gives, namely equations (6) or (7), are all right and only the unitary condition (ö) is unsatisfactory. Presumably, if one could make some more thorough investigation of the connection between the classical contact transformation theory and the present quantum transformation theory, one would be able to see just where the classical symmetry property that we are interested in gets lost and how it is to be restored*.

ON THE DIFFERENCE $\pi(x)-\text{li}(x)$ (I).

S. SKEWES†.

1. Let $\pi(x)$ be the number of primes less than x, and let $\text{li}(x)$ denote as usual the "logarithmic integral"

$$\int_0^x \frac{dx}{\log x} \ddagger.$$

The "prime number theorem" states that

$$\lim_{x\to\infty} \pi(x)/\text{li}(x) = 1.$$

Numerical evidence suggests that $\pi(x)/\text{li}(x)$ tends to unity from below, so that

$$\pi(x)-\text{li}(x) < 0.$$

Littlewood§ has proved, however, that the number

$$P(x) = \pi(x)-\text{li}(x)$$

actually changes sign infinitely often as x increases to infinity.

Ingham§ has pointed out that Littlewood's method provides, even in principle, no definite number x_0, such as a repeated exponential like 9^{9^9}, before which $P(x)$ has changed sign. My object is to obtain, assuming

* A paper by P. Jordan, *Zeitschrift für Phys.*, 80 (1933), 285, gives an interesting attempt to improve the present quantum formalism by the introduction of non-associative algebra. This work, though, does not seem to lead to any transformation theory having analogies with the classical contact transformation theory and is therefore not the solution of our problem.

† Received and read 15 June, 1933.

‡ More precisely, $\text{li}(x) = \lim_{\eta\to+0}\left(\int_0^{1-\eta} + \int_{1+\eta}^x\right)\frac{du}{\log u}.$

§ See A. E. Ingham, "The distribution of prime numbers", *Cambridge Math. Tracts*, No. 30, 1932.

the truth of the Riemann hypothesis, such a number x_0; that is to obtain an x_0 such that, for *some* $x \leqslant x_0$, the inequality $P(x) > 0$ is satisfied. I prove, in fact, that

$$x_0 = 10^{10^{10^{34}}}$$

is such a number.

The failure of Littlewood's method to furnish a definite number x_0 is traceable to a corresponding failure in the "Phragmen-Lindelöf theorem". The usual form of this theorem, modified to apply to harmonic functions, is as follows.

THEOREM I. *Let $u(s) = u(\sigma, t)$ satisfy the following conditions:*

(i) *u is a harmonic function of σ and t in the open infinite strip $a < \sigma < b$;*

(ii) *u is continuous at every (finite) point of the lines $\sigma = a$ and $\sigma = b$ bounding the strip;*

(iii) *u is majorized by a function $M(t)$, i.e.*

$$u \leqslant M(t), \tag{B}$$

everywhere in the open strip, where $M(t)$ is indevendent of σ and satisfies the inequality

$$M(t) < Ke^{\vartheta\pi|t|/(b-a)}, \tag{1}$$

where K and ϑ are constants, and $\vartheta < 1$. Suppose now that the inequality

$$u < C \tag{2}$$

is satisfied at all points of the boundary of the strip. Then (2) *is satisfied at all interior points of the strip.*

The condition (1) to be satisfied by the majorant $M(t)$ can be made slightly less restrictive, but not to the extent of taking $\vartheta = 1$.

The theorem may be restated as follows:

THEOREM II. *Let $u(s)$ satisfy* (i), (ii), *and* (iii). *Suppose now that $u(s_0) > C_1$ at an interior point s_0 of the strip. Then, if $C_2 < C_1$, the inequality $u > C_2$ is satisfied at some point of the boundary of the strip.*

Since C_2 is an arbitrary number less than C_1, it follows, of course, that [if $u(s_0) > C_1$] the inequality $u > C_1$ also is satisfied at some boundary point. The two forms of statement are therefore equivalent, but the form II must be adopted if we are to advance any further.

Mr. Ingham's criticism may now be stated in more drastic form: *the usual proof of the theorem does not enable us to say definitely how soon, as $|t|$ increases, a boundary point must occur at which $u > C_2$.* This is the problem which we have to solve.

The solution is in two stages. First, we show that the condition (B) is sufficient to secure that u is majorized by the Poisson integral of the function represented by the boundary values $u(a, t)$ and $u(b, t)$. Secondly, a study of the Poisson integral enables us to obtain a value T, determined by an explicit equation involving s_0, C_1, C_2, and the function $M(t)$, such that the inequality $u > C_2$ must be satisfied at some boundary point with a t satisfying $|t| < T$. Incidentally, we need to assume only a weakened form of the continuity condition (ii) (a fact which is of material value when we come to apply the theorem), and a form of condition (iii) which is in its way "best possible". (See the statement of Theorem 4 below.)

2. In view of the greater familiarity of the Poisson integral for a circle, we first state and prove our theorems for the case of a circle (with one critical point on the boundary corresponding to the point at infinity on the strip).

THEOREM 1. *Suppose that $u(\rho, \theta)$ satisfies the following conditions:*

(i) *$u(\rho, \theta)$ is harmonic in the unit circle $\rho < 1$;*

(ii) *$\lim_{\rho\to 1} u(\rho, \theta) = U(\theta)$ almost everywhere;*

(iii) *$u(\rho, \theta) \leqslant M(\theta)$ for all $\rho < 1$,*

where $M(\theta)$ is independent of ρ and integrable in the sense of Lebesgue.

Then U is integrable L, and

$$u(\rho, \theta) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta)\, P(\rho, t-\theta)\, dt,$$

where
$$P(\rho, t-\theta) = \frac{1-\rho^2}{1-2\rho\cos(t-\theta)+\rho^2};$$

i.e. u is majorized by the Poisson integral of U.*

Let r satisfy $\tfrac{1}{2}(1+\rho) < r < 1$, say. Then

$$(1) \qquad 2\pi u(\rho, \theta) = \int_{-\pi}^{\pi} u(r, t)\, P(r, \rho, t-\theta)\, dt,$$

* The hypothesis (iii) asserts only that u is bounded *above* by $M(\theta)$. If it is replaced by the hypothesis $|u| \leqslant M(\theta)$, we can change the signs of u and U in the conclusion of the theorem, and can conclude that u is actually equal to the Poisson integral of U.

where $$P(r,\rho,t-\theta)=\frac{r^2-\rho^2}{r^2-2r\rho\cos(t-\theta)+\rho^2}.$$

Since $$P(r,\rho,t-\theta)\leqslant\frac{r+\rho}{r-\rho}\leqslant\frac{4}{1-\rho},$$

the integrand in (1) is less than or equal to $4M(\theta)(1-\rho)^{-1}$. Applying Fatou's lemma to (1), we have

$$2\pi u(\rho,\theta)=\overline{\lim_{r\to 1}}\int_{-\pi}^{\pi}u(r,t)\,P(r,\rho,t-\theta)\,dt$$

$$\leqslant\int_{-\pi}^{\pi}\overline{\lim_{r\to 1}}\,\{u(r,t)\,P(r,\rho,t-\theta)\}\,dt$$

(2) $$=\int_{-\pi}^{\pi}U(t)\,P(\rho,t-\theta)\,dt,$$

which is the inequality desired. It follows incidentally, of course, that U is integrable. For, by taking $\rho=0$ in (2), we have

$$\int_{-\pi}^{\pi}U(t)\,dt\geqslant 2\pi u(0)>-\infty.$$

Also, since $U(t)\leqslant M(t)$, we have $|U|\leqslant 2M-U$. Hence

$$\int_{-\pi}^{\pi}|U|\,dt\leqslant 2\int_{-\pi}^{\pi}M\,dt-\int_{-\pi}^{\pi}U\,dt<+\infty.$$

THEOREM 2. *Let $C_2<C_1$, and let $u(\rho,\theta)$ satisfy the conditions* (i), (ii), *and* (iii) *of Theorem* 1. *Suppose now that, for a point $z_0=\rho_0 e^{i\theta_0}$ inside the unit circle, we have*

$$u(z_0)>C_1.$$

Then there exists an α such that, for some θ satisfying $|\theta|>\alpha$, we have

$$U(\theta)>C_2.$$

Also α may be either of α_1 or α_2, where α_1 is the greatest root (less than π) of the equation

(3) $$\int_{-\alpha_1}^{\alpha_1}\{U(t)-C_2\}\,P(\rho_0,t-\theta_0)\,dt=2\pi(C_1-C_2)$$

and α_2 is the greatest root (less than π) of the equation

(4) $$\int_{-\alpha_2}^{\alpha_2}\{M(t)-C_2\}\,P(\rho_0,t-\theta_0)\,dt=2\pi(C_1-C_2).$$

The equations (3) *and* (4) *certainly possess each at least one positive root less than* π. *Finally, in the special cases when respectively* (a) $C_2 \geqslant 0$, *and* (b) $C_2 \geqslant 0$ *and* $M(t) \geqslant 0$ *for all* t, *permissible values of* a *are* a_3 *and* a_4 *respectively, where* a_3 *is the greatest root (less than* π*) of*

$$(5) \qquad \int_{-a_3}^{a_3} M(t)\, P(\rho_0, \theta_0 - t)\, dt = 2\pi(C_1 - C_2),$$

and a_4 *the greatest root (less than* π*) of*

$$(6) \qquad \int_{-a_4}^{a_4} M(t)\, dt = 2\pi(C_1 - C_2)\frac{1-\rho_0}{1+\rho_0}.$$

Of these values $a_4 \leqslant a_3 \leqslant a_2 \leqslant a_1$; *but* a_2 *depends only on* C_1, C_2, z_0 *and the function* $M(\theta)$; a_3 *depends only on* $C_1 - C_2$, z_0, *and* $M(\theta)$; *and* a_4 *depends only on* $C_1 - C_2$, ρ_0, *and* $M(\theta)$.

Suppose that $U(\theta) \leqslant C_2$ except in the range $-a \leqslant \theta \leqslant a$. Then

$$\frac{1}{2\pi}\left(\int_{-\pi}^{-a} + \int_{a}^{\pi}\right)\{U(t) - C_2\}\, P(\rho_0, \theta_0 - t)\, dt \leqslant 0.$$

Also

$$\int_{-\pi}^{\pi}\{U(t) - C_2\}\, P(\rho_0, \theta_0 - t)\, dt \geqslant 2\pi\{u(z_0) - C_2\} > 2\pi(C_1 - C_2),$$

by Theorem 1. Subtracting, we have

$$F_1(a) = \int_{-a}^{a}\{U(t) - C_2\}\, P(\rho_0, \theta_0 - t)\, dt > 2\pi(C_1 - C_2).$$

If we define $F_2(a)$, $F_3(a)$, $F_4(a)$ by the formulae

$$F_2(a) = \int_{-a}^{a}\{M(t) - C_2\}\, P(\rho_0, \theta_0 - t)\, dt,$$

$$F_3(a) = \int_{-a}^{a}\{M(t)\}\, P(\rho_0, \theta_0 - t)\, dt,$$

$$F_4(a) = \frac{1+\rho_0}{1-\rho_0}\int_{-a}^{a} M(t)\, dt,$$

we have $F_2(a) \geqslant F_1(a)$ in any case, and $F_4(a) \geqslant F_3(a) \geqslant F_2(a)$ in the special cases. Hence the hypothesis that $U(\theta) \leqslant C_2$ except in $(-a, a)$ implies that $F(a) > 2\pi(C_1 - C_2)$, where F is either of F_1 or F_2, or again, in the special cases, either of F_3 or F_4. It follows that, if a is such that

$$F(a) \leqslant 2\pi(C_1 - C_2),$$

where F is one of F_1, F_2, or, in the special cases, F_3 or F_4, then $U(\theta) > C_2$ for some θ satisfying $|\theta| \geqslant a$. Now $F_1(a)$ is continuous in a, vanishes at

$a = 0$, and is greater than $2\pi(C_1 - C_2)$ at $a = \pi$, its value being not less than $2\pi\{u(z_0) - C_2\}$. The same things are true of F_2, F_3, and F_4, their values at $a = \pi$ being still greater. It follows that the various equations $F(a) = 2\pi(C_1 - C_2)$ have each a solution in $(0, \pi)$, and the proof of the theorem is completed*.

3. The theorems in the case of the strip are as follows.

THEOREM 3. *Suppose that $u(\sigma, t)$ satisfies the following conditions* :

(i) *$u(\sigma, t)$ is harmonic in the strip $0 < \sigma < \pi$;*

(ii) $\lim_{\sigma \to 0} u(\sigma, t) = U(0, t)$, $\lim_{\sigma \to \pi} u(\sigma, \tau) = U(\pi, t)$, *both for almost all t;*

(iii) *$u(\sigma, t) \leqslant M(t)$ for all values of σ satisfying $0 < \sigma < \pi$, where $M(t)$ is independent of σ, and*

$$\int_{-\infty}^{\infty} M(t) e^{-|t|} dt < \infty.$$

Then $Ue^{-|t|}$ is integrable L over the boundary of the strip, and

$$u(\sigma, t) \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin\sigma\, U(0, y)\, dy}{\cosh(t-y) - \cos\sigma} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin\sigma\, U(\pi, y)\, dy}{\cosh(t-y) + \cos\sigma}.$$

THEOREM 4. *Let $C_2 < C_1$, and let $u(\sigma, t)$ satisfy the conditions* (i), (ii), *and* (iii) *of Theorem* 3. *Suppose now that, for a point $s_0 = \sigma_0 + it_0$ inside the strip, we have*

$$u(\sigma_0, t_0) > C_1.$$

Then there exists an a such that, for some t satisfying $|t| < a$, we have one of

$$U(0, t) > C_2, \quad U(\pi, t) > C_2.$$

If a_1 is the least root of the equation, corresponding to equation (3) *of Theorem* 2,

$$(1) \quad \frac{1}{2\pi}\left(\int_{-\infty}^{-a_1} + \int_{a_1}^{\infty}\right) \frac{\sin\sigma_0 \{U(0, y) - C_2\}\, dy}{\cosh(t_0 - y) - \cos\sigma_0} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin\sigma_0 \{U(\pi, y) - C_2\}\, dy}{\cosh(t_0 - y) + \cos\sigma_0} = 2\pi(C_1 - C_2),$$

then a may have the value a_1. The equations corresponding to (4), (5), *and* (6) *can be written down by analogy (and their least roots are to be taken).*

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* My original proof was somewhat more clumsy, and involved integration round a circle interior to the unit circle. For this improved form of the proof I am indebted to Professor Littlewood.

The strip which we have taken has boundaries $\sigma = 0$ and $\sigma = \pi$, but the alterations in the formulae necessary to suit any other strip can be made without difficulty.

4. The formula actually used for our application to the problem of $P(x)$ is equation (1) in Theorem 4. We thereby avoid the complications involved in determining *precisely* a majorizing function $M(t)$; it is enough to know that there is one, so that we can apply Theorem 3. We then integrate (1) by parts, after which we are concerned only with the behaviour of $\int U\,dt$, not that of U, and this is more or less known. No further difficulties of principle arise in the calculations, and what remains to be done is the arithmetical development of ground already covered. This is a matter of some intricacy, and it is perhaps capable of refinements. What would be more important, it is possible that the restriction of the Riemann hypothesis can be removed. I propose, therefore, to postpone the details to a later paper; in the meantime, I have obtained a value of x_0 which, though possibly capable of improvement, undoubtedly provides a solution to the original problem. The value of this x_0 is

$$e^{e^{e^{79}}} = 10^{10^{10^{34}}}, \text{ approximately.}$$

HOMOGENEOUS SYSTEMS OF DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS

S. BOCHNER*.

Let the coefficients $a_{\mu\nu}(x)$, $b_\mu(x)$ of the system of differential equations

$$\frac{dy_\mu(x)}{dx} = \sum_{\nu=1}^{m} a_{\mu\nu}(x)\, y_\nu(x) + b_\mu(x) \quad (\mu = 1, 2, \ldots, m) \tag{1}$$

be a.p.† functions (in the sense of H. Bohr). Naturally, the question arises, under what supplementary conditions the solutions are also a.p. When the coefficients $a_{\mu\nu}(x)$ are constant, fairly satisfactory conditions are known‡, relating not only to differential equations but also to other functional equations. The case of variable coefficients $a_{\mu\nu}(x)$ seems to be less favourable to results of a similarly definite type. However, J. Favard§ has attacked the problem successfully, and his method, which

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* Received and read 15 June, 1933.

† a.p. = almost periodic.

‡ S. Bochner, *Math. Annalen*, 104 (1931), 579-587.

§ J. Favard, *Acta. Math.*, 51 (1928), 31-81.