The Bernoulli Conundrum

Abstract

Bernoulli's principle is typically stated in the form that increasing the speed of a gas lowers the pressure. This illogical interpretation casts aspersions on Bernoulli's equation, which is a direct application of Newton's second law. Consequently, authors have sought alternative explanations, including an isoergic model and the "bounce" model, inconsistent with physics. The difficulties are removed by recognizing that Bernoulli's equation tells us that a pressure difference causes a change in speed, and pressure differences are caused by curvature of flow, interpreted locally as producing a centrifugal force.

Subject headings:

- Bernoulli
- gases
- lift
- centrifugal force

The Bernoulli Conundrum

Robert P. Bauman
Professor of Physics Emeritus
University of Alabama at Birmingham

Occasionally, phenomena are discovered in nature that seem to be counterintuitive — apparently correct yet seemingly impossible. Bernoulli's equation is an example of such a conundrum. The principle, first stated by Daniel Bernoulli in 1738 in his *Hydrodynamica*, is often stated in the abbreviated form

Moving fluids have a lower pressure.

The explanatory equation,

$$P + \frac{1}{2}\rho u^2 = \text{constant} \tag{1}$$

in which P is the pressure of the fluid of density ρ and speed u, can be easily derived from Newton's second law, which states that the net force acting on any body is equal to the mass of the body times its acceleration. Even so, the result *appears* unexpected, if not absurd. Many physicists have denied its validity and/or spent much time looking for alternative explanations.[1] The equation is important in practice, with applications running from reading a newspaper to sailing and flying.

Background

Daniel Bernoulli was a member of a distinguished family of mathematicians from Basle, Switzerland. Jacques (1654-1705) and Jean (1667-1748), his brother, working together and in competition, contributed to the development of calculus. Daniel (1700-1782), son of Jean, applied Newtonian mechanics and Leibnitz' *vis viva* to hydrostatics and hydrodynamics In part 5 of section 12 of *Hydrodynamica* he gave the relationship between the pressure of a gas and the speed of the gas, now known as Bernoulli's theorem. This was accomplished more than 60 years before "energy" was defined (by Thomas Young) and nearly a century before Coriolis properly

---

[1] An attempted simplification of the equation suggests that it is a sum of energy terms and that because energy is preserved in the flow, the sum should be constant. The physics is seriously flawed. The integral of (constant) volume times the change in pressure is *not* equal to work and is not a change of energy. As shown many times, it is *enthalpy*, $H = E + PV$, that is constant in the flow, not energy. One cannot have both H and E constant when PV is changing.

defined kinetic energy.

Derivation

Derivation of Bernoulli's equation is simply a sum, or integration, of mass times acceleration, under the action of a pressure differential, or force. Newton's second law, applied to a fluid element of density ρ, cross-sectional area A, and length Δz, gives

$$f = -A\,\Delta P = ma = (\rho A\,\Delta z)\frac{\Delta u}{\Delta t} = \rho A\frac{\Delta z}{\Delta t}\Delta u = \rho A u\,\Delta u \qquad (2)$$

where u is the speed of the fluid along the z axis. Pressure gradient is opposite in direction to the force, requiring the change of sign before ΔP in the equation. This is generally known as Euler's equation for one-dimensional flow. Dividing by area gives

$$\Delta P = -\rho u \; \Delta u \qquad (3)$$

for the change in speed Δu (or du) over a distance for which the pressure changes by ΔP (or dP).

If we make the typical, but unnecessary, assumption that density, ρ, is constant, we can integrate over speed and pressure to obtain

$$P + \frac{1}{2}\rho u^2 = \text{constant} \qquad (1)$$

which is Bernoulli's equation.

Adding a change in hydrostatic pressure with change in height gives the change in pressure, dP, in terms of the density of the fluid, ρ, the speed, u, and change in speed, du, and the change in height, dh.

$$-\,dP = \rho\, u\, du + \rho g\, dh \qquad (4)$$

and therefore, again assuming constant density,

$$P + \frac{1}{2}\rho u^2 + \rho g h = \text{constant} \qquad (5)$$

with g the gravitational field strength (or "free-fall acceleration due to gravity", 9.8 m/s per second, or 32 ft/s^2).

Compressible Fluids

Although this is adequate for liquids, it is an unsatisfactory derivation for gases because gases are compressible. A complete analysis should at least consider, in addition to isochoric (*i.e.*, constant volume) flow, flows that are isothermal and adiabatic. To do so it is convenient to define a parameter, r, which is essentially of the form $r = \Delta E/E$, the change in energy divided by a reference energy level,

$$r = \frac{\frac{1}{2}mu^2 + mgh - \left(\frac{1}{2}mu_0^2 + mgh_0\right)}{kT_0} \tag{6}$$

and then integrate Euler's equation with appropriate assumptions about the flow conditions.

If the flow is isochoric (incompressible fluid), the integration gives:

$$P = P_0(1 - r) \tag{7}$$

which is the previous result (5) when we replace P_o with NkT_o/V. If the flow is isothermal (difficult to achieve in practice),

$$P = P_0 e^{-r} \approx P_0(1 - r + \cdots) \tag{8}$$

If the flow is adiabatic ($Q = 0$; no "heat" exchange, or transfer of thermal energy, which is most probable),

$$P = P_0\left(1 - \frac{\gamma - 1}{\gamma}r\right)^{\gamma/(\gamma-1)} \approx P_0(1 - r + \cdots) \tag{9}$$

with γ the ratio of heat capacities, C_P/C_V.

For adiabatic flow, temperature varies as

$$T = T_0\left(1 - \frac{\gamma - 1}{\gamma}r\right) \tag{10}$$

and density varies as

$$\rho = \rho_0 \left(1 - \frac{1}{\gamma} r + \cdots \right) \tag{11}$$

The difference between these results is small for speeds that are not close to the speed of sound. Calculated for a diatomic gas ($\gamma = 7/5$), convergence of formulas occurs at approximately $r = 0.25$ (speed along the direction of flow). Obviously, agreement between expressions for pressure under isochoric, isothermal, and adiabatic conditions does *not* imply similarity of isochoric, isothermal, and adiabatic conditions.[2]

Derivation by Equivalence

A second, alternative derivation shows there should be a pressure decrease along the direction of flow, and shows a form of symmetry between the two otherwise disparate terms (speed and height) of the equation.[3] If a fluid of density ρ, initially at rest, is given a uniform acceleration, a, over a distance s, to a final speed u, the standard kinematic equations tell us that

$$a = u^2/2s \tag{12}$$

Replacing the gravitational field strength, g, with acceleration we obtain, by analogy with the equation for change of density with depth,

$$-\,\Delta P = \rho g h \tag{13}$$

an additional term which adds to give

$$-\,\Delta P = \rho g h + \rho a s = \rho g h + \tfrac{1}{2}\rho u^2 \tag{14}$$

It is also clear that ΔP increases in magnitude with a and s, so that P may (in principle) go as low as zero. (A negative pressure would be meaningless. A zero pressure would require zero ambient pressure opposing the flow.)

We need not assume acceleration is constant. More generally,

$$-\,dP = \rho\, a\, ds = \rho\, du/dt\, ds = \rho\, u\, du \tag{15}$$

---

[2] For more details see Robert P. Bauman and Rolf Schwaneberg, "Interpretation of Bernoulli's Equation", *Phys. Teach.* **32**, 478-488 (November 1994).

[3] Robert P. Bauman, "An Alternative Derivation of Bernoulli's Principle", *Am. J. Phys.*, **68**, 288-289 (March, 2000).

and integration with constant density gives

$$- \Delta P = \tfrac{1}{2} \rho \, \Delta(u^2) \qquad (16)$$

Criticisms

There are many criticisms of Bernoulli's equation that are frequently expressed. For example:

1. The equation is derived for motion along a *streamline*, a line of flow of fluid without turbulence or viscosity, so it should only be applied along a single streamline.

However, in almost every instance, the fluid at either end of the flow has a uniform pressure, so the initial and final pressures along different streamlines are the same and there is no error in comparing intermediate pressures along different flow lines.

2. Real fluids (liquids or gases) do have viscosity (fluid drag).

However, a fluid without viscosity may be considered a limiting case, and corrections can be added for the (generally small, although not negligible) effects of viscosity.

3. One should distinguish between pressure measured in different directions in the moving fluid. But we must keep in mind the principle of relativity, which tells us that absolute motion cannot be detected. The effects must be the same whether a wing is passing through stationary air or the wing is stationary in air flowing through a wind tunnel. This, again, is confirmed by experiment. We know that pressure in stationary air must be isotropic (the same in all directions), so the same must be true for pressure in freely flowing air.

4. It doesn't make sense that just because a fluid gains speed, it loses pressure. This is probably the main reason that many people disbelieve Bernoulli's equation, even though Bernoulli's equation is a necessary conclusion from Newton's fundamental law of physics.

Re-examination of the equation, as expressed above, shows that such uncertainties of reference speed, or reference frame, are inherently absorbed into the constant on the right hand side of the equation, so the equation expresses only *changes* of pressure associated with *changes* of speed.

5. Bernoulli's theorem is often obscured by demonstrations involving non-Bernoulli forces. For example, a ball may be supported on an upward jet of air or water, *because* any fluid (the air and water) has *viscosity*, which retards the slippage of one part of the fluid moving past another part of the fluid. In the absence of viscosity — often loosely referred to as "thickness" of the moving fluid, as exhibited by syrup or by cold motor oil — all parts of the stream in linear motion would move at the same speed and there would be no force exerted by the fluid along the direction of flow.[4]

Bernoulli's effect is important for the supported body, but in a much more subtle way, as discussed below.

Experimental Evidence for Bernoulli's Equation

1. The Wright brothers, and earlier experimenters with gliders, gave us the prime example of the Bernoulli effect. Air pressure above a wing surface (where the air is moving faster relative to

[4] This was first recognized by D'Alembert, about 1746.

the wing) is lower than the pressure below the wing. Pressure difference multiplied by wing area is equal to a force, which for an airplane is directed upward. This force is called “lift” and is the explanation for why airplanes can fly.[5]

2. If a large beach ball is held in a nearly horizontal flow of air, as the ball sinks it is pushed back upward into the flow region, by the higher pressure beneath, out of the flow, and by lower pressure in the main stream of the flow. This provides lift that keeps the ball suspended in the nearly horizontal flow.

Support of the ball on a vertical jet of fluid provides a variation. The push of the fluid on the ball, in the direction of the flow (which keeps the ball from balling straight downward) arises from the non-Bernoulli viscous forces, but it is the Bernoulli effect that keeps that ball more or less centered along the primary line of flow, so it doesn’t “fall off” the jet.

Interpretation: Why Does it Work?

If you ask the wrong question, you may force the wrong answer. The question has often been asked (especially in recent years), “Why does increasing the speed of a gas lower its pressure?” Because the question seems to have no logical answer, many physicists have attacked the equation, which is equivalent to saying that Newton was wrong and that the conclusions drawn from Bernoulli’s equation in aerodynamics are wrong. (Can planes *really* fly?)

The difficulty lies in the assumption implied in formulating the question. The appropriate question is, “Why does lowering pressure in a gas cause an increase in gas speed?” The answer is then apparent from Newton’s second law. The pressure difference between two points in a fluid causes an acceleration.[6]

Only quite recently has the equation been studied in sufficient depth to lead to a clear understanding. We make the usual assumptions that the effects of viscosity can be ignored (certainly not rigorously true, but a satisfactory first approximation) and that we can ignore boundary effects — that is, ignore effects related to the presence of wind tunnel walls or the ground (usually a reasonable starting point).

Careful analysis shows that if the fluid is a gas, a significant effect is the conversion of random kinetic energy of the molecules to collective flow. That is, there is a drop in temperature of the air as the speed increases. That temperature drop lowers the pressure.[7] However, this effect is small compared to the very large pressure changes typically recorded, and is absent in liquids. So this conversion is not a major part of “the Bernoulli effect”.

---

[5] The *reducto ad absurdum* is given by Landis (Fred Landis, *Encarta*, 2007; *Bernoulli’s Principle*) who states “...the principle technically only applies to systems that do not produce a net force.”

[6] An accurate qualitative argument of pressure as cause and speed as effect was given by S. Brusca, “Buttressing Bernoulli”, *Phys. Educ.* **21**, 14-12, 262-263 (1986).

[7] See reference 2, or Robert P. Bauman, *Modern Thermodynamics with Statistical Mechanics*, Macmillan Publishing Co., New York, 1992; p. 150.

The remaining question, then, is what is it, along the path of a flowing liquid or gas, that causes the pressure to drop (and thus causes the fluid to gain speed)? A typical cause of a change in speed is a simple fan, or the equivalent. Moving fan blades push the (viscous) fluid, increasing its speed as it escapes from the high-pressure region. Further down stream, a restriction in the flow may cause a decrease in speed and increase in pressure, followed by an increase in speed and decrease in pressure as the fluid passes beyond the constriction. The flow stops when the opposing pressure is sufficient to decelerate the fluid. This is called the *static pressure* or the *ram pressure*. It is approximately equal to the pressure that initiated the flow.

The key to understanding Bernoulli's equation in typical applications is a pair of experiments with a simple manometer, or pressure gauge. If one leg of a U-tube manometer (*e.g.*, with water as the indicating fluid) is thrust into the air flow from a fan, as shown in Figure 1, the reaction is strong, showing that the pressure read within the flowing air is much lower than the pressure of ambient air outside the flow.

If, however, the end of the manometer is fitted inside a flat surface so that the air flows smoothly across the open end, no pressure difference is shown. The difference is a contrast between linear flow and curved flow.[8]

Figure 1. A simple open-end manometer, thrust into moving air, shows a much lower pressure in the region of higher speed. The effect disappears if curvature of flow around the end of the tube is eliminated. (Curvature of flow is important but is ignored in the schematic drawing.)

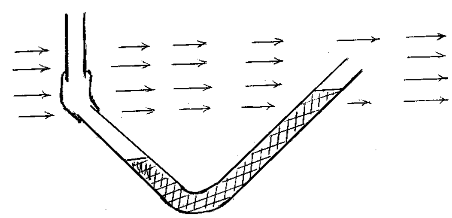

Figure 1. A simple open-ended manometer, thrust into moving air, shows a much lower pressure in the region of higher speed.

Lift

As is well known, and understood, moving bodies, including fluids, tend to move at a constant speed in a straight line. If the flow is across a curved surface, the "preferred" straight-line trajectory of the fluid carries it away from the curved surface, leaving a relative vacuum (*i.e.*, a low-pressure region) along the surface.[9] The low-pressure region is precisely what is required to

---

[8] This distinction has been drawn by Evan Jones, private communication. An equivalent point has been made by Holger Babinsky, "How do wings work?", *Phys. Educ.* **32**, 497-507 (November, 2003); see especially the Appendix.

[9] A more sophisticated description of the same process assigns a centrifugal force ("fleeing the center") to the moving fluid in what appears to be a rotating reference frame and, because there is no counterbalancing central force (Latin "centripetal" force), a pressure

cause acceleration of the fluid coming behind. Thus a low-pressure region is produced by the curvature of flow and has caused the fluid to move faster, all in agreement with Newton's laws of motion and hence with Bernoulli's equation (Figure 2). Over an airfoil, the decrease in pressure may be sufficient that the flow may reach supersonic speeds. Calculated pressures show that any other major cause of pressure drop or speed change is likely to lead to excessive values when added to this anticipated effect of curvature of flow. The art and science of aerodynamics is to refine the calculations to include both the effect of wing shape and the known secondary effects of turbulence, viscosity, and higher speeds (leading to deviations from equation 9).

Figure 2. Points A correspond to ambient pressure. Point B is a low pressure region. Centrifugal force on flowing air creates a low-pressure region around B which accelerates the air flow and causes lift on the wing. Air reaches the normal plane, N, at C before it reaches C'. (Representation is schematic.)

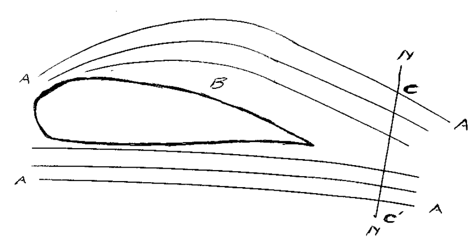

Figure 2. Points A correspond to ambient pressure. Point B is a low pressure region.

Superficial arguments claiming that airplanes should not be able to fly with flat wings (even when made of balsa or paper) or upside down, ignore the curvature of flow that is determined by the angle of attack. The argument is not as pretty as with cambered wings, but it is basically the same argument.

Complications

It is tempting to argue that the time of flow should be the same along each of the stream lines, starting from a vertical line perpendicular to the air flow (or relative flow, for a wing in stationary air) and ending, for investigative purposes, along a line perpendicular to the flow below and behind the wing where the air is again moving uniformly. This can be tested in wind tunnels by injecting small bubbles or puffs of smoke as tracers. In a typical experiment, the upper stream arrives at the "termination" plane *before* the other streams, showing that it travels sufficiently faster to go farther but arrive sooner.

There is nothing in Bernoulli's equation that implies equal times of transit. Therefore failure to find equal times casts no aspersions on the equation. On the other hand, the time difference is generally small, and may be a second-order effect. To maintain steady flow it seems reasonable

---

difference is produced.

that the flow times above and below should be the same. In practice, viscosity enters for real air to eventually dampen the disturbance behind the wing and avoid accumulation of air along either path.

Perhaps the most common fallacy, which seems to reappear with each generation that studies Bernoulli's equation, is that lift arises primarily from "air bounce" off the lower wing surface.[10] There is such an effect (although it is better represented by the smooth curved flow of Figure 3), but it is smaller than usually expected. Furthermore, the downward deflection has already been fully included in the analysis based on Bernoulli's equation. Because of decreased pressure above, where the air is moving faster, the air pressure below pushes the wing up. This, of course, produces a reaction force on the air, causing the air to move downward. The Bernoulli effect is a Newtonian force and subject to all the Newtonian rules. This *downwash* is particularly obvious from the airfoils of helicopter blades, or to pilots as they attempt to land and experience reflection of the air from the ground.

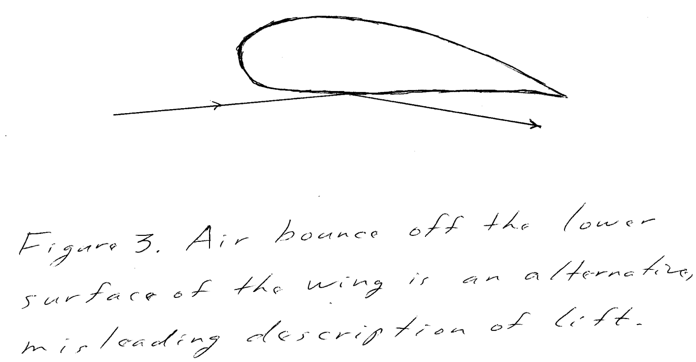

Figure 3. Air bounce off the lower surface of a wing is an alternative, misleading description of lift. The pressure increase caused by the deflection off the lower surface is much less than the pressure decrease above the wing surface associated with increased fluid speed. The effect of the air deflection below is already included in Bernoulli's equation which considers the pressure difference between surfaces.

To observe lift, it is not even necessary that the air flow include the lower surface of the wing, or of a beach ball suspended in a nearly horizontal air flow. Because the fluid curves as it passes over the upper surface (*e.g.*, in the popular demonstration of blowing over the top of a sheet of paper held "horizontally"), the pressure is less than ambient above the curved surface and the air foil or ball is subject to lift. If the "bounce" were in addition to the pressure differential, the lift on the wing would be substantially greater than expected or observed.

The Coanda effect is another "add on" that must be considered in quantitative analyses of airfoil performance, but is typically neglected in initial analyses. As for syrup pouring from a pitcher, viscous drag at the surface gives a rotation to the fluid, producing a typical curved path as it leaves the surface. This is included among the effects of viscosity in real air. Viscosity adds other important side effects, including drag on a wing or other object (a force in the direction of

---

[10] An explicit example is given by *Encarta* (*Aerodynamics*, unsigned article; Microsoft, 2007), where lift is ascribed to Newton's third law and the illustration shows the flow passing entirely *below* the wing.

the flow) and turbulence.

Demonstrations

Bernoulli's equation explains why you can separate two sheets of paper by blowing at the edge. The pressure is less on the outside, where the air is moving, than between sheets where the air is static. To us, the distance from static air at the center to flowing air around the outside seems very small, but to an air molecule the distance is far from negligible, and the path around the outside necessarily involves curvature.

A propeller in water causes high speeds of water around the tips of the blades, which may produce pressures below the vapor pressure of the water (about 1/30 atm at room temperature). Bubbles of water vapor form, an effect called *cavitation*. Resulting forces cause severe wear on propeller blades, including possible destruction.

The aerodynamic "lift" of a sail is well known to most sailors. The sail assumes an airfoil contour that gives a force roughly perpendicular to the direction of air flow past the sail. Bulging tops of convertibles are a similar familiar sight, produced by curvature of air flow over the top. Lift on racing cars is well known and counteracted by wings set for negative lift.

Aspirator vacuum pumps in laboratory sinks have effectively produced low pressures for many decades, and spray cans and bottles with pumps have delivered insecticides, waxes, and perfumes on demand, although Evans has shown that if the curvature of flow is eliminated, an aspirator no longer works.

Conclusion

In short, Bernoulli's equation is a straight-forward application of Newton's laws, often misinterpreted. There are other details that must be included in a full treatment of aerodynamic/hydrodynamic flow, including viscous drag and turbulence, but there should be no surprises in Bernoulli's equation.

The difficulties are removed by recognizing that Bernoulli's equation tell us that a pressure difference causes a change in speed, and pressure differences are caused by curvature of flow, interpreted locally as producing a centrifugal force.

For those who wish to avoid the details, it is only necessary to point out that where there is curved fluid flow, there is a pressure difference (*i.e.*, lift), and from Bernoulli's equation (simply and properly interpreted) the existence of the pressure difference tells us there must be a speed difference.