TUTORIAL on QUATERNIONS
Part I

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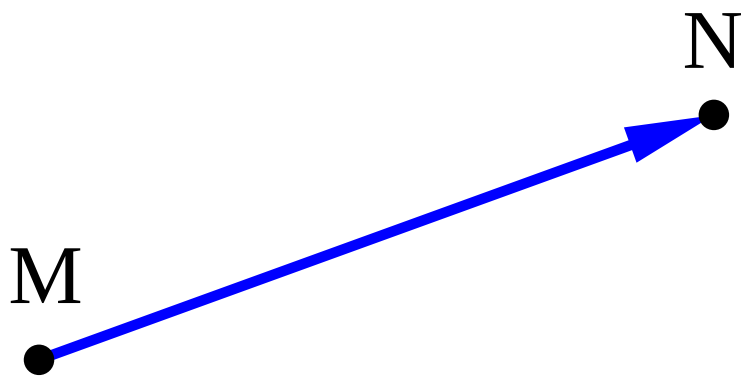
This document was created using LyX and the LaTeX Seminar style.

Introduction

- Quaternions are commonly used to represent rotations.
- They were introduced by William Hamilton (1805-1865) [1]
- Quaternions were conceived as Geometrical Operators
- A Complete Calculus of Quaternions was introduced by Hamilton [2]

Definition of Vector



A *Vector* is a line segment with orientation

Vector $\overrightarrow{MN}$ represents the relative position of point N with respect to point M

Hamilton's Motivation for Quaternions

Create a Mathematical Concept to represent

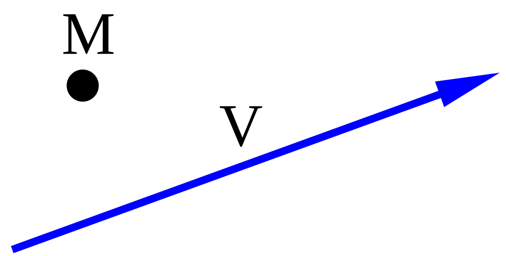
The **RELATIONSHIP** between two **VECTORS**.

In the same way that a Vector represent

The **RELATIONSHIP** between two **POINTS**.

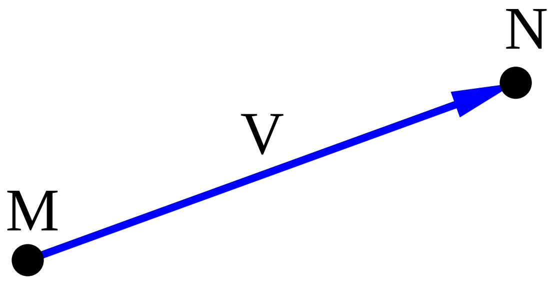
Vector applied to Point

Given :



Point *M* and a Vector *V*

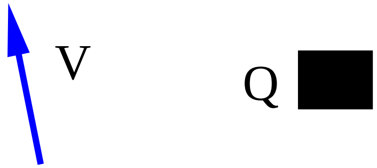
The application of the Vector over the Point Results in a



Unique Point *N*

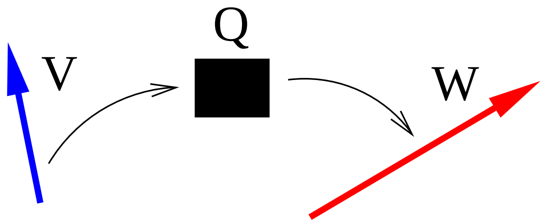
Quaternion applied to Vector

In the same way, Hamilton wanted that given



Vector V and a Quaternion Q

The application of the Quaternion over the Vector Results in a



Unique Vector W

Quatenion Rationale

A vector is completly defined by

- Length
- Orientation

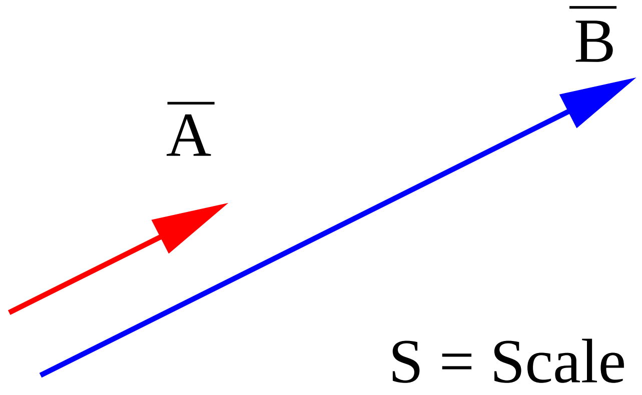
In order to define a vector in terms of another vector
a Quaternion has to represent

- Relative Length
- Relative Orientaton

Definition of Scalar

A *Scalar* is defined as

The ratio between the lengths of two **PARALLEL** vectors $\overrightarrow{A}$ and $\overrightarrow{B}$



It represents the **RELATIVE LENGTH** of one vector with respect to the other.

Note that in programming jargon *scalar* has mistakenly taken the place of *rea*l

Scalar - Vector Operations

$$S = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

A *Scalar* S is the Quotient between two **PARALLEL** vectors $\overrightarrow{A}$ and $\overrightarrow{B}$

$$\overrightarrow{A} = S \diamond \overrightarrow{B}$$

A *Scalar* is an Operator that

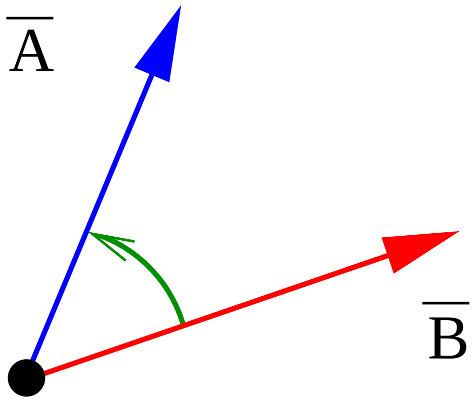
- Changes the **SCALE** of the vector
- Keeps its orientation unchanged

Application the Scalar Operator is noted by the symbol $(\diamond)$.

Definition of a Versor

A *Versor* is defined as

The quotient between two non-parallel vectors of **EQUAL LENGTH**



It represents the **RELATIVE ORIENTATION** of one vector with respect to the other.

Versor - Vector Operations

$$V = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

A *Versor* V is the Geometric Quotient between two non-parallel vectors of **EQUAL LENGTH** $\overrightarrow{A}$ and $\overrightarrow{B}$

$$\overrightarrow{A} = V \diamond \overrightarrow{B}$$

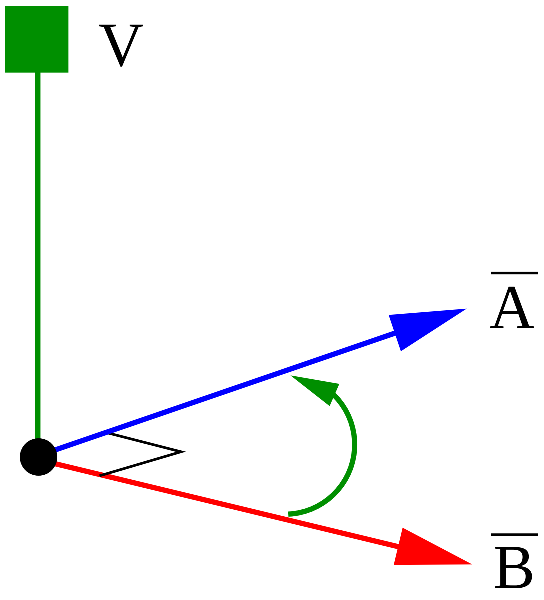
A *Versor* is an operator that

- Changes the **ORIENTATION** of the vector
- Keeps its length unchanged

Application of the Versor Operator is noted by the symbol ($\diamond$).

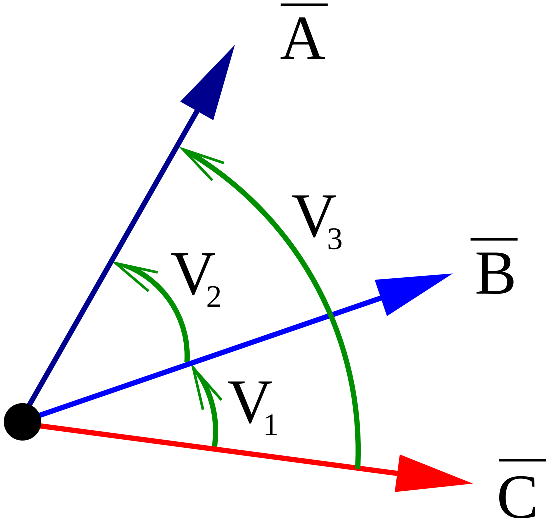
Right Versors

A *Right Versor* is a Versor that applies a 90° rotation



Vector length is left unchanged as in any other Versor application

Composing Versors



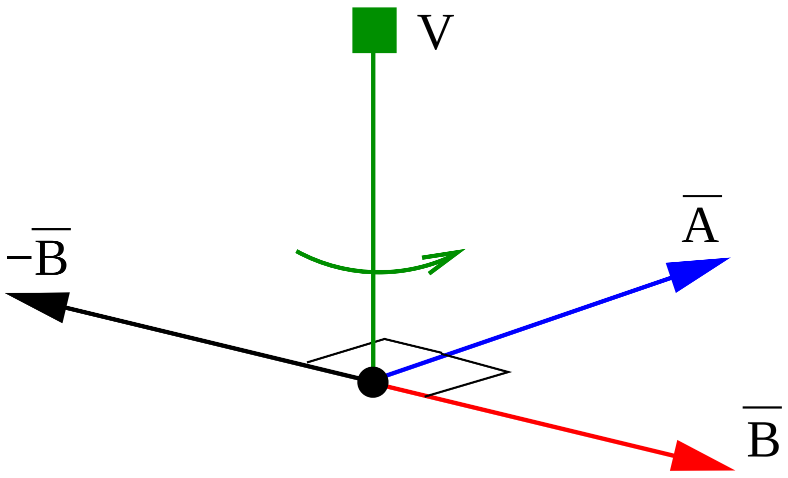
$$V_3 = V_2 \diamond V_1$$

$$\begin{aligned}
\overrightarrow{A} &= V_2 \diamond \overrightarrow{B} \\
\overrightarrow{B} &= V_1 \diamond \overrightarrow{C} \\
\overrightarrow{A} &= V_2 \diamond V_1 \diamond \overrightarrow{C} \\
\overrightarrow{A} &= V_3 \diamond \overrightarrow{C}
\end{aligned}$$

Versor composition is the consecutive application of two versors operators.

It is noted by the symbol $(\diamond)$

Composing Right Versors



$$\overrightarrow{A} = V \diamond \overrightarrow{B}$$
$$-\overrightarrow{B} = V \diamond \overrightarrow{A}$$
$$-\overrightarrow{B} = V \diamond V \diamond \overrightarrow{B}$$
$$-1 = V \diamond V$$

(-1) is the **INVERSION** operator that inverts the direction of a vector.

The double application of a right versor to a vector, inverses the vector.

Definition of Quaternion

$$Q = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

A *Quaternion* is the Geometrical Quotient of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$

$$\overrightarrow{A} = Q \diamond \overrightarrow{B}$$

A *Quaternion* is an operator that

- Changes the **ORIENTATION** of the vector
- Changes the **LENGTH** of the vector

Application of the Quaternion Operator is noted by the symbol ($\diamond$)

Quaternion Characteristics

- **Axis(Q)** = Unit Vector perpendicular to the plane of rotation
- **Angle(Q)** = Angle between the vectors in the quotient
- **Index(Q)** = In a Right Quaternion is the Axis(Q) multiplied by the length ratio of the two vectors in the quotient.

Representation of Quaternions

Quaternion = "*A set of Four*"

From

- the Latin *Quaternio*
- the Greek τετρακτυς

The combined operation of *Scalar* and *Versor* requires 4 numbers:

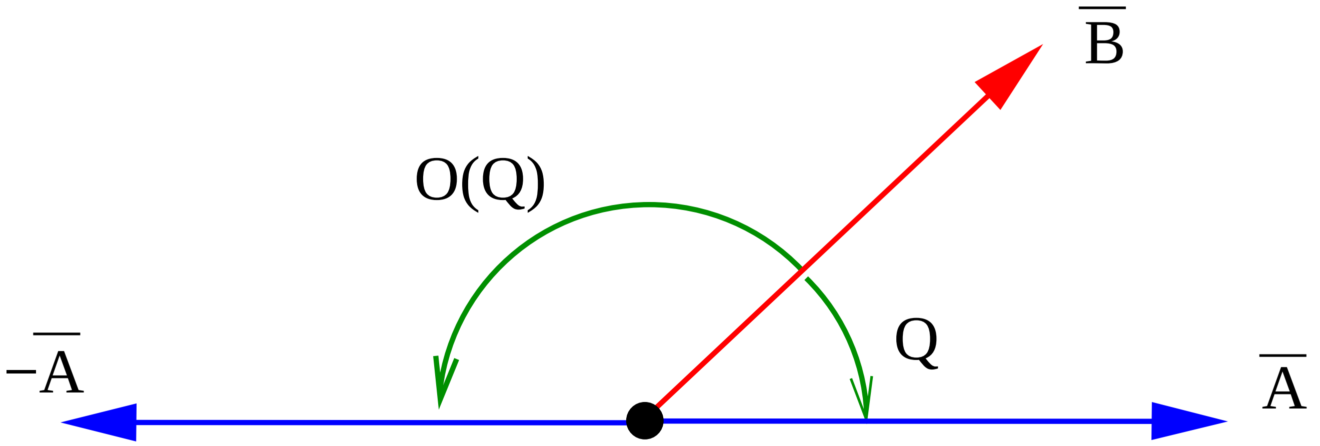
- 1 for Scale
- 1 for Angle
- 2 for Orientation (common plane)

Quaternion = Scalar combined with Versor

Opposite Quaternions

The quaternion Q has an *Opposite* quaternion $O(Q)$

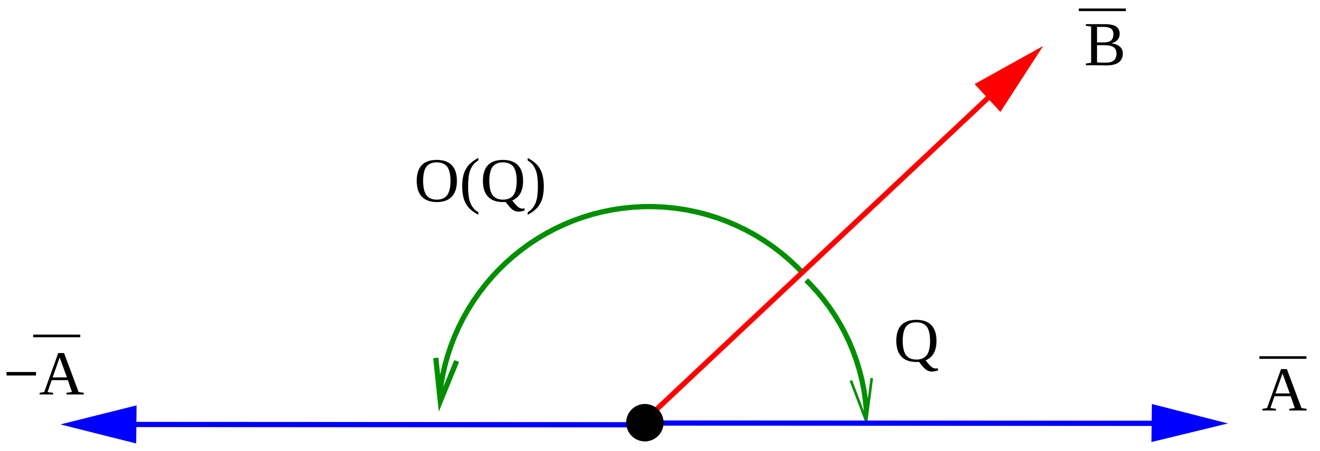
$$Q = \frac{\overrightarrow{A}}{\overrightarrow{B}} \qquad O(Q) = \frac{-\overrightarrow{A}}{\overrightarrow{B}} = -Q$$



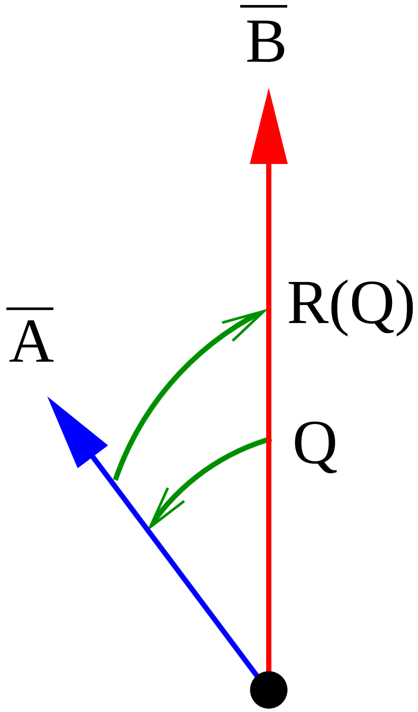
Opposite Quaternion Properties

$$Angle(Q) + Angle(O(Q)) = \pi$$

$$Axis(Q) = -Axis(O(Q))$$



Reciprocal Quaternions



$$Q = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

The quaternion Q has a *Reciprocal* quaternion $R(Q)$

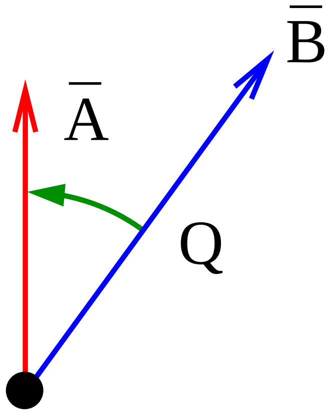
$$R(Q) = Q^{-1} = \frac{\overrightarrow{B}}{\overrightarrow{A}}$$

Their composition (one quaternion applied after the other) is

$$Q \diamond R(Q) = 1$$

The (1) operator is an Identity Operator that leaves vectors unchanged.

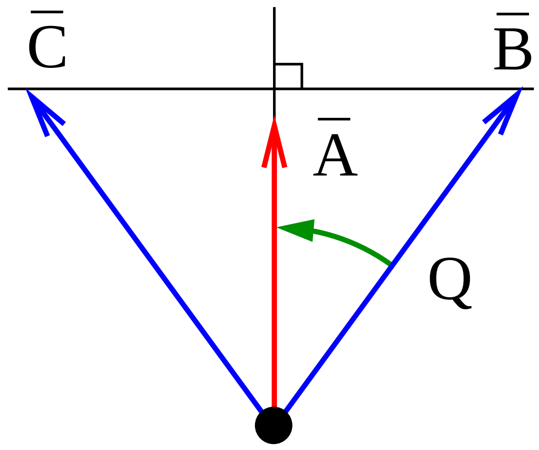
Conjugate Quaternion



The geometric reflection of vector $\overrightarrow{B}$ (the denominator) over vector $\overrightarrow{A}$ (the numerator) will be vector $\overrightarrow{C}$

Given the pair of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ and their quotient

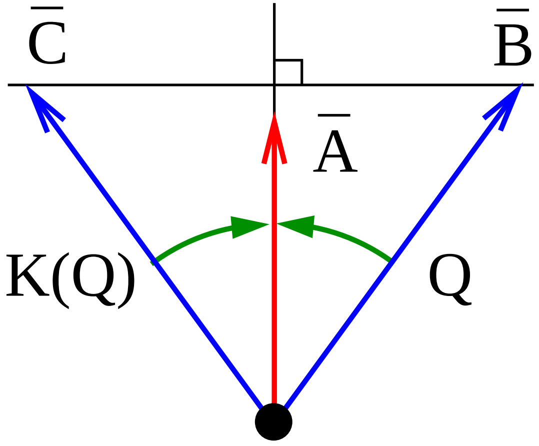
$$Q = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$



Conjugate Quaternion

The *Conjugate* of Quaternion Q is defined as the quotient $K(Q)$

$$K(Q) = \frac{\overrightarrow{A}}{\overrightarrow{C}}$$



$$Angle(Q) = Angle(K(Q))$$

$$Axis(Q) = -Axis(K(Q))$$

Norm of a Quaternion

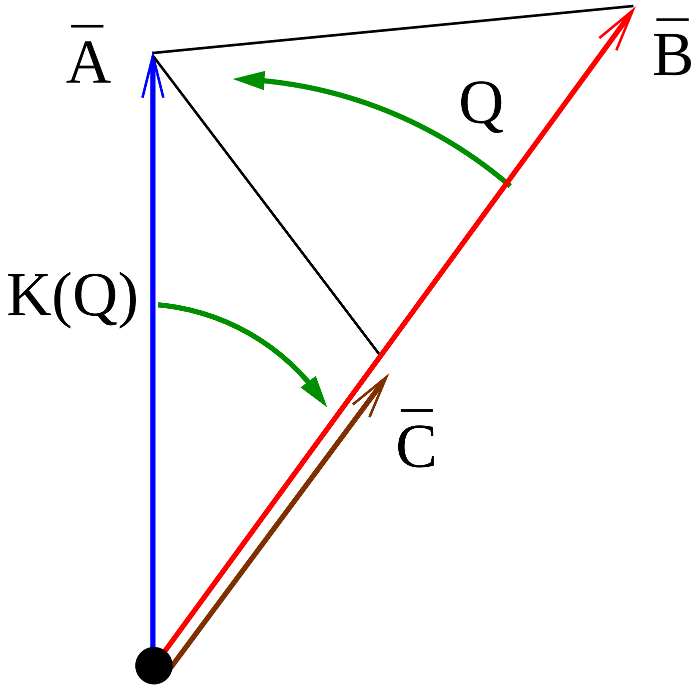
The *Norm* is the composition of a Quaternion with its Conjugate

$$N(Q) = Q \diamond K(Q)$$

$$Q = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

$$K(Q) = \frac{\overrightarrow{C}}{\overrightarrow{A}}$$

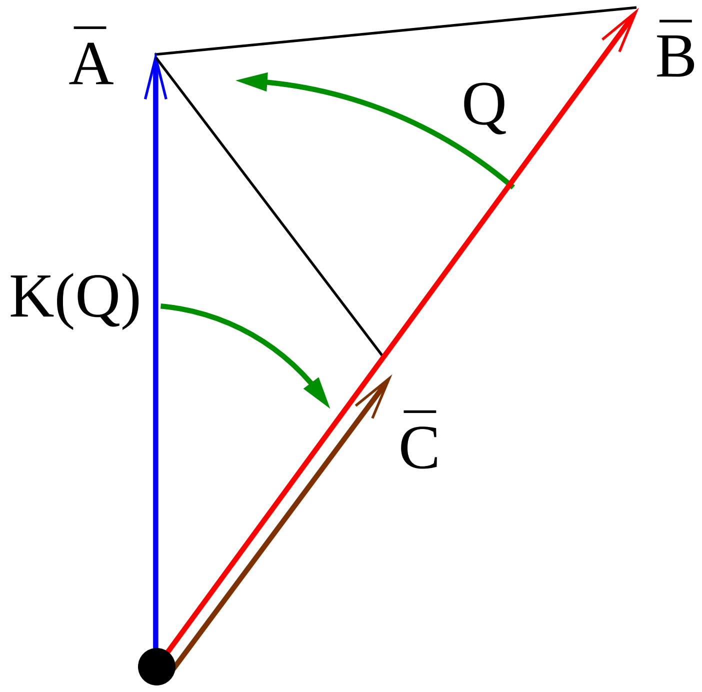
$$N(Q) = \frac{\overrightarrow{A}}{\overrightarrow{B}} \diamond \frac{\overrightarrow{C}}{\overrightarrow{A}} = \frac{\overrightarrow{C}}{\overrightarrow{B}} = \left[\frac{\left\|\overrightarrow{A}\right\|}{\left\|\overrightarrow{B}\right\|}\right]^2$$



Norm of a Quaternion

The rotation of the Conjugate $K(Q)$ compensates the rotation of the quaternion Q.

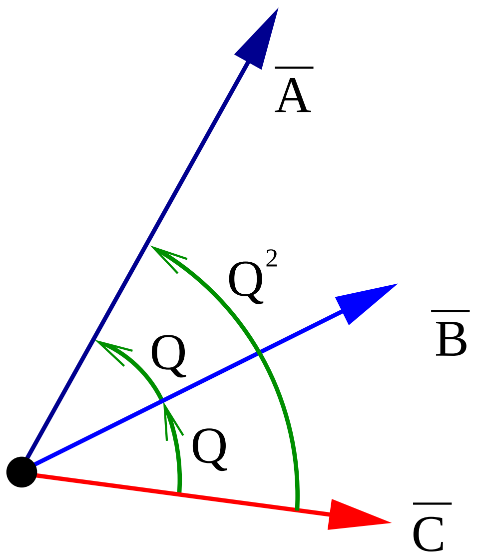
The operator $N(Q)$ produce a parallel vector, hence $N(Q)$ is always a positive *Scalar* operator



Square of a Quaternion

The *Square* of a Quaternion is defined as :

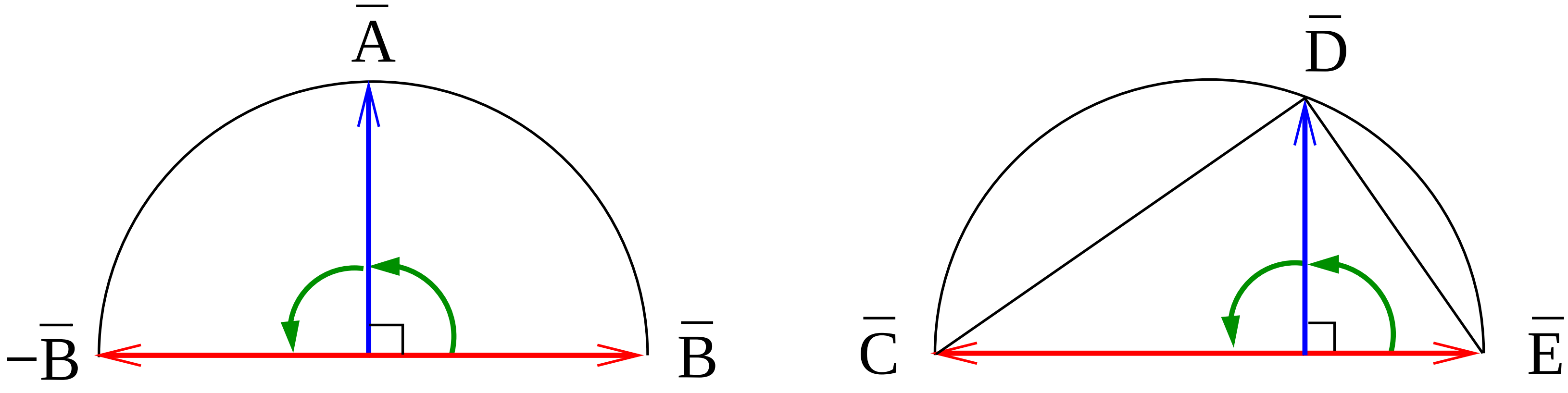
Applying the quaternion twice



$$\begin{aligned}\overrightarrow{B} &= Q \diamond \overrightarrow{C} \\ \overrightarrow{A} &= Q \diamond \overrightarrow{B}\end{aligned}$$

$$\begin{aligned}\overrightarrow{A} &= Q \diamond \left(Q \diamond \overrightarrow{C}\right) \\ &= Q \diamond Q \diamond \overrightarrow{C} \\ &= (Q \diamond Q) \diamond \overrightarrow{C} \\ &= (Q)^2 \diamond \overrightarrow{C}\end{aligned}$$

Composing Right Quaternions



The succesive application of a Right Quaternion over a Vector results in a Vector in the opposite direction.

$$\left(\frac{\overrightarrow{A}}{\overrightarrow{B}}\right)^2 = \frac{-\overrightarrow{B}}{\overrightarrow{B}} - 1$$

$$\frac{\overrightarrow{C}}{\overrightarrow{E}} = \left(\frac{\overrightarrow{D}}{\overrightarrow{E}}\right)^2 = -\left(\frac{\left\|\overrightarrow{D}\right\|}{\left\|\overrightarrow{E}\right\|}\right)^2$$

The square of any right quaternion is a **NEGATIVE** scalar operator

Versor of a Quaternion

Versor of a Vector = Unit vector parallel to the vector

$$U\left(\overrightarrow{A}\right) = \frac{\overrightarrow{A}}{\left\|\overrightarrow{A}\right\|} = \widehat{A}$$

Versor of a Quaternion = Quotient of the Versors of the vectors

$$U(Q) = U\left(\frac{\overrightarrow{A}}{\overrightarrow{B}}\right) = \frac{U\left(\overrightarrow{A}\right)}{U\left(\overrightarrow{B}\right)} = \frac{\widehat{A}}{\widehat{B}}$$

It is the part of the Quaternion that represents Relative Orientation

Tensor of a Quaternion

Tensor of a Vector = Length of the vector

$$T\left(\overrightarrow{A}\right) = \left\|\overrightarrow{A}\right\|$$

Tensor of a Quaternion = Quotient of the tensor of the vectors

$$T(Q) = T\left(\frac{\overrightarrow{A}}{\overrightarrow{B}}\right) = \frac{T\left(\overrightarrow{A}\right)}{T\left(\overrightarrow{B}\right)} = \frac{\left\|\overrightarrow{A}\right\|}{\left\|\overrightarrow{B}\right\|}$$

It is the part of the Quaternion that represents Relative Sscale

Tensor and Versor of a Quaternion

Versor operator applies ***VERSION*** to a vector

Changes vector's orientation

Tensor operator applies ***TENSION*** to a vector

Stretches the vector and change its length

Tensor and Versor of a Quaternion

A **Vector** can be decomposed in **Versor** and **Tensor** parts

$$\overrightarrow{A} = T\left(\overrightarrow{A}\right) \diamond U\left(\overrightarrow{A}\right) = \left\|\overrightarrow{A}\right\| \diamond \widehat{A}$$

A **Quaternion** can be decomposed in **Versor** and **Tensor** parts

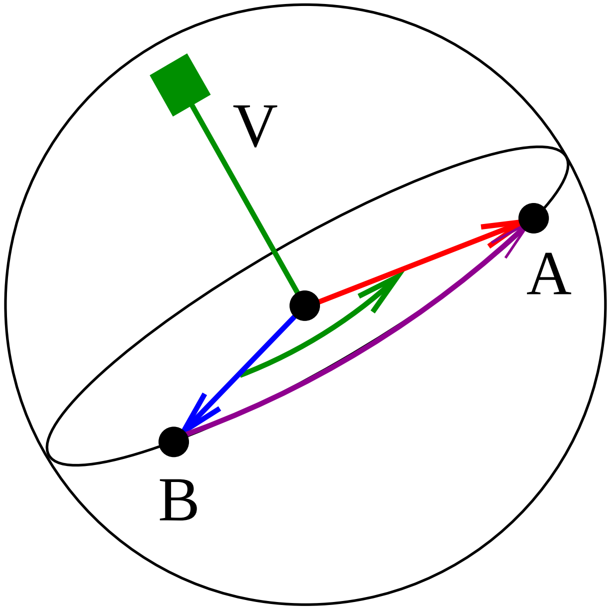
$$T(Q) = T(Q) \diamond U(Q) = \left[\frac{T\left(\overrightarrow{A}\right)}{T\left(\overrightarrow{B}\right)}\right] \diamond \left[\frac{U\left(\overrightarrow{A}\right)}{U\left(\overrightarrow{B}\right)}\right]$$

Vector - Arcs

Versors can be represented on the surface of a unit sphere.

$$V = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

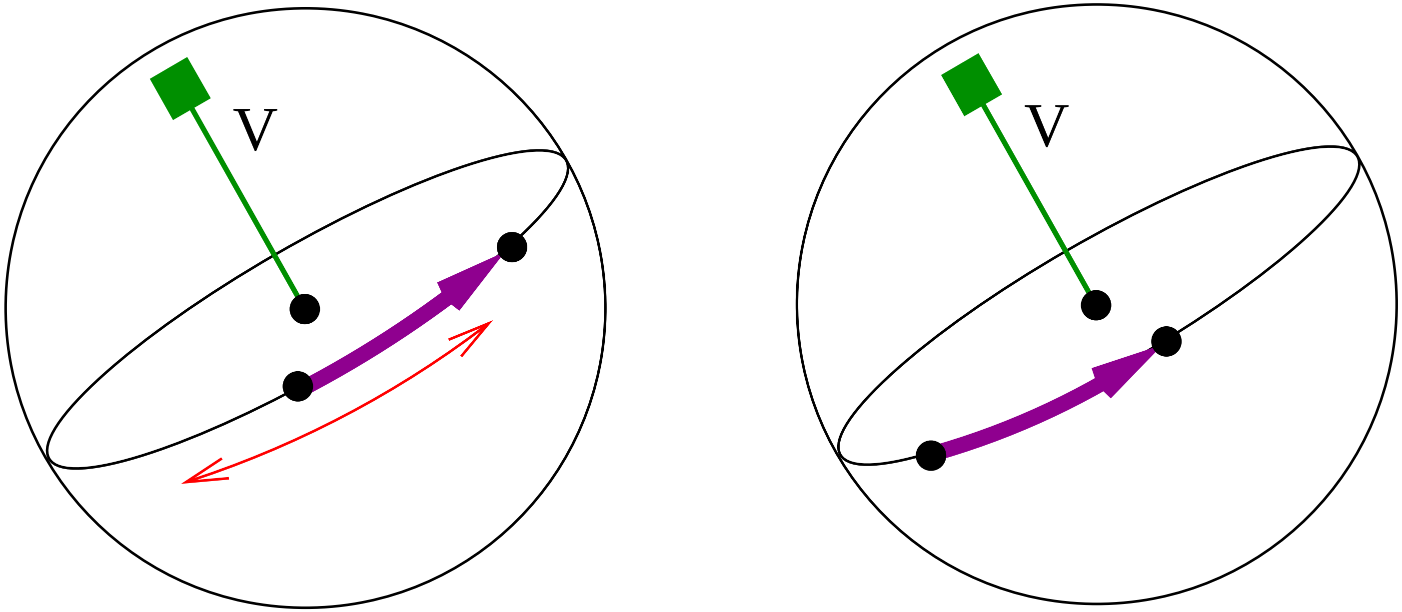
Application of versor V will move point B to point A



The Maximum Arc joining points B and A is defined as ***Vector-Arc***

Sliding Vector - Arcs

In the same way that Vectors can be translated on a plane



Vector arcs can freely **slide** along the great circle
and still represent the **SAME** *Versor*.

Composition of Biplanar Versors

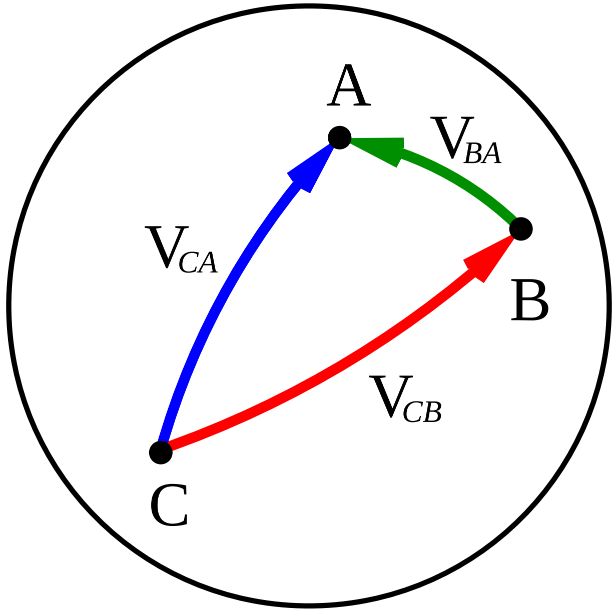
$$V_{BA} = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$

composed with

$$V_{CB} = \frac{\overrightarrow{B}}{\overrightarrow{C}}$$

results in the versor

$$V_{CA} = \frac{\overrightarrow{A}}{\overrightarrow{B}} \diamond \frac{\overrightarrow{B}}{\overrightarrow{C}} = \frac{\overrightarrow{A}}{\overrightarrow{C}}$$



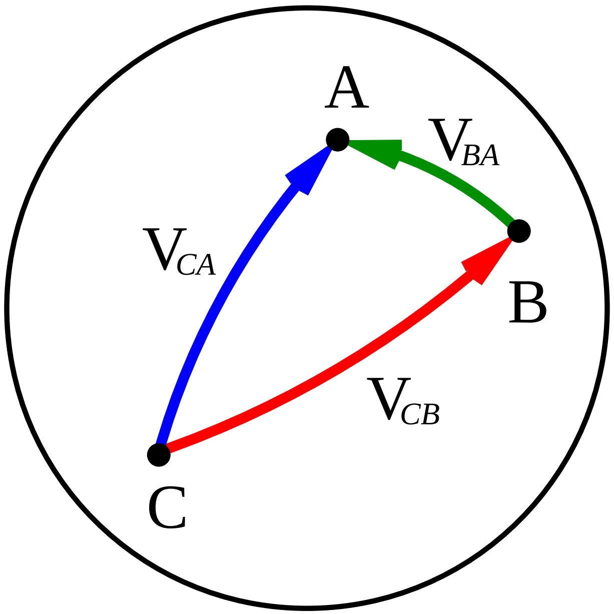
Multiplication and Division of Diplanar Versor

The *Spherical Triangle* *ABC* is used to define versor operations analogously to how the parallelogram is used for vector operations

Multiplication of Versors as

$$V_{CA} = V_{BA} \cdot V_{CB}$$

like the sum of vectors

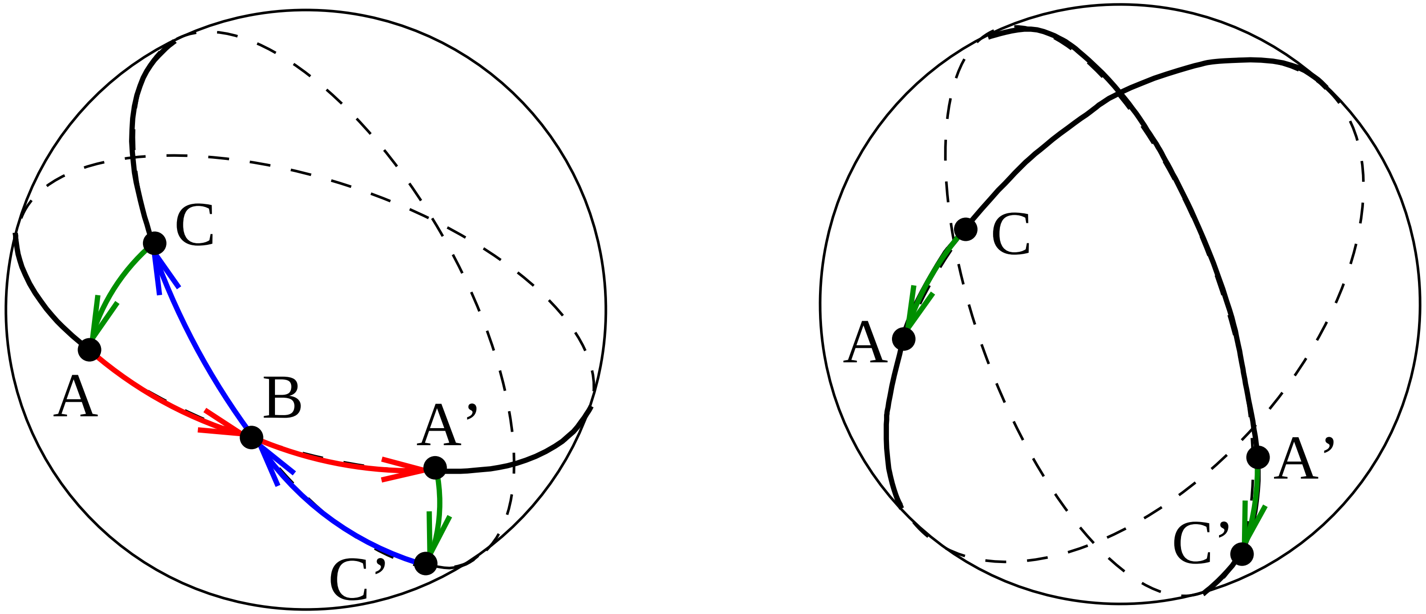


Division of Versors as

$$V_{BA} = \frac{V_{CA}}{V_{CB}}$$

like the difference of vectors

Versor Composition is Non-Commutative



The resulting versors V_{CA} and $V_{A'C'}$ have the same angle

but different axis (and so, different planes)

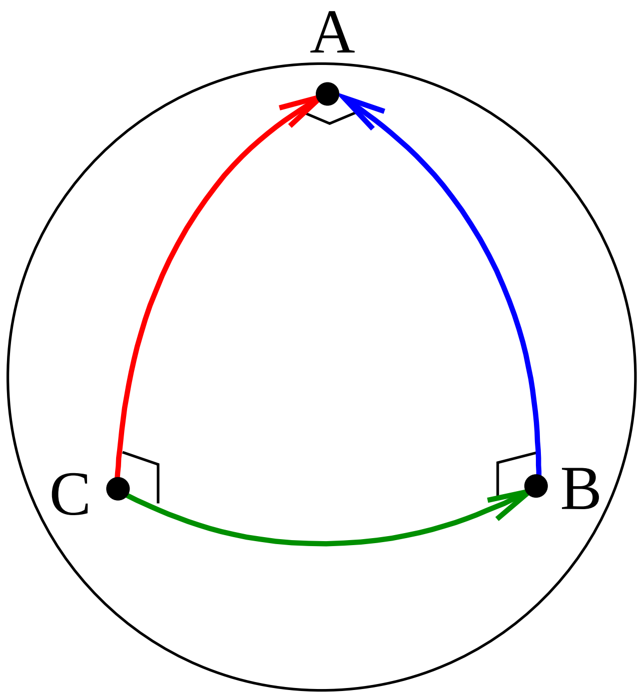
Composition of two Orthogonal Right Versors

The multiplication of two orthogonal Right Versors produce a Right Versor orthogonal to them

$$V_{CB} \cdot V_{BA} = V_{CA}$$

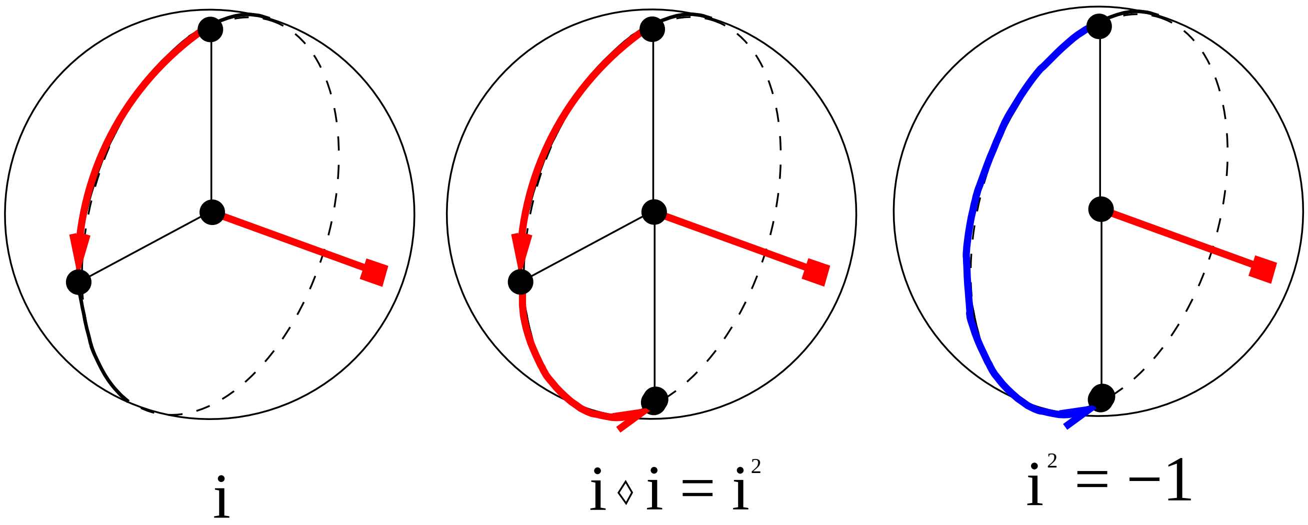
An when the order is reversed

$$V_{BA} \cdot V_{CB} = -V_{CA}$$



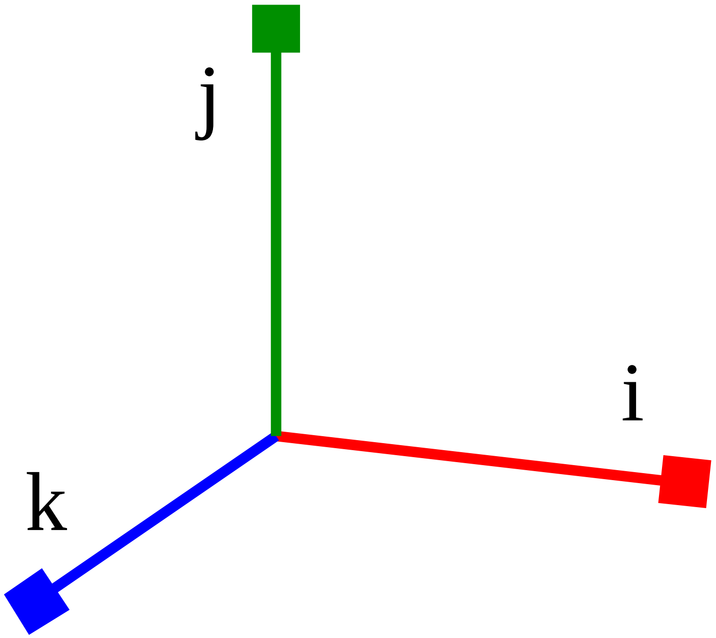
Square of Elementary Versors

The *Square of* an operator is the operator applied twice



The square of Right Versors is always the (-1) Operator

Elementary Versors

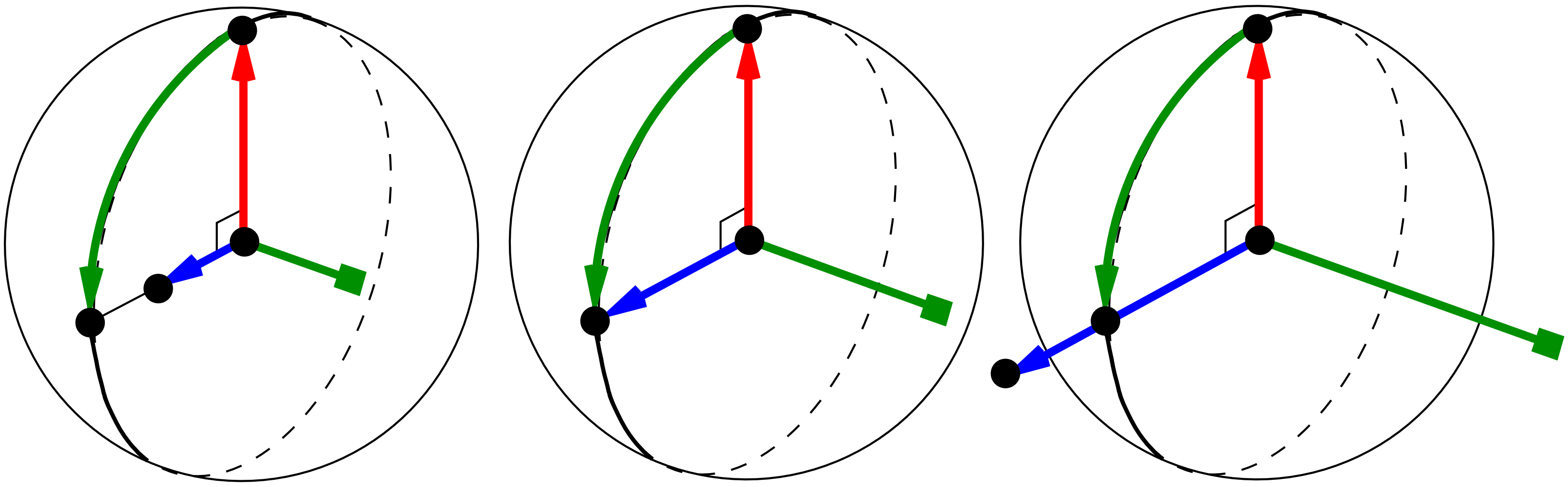


Composition of Elementary Versors

right-hand	self
$i \cdot j = k$	$i \cdot i = -1$
$j \cdot k = i$	$j \cdot j = -1$
$k \cdot i = j$	$k \cdot k = -1$

Index of Right Quaternions

The *Index of* a Right Quaternion is



the Axis of the quaternion Scaled by the ratio of lengths

Sum of Versors

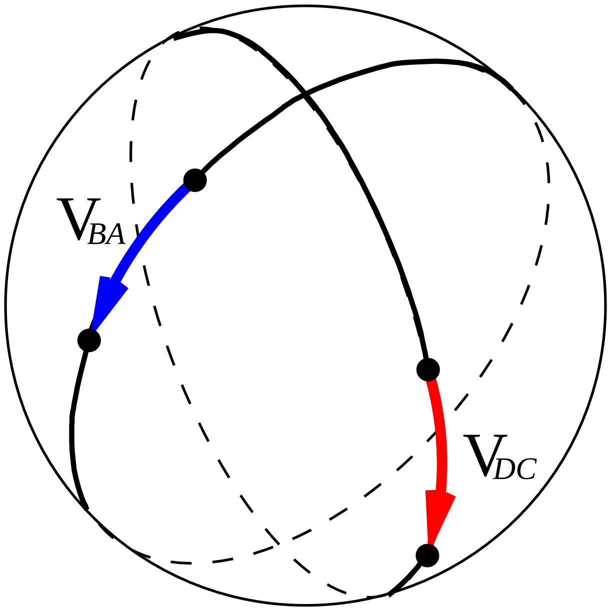
Versors are Quotients.

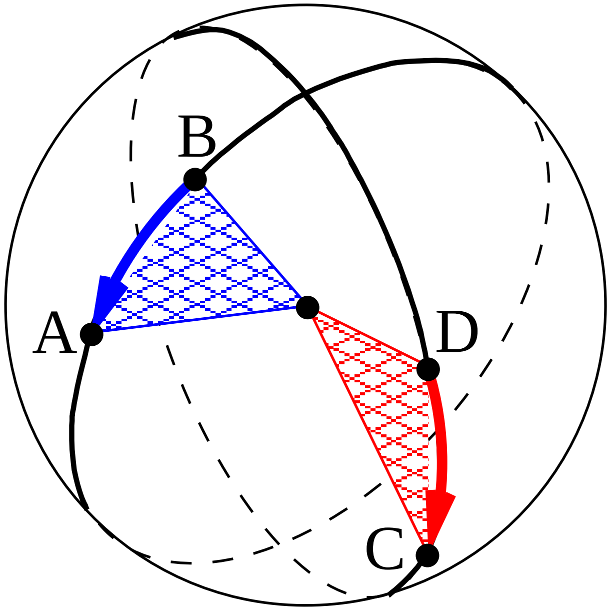
They can be summed **ONLY** when they have
a **COMMON DENOMINATOR**

$$\left.\begin{array}{l} V_{BC} = \frac{\overrightarrow{C}}{\overrightarrow{B}} \\ V_{BA} = \frac{\overrightarrow{A}}{\overrightarrow{B}} \end{array}\right\} V_{BC} + V_{BA} = \frac{\overrightarrow{C}}{\overrightarrow{B}} + \frac{\overrightarrow{A}}{\overrightarrow{B}} = \frac{\overrightarrow{C} + \overrightarrow{A}}{\overrightarrow{B}}$$

A Common Denominator can **ALWAYS** be found

Getting a Common Denominator

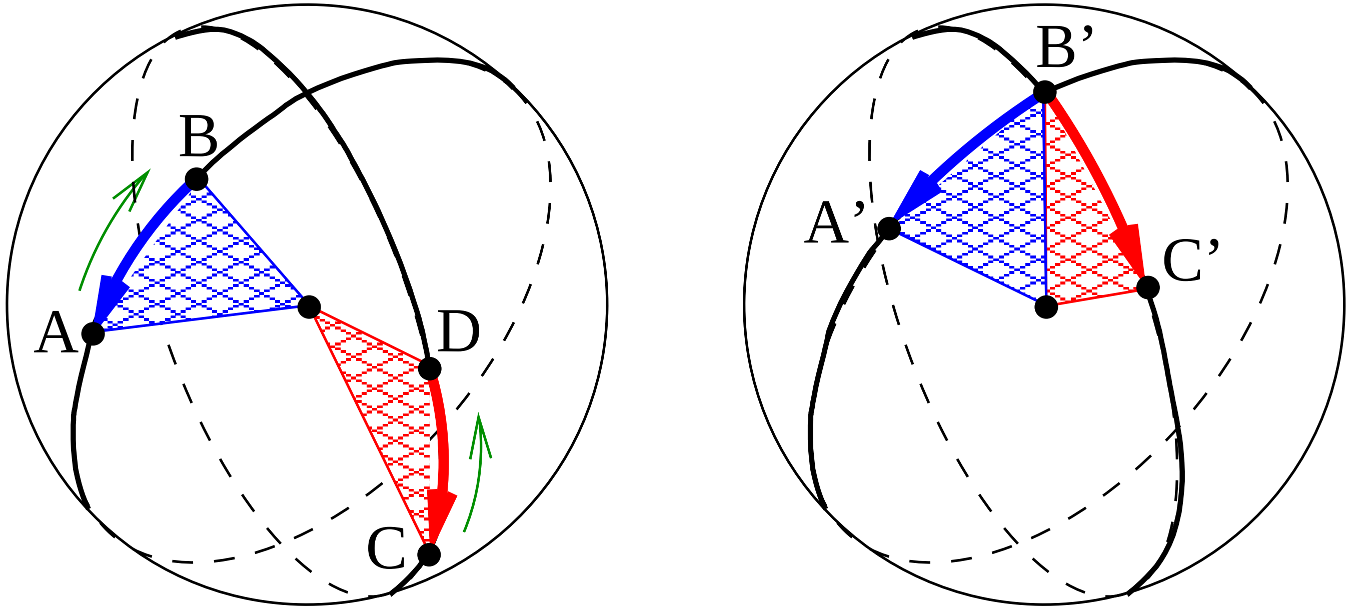




$$V_{BA} = \frac{\overrightarrow{A}}{\overrightarrow{B}} \qquad V_{DC} = \frac{\overrightarrow{C}}{\overrightarrow{D}}$$

Getting a Common Denominator

Slide both versors along their great circles

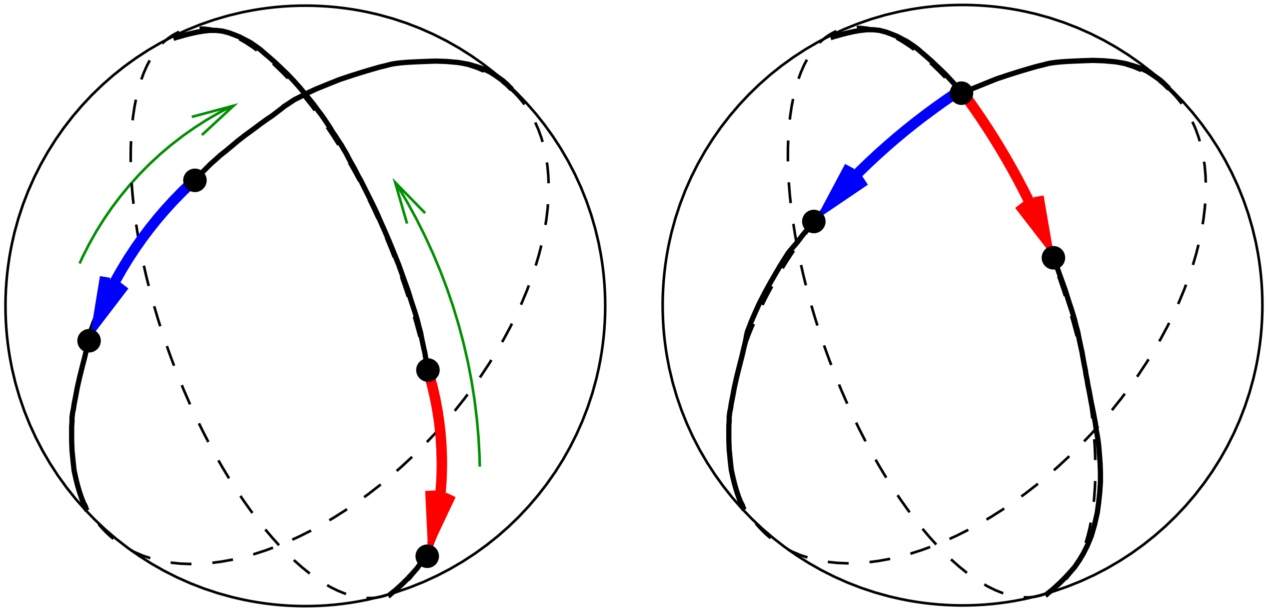


Until their origins coincide

The vector $\overrightarrow{B'}$ in the intersection is the common denominator

Geometrical Interpretation of the Sum

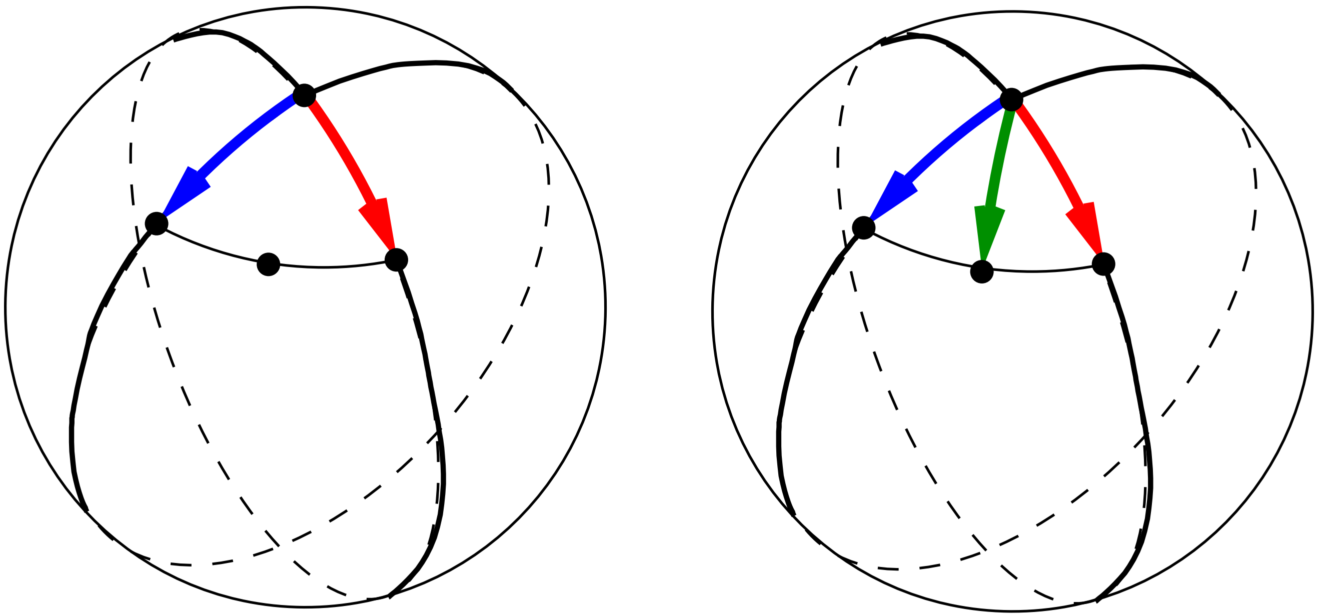
As with Vectors, first **SLIDE** both
Vector-Arcs to a common origin



In order to get a common denominator

Geometrical Interpretation of the Sum

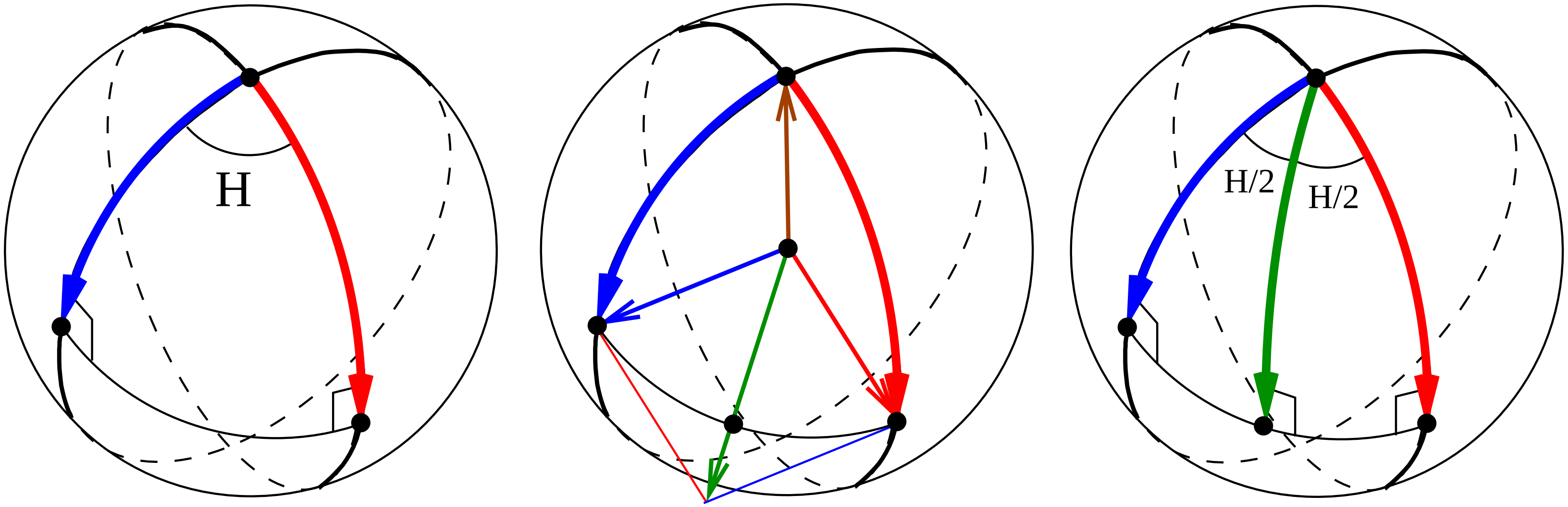
Add the two vectors in the numerator



Finally get the new Quotient

Sum of two Right Versors

It is **always** a right quaternion

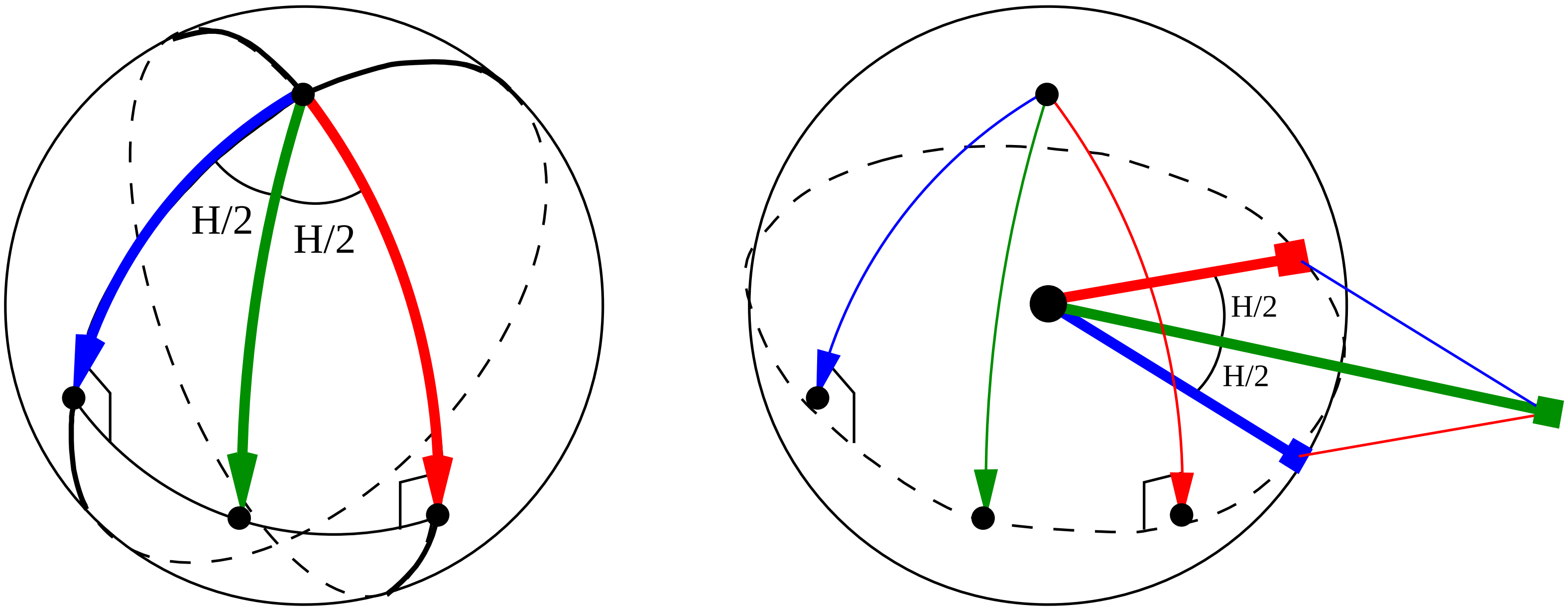


Its plane **BISECTS** those of the original two versors

and has a **Scalar** characteristic > 1

Sum of two Right Versors

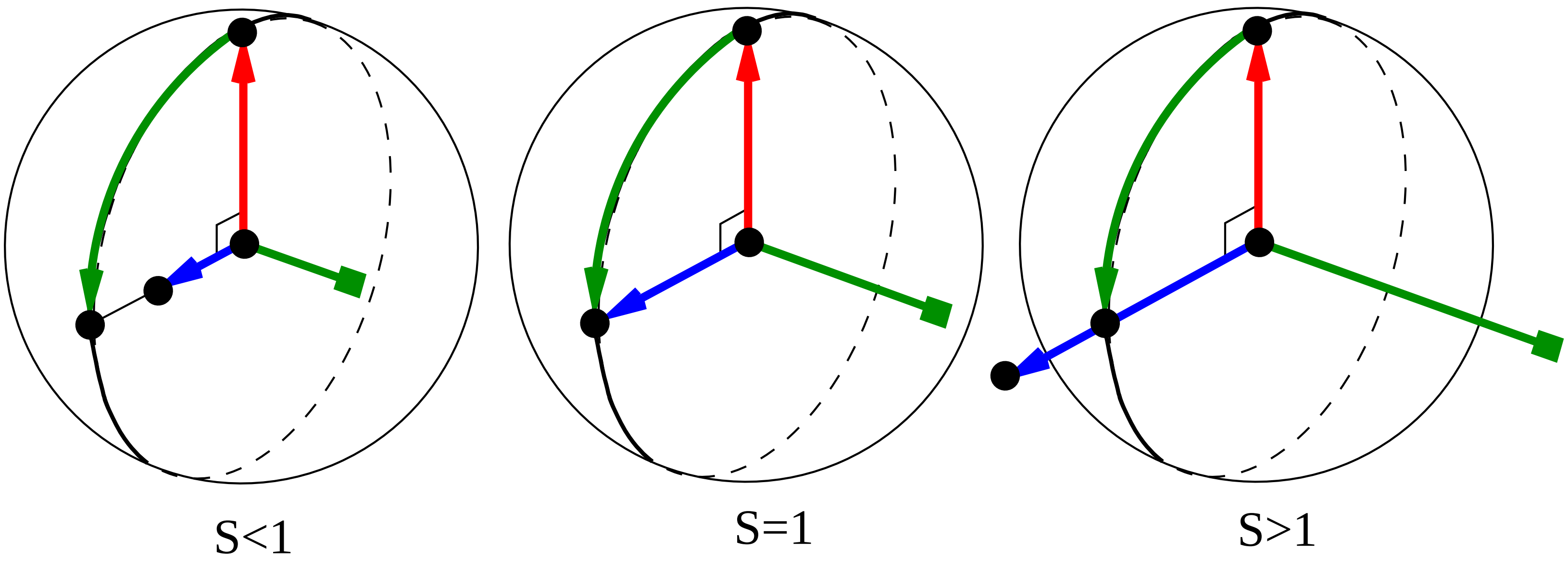
The *Index* of the resulting Versor



is equal to the **sum** of indices of the two versors

Multiplying a Right Versor by a Scalar

Multiplication by a Scalar affects
only the Scalar part of the Right Versor

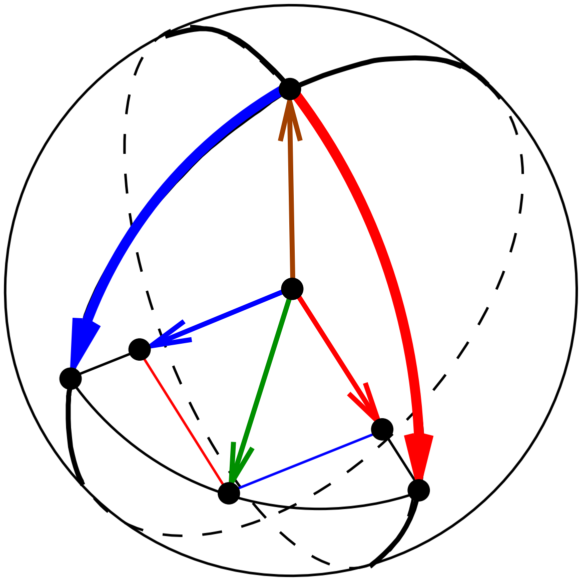


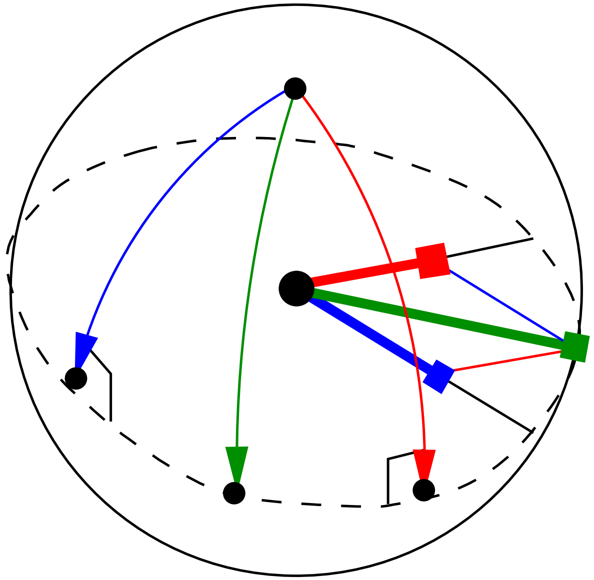
It modifies the length ration of the vectors in the Quotient

Right Versor in terms of Orthogonal Right Versor

If the three Orthogonal Right Versors i, j, k are multiplied by Scalars x, y, z

$$Q = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \qquad x^2 + y^2 + z^2 = 1$$



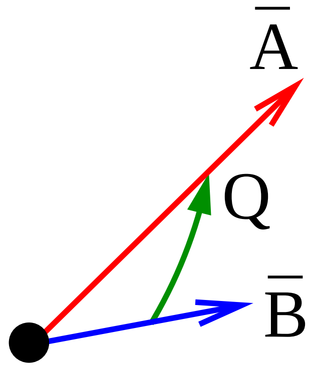


Their sum will be a Right Versor whose axis has (x, y, z) as componets.

Scalar and Right Parts of Quaternions

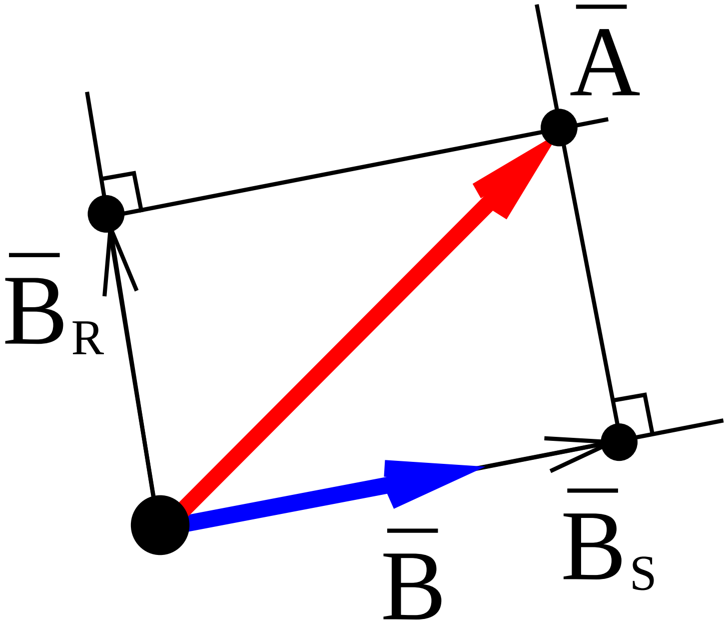
A Quaternion operator applied to a vector $\overrightarrow{B}$
performs an operation that produces another vector $\overrightarrow{A}$

$$Q = \frac{\overrightarrow{A}}{\overrightarrow{B}}$$



$$\overrightarrow{A} = Q \diamond \overrightarrow{B}$$

Scalar and Right Parts of Quaternions

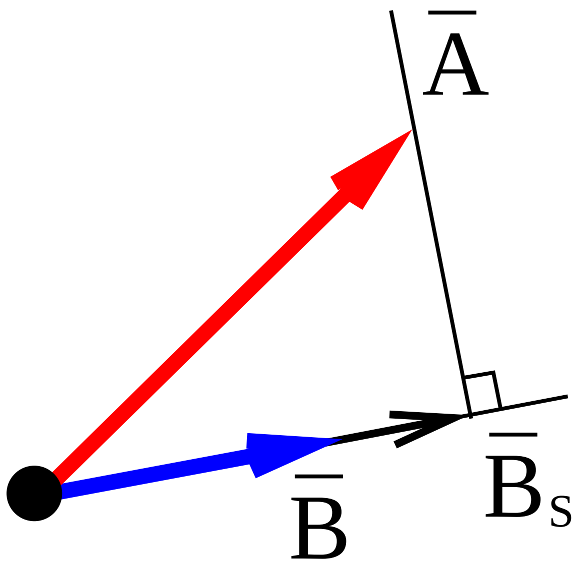


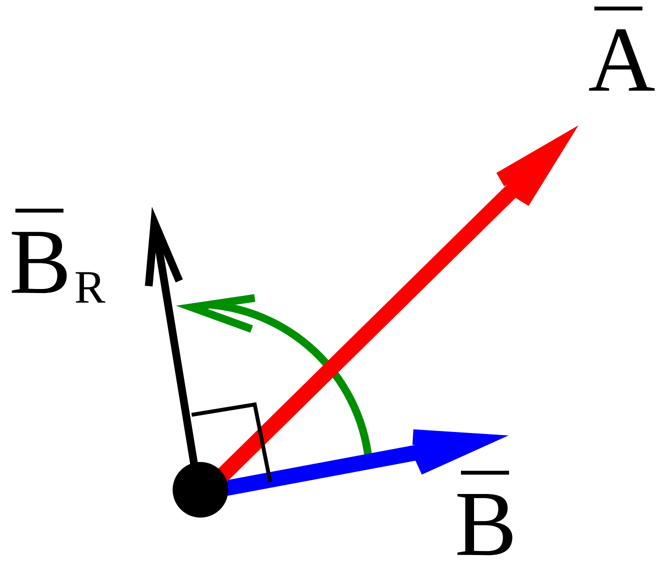
The new vector $\overrightarrow{A}$ can be expressed as a sum of two orthogonal vectors

$$\overrightarrow{A} = \overrightarrow{B}_S + \overrightarrow{B}_R$$

One parallel to $\overrightarrow{B}$ and another orthogonal to $\overrightarrow{B}$

Scalar and Right Parts of Quaternions



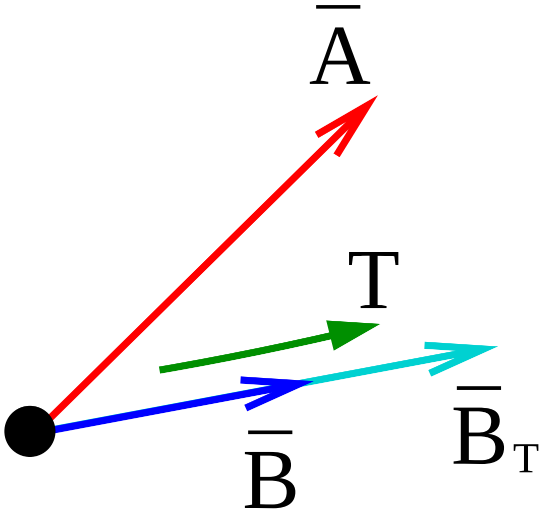


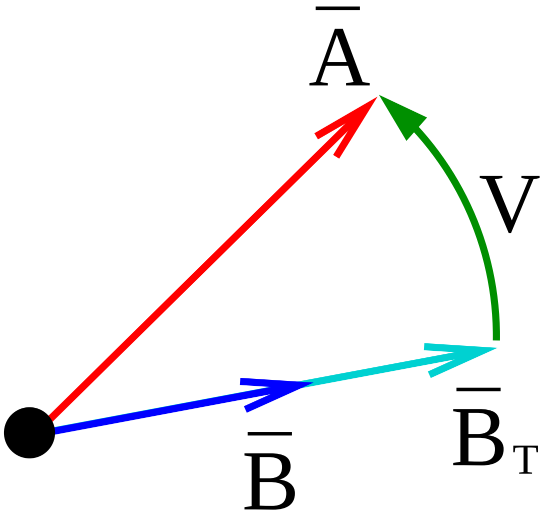
$\overrightarrow{B}_S$ is obtained by applying an Scalar Operator to $\overrightarrow{B}$

$\overrightarrow{B}_R$ is obtained by applying a Right Quaternion to $\overrightarrow{B}$

Tensor and Versor Part of a Quaternion

The same operation can be decomposed in
a Tensor Operator and a Versor Operator





$\overrightarrow{B}_T = T \diamond \overrightarrow{B}$

$\overrightarrow{A} = V \diamond \overrightarrow{B}_T$

Scalar and Right versus Tensor and Versor

SCALAR and **RIGHT** parts are a

Representation in **RECTANGULAR** coordinates

TENSOR and **VERSOR** parts are a

Representation in **POLAR** coordinates

Quaternions as Four Coefficients

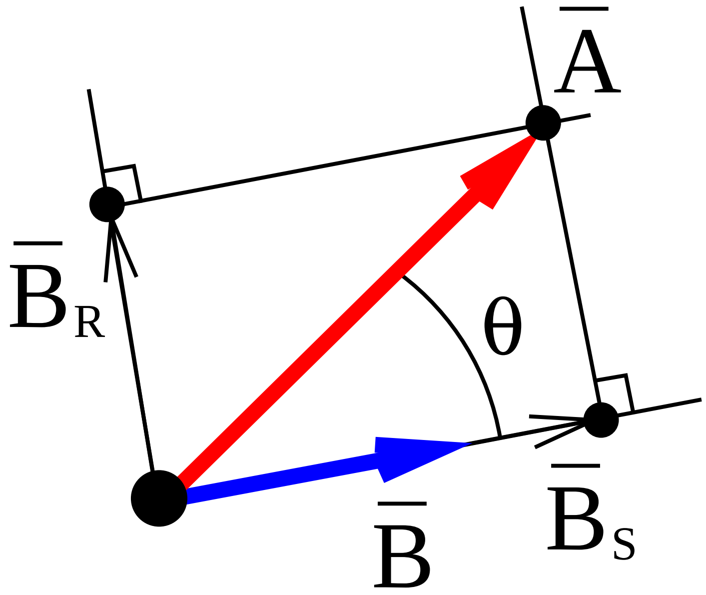
Let L be the **Ratio** of lenghts between vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is $L = \frac{\left\|\overrightarrow{A}\right\|}{\left\|\overrightarrow{B}\right\|}$

The **Scalar** factor

$$S = \frac{\left\|\overrightarrow{B}_S\right\|}{\left\|\overrightarrow{B}\right\|}$$

should be equal to

$$L \cos\theta$$



Quaternions as Four Coefficients

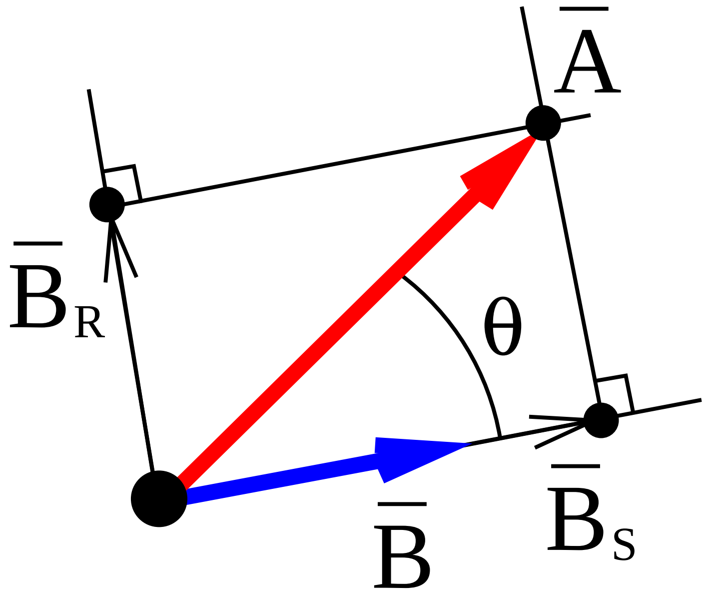
Let L be the **Ratio** of lenghts between vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is $L = \frac{\left\|\overrightarrow{A}\right\|}{\left\|\overrightarrow{B}\right\|}$

The **Tensor** of he **Right** part

$$R = \frac{\left\|\overrightarrow{B}_R\right\|}{\left\|\overrightarrow{B}\right\|}$$

should be equal to

$$L \sin\theta$$



Quaternions as Four Coefficients

The **Quaternion** Q can then be written as

$$Q = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} + w$$

Where

$$\begin{aligned} w &= L\cos\theta \\ \sqrt{x^2+y^2+z^2} &= L\sin\theta \end{aligned}$$

- The real number w represents the **Scalar** part,
- The sum $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ represents the **Right** part.

Product of Quaternions

Two quaternions Q_1 and Q_2 are composed by

$$Q_1 \diamond Q_2 = T(Q_1)\,U(Q_1) \diamond T(Q_2)\,U(Q_2)$$

That is equivalent to

$$Q_1 \diamond Q_2 = T(Q_1)\,T(Q_2) \cdot U(Q_1) \diamond U(Q_2)$$

Product of Quaternions

Given a Quaternion Q resulting from the composition

$$Q = Q_1 \diamond Q_2$$

Its Tensor is

$$T\left(Q\right) = T\left(Q_1\right) T\left(Q_2\right)$$

Its Versor is

$$U\left(Q\right) = U\left(Q_1\right) \diamond U\left(Q_2\right)$$

Representation by four coefficients

Let P and Q be two Quaternions, represented by four coefficients

$$\begin{aligned} P &= x_p\,\mathbf{i} + y_p\,\mathbf{j} + z_p\,\mathbf{k} + w_p \\ Q &= x_q\,\mathbf{i} + y_q\,\mathbf{j} + z_q\,\mathbf{k} + w_q \end{aligned}$$

Their composition $P \diamond Q$ can be expressed by

$$P \diamond Q = \mathbf{L}(P)Q = \begin{bmatrix} w_p & -z_p & y_p & x_p \\ z_p & w_p & -x_p & y_p \\ -y_p & x_p & w_p & z_p \\ -x_p & -y_p & -z_p & w_p \end{bmatrix} \begin{bmatrix} x_q \\ y_q \\ z_q \\ w_q \end{bmatrix}$$

Representation by four coefficients

Let P and Q be two Quaternions, represented by four coefficients

$$
\begin{aligned}
P &= x_p\,\mathbf{i} + y_p\,\mathbf{j} + z_p\,\mathbf{k} + w_p \\
Q &= x_q\,\mathbf{i} + y_q\,\mathbf{j} + z_q\,\mathbf{k} + w_q
\end{aligned}
$$

Their composition $P \diamond Q$ can be expressed by

$$
P \diamond Q = \mathbf{R}(Q)P = \begin{bmatrix} w_q & z_q & -y_q & x_q \\ -z_q & w_q & x_q & y_q \\ y_q & -x_q & w_q & z_q \\ -x_q & -y_q & -z_q & w_q \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \\ w_p \end{bmatrix}
$$

Rotating a Vector (Finally !!)

A Quaternion $q = (x, y, z, w)$ rotates a Vector v by using the product

$$v' = q \diamond v \diamond q^{-1}$$

Which can be reduced to a Matrix-Vector multiplication $\mathbf{L}(q)\,\mathbf{R}\left(q^{-1}\right)\ v$

$$\begin{bmatrix} \left(w^2+x^2-y^2-z^2\right) & (2xy-2wz) & (2xz+2wy) & 0 \\ (2xy+2wz) & \left(w^2-x^2+y^2-z^2\right) & (2yz-2wx) & 0 \\ (2xz-2wy) & (2yz+2wx) & \left(w^2-x^2-y^2+z^2\right) & 0 \\ 0 & 0 & 0 & \left(w^2+x^2+y^2+z^2\right) \end{bmatrix}$$

References

[1] W.R. Hamilton. *Elements of Quaternions*, volume I. Chelsea Publishing Company, third edition, 1969. The original was published in 1866.

[2] C.J. Joly. *A Manual of Quaternions*. MacMillan and Co., Limited, 1905.