig. 2 shows an origin **O**, a point **C** where a number of lines converge. The extremes have been put at 23.43° apart to cover the solstices, and lines added at five degree intervals to give declination lines for those angles. A quarter arc protractor is included about **O** which allows the style height to be set. The distance **OC** is the distance L of the nodus from the origin along the sloping style. The resulting dial can be scaled from the actual value of the distance of the nodus from the origin.

The simplest way to use the diagram is to have a piece of tracing paper with a line across the centre, which will be the sub-style line and a line at right angles to it near the left side of the page. Push a drawing pin through **O** from the back of the page and pierce the tracing paper where the two lines intersect. Rotate the tracing paper so that the central line is set with the correct style height SH on the protractor. Mark along the central line the points $\mathbf{W_0}$, $\mathbf{E_0}$ and S_0 at which the lines $\mathbf{CS_1}$, **CE** and $\mathbf{CS_2}$ intersect. The line through $\mathbf{E_0}$, the centre of these, at right angles to the central line, is the equinox line. The other two points are where the two solstice lines cross the sub-style line. The central line and the equinox line are shown dashed in Fig. 3. Now rotate the tracing paper a few degrees and mark with a straight edge, shown dotted on Fig. 3, where the line from **O** to the intersection with the equinox line cuts the lines $\mathbf{CS_1}$ and $\mathbf{CS_2}$. These points are $\mathbf{D_1}$ and $\mathbf{J_1}$ and they are two more points on the solstice declination lines. Rotate the sheet a few more degrees and repeat the process for the next two points. Finally join up all the points $\mathbf{D_n}$ and $\mathbf{J_n}$ to give the solstice lines. Declination lines for other angles can be drawn in the same way using the 5, 10, 15 and 20 degree lines in Fig. 2.

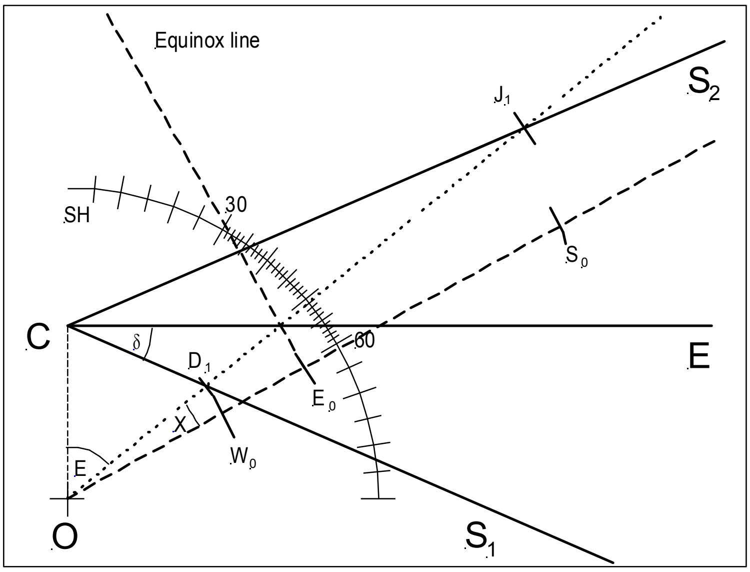

Fig. 3. Use of the protractor shown in Fig. 2 for drawing declination lines.

CARTESIAN CO-ORDINATES

In Fig. 4, an hyperbola with offset origin (-c, 0) is drawn and the distances a, b, and c are indicated. The formula for the hyperbola is

$$\frac{(x-c)^2}{a^2} - \frac{y^2}{b^2} = 1$$

or

$$y = \pm b \sqrt{\left(\frac{(x-c)^2}{a^2} - 1 \right)}$$

The asymptotes are

$$y = \pm \frac{b}{a}(x-c)$$

For declination lines on a planar sundial the values of a, b, and c in terms of style height and declination are:-

$$a = \frac{L \times \sin SH \times \tan \delta}{\cos^2 SH - \sin^2 SH \times \tan^2 \delta}$$

$$b = \frac{L \times \sin SH}{\sqrt{(\cos^2 SH - \sin^2 SH \times \tan^2 \delta)}}$$

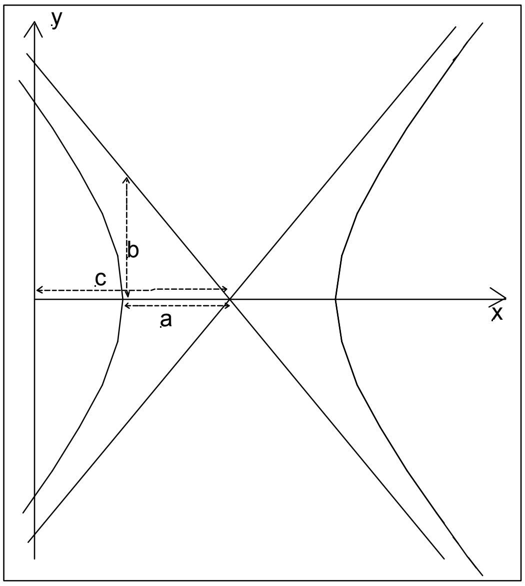

Fig. 4. Hyperbola and asymptotes indicating a, b *and* c.

$$c = \frac{L \times \cos SH}{\cos^2 SH - \sin^2 SH \times \tan^2 \delta}$$

This hyperbola can be drawn based on the origin of the dial and with the x axis as the sub-style line for any style height and declination using the cartesian co-ordinates x and y with the origin at the origin of the dial**.** This method may be found to be a more convenient way of plotting declination lines than the polar co-ordinate method above. The hour line angles are still calculated according to equation 5 above.

EXISTING DIALS

It is possible to check the correctness of the solstice lines on an existing dial such as that in Fig. 5 by measuring the distances along the sub-style line from the origin to the equinox line **OE** and the solstice lines $\mathbf{OS_1}$ and $\mathbf{OS_2}$.

The following relationships follow from the above formulae:-

$$\frac{1}{OS_1} + \frac{1}{OS_2} = \frac{2}{OE}$$

This is always true. In addition we have:-

$$\tan SH = \frac{OE}{2\tan\delta}\left(\frac{1}{OS_1} - \frac{1}{OS_2}\right)$$

$$L = OE \cos SH$$

$$ON_1 = \frac{L}{\tan\delta}$$

$$\frac{S_1 S_2}{N_1 N_2} = \tan SH \times \tan\delta$$

where the line N_1N_2 is parallel to the sub-style line OS_2.
So for any existing dial the accuracy of its declination lines can easily be checked and its style height and nodus distance found with a few simple measurements.

CONCLUSIONS

It has been shown that the pattern of declination lines on a planar sundial is dependent only on the style height and it is symmetrical about the sub-style line.

Three ways of constructing declination lines are given, all based on the origin of the dial.

a) Polar co-ordinates using SH and h_0.

b) A graphical method which can be scaled to any size required.

c) Cartesian co-ordinates using only SH.

A simple method of checking declination lines on existing dials is also given which allows the value of SH and L to be found from a few simple measurements.

REFERENCES

1. J. Davis (Ed.): *BSS Sundial Glossary.* British Sundial Society, (2004).

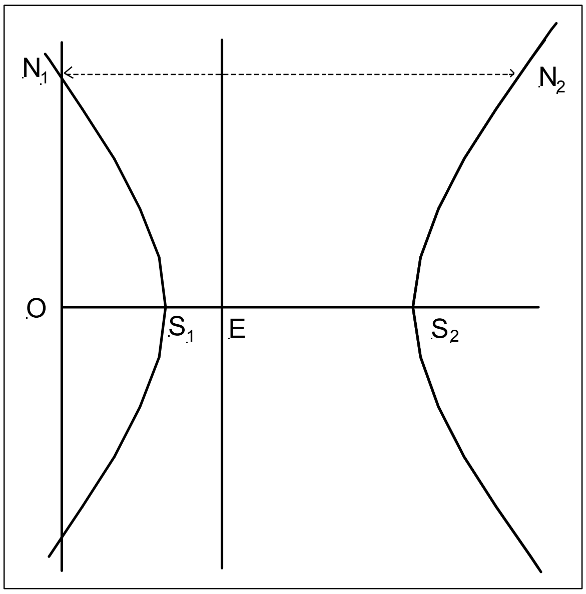

Fig. 5. Measurement points for checking declination lines on an existing dial.

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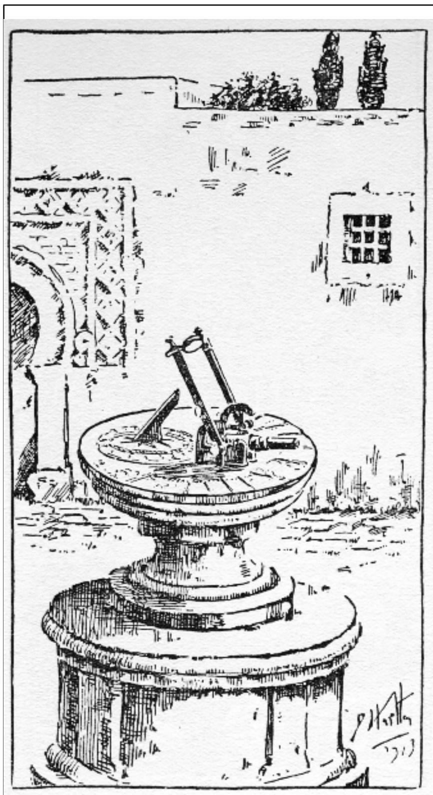

Another noon cannon, this one from Henslow. It looks very similar to the Maidstone museum one illustrated by Warrington Hogg in Gatty so it seems likely that the location here is fictitious.