TUTORIAL on QUATERNIONS
Part II

Luis Ibáñez

August 13, 2001

This document was created using LYX and the LATEX Seminar style.

Contents

- Differentials of Quaternions
- Developments of Functions of Quaternions

Differentials of Quaternions

The difficult point in defining Differentials over Quaternions is

Lack

of the Conmutative Property.

$$P \diamond Q \neq Q \diamond P$$

Adopted Definition

Following Newton's definition of *Fluxions*

Hamilton [2] defined *Simultaneous Differentials* as

Limits of Equi-multiples of
Simultaneous and *Decreasing Differences*.

What it means ?

Given a system of connected *Quaternions*

$$q,\ r,\ s$$

the symbols

$$\triangle q,\ \triangle r,\ \triangle s$$

represent their ***Simultaneous Differences***

What it means ?

The sums

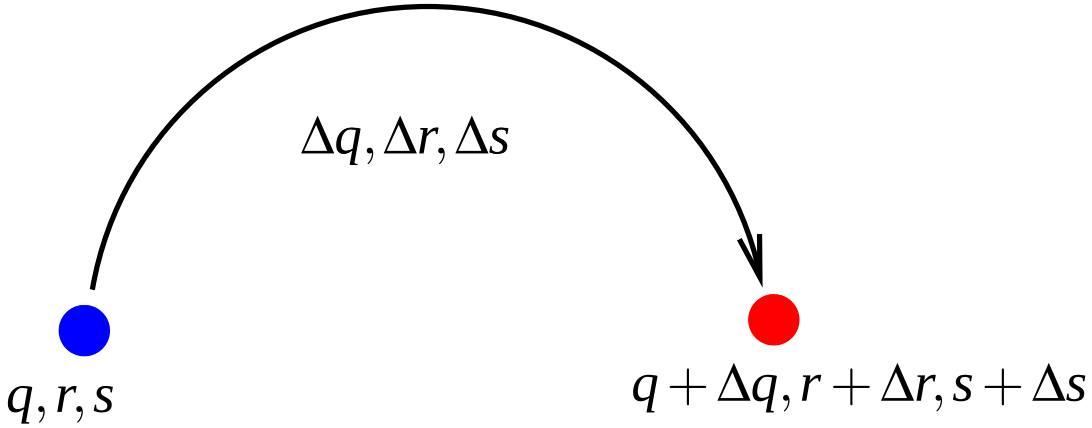
$$q + \triangle q,\ r + \triangle r,\ s + \triangle s$$

are a **New** system of Quaternions

satisfying the **Same Laws** of connexion as the **Old** system.

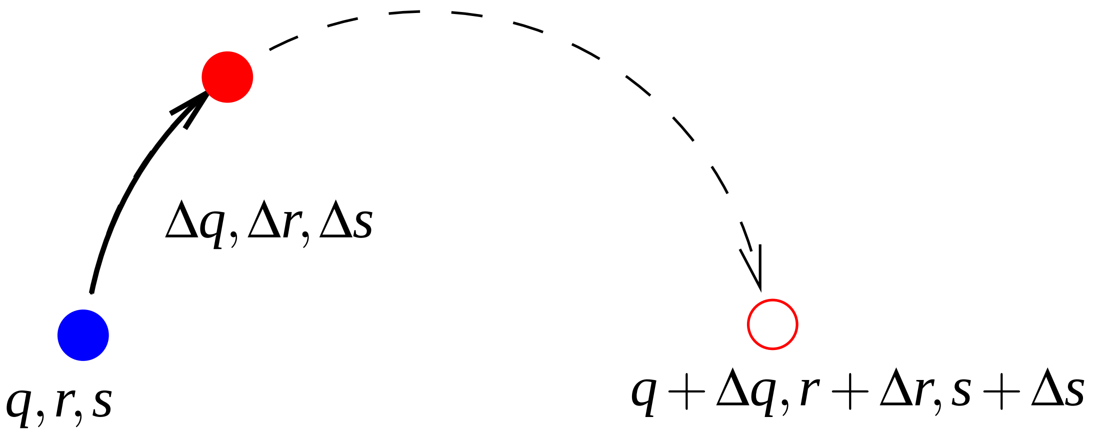
What it means ?

The differences $\Delta q, \Delta r, \Delta s$ start at an arbitrary size



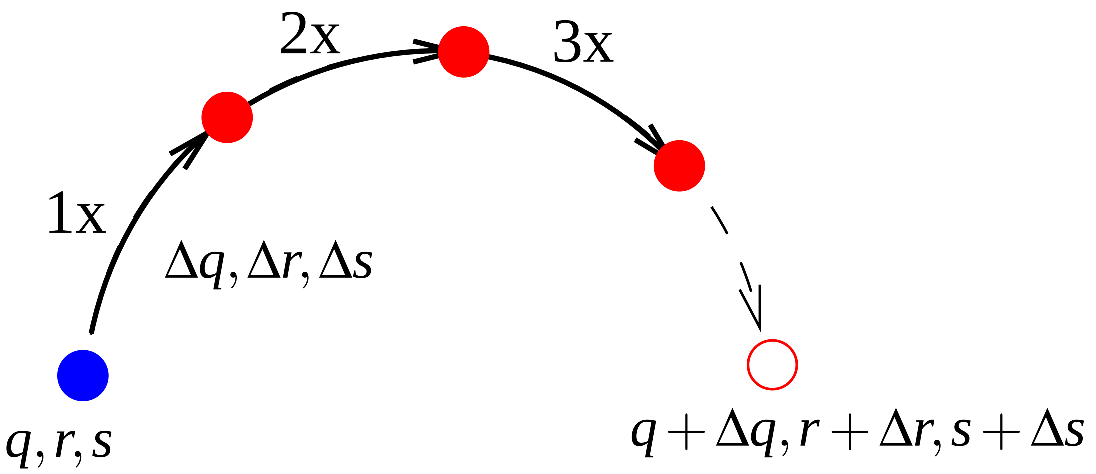
What it means ?

Then, they are *simultaneously decreased* by a factor



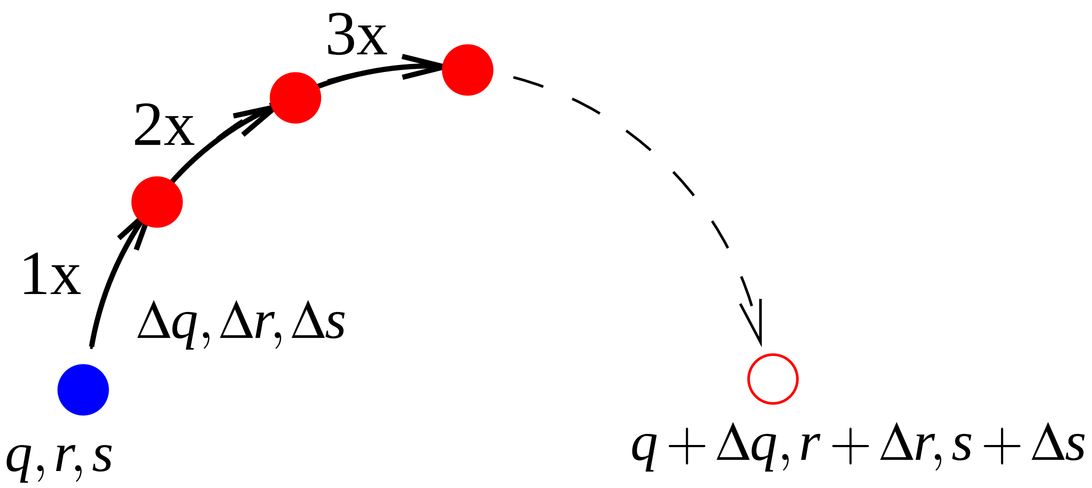
What it means ?

Integer multiples of the new differences are considered



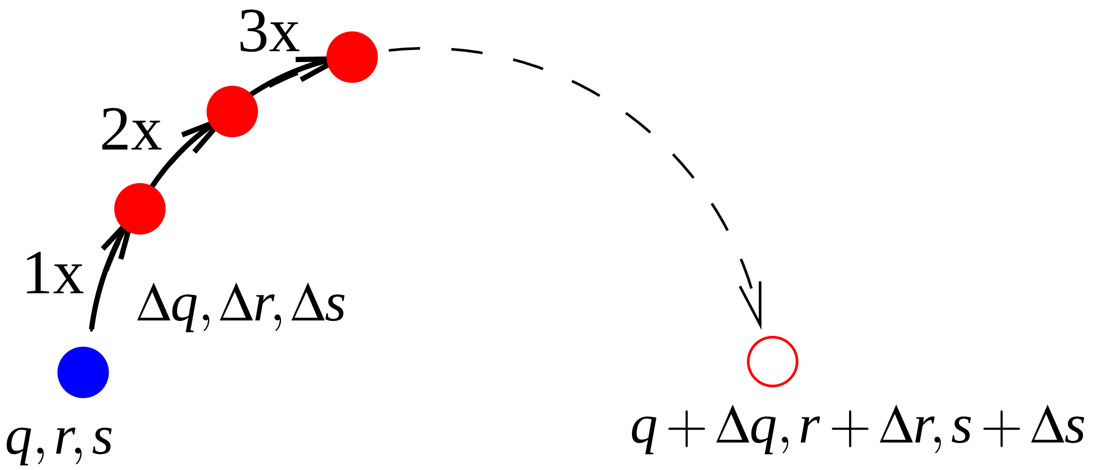
What it means ?

...the factor is further decreased



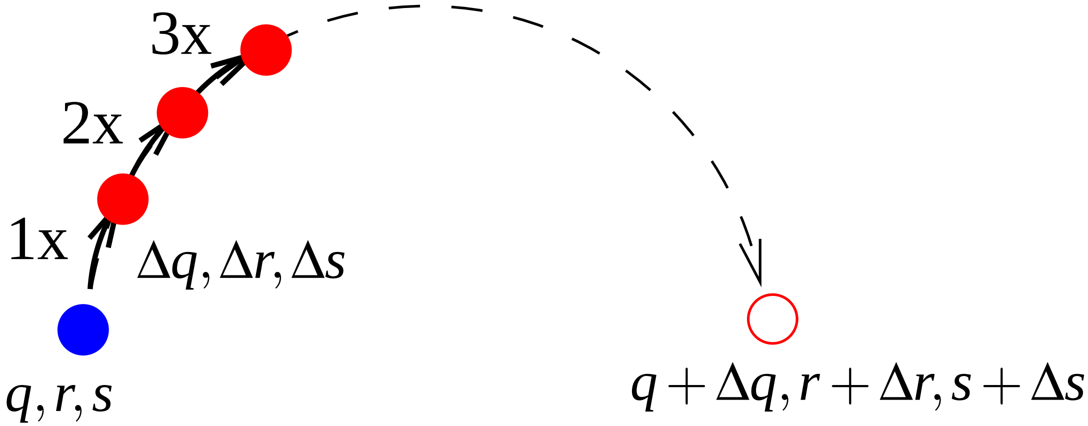
What it means ?

...and decreased...



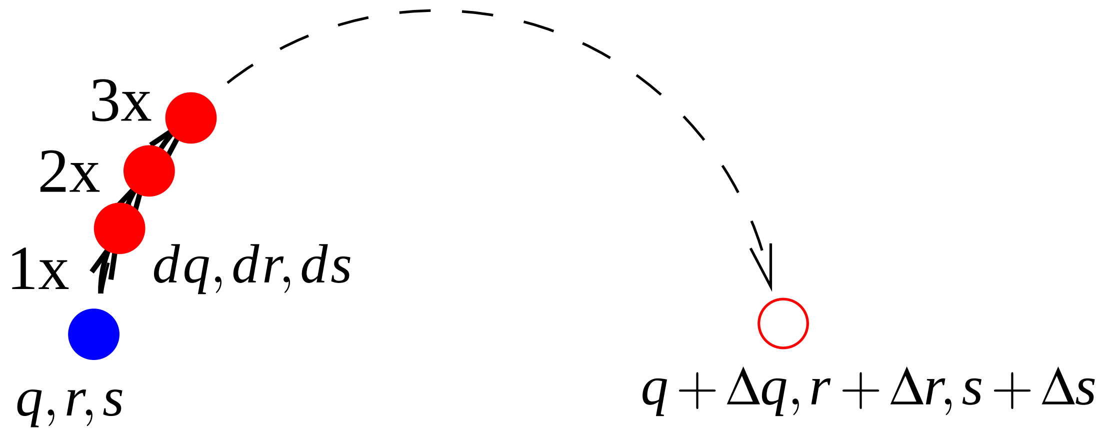
What it means ?

...and decreased...



What it means ?

...and taken **to the limit !**



Definition Revisited

If all the multiples $n\Delta$ converge to the same value

when the factor Δ is decreased,

Then the *Limit* of Δ 's exist and they are called

Simultaneous Differentials

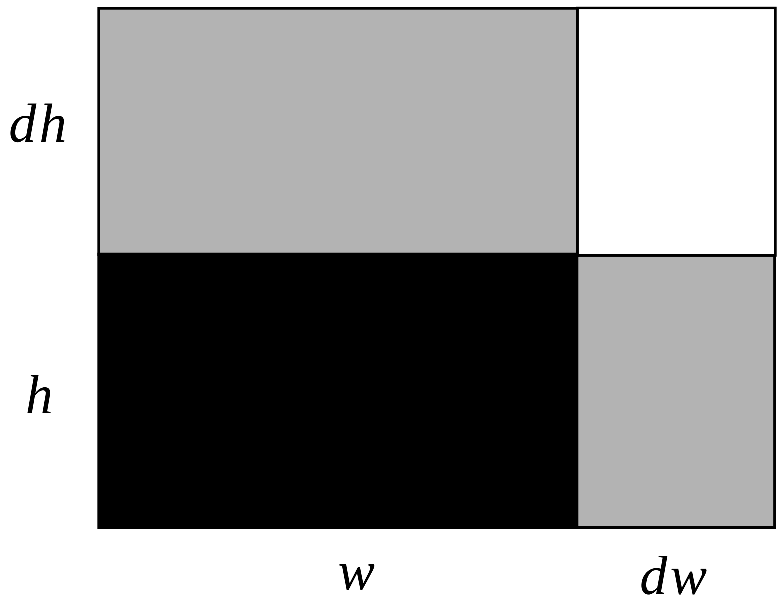
$$dq,\ dr,\ ds$$

Again....

Limits of Equi-multiples of
Simultaneous and *Decreading* *Differences*.

Consequence of this Definition

The Surface *Differentials* of this black Rectangle (w,h)



$$dS = w \cdot dh + h \cdot dw$$

is the sum of shaded rectangles at the sides $(h \cdot dw)$ and $(w \cdot dh)$

Lies my Calculus Teacher told me...

Differentials are *infinitesimally small*...

The Truth is

What they have to be is *linearly related*

$$dS = h \cdot dw + w \cdot dh$$

NOT because

$$dw \cdot dh \longrightarrow 0$$

BUT because

$$dw \cdot dh$$

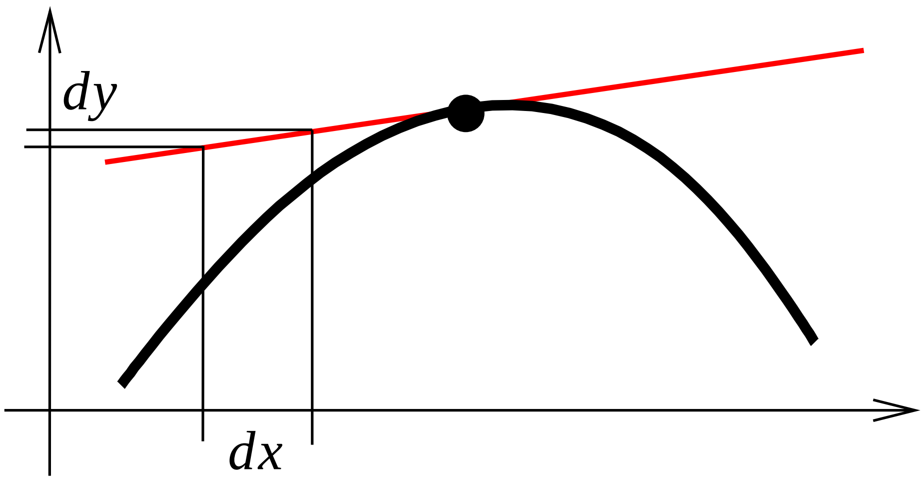
is **NOT LINEAR** with respect to a factor applied

simultaneously to dh, dw and dS

Differentials as Linear Approximations

Differentials don't need to be SMALL

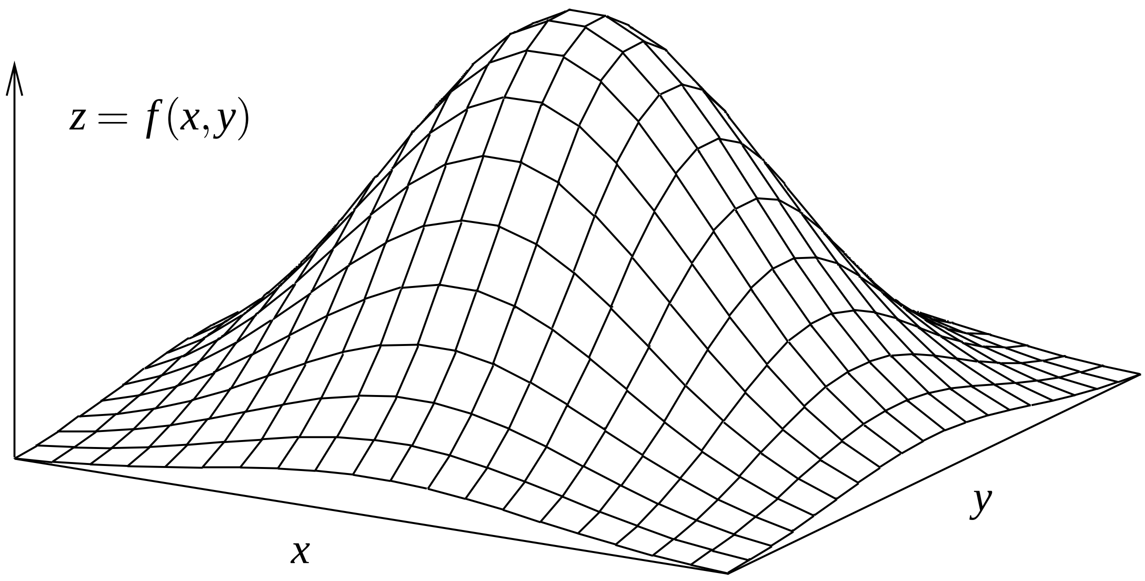
They are a LINEAR APPROXIMATION [1].



$$dy = A \cdot dx + B$$

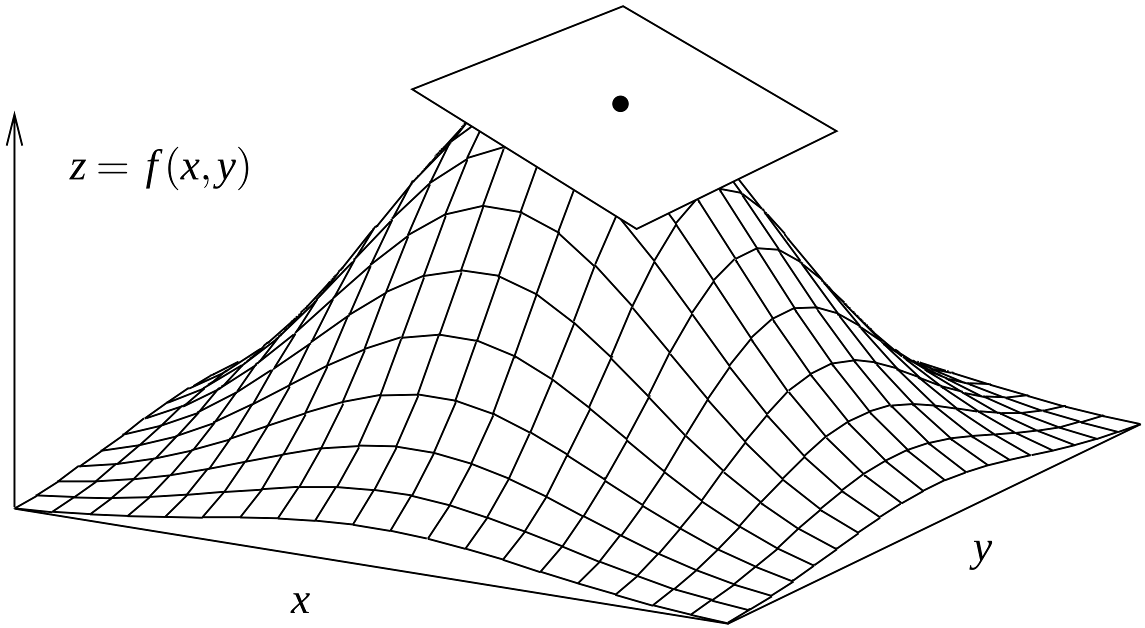
Differentials as Linear Approximations

In a 2D function $z = f(x,y)$ the Linear Approximation is a *Plane*.



Differentials as Linear Approximations

$$dz = A \cdot dx + B \cdot dy + C$$



Differentials as Linear Approximations

The *Differentials*

$$dx \quad dy \quad dz$$

Can be as Large as you want

but

They have to be related by a Linear Equation

Differential of Functions of Quaternions

Let Q be a *Function* of the Quaternion variables { q, r,... }

$$Q = F\,(q, r, \ldots)$$

and let

$$dq, dr, \ldots$$

be any *Simultaneous Differentials* of { q, r, ... }

Differential of Functions of Quaternions

The *Simultaneous Differential* of function Q is

$$dQ = \lim_{n \to \infty} n \cdot \left[F\left(q + \frac{dq}{n}, r + \frac{dr}{n}, \ldots \right) - F\left(q, r, \ldots\right) \right]$$

where n

is an integer multiple of a particular real value.

Differentials in one Dimension

The well known equation

$$\frac{d f(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Expressed according to the new definition

$$d f(x) = \lim_{n \to \infty} n \left[f(x + \frac{dx}{n}) - f(x) \right]$$

Example

$$f(x) = (x)^2$$

$$d\,f(x) = \lim_{n \to \infty} n \left[\left(x + \frac{dx}{n} \right)^2 - (x)^2 \right]$$

$$d\,f(x) = \lim_{n \to \infty} n \left(x^2 + 2\,x\,\frac{dx}{n} + \frac{(dx)^2}{n^2} - x^2 \right)$$

$$d\,f(x) = \lim_{n \to \infty} \left(2\,x\,dx + \frac{(dx)^2}{n} \right) = 2\,x\,dx$$

Differential of a Function of One Variable

The Differential dx is like another variable

$$d\,f\,(x) = g\,(x, dx)$$

for example, given

$$f\,(x) = x^2$$

the differential is a function of Two Independent Variables x and dx

$$d\,f\,(x) = g\,(x, dx) = 2\,x\,dx$$

Differentials of Functions of Quaternions

Quaternions composition (*multiplication*) is **NOT** commutative

$$f(q) = q^2 = q \diamond q$$

The differential

$$df(q) = \lim_{n \to \infty} n \left[\left(q + \frac{dq}{n} \right)^2 - (q)^2 \right]$$

Results in

$$df(q) = q \diamond dq + dq \diamond q$$

Differentials of Functions of Quaternions

The Quotient

$$\frac{df(q)}{dq} = df(q) \diamond dq^{-1}$$

For the current example $f(q) = q^2$

$$\frac{df(q)}{dq} = q \diamond dq \diamond dq^{-1} + dq \diamond q \diamond dq^{-1}$$

$$\frac{df(q)}{dq} = \quad q \ + \ dq \diamond q \diamond dq^{-1}$$

Which is a function of q and dq

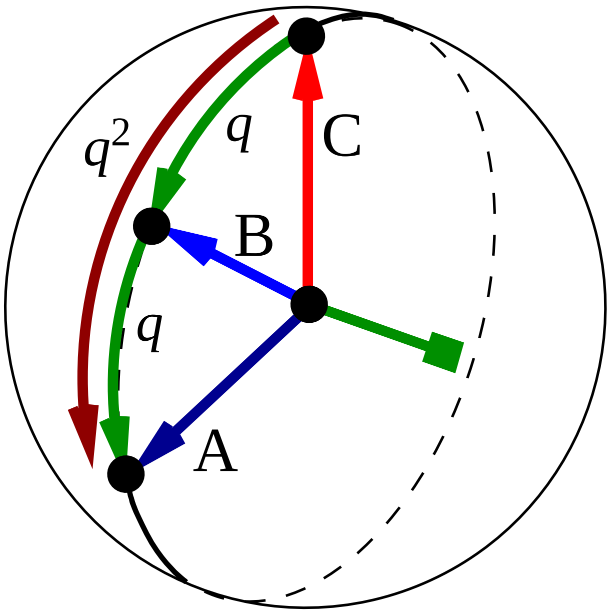
Comparison with Traditional Differentials

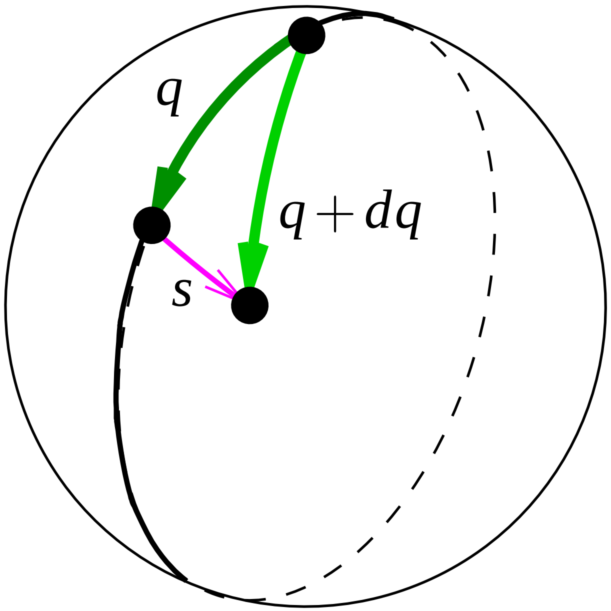
	Function	Quotient of Differentials
Scalars	$f(x) = x^2$	$\frac{df(x)}{dx} = 2x$
Quaternions	$f(q) = q^2$	$\frac{df(q)}{dq} = q + dq \diamond q \diamond dq^{-1}$

The Quotient of Quaternion Differentials is a new Function of TWO INDEPENDENT variables :

q and dq

Geometric Interpretation of the Differential of the Square Function

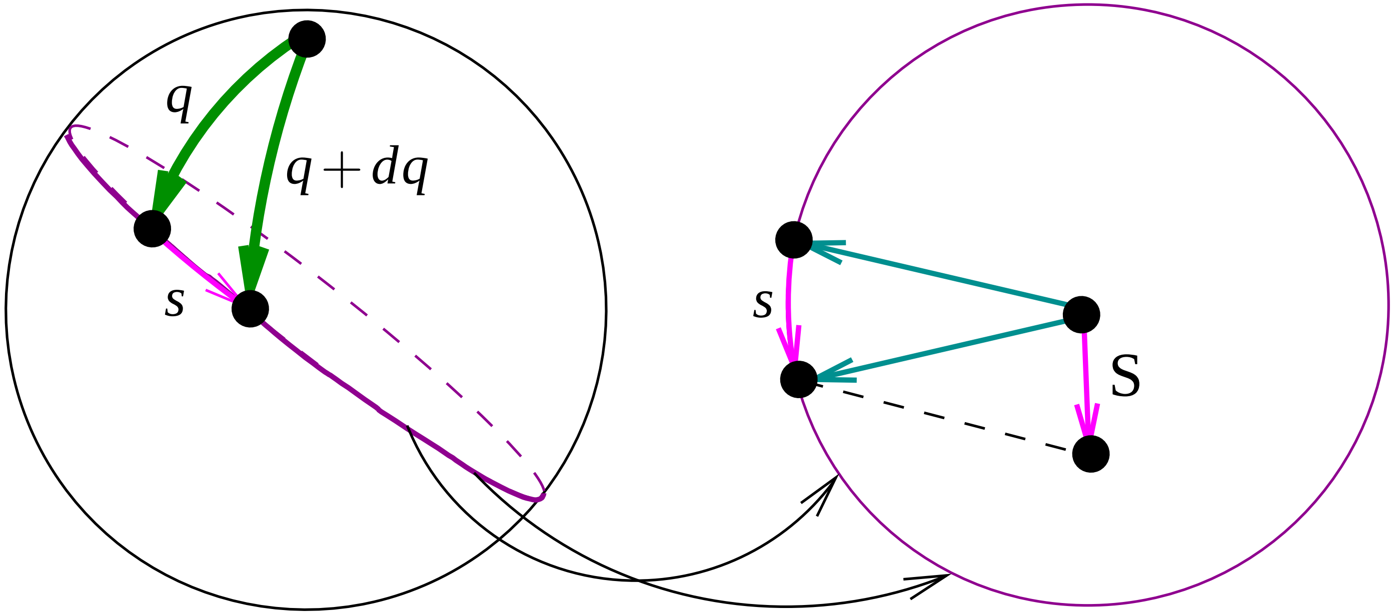




$$q = \frac{\vec{A}}{\vec{B}} = \frac{\vec{B}}{\vec{C}} \qquad q^2 = \frac{\vec{A}}{\vec{C}} \qquad s = \frac{q+dq}{q} = 1 + \frac{dq}{q}$$

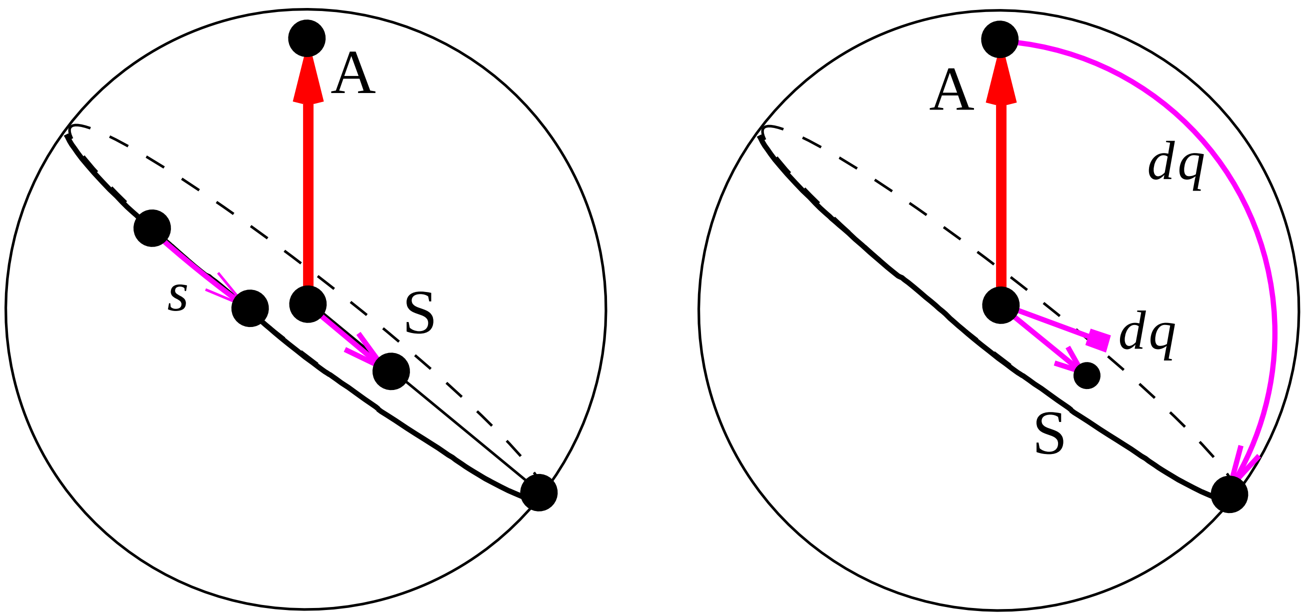
a variation in q is represented by the Differential dq

Geometric Interpretation of the Differential of the Square Function



In order to find the Differential dq, the vector $\overrightarrow{S}$ corresponding to the chord of Vector-Arc s is taken and shifted to the origin of the sphere.

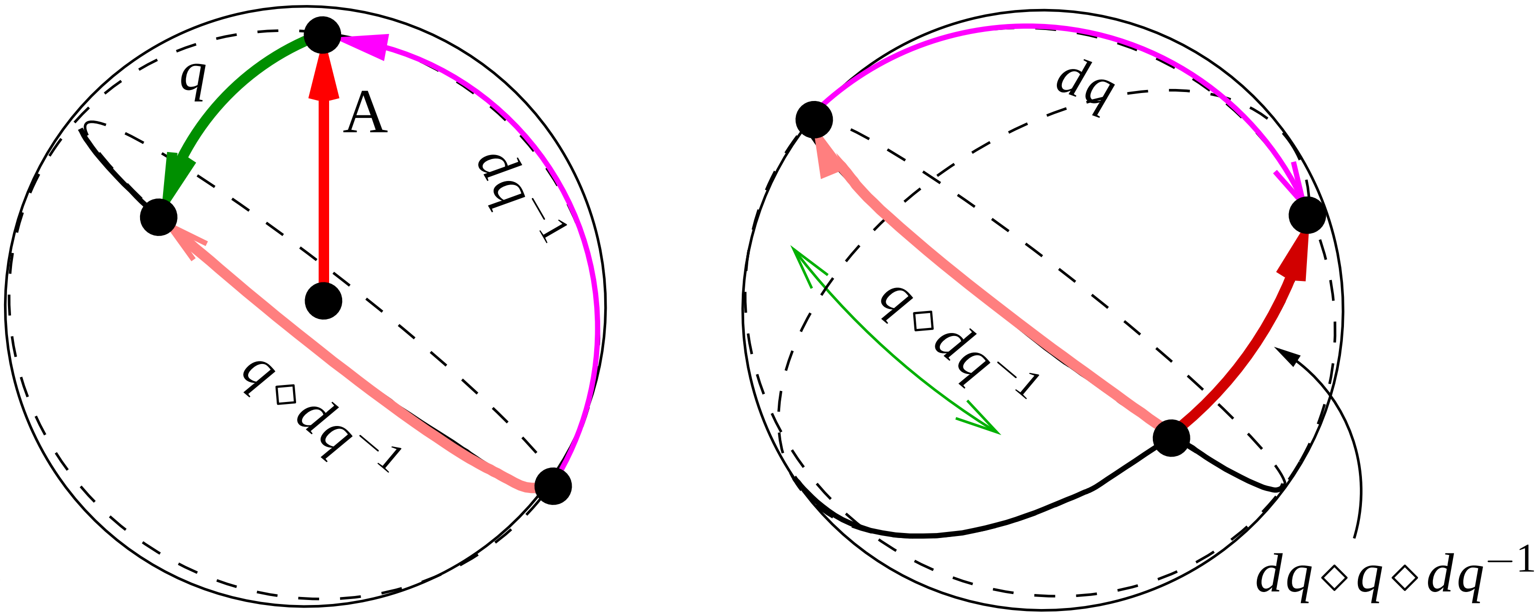
Geometric Interpretation of the Differential of the Square Function



The Quotient between vectors $\overrightarrow{S}$ and $\overrightarrow{A}$ is the Quaternion dq

$$dq = \frac{\overrightarrow{S}}{\overrightarrow{A}}$$

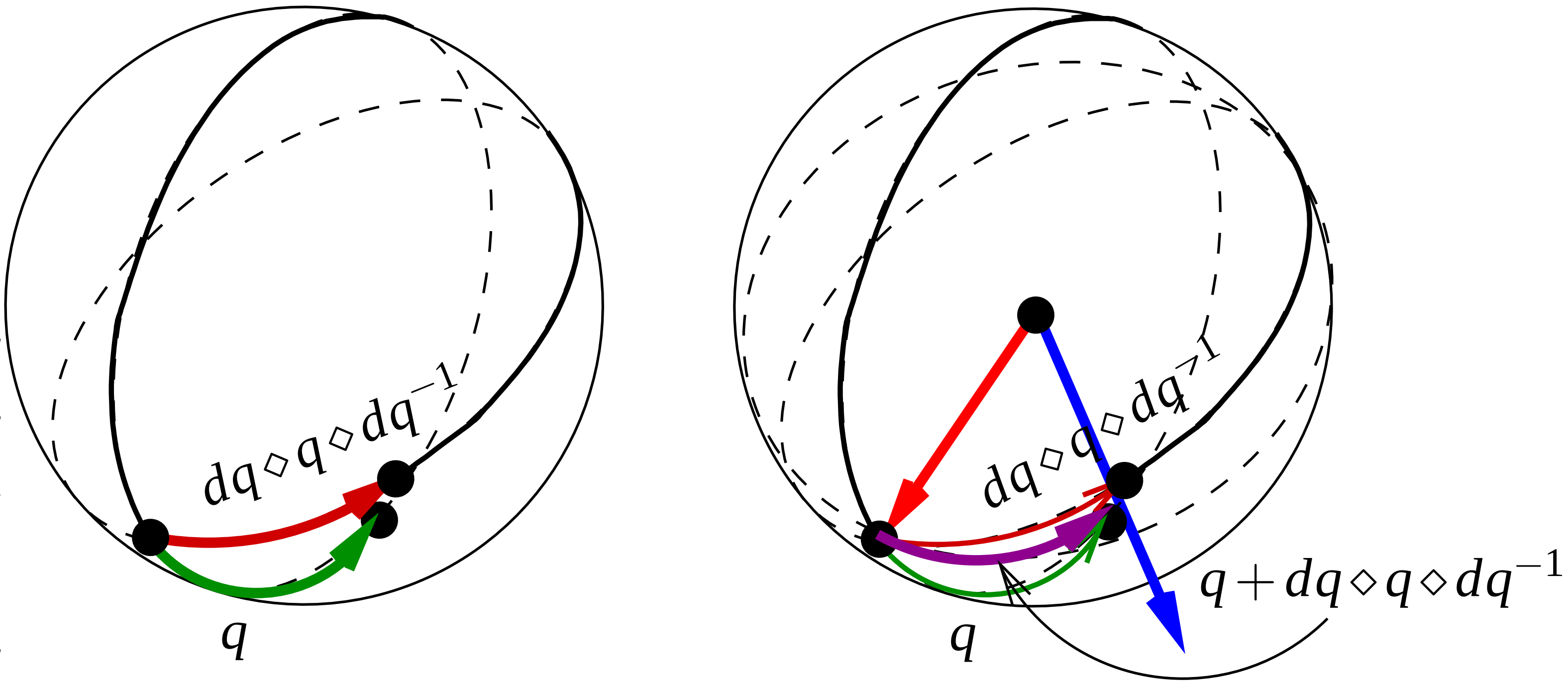
Geometric Interpretation of the Differential of the Square Function



q is composed with dq^{-1} by forming a spherical triangle with their corresponding Vector-Arcs

The resulting Vector-Arc $q \diamond dq^{-1}$ can be further composed with dq by using another spherical triangle.

Geometric Interpretation of the Differential of the Square Function



The sum of q and $dq \diamond q \diamond dq^{-1}$ is performed by finding first a common denominator vector, then adding the two vector in the numerator.

Sum of Reciprocals (a property)

$$R(q) = q^{-1} = \frac{1}{q}$$

$$R(p) + R(q) = p^{-1} + q^{-1} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{p} + \frac{1}{q} = \left(\frac{1}{q} \diamond q\right) \diamond \frac{1}{p} + \frac{1}{q} \diamond \left(p \diamond \frac{1}{p}\right) = \left(\frac{1}{q}\right) \diamond (p+q) \diamond \left(\frac{1}{p}\right)$$

$$\frac{1}{p} + \frac{1}{q} = \left(\frac{1}{q}\right) \diamond (p+q) \diamond \left(\frac{1}{p}\right)$$

Differential of the Reciprocal

$$f(q) = R(q) = q^{-1}$$

$$df(q) = \lim_{n\to\infty} n \left[\left(q + \frac{dq}{n} \right)^{-1} - (q)^{-1} \right]$$

$$df(q) = \lim_{n\to\infty} n \left[\left(q + \frac{dq}{n} \right)^{-1} \diamond \left(q - \left(q + \frac{dq}{n} \right) \right) \diamond (q)^{-1} \right]$$

$$df(q) = -q^{-1} \diamond dq \diamond q^{-1} = -q^{-1}\, dq\, q^{-1}$$

Comparison with Traditional Differentials

	Function	Quotient of Differentials
Scalars	$f(x) = x^{-1}$	$\frac{df(x)}{dx} = -x^{-2}$
Quaternions	$f(q) = q^{-1}$	$\frac{df(q)}{dq} = -q^{-1} \diamond dq \diamond q^{-1} \diamond dq^{-1}$

The Quotient of Quaternion Differentials is a new Function of TWO INDEPENDENT variables :

q and dq

Quotient of Differentials

The Quotient between two Differentials can be separated in
Tensor and Versor parts

$$\frac{d\,f(q)}{dq} = \frac{T\,(d\,f(q))}{T(dq)}\;\frac{U\,(d\,f(q))}{U(dq)}$$

The Differentials $d\,f(q)$ and dq are Equi-Multiples,

so scaling dq will scale $d\,f(q)$ by the same factor.

Quotients of Differentials are Invariant to Scale changes in their Tensors

Quotient of Differentials

In the example $f(q) = q^2$

The Quotient of Differentials

$$\frac{d\,f(q)}{dq} = q + dq \diamond q \diamond dq^{-1}$$

Can be reduced to

$$\frac{d\,f(q)}{dq} = q + U(dq) \diamond q \diamond U(dq)^{-1}$$

That only depends on (dq)'s Direction represented by the Versor $U(dq)$

Partial Differentials

Given a function of several quaternion variables

$$Q = f(q, r, s)$$

Its Differential satisfies

$$dQ = d_q Q + d_r Q + d_s Q$$

each Partial Differential $d_x Q$ is obtained by differentiating with respect to x as if the other variables were constant.

Succesive Differentials

For example, given the Quaternion function

$$f(q) = q^2 = q \diamond q$$

The first Differential is

$$d f(q) = q \diamond dq + dq \diamond q$$

Taking the Differential of this last expression,
where q and dq are considered as two independent variables

$$d^2 f(q) = dq \diamond dq + q \diamond d^2 q + d^2 q \diamond q + dq \diamond dq$$

Successive Differentials

The Second Differential of

$$f(q) = q^2$$

is then reduced to

$$d^2 f(q) = q \diamond d^2 q + d^2 q \diamond q + 2\, dq \diamond dq$$

Which is a function of **THREE** independent Quaternion variables

$$\{q, dq, d^2 q\}$$

None of them necessarily SMALL

Taylor's Series Extended to Quaternions

Having that

$$d^m f(q) = d(d^{m-1} f(q))$$

The *Taylor's Series* Expansion can be applied to functions of Quaternions

$$f(q+dq) = f(q) + \frac{d f(q)}{1!} + \frac{d^2 f(q)}{2!} + \frac{d^3 f(q)}{3!} + \frac{d^4 f(q)}{4!} + \cdots$$

Where $f(q+dq)$ will be a function of $\{q, dq, d^2q, d^3q, \ldots\}$ Quaternion variables NOT necessarily SMALL

Taylor's Series Approximation

The Tensor of the Quaternion Variables

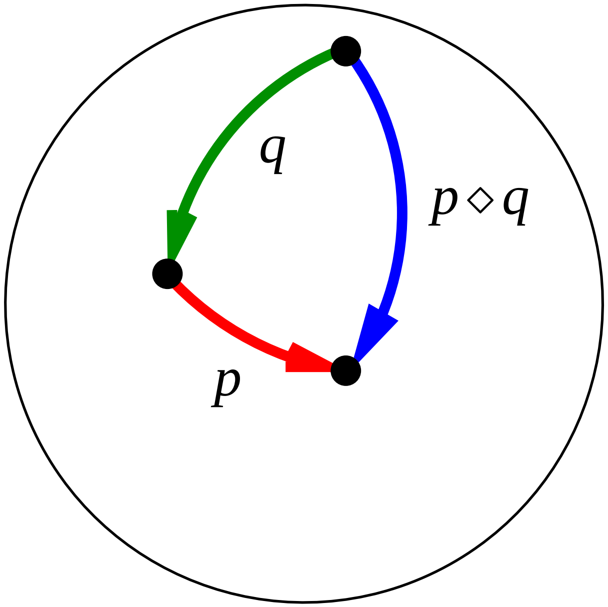
$$\{q, dq, d^2q, d^3q, \ldots\}$$

can be scaled by a Scalar factor x to produce an **Approximation**

$$Fx = f(q + xdq) - f(q) - \frac{x}{1!} df(q) - \frac{x^2}{2!} d^2 f(q) - \frac{x^3}{3!} d^3 f(q) - \cdots$$

Operations in Versor Space

Composition of Versors is equivalent to



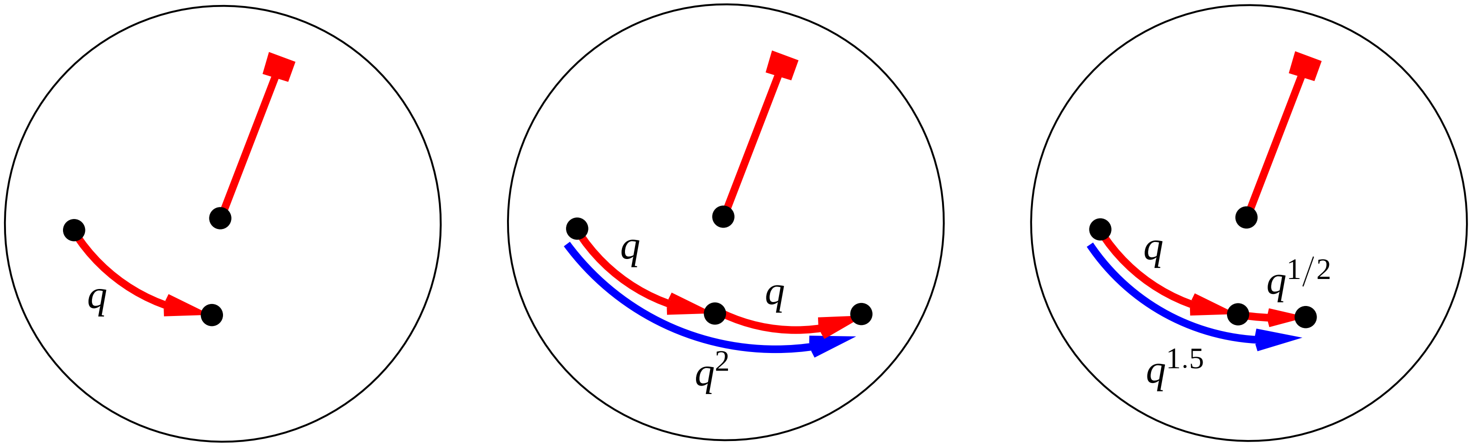
sum of their **Vector-Arcs** on the Unit Sphere Surface.

This is a Non-Commutative operation

Operations in Versor Space

Increments of Versor's Angle is equivalent to Exponentiation

for example, in order to double the angle
the versor is applied twice, which is $q \diamond q = q^2$



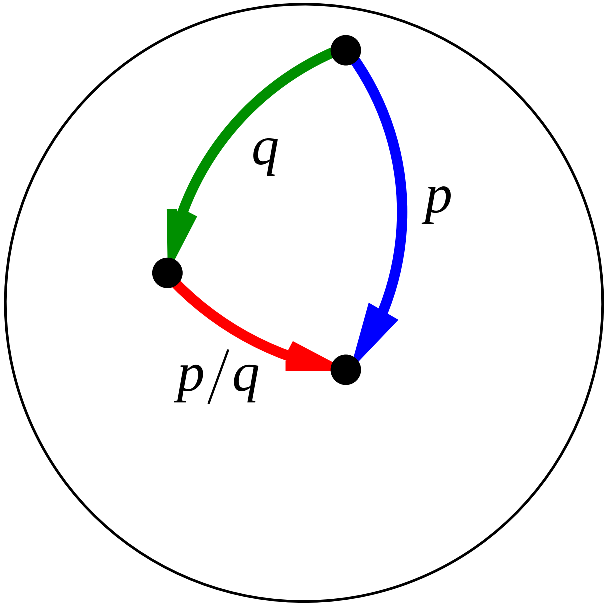
Like in Complex numbers

$$e^{\mathbf{i}\theta} = cos\theta + \mathbf{i}sin\theta$$

Operations in Versor Space

Subtraction of Vector-Arcs is equivalent to

a Quotient of Versors

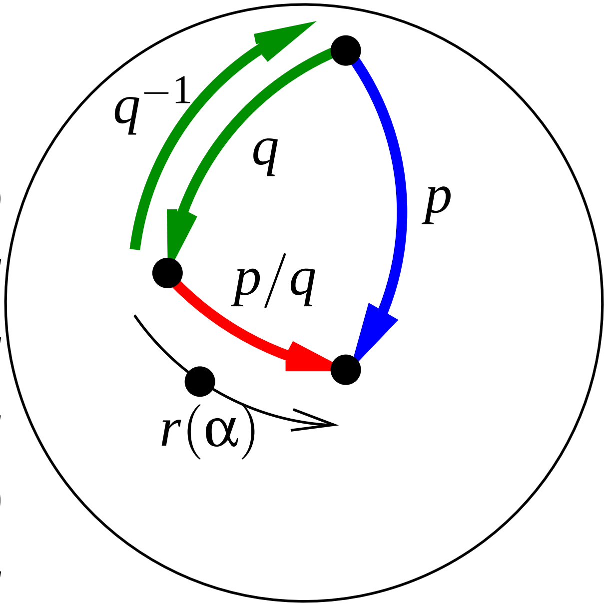


$$\frac{p}{q} = p \diamond q^{-1}$$

$$\begin{aligned} \frac{p}{q} \diamond q &= \left(p \diamond q^{-1}\right) \diamond q \\ &= p \diamond \left(q^{-1} \diamond q\right) \\ &= p \end{aligned}$$

Versor Spherical Linear Interpolation

Sperical Linear Interpolation (***Slerp***)



$$r(\alpha) = \left(\frac{p}{q}\right)^{\alpha} \diamond q$$

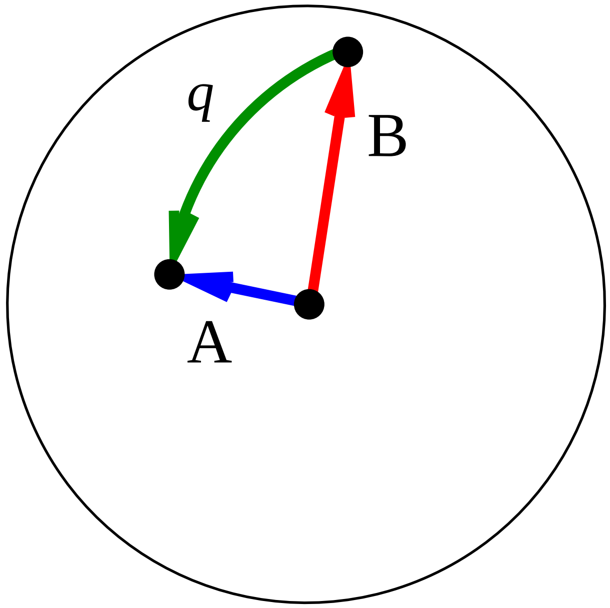
$$\begin{aligned} r(0) &= q \\ r(1) &= p \end{aligned}$$

$$\alpha \in [0,1]$$

The Quotient $\frac{p}{q}$ produce the Quaternion that relates p with q.
Exponent α allows to regulate how much of this Quotient is applied

Optimization of Versor Functions

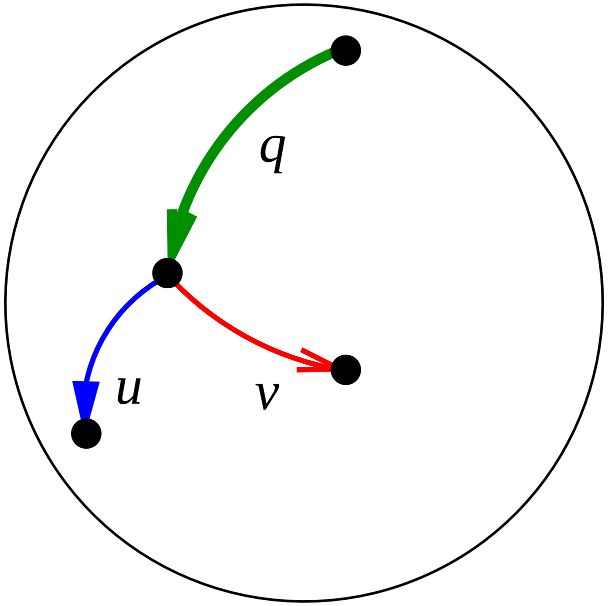
If the Space of Quaternions is restricted to **Versors**



The only valid operations are those that keep the end of vectors in the surface of the **Unit Sphere**

Optimization of Versor Functions

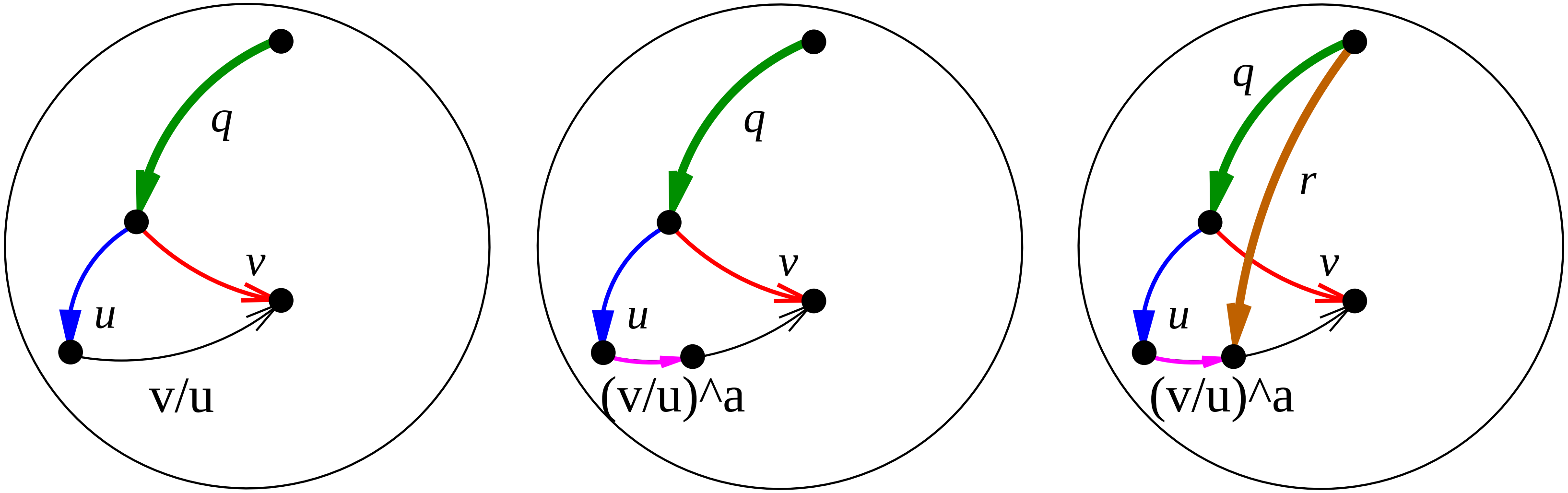
The Versor Space is a **2D** Space



In order to enforce that Variations of a Versor result in another versor, the only valid operations are compositions with Versors, (e.g. u and v)

Optimization of Versor Functions

Gradient Descent-like Optimization Method



$$r = \left(\frac{v}{u}\right)^{\alpha} \diamond u \diamond q$$

$$a = \frac{f(u \diamond q)}{f(u \diamond q) + f(v \diamond q)}$$

References

[1] C. T. J. Dodson and T. Poston. *Tensor Geometry*. Graduate Texts in Mathematics. Springer-Verlag, second edition, 1990.

[2] W.R. Hamilton. *Elements of Quaternions*, volume I. Chelsea Publishing Company, third edition, 1969. The original was published in 1866.