<u>**Some Derivations of $E = mc^2$**</u>

Michael Good[1]

1 Rest Energy Using Quantum

Imagine a photon with a velocity c, effective mass m, momentum p, energy E, wavelength λ, frequency f. From the classical definition of momentum, and from the classical definition of frequency and wavelength we have

$$p = mc$$

and

$$c = \lambda f.$$

From quantum mechanics, Planck's energy-frequency equation and De-Broglie's wavelength-momentum equation

$$E = hf$$

and

$$\lambda = h/p.$$

These four equations can be used to obtain $E = mc^2$. Plug and chug and these equations mesh nicely:

$$c/f = \lambda = h/p = h/mc$$

and

$$c/f = c/(E/h) = hc/E$$

therefore

$$hc/E = h/mc$$

thus

$$c/E = 1/mc$$

yields

$$E = mc^2$$

This quantum hand-waving derivation has little physical grounds backing it my opinion. Though its simplicity is hard to beat.

2 Rest Energy Using Binomial Expansion

This short derivation is dependent upon the reader's acceptance of the Lorentz factor and that mass is a relativistic quantity.

$$\gamma = \sqrt{1 - \beta^2}$$

[1]Undergraduate of Physics, Georgia Institute of Technology

and

$$m` = m\gamma$$

With these two formulae in hand, an approximation using the binomial expansion is made. This leads to the strong suggestion that the rest energy of a particle is $E = mc^2$.

$$\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}} = 1 + \frac{1}{2}(\frac{v^2}{c^2}) - \frac{1}{2}(-\frac{1}{2} - 1)(-\frac{v^2}{c^2})^2(\frac{1}{2}) + ...$$

This can be approximated to:

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Using relativisitc mass we can say:

$$m` = m\gamma = m(1 + \frac{1}{2}\frac{v^2}{c^2})$$

$$m`c^2 = mc^2 + \frac{1}{2}mv^2$$

This is total relativisitc mass equal to the kinetic energy plus the rest energy of the particle. Thus the rest energy part is:

$$E = mc^2$$

I find this particularly simple and mindless derivation of the famous rest energy formula. For teaching purposes there exists better derivations, though this can be used to hit the target quickly.

3 Rest Energy using Calculus

This derivation of $E = mc^2$ can be very enlightening to the student who has not been introduced to special relativity. For this derivation, both the Lorentz factor and relativistic mass must be accepted before we can proceed.

$$\gamma = \sqrt{1 - \beta^2}$$

and

$$m = m_0\gamma$$

Where

$$\beta = \frac{v}{c}$$

With non-relativistic mass, or Newtonian mechanics, kinetic energy equal to $\frac{1}{2}mv^2$ can be shown to come from this sloppy mathematical derivation but physically consistent result:

$$E = \int F dx = \int \frac{d}{dt}(mv)dx$$

$$E = \int vd(mv) = \int (vm)dv = \frac{1}{2}mv^2$$

Now, proceeding the same way only with relativistic masss, we can see one method of obtaining $E = mc^2$ by treating mass as a variable under the differential:

$$E = \int F dx = \int \frac{d}{dt}(mv)dx = \int d(mv)v$$

Where this time the mass cannot move outside the differential, but is variable under the integral sign.

$$E = \int d(mv)v = \int (mdv + vdm)v = \int vmdv + v^2 dm$$

At this point it would be nice to find a substitution for the $vmdv + v^2dm$ and this can found by acceptance of the Lorentz transformation for mass.

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Some simple algebraic manipulation, square both sides, and times by c^2 will yield:

$$m^2c^2 - m^2v^2 = m_0^2c^2$$

Take the differential:

$$d(m^2c^2 - m^2v^2) = d(m_0^2c^2)$$

$$2mc^2dm + -m^2(2v)dv + v^2(-2m)dm = 0$$

$$2mc^2dm = 2vm^2dv + 2mv^2dm$$

$$c^2dm = vmdv + v^2dm$$

This is the substitution we where looking for, plug this in and the integral changes to something more familiar:

$$E = \int vmdv + v^2dm = \int c^2dm$$

$$E = mc^2$$