An Overview of Interest-Rate Option Models

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Abstract

This presentation reviews interest-rate modeling since its advent in the mid 70s, with particular focus on fixed-income derivatives and their valuation. We highlight the crucial role played by practitioners in the evolution of modeling to the present day. As we argue, the major theoretical advances were incited by traders needs and insistence on model compatibility with liquid instruments. The Gaussian, BDT, HJM, and Libor Market Models are discussed in some detail and many other models remarked on. We conclude with some outstanding issues of great practical importance that still defy a satisfactory theoretical solution.

Earlier Models and shortcomings

- In mid 80's banks and institutional investors were interested in evaluating **callable bonds**.
- How to price the **embedded option**?
- The bond's **call-adjusted duration**?
- There were many term structure models.
- But, they were unacceptable to traders:
- They did *not* incorporate the initial yield curve and so priced non-callable bonds **inconsistently** with the market.
- They were (partial) equilibrium rather than pure-arbitrage models like Black-Scholes.

Earlier Short-Rate Models

The instantaneous interest rate r_t follows

$$dr_t = a(r_t)dt + b(r_t)dW_t,$$

where W_t is a Brownian motion under the risk-neutral measure. Then, zero-coupon bonds and European bond option prices are function $C(r,t)$ of r_t and time that satisfy the **PDE**

$$\frac{\partial C}{\partial t} + a(r)\frac{\partial C}{\partial r} + \frac{1}{2}b^2(r)\frac{\partial^2 C}{\partial r^2} - rC = 0.$$

- **Vasicek** : $dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$.
- **CIR** : $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$.
- **Brennan-Schwartz**: $dr_t = \kappa(\theta - r_t)dt + r\sigma dW_t$.
- **Dothan** : $dr_t = \sigma dW_t$.

Earlier multifactor models:

- Brennan-Schwartz two-factor short rate and long rate model
- CIR multi-state variable model: r_t is a given function of a multidimensional diffusion process.

The problem with all these models was that they did *not* **calibrate to the yield curve**. In fact, they could not fit an arbitrary prescribed yield curve because the SDE coefficients were assumed constant.

Practitioner ad hoc models: mimic the Black-Scholes or the binomial model with yield rather than price as the underlying variable:

- Lognormal bond yield
- Binomial bond yield

Arbitrage-Free Curve-Fitting Models

- CIR (1985) (Cox, et. al.) mentioned an extension of their model with time-varying drift to incorporate a given yield curve, but the idea was not pursued.
- The **Ho-Lee Model** (1986) was the first paper that developed this idea prominently.
- It was a **binomial model** exhibiting explicitly the curve at each state of the world in terms of an initially prescribed curve.
- Jamshidian and independently HJM showed in 1987 that its continuous-time limit was a term-structure model in which the forward-rate volatility $\sigma(t, T)$ is a constant:

$$\sigma(t, T) = \sigma.$$

- Equivalently, in the continuous-time limit of Ho-Lee model, the T-maturity **forward rate** $f(t,T)$ was given by

$$f(t,T) = \frac{1}{2}\sigma^2 t^2 + f(0,T) + \sigma W_t.$$

- Equivalently, the short rate $r_t = f(t,t)$ followed the Gaussian process

$$dr_t = (\sigma^2 t + \frac{df(0,t)}{dt})dt + \sigma dW_t.$$

- **Jamshidian (1987)** also generalized this to include a mean-reversion κ, namely, introduced the extension of Vasicek model with r_t having a time-varying drift θ_t:

$$dr_t = \kappa(\theta(t) - r_t)dt + \sigma dW_t.$$

- He showed $\theta(t)$ is given in terms of initial forward curve $f(0,t)$ by

$$\theta(t) = e^{-\kappa t}\frac{d}{dt}(e^{\kappa t}(f(0,t) + \frac{\sigma^2}{2\kappa^2}(1 - e^{-\kappa t})^2)).$$

- In HJM-terms, $\sigma(t,T) = e^{-\kappa(T-t)}$.

- This mean-reverting Gaussian model also appeared in the 1988 version of HJM, and was also studied in 1989 by **Hull and White** (who named it after themselves).
- Introducing the concept of **forward-risk adjustment** (reportedly also arrived at by H. Geman in 1989), Jamshidian (1987) also derived the pricing formula for an **option on a zero-coupon bond**, which was of Black-Scholes/Merton (1973) form. (Also applied it to options in CIR model).
- Restated in the more polished terminology of **forward measure**, he essentially showed *forward prices and forward ares martingales under the forward measure.*
- He also showed that (in a single-state variable model) *an option on a coupon bond* ***decomposes*** *into a sum of options on zero-coupon bond*s with appropriate strikes.

The HJM Approach

- In 1987, **Heath, Jarrow and Morton** formulated the joint **dynamics of the entire forward-rate curve** $f(t,T)$ starting from a given initial curve $f(0,T)$.
- They showed that the curve's arbitrage-free *risk-neutral* dynamics is determined by the forward-rate volatilities $\sigma(t,T)$:

$$df(t,T) = \int_t^T \sigma(t,s)ds\sigma(t,T)dt + \sigma(t,T)dW_t.$$

- This is effectively an *infinite dimensional SDE*, driven by a *finite-dimensional* Brownian motion. ($\sigma(t,T)$ can be stochastic.)
- The model is generally "path dependent", requiring Monte-Carlo simulation for derivatives valuation - otherwise discretization results in non-recombining trees.

- HJM also showed the flat volatility model $\sigma(t,T) = \sigma$ is the continuous-time limit of the Ho-Lee Model.
- They pointed out that the "lognormal" volatility function $\sigma(t,T) = \sigma f(t,T)$ leads to explosion of the forward rates, and is hence inadmissable.
- In a 1988 version, they extended the approach to multifactors (finite-dimensional Brownian motion) and exhibited a two-factor model, with a flat volatility for one factor and a Vasicek $e^{-\kappa(T-t)}$ for the other.
- By a lengthly calculation (without use of forward-risk-adjustment) they derived the zero-coupon bond option formula for the above Gaussian example.

The Gaussian interest-rate model

- The Gaussian model has been studied by many, particularly by Jamshidian in late 80s and early 90s whom we follow here.
- Its advantage is **analytic tractability** and **efficient valuation** implementation.
- Its drawback is **interest-rates are normally distributed**, so can get negative.
- In its general form it assumes all volatilities (e.g., $\sigma(t, T)$ and all correlations, including with any exchange rates, equities, and other economies) are deterministic.
- **Deterministic volatility** results in nice and simple **pricing formulae for European options, futures convexity and quantos** (Jamshidian (1993)).

- As discretization is in general non-recombining, efficient Bermudan option valuation requires a special form of $\sigma(t,T)$: in 1-factor case,
$$\sigma(t,T) = \sigma(t)e^{-\int_t^T \kappa_s ds}.$$
Equivalently, $\sigma(t,T)$ is "separable", i.e., of form $\sigma(t,T) = a(t)b(T)$.
- In this case, **Jamshidian (1991)** showed

$$dr_t = (\frac{df(0,t)}{dt} + v(t) + \kappa(t)(f(t,0) - r_t))\,dt \\ +\sigma(t)dW_t,$$

where
$$v(t) := \int_0^t \sigma^2(s,t)ds.$$
- This implies r_t is a diffusion process.

- Hence bond and option prices C are functions $C(r,t)$ of (r,t) and satisfy the **PDE**

$$\frac{\partial C}{\partial t} + (\frac{df(0,t)}{dt} + v(t) + \kappa(t)(f(t,0) - r))\frac{\partial C}{\partial r} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 C}{\partial r^2} - rC = 0.$$

- This enables efficient **finite-difference** implementation.
- Explicit formulae were derived for forward rates and bond prices as functions of (t, r_t):

$$f(t,T) = f(t,0) + e^{-\int_t^T \kappa_s ds}(r_t - f(0,t) + v(t)\beta(t,T)),$$

where

$$\beta(t,T) := \int_t^T e^{-\int_t^s \kappa_u du} ds.$$

- As for zero-coupon bond prices $P(t,T)$, they are by definition related to forward rates by

$$f(t,T) = -\frac{\partial \log(P(t,T))}{\partial T}. \quad (P(T,T) = 1)$$

- One has

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{\beta(t,T)(f(t,0)-r_t)-\frac{1}{2}v(t)\beta^2(t,T)}.$$

- This formula makes valuation of a coupon bond (as required at each exercise date of a bermudan option) efficient and exact.
- The fundamental solution of the PDE was also derived. Namely, for $t \leq T$, any solution $C(t,r)$ of the PDE satisfies

  $$C(r_t, t) = \int_t^T G(t,R,T)C(R,T)dR,$$

  where

$$G(t,R,T) = P(t,T)\frac{e^{-\frac{(R-f(t,T))^2}{2(V(T)-V(t))}}}{\sqrt{2\pi(v(T)-v(t))}}.$$

BDT Model and Forward Induction

- To remedy the negative rates in the Ho-Lee model, **Black, Derman and Toy** proposed in 1990 a binomial model with the volatility of short rate r_t proportional to r_t. (An extension with mean reversion was later proposed by Black and Karisinski.)
- The BDT model no longer had the analytical tractability of Ho and Lee, and the authors did not discuss any method for efficient calibration of the model to a prescribed initial curve.
- **Jamshidian (1991**) showed the continuous time limit of the BDT model was **lognormal** of the form
$$r_t = e^{A(t)+\sigma(t)W_t},$$
for some functions $A(t)$ and $\sigma(t)$.

- For discretization, he proposed binomializing of W_t in the natural way: $W_0 = 0$,
$$W_{t+\Delta t} - W_t \approx \pm\sqrt{\Delta t},$$
equal probabilities of 1/2 of up or down.
- At period n (time $t = n\Delta t$) there are thus $n+1$ states $i = -n, -n+2, \cdots, n-2, n$.
- The discrete version of the fundamental solution of the pricing PDE is known as **Arrow-Debreu prices**. Specifically, $G(n, i, m, j)$ is the price at period n and state i of the contingent claim with period m payoff of 1 at state j and 0 at other states ($n \leq m$).
- Being a price process, it was known $G(n, i, m, j)$ satisfies the **backward induction**, namely the **backward equation**
$$G(n, i, m, j) =$$
$$\frac{1}{2}P(n, i)(G(n+1, i+1, m, j) + G(n+1, i-1, m, j)).$$

- Jamshidian showed $G(n, i, m, j)$ also satisfies a **forward equation**, namely,

$$G(n,i,m{+}1,j) = \frac{1}{2}P(m,j{+}1)G(n,i,m,j{+}1) + \frac{1}{2}P(m,j-1)G(n,i,m,j-1).$$

- This enabled **Forward Induction**: an efficient determination of $A(t)$ consistent with a given initial curve by a *single* forward sweep of the binomial tree (given σ_t).
- Given an initial curve and an initial yield volatility curve, it similarly enabled in an efficient single sweep the joint determination of a consistent $A(t)$ and $\sigma(t)$.
- The method was equally applicable to a larger class of models of the form $r_t = f(A(t) + \sigma(t)W_t)$ with $f(x)$ an increasing function (or more generally of the form $r_t = g(W_t, A(t), \sigma(t))$).

Other analytically tractable models

While, prior to the Libor Market Model, the Gaussian and BDT models were the most popular among practitioners due to their intuitiveness and efficiency, several other models have been of theoretical interest, including

- **Quadratic interest-rate model**: Analytic solutions for bond prices and bond options, Riccati Equation for curve fitting: Beaglehole Tenney (1991), El Karoui et. al. (1992), Jamshidian (1996).
- **Simple square-root model**: An extended CIR model *calibrating the curve analytically* and pricing bond option in terms of noncentral chi-squared distribution: Jamshidian (1995).

- **Affine Yield Model**: A generalization of Vasicek and CIR models, Riccati-Equation based: Duffie and Kan (1996).
- **Positive Interest Model**: An approach for constructing positive-rate models: Flesaker and Hughston (1993).
- **The potential approach** : An approach for deriving bond pricing formulae in several models: Rogers (1996).
- **Markov-functional model** : A low dimensional alternative to the Libor-Market Model: Hunt, et. al. (2000).
- **Models with jump**: Quite a few papers since mid 90s, including extensions of HJM and Libor Market Models to Lévy processes and general semimartingales.

Libor Market Model

- As early as 1990, **Neuberger** provided an intuitive argument that led to a European **swaption** (and as a special case, **caplet**) pricing formula destined to become the industry standard.
- In fact, viewing the swaption as an option to exchange the swap's floating and fixed legs, leads, via *Margrabe formula*, to the desired Black-type formula, subject only to the assumption that the ratio of the two legs, which is non other than the **forward swap rate, has deterministic volatility.**

- However, it was not generally recognized until mid 90's that this Black-type pricing formula for caplets and swaptions was in fact theoretically consistent and sound.
- **Miltersen et. al.** (1997) derived the formula for caplets under a theoretically recognized approach, demonstrating that it is arbitrage free and consistent.
- Using an HJM setting under the risk-neutral measure (which actually turns out to be inappropriate for LMM), **Brace et. al. (1997) (BGM)** formulated a **forward-Libor term structure** for Libor rates in which the Black formula held for caplets.

- **Musiela and Rutkowski (1997)** took the more satisfactory approach of examining forward Libor rates under the **forward measure**.
- They observed the n-th forward Libor rate L_t^n is a martingale under the T_{n+1}-forward measure, leading easily to the Black-type formula if its volatility σ_t^n is deterministic.
- They also derived the drift of L_t^{n-1} under the T_{n+1}-forward measure, which depended only on L_t^n, thus iteratively constructing of the Libor market model.
- But, they neglected actually writing the SDE system for the forward Libor rates under the final maturity forward measure.

- Highlighting a *finite* tenor structure, $0 = T_0 < T_1 < \cdots < T_m$, **Jamshidian (1997)** derived the **forward Libor SDE system** under the **terminal measure** (T_m-forward measure).
- He introduced **the spot-Libor measure** as the proper analog of risk-neutral measure for the discrete-tenor model, and derived the SDE system under this measure.
- He introduced the **forward swap measure** (useful for valuation of European swaptions) and the **swap market model**, which models the volatilities of forward swaps rates and is useful for European calibrated Bermudan swaption valuation.
- He also described various major Libor and swap derivatives of the time and discussed the model's applicability to them.

- An attraction of the Libor Market Model is that it can easily calibrate to at-the-money caplets.
- The simplest way is to equate for all t the forward Libor volatility σ_t^n to the (stripped) implied volatility of the n-th caplet. (Of course, this might not yield the best model - for one thing it allows no caplet smile.)
- Following Jamshidian, let B^n denote the T_n-maturity zero-coupon bond price process, and $\delta_n \approx T_{n+1} - T_n$ be given day-count fractions. For $n < m$, the **forward Libor rate** process L^n is defined by

$$L_t^n := \frac{1}{\delta_n}(\frac{B_t^n}{B_t^{n+1}} - 1).$$

- Since the numeraire of the T_n-forward measure is by definition B^n, it follows that L^{n-1} is a martingale under the T_n-forward measure.
- Assuming L^n are continuous, the above easily implies that the drift of L^n under the terminal measure is given by

$$-\sum_{i=n+1}^{m} \frac{d[L^n, L^i]_t}{1+\delta_i L_t^i}.$$

- Thus, e.g., in the 1-factor case, **the SDE under terminal measure** is given by

$$\frac{dL_t^n}{L_t^n} = -\sum_{i=n+1}^{m} \frac{\sigma_t^n \sigma_t^i L_t^i}{1+\delta_i L_t^i} dt + \sigma_t^n dW_t^m,$$

where W_t^m is a Brownian motion under the terminal measure.

- Next, consider investing 1 dollar at time 0 in the T_1-maturity zero-coupon bond, and at T_1 investing the received principal in the T_2-maturity bond, and continuing so on to roll over the principal to the next maturity.
- Price process B^* of this asset is clearly

$$B_t^* = B_t^{\eta(t)} \prod_{j=1}^{\eta(t)} \frac{1}{B_{T_{j-1}}^j},$$

where $\eta(t)$ is the integer such that

$$T_{\eta(t)} - 1 < t \leq T_{\eta(t)}.$$

- The **spot Libor measure** is defined as the equivalent martingale measure induced by the above asset as numeraire. So, B^n/B^* is a martingale under spot Libor measure.

- Assuming continuity, the drift of L^n under the spot Libor measure turns out to be
$$\sum_{i=\eta(t)}^{n} \frac{d[L^n, L^i]_t}{1+\delta_i L_t^i}.$$
- Thus, the **forward Libor SDE under the spot Libor measure** is given by

$$\frac{dL_t^n}{L_t^n} = \sum_{i=\eta(t)}^{n} \frac{\sigma_t^n \sigma_t^i L_t^i}{1+\delta_i L_t^i} dt + \sigma_t^n dW_t^*,$$

where W_t^* is a Brownian motion under the spot Libor measure.

- For fixed $n < m$, the forward swap measure is defined as the equivalent martingale measure induced by taking the annuity

$$\sum_{i=n+1}^{m} \delta_{i-1} B_t^i$$

as numeraire.

- Hence, **the forward swap rate**

$$\frac{B^n - B^m}{\sum_{i=n+1}^{m} \delta_{i-1} B_t^i}$$

is a martingale under forward swap measure.

- The Black formula for a European swaption now follows easily if the forward swap rate volatility is assumed deterministic.

Some Outstanding Issues

Calibrating to Caplet/swaption volatility smile. One of the most outstanding challenges remains calibration to the volatility smile. Even in the equity case the problem is not easy. But, in the fixed income case the problem is compounded by the added dimensionality of the swaption tenor. As such, one now has to calibrate to a 3-dimensional "volatility cube" rather than just a 2-dimensional volatility surface.

Exact fit is practically out of question. The tradeoff is between the goodness of fit and efficiency. But, whereas many models have been proposed that produce a smile, including a CEV type volatility model by Andersen and Andreasen (2000), other stochastic volatility models, and not least, models with jumps, there still appears no consensus as to their adequacy and suitability for smile calibration. Fortunately, certain practical improvisations are possible.

Valuation of Bermudan options in the Libor Market Model. Viewing each forward Libor rate L^n as a state variable results in a Markovian setting and an associated PDE for prices of options. But the dimensionality will be too large for the finite difference method. So, save for a low-dimensional Markovian approximation, the only way to handle American and Bermudan options in LMM is by Monte-Carlo simulation. Although the past two decades have witnessed important strides in this direction, to our knowledge the jury is still not out on their effectiveness in the Libor Market Model.

For example, one of the most promising Monte-Carlo techniques is the regression method for calculating conditional expectation. For this purpose a suitable set of "basis" functions of the state variables is chosen to project on the span of. For two or three state variables, low degree polynomials are found to provide an adequate approximation. But, when there are dozens of state variables as in the Libor Market Model, such a set will be too large for efficient regression. On the hand hand, too small a set of basis functions will fail to provide an adequate approximation. We are not aware of any resolution of this difficulty.

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