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CLASS NUMBERS, IWASAWA INVARIANTS AND MODULAR FORMS

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1. Iwasawa invariants

K, a number field

p, an prime

$$q := \begin{cases} p & \text{if } p \neq 2 \\ 4 & \text{if } p = 2 \end{cases}$$

$\mathbb{Q}_n$, the unique subfield of $\mathbb{Q}(\zeta_{qp^n})$ of degree p^n over $\mathbb{Q}$ (unless $p = 2$, $n = 1$)

$K_n := K\mathbb{Q}_n$

Cl_n, the p-part of the class group of K_n

Iwasawa. For sufficiently large n,

$$\sharp Cl_n = p^{p^n \mu(K,p) + n\lambda(K,p) + \nu(K,p)}.$$

Geenberg conjecture. If K is a totally real number field, then

$$\lambda(K,p) = \mu(K,p) = 0$$

for any prime p.

Ferrero-Washington. If K is an abelian number field, then

$$\mu(K,p) = 0$$

for any prime p.

2. Iwasawa λ-invariants of quadratic fields

Real quadratic fields

p, a prime

$D > 0$, a positive fundamental discriminant

$\lambda(D,p) := \lambda(\mathbb{Q}(\sqrt{D}), p)$

Question.

$$\sharp\{0 < D < X \mid \lambda(D,p) = 0\} >?$$

$p = 2$**:** (Gauss' genus theory + a theorem of Iwasawa [5])

$$\sharp\{0 < D < X \mid \lambda(D,2) = 0\} \gg X/\log X.$$

$p = 3$**:** (Davenport-Heilbronn theorem [4] refined by Horie and Nakagawa [7] + a theorem of Iwasawa [5])

$$\sharp\{0 < D < X \mid \lambda(D,3) = 0\} \gg X.$$

$p > 3$**:** (Ono [8] and Byeon [1])

$$\sharp\{0 < D < X \mid \lambda(D,p) = 0\} \gg \sqrt{X}/\log X.$$

Imaginary quadratic fields

p, a prime

$D < 0$, a negative fundamental discriminant

If $(\frac{D}{p}) = 1$, then $\lambda(D,p) \geq 1$.

Question. How often do trivial λ-invariants occur?

1. $(\frac{D}{p}) \neq 1$ and $\lambda(D,p) = 0$

One can have similar results to the case of real quadratic fields.

2. $(\frac{D}{p}) = 1$ and $\lambda(D,p) = 1$

$p = 2$**:** (Ferrero and Kida's formula))

$$\sharp\{0 < D < X \mid \lambda(D,2) = 0 \text{ and } (\frac{D}{p}) = 1\} \gg X/\log X.$$

$p \geq 3$: (Jochnowitz [6])

For any prime p, if there is at least one imaginary quadratic field $\mathbb{Q}(\sqrt{D_0})$ ($D_0 < 0$) such that $\lambda(D_0, p) = 1$ and $(\frac{D_0}{p}) = 1$, then there are infinitely many such fields.

Main Theorem of this talk: (Byeon [2] 2005)

Let p be an odd prime.

$$\sharp\{-X < D < 0 \mid \lambda(D,p) = 1 \text{ and } (\frac{D}{p}) = 1\} \gg \sqrt{X}/\log X.$$

The aim of this talk is to explain how to obtain the main theorem.

3. Existence of at least one

Proposition 1

(i) Let p be an odd prime and $D_0 < 0$ be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{1-p^2})$. Then $\chi_{D_0}(p) = 1$ and $\lambda_p(\mathbb{Q}(\sqrt{D_0})) = 1$ if and only if $2^{p-1} \not\equiv 1 \pmod{p^2}$, that is, p is not a Wieferich prime.

(ii) Let p be a Wieferich prime. If $p \equiv 3 \pmod 4$, let $D_0 < 0$ be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{1-p})$ and if $p \equiv 1 \pmod 4$, let $D_0 < 0$ be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{4-p})$. Then $\chi_{D_0}(p) = 1$ and $\lambda_p(\mathbb{Q}(\sqrt{D_0})) = 1$.

Proof: This theorem follows from the following lemma.

Lemma (Gold)

Let p be an odd prime and $D < 0$ be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ such that $\chi_D(p) = 1$. Let $(p) = \mathbf{P}\bar{\mathbf{P}}$ in $\mathbb{Q}(\sqrt{D})$. Suppose that $\mathbf{P}^r = (\pi)$ is principal for some integer r not divisible by p. Then $\lambda_p(\mathbb{Q}(\sqrt{D})) = 1$ if and only if $\pi^{p-1} \not\equiv 1 \pmod{\bar{\mathbf{P}}^2}$.

4. Existence of infinitely many

Proposition 2

Let p be an odd prime and $D < 0$ be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ such that $\chi_D(p) = 1$. Then $\frac{L(1-p,\chi_D)}{p}$ is p-integral and

$$\lambda_p(\mathbb{Q}(\sqrt{D})) = 1 \Longleftrightarrow \frac{L(1-p,\chi_D)}{p} \not\equiv 0 \pmod{p},$$

where $L(s,\chi_D)$ is the Dirichlet L-function.

Proof: This theorem follows from the following lemma.

Lemma (Washington)

Let $D < 0$ be the fundamental discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$.

$$\lambda(D,p) = 1 \Longleftrightarrow L_p(0,\chi_D\omega) \not\equiv L_p(1,\chi_D\omega) \pmod{p^2}.$$

Proof of Main Theorem:

Cohen modular forms

r, N, non-negative integers with $r \geq 2$

Define Cohen number $H(r,N)$ by

$$H(r,N) := \begin{cases} 0 & \text{if } N \not\equiv 0,1 \pmod 4 \\ \zeta(1-2r) & \text{if } N = 0 \\ Ł(1-r,(\frac{D}{\cdot}))\cdot * & \text{if } (-1)^r N = Df^2. \end{cases}$$

(For the detail of $*$, see [3].)

Cohen. $F_r(z) := \sum_{N=0}^{\infty} H(r,N)q^N \in M_{r+1/2}(\Gamma_0(4),\chi_0)$.

Consider the following modular form

$$G_p(z) := \sum_{(\frac{-N}{p})=1} \frac{H(p,N)}{p} q^N \in M_{p+1/2}(\Gamma_0(4p^4),\chi_0).$$

By Proposition 1, we have

$$G_p(z) \not\equiv 0 \pmod{p},$$

by proposition 2, we have

$$H(p,D)/p \not\equiv 0 \pmod{p} \text{ iff } \lambda(D,p) = 1.$$

Finally applying Sturm's theorem to the modular form $G_p(z)$, we have

$$\sharp\{-X < D < 0 \mid H(p,D)/p \not\equiv 0 \text{ and } (\frac{D}{p}) = 1\} \gg \sqrt{X}/\log X$$

and complete the proof of main theorem.

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