GEOMETRIZATION OF 3-MANIFOLDS WITH SYMMETRIES

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1. Thurston's geometrization conjecture

The main subject of these lectures is the Geometry and Topology of 3-dimensional orientable manifolds and orbifolds. The point of view adopted is to emphasize their geometric properties.

A major result of the late 19th century is Riemann uniformization theorem for surfaces, i.e. compact connected (orientable) 2-dim manifolds.

Theorem 1. *Any surface admits a Riemannian metric of constant curvature $k = +1, 0, -1$ i.e.*

- $F = S^2/\Gamma \qquad \Gamma \subset_{finite} SO(3)$
- $F = E^2/\Gamma \qquad \Gamma \subset_{discrete} \mathrm{Isom}^+ \mathbb{E}^2$
- $F = H^2/\Gamma \qquad \Gamma \subset_{discrete} PSL_2(\mathbb{R})\,.$

In other words, any compact surface is either elliptic, Euclidean or hyperbolic. Moreover a surface belongs to a unique geometric type according to the Gauss-Bonnet formula $2\pi\chi(F) = \int_F kds$ which determines the Euler characteristic of the surface.

In some sense the topology determines the geometric type. In particular $\pi_1 F$ determines the geometric type, since it determines $\chi(F)$.

In dimension 3 the situation is much more complicated. It is only in the late 25 years, due mainely to the work of Thurston that an analogous but much more involved theory has emerged. This theory can be very well summarized by the following Geometrization conjecture, due to Thurston:

Conjecture 2 (Thurston)**.** *The interior of any compact orientable 3-manifold can be split along a finite collection of* essential *disjoint embedded spheres and tori into canonical submanifolds whose interior admit a complete homogeneous Riemannian metric after capping off the sphere components by balls.*

A particular case of this conjecture is the well known Poincaré's conjecture:

Conjecture 3 (Poincaré)**.** *A closed simply connected* 3*-manifold is homeomorphic to* S^3*.*

Thurston's approach to the study of 3-manifolds put forwards a geometric point of view, in which the Poincaré conjecture becomes a uniformization problem. Thurston has classified the homogeneous 3-dimensional geometries that may endow the interior of a compact 3-manifold. There are only height possible geometries: three geometries with constant curvature, corresponding to the elliptic 3-sphere S^3, the Euclidean space E^3 and the hyperbolic space H^3; two product geometries modeled on $E^1 \times S^2$, and $E^1 \times H^2$; three fibered geometries modeled on the Lie groups Nil, $\widetilde{SL_2(\mathbb{R})}$ and Sol.

Among these height 3-dimensional geometries the hyperbolic geometry is the only one that does not collapse to a 2-dimensional or a 1-dimensional geometry. Thurston's work has shown that this robust hyperbolic geometry is the most common one among geometric 3-manifolds. The topological background for Thurston's geometrization conjecture is given by the following splitting theorem for an orientable compact 3-manifold:

Theorem 4 (Canonical Decomposition)**.** *Let M be a compact orientable* 3*-manifold different from* S^3*.*

a) There is a finite (perhaps empty) family of disjoint essential embedded spheres in M which splits M into prime manifolds different from S^3. Moreover, these prime factors M_i are unique up to homeomorphism.

b) For each irreducible factor M_i, there is a finite (perhaps empty) family of essential disjoint and non-parallel tori $\mathcal{T}_i$ which splits M_i into either Seifert fibred *or* atoroidal *pieces. Moreover, a minimal such family $\mathcal{T}_i$ is unique up to isotopy in M_i.*

Definition 5.
- *An embedded 2-sphere $S^2 \hookrightarrow M^3$ is essential if it does not bound a ball in M or is not parallel to a sphere in ∂M^3.*
- *An orientable* 3*-manifold M is* irreducible *if any embedding of the* 2*-sphere into M extends to an embedding of the* 3*-ball into M.*

The first stage a) of the Canonical decomposition, due to H. Kneser, expresses the 3-manifold M as the connected sum of prime factors: $M = M_1 \sharp \ldots \sharp M_h \sharp S^1 \times S^2 \sharp \ldots \sharp S^1 \times S^2$, where the M_i are irreducible. The irreducible factor are unique (cf. Hempel's book [He]).

Definition 6.
- *An embedded torus $T \hookrightarrow M$ is called* essential *if $\pi_1 T \rightarrowtail \pi_1 M$ is injective (T is incompressible) and T is not parallel to ∂M.*
- *M is* atoroidal *if any $\mathbb{Z}\oplus\mathbb{Z} \hookrightarrow \pi_1 M$ is conjugated to the fundamental group of a boundary component of ∂M.*

- *A compact orientable* 3*-manifold is* Seifert fibred *if it admits a foliation by circles such that each circle has a saturated tubular neighborhood. Such a manifold has a geometric structure modelled on one of the six geometries:* S^3, E^3, $E^1 \times S^2$, $E^1 \times H^2$, Nil, $\widetilde{SL_2}(\mathbb{R})$.

The minimal family of tori $\mathcal{T}_i$ in M_i is called the JSJ-family for M_i (Jaco-Shalen-Johannson). The stage b) of the Canonical decomposition is a result which follows from the works of F. Waldhausen [Wa2] (1968), W. Jaco and P. Shalen [JS], K. Johannson [Joh] (1975/76), P. Scott [Sco1, Sco2](1981), G. Mess [Me] (1986), P. Tukia [Tu] (1986), A. Casson and D. Jungreis [CJ], D. Gabai [Ga](1990). In particular it implies the following:

Theorem 7 (Torus Theorem)**.** *Let M be a compact orientable 3-mad. If M is not atoroidal then either the JSJ-family of tori $\mathcal{T} \neq \emptyset$ or M is Seifert fibered.*

From the Kneser-Milnor prime decomposition, the JSJ-decomposition and the torus theorem one sees that Thurston's Geometrization Conjecture reduces to the following uniformization Conjecture.

Conjecture 8 (Uniformization)**.** *Let M be a compact orientable irreducible atoroidal* 3*-manifold. Then:*

- *M is hyperbolic if and only if $\pi_1 M$ is infinite.*
- *M is spherical if and only if $\pi_1 M$ is finite.*

Theses conjectures are still widly open. The main and fundamental contribution of Thurston to his conjecture is the following hyperbolization theorem:

Theorem 9 (Thurston's hyperbolization)**.** *Let M be a compact orientable irreducible and atoroidal 3-manifold. If $\pi_1 M$ is not virtually abelian, and M contains a properly embedded, π_1-injective surface (i.e. M is Haken), then the interior of M admits a complete hyperbolic structure.*

For the proof of this theorem we refer to [Th3, Th4, Th5], [McM1, McM2], [Ka], [Ot1, Ot2].

Example 10. *A knot space $k \subset S^3$ is hyperbolic iff k is not a (p/q)-torus knot and is not a satellite.*

Definition 11. *A 3-manifold is small if it is closed, irreducible, and does not contain any incompressible surface.*

Thurston's conjecture is open only for small atoroidal 3-manifolds.

Conjecture 12. *Every small 3-manifold is* geometric*.*

It is a deep obsevation by Thurston that in some cases Seifert geometries may appear as the result of collapsing certain kind of "hyperbolic singular structures". This is one of the key points in the proof of the following theorem which is the goal of theses lectures:

Theorem 13 (Orbifold Theorem)**.** *Let M be a* small *3-manifold. Let $\varphi \in \mathrm{Diff}^+(M)$ a non trivial periodic diffeomorphism ($\varphi \neq \mathrm{Id}$, $\varphi^n = \mathrm{Id}$) with a non empty fixed point set, $Fix(\varphi) \neq \emptyset$. Then M admits a $\langle\varphi\rangle$-invariant Hyperbolic or Seifert fibred structure.*

Remark 14. *The theorem remains true for a atoroidal* 3*-manifold instead of a small one.*

Here are some corollaries of the orbifold theorem:

Corollary 15. *Thurston's Geometrization Conjecture is true, provided that there is an orientation preserving, homeomorphism $\phi\colon M \to M$ with $\phi \neq id$, $\phi^n = id$ for some $n \geq 2$ and $Fix(\phi) \neq \emptyset$.*

Corollary 16. *Any compact orientable* 3*-manifold of Heegaard genus two has a geometric decomposition in the sense of Thurston.*

Corollary 17. *Let $k \subset S^3$ be a hyperbolic knot (i.e. $S^3 \setminus k$ admits a complete hyperbolic structure with finite volume). For $n \geq 3$ every n-fold cyclic covering of S^3 branched along k is a hyperbolic 3-manifold except when $n = 3$ and k is the figure eight knot where it is Euclidean. Moreover the covering translations act always isometrically.*

Short History. This theorem is a particular case of a theorem on the geometrization of 3-orbifolds announced by Thurston in the late 1981 [Th2, Th6]. Unfortunatly he never published a proof.

Recently in 2000, two different proofs of the general case have been announced. One by D. Cooper, C. Hodgson and S. Kerckhoff: in [CHK] they present the background material and give an outline of their proof. The other proof is due to M. Boileau, B. Leeb and J. Porti: it has been announced in [BLP1] and the complete proof can be found in [BLP2]. A proof of the cyclic case already appeared in [BoP].

2. Thurston's eight geometries

There are only eight relevant homogeneous geometries involved in Thurston's geometrization conjecture.

Definition 18. *A n-dimensional geometry is a simply connected complete homogeneous Riemannian n-dimensional manifold X which is maximal. (i.e.* $\mathrm{Isom}(X)$ *acts transitively on X) and that there is no other* $\mathrm{Isom}(X)$*-invariant-Riemannian metric on X with bigger isometry group).*

Two geometries are equivalent if there is a diffeomorphism $\varphi\colon X \to X'$ which conjugates the action of $\mathrm{Isom}(X)$ and $\mathrm{Isom}(X')$.

Given a n-dimensional geometry X, a n-manifold M admits a X-structure if $\mathrm{int}\, M \cong X/\Gamma$, where $\Gamma \subset \mathrm{Isom}^+(X)$ is a discrete (torsion free) subgroup without fixed points.

We say that X is *relevant* if it has a quotient of finite volume.

In dimension 2 the homogeneous Riemannian metrics are the metrics with constant sectional curvature: so, up to rescalling, they are modelled on S^2, E^2 or H^2.

Proposition 19 (Thurston)**.** [Th7] *There are only (up to equivalence) 8 relevant homogeneous geometries in dimension 3.*

Here is a list of these geometries according to the size of the stabilizer $G_x \subset \mathrm{Isom}_0^+(X)$ of a point $x \in X$, where $\mathrm{Isom}_0^+(X)$ is the the component of the identity in $\mathrm{Isom}^+(X)$.

Isotropic geometries:$G_x \cong SO(3)$

There are, up to equivalence, three isotropic geometries modelled on S^3, E^3 or H^3.

- Spherical geometry $X = S^3$. Then M is finite quotient of S^3 by a subgroup of $O(4)$.
- Euclidean geometry $X = E^3$. Example: T^3. Then by Bieberlach's theorem, every compact Euclidean 3-manifold is a quotient of T^3 or $T^2 \times [0,1]$ by a finite subgroup of $\mathrm{Isom}(E^3)$.
- Hyperbolic geometry $X = H^3$. Example: the Seifert-Weber dodecahedral space.

Anisotropic geometries with $G_x \cong SO(2)$

Trivial products

- $X = S^2 \times E^1$. There are only two orientable examples: $S^2 \times S^1$, $\mathbb{R}P^3 \sharp \mathbb{R}P^3$, and $S^1 \times D^2$.
- $X = H^2 \times E^1$. Every compact orientable manifold with this geometry is finitely covered by the product of S^1 with a surface of negative Euler characteristic.

Twisted products

- X = Nil. Nil is the nilpotent real Lie group of dimension three (Heisenberg Matrix group)

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \quad x, y, z \in \mathbb{R} \right\}$$

  $\mathbb{R} \to \mathrm{Nil} \to \mathbb{R}^2$. Every nilpotente 3-manifold is finitely covered by a S^1-bundle over T^2 with non-zero Euler class.
- $X = \widetilde{SL_2}(\mathbb{R})$. Then M is a finitely covered by a S^1-bundle with non zero Euler class, over a closed surface F of negative Euler characteristic.

Anisotropic geometry with $G_x \simeq \{id\}$

Then X is a unimodular Lie group, and from the classification of 3-dimensional unimodular Lie groups there is only one possibility.

- X = Sol. Sol is the solvable Lie group given by the split extension $\mathbb{R}^2 \to \mathrm{Sol} \to \mathbb{R}$, where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

  Every solvable 3-manifold is finitely covered by a torus bundle with an Anosov monodromy (i.e. given by a matrix in $SL_2(\mathbb{R})$ having two distinct real eihgen values).

A proof of this classification can be found in [Sco2], [Th7]. In these notes, we will not use the classification, but mainly the fact that a Seifert fibered 3-manifold admits one (and only one) of the following six geometries: S^3, E^3, $\widetilde{SL_2(\mathbb{R})}$, Nil, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$.

Typical representants (up to a finite covering) for hyperbolic 3-manifolds are not known. According to Thurston, these have to be surface bundles:

Conjecture 20 (Thurston)**.** *Any complete orientable hyperbolic 3-manifold with finite volume is finitely covered by a surface bundle with pseudo-Anosov monodromy.*

3. Orbifolds

The natural object to consider in the proof of Theorem 13 is the quotient $\mathcal{O} = M/\langle\varphi\rangle$, equipped with its *orbifold* structure: that means that we keep track of the non free group action induced by $\langle\varphi\rangle$.

Definition 21. *A (smooth) n-*orbifold *is a metrizable topological space $\mathcal{O}$ endowed with a collection $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_i$, called an* atlas, *where for each i, U_i is an open subset of $\mathcal{O}$, $\tilde{U}_i$ is an open subset of $\mathbb{R}^{n-1}\times[0,\infty)$, $\psi_i : \tilde{U}_i \to U_i$ is a continuous map (called a* chart*) and Γ_i is a finite group of orientation preserving diffeomorphisms of $\tilde{U}_i$ satisfying the following conditions:*

- *The U_i's cover $\mathcal{O}$.*
- *Each ψ_i factors through a homeomorphism between $\tilde{U}_i/\Gamma_i$ and U_i.*
- *The charts are compatible in the following sense: for every $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ with $\psi_i(x) = \psi_j(y)$, there is a diffeomorphism f between a neighborhood V of x and a neighborhood W of y such that $\psi_j(f(z)) = \psi_i(z)$ for all $z \in V$.*

Roughly speaking a n-dimensional orbifold is a metrizable space with a coherent atlas of neighbourhoods which are diffeomorphic to quotients of $\mathbb{R}^n$ by a finite diffeomorphism group preserving the orientation.

In our case an orbifold is a metrizable space in which each point $x \in \mathcal{O}$ has a "neighbourhood modelled" on a ball B^3 or on a *football* $B^3/\mathbb{Z}/p\mathbb{Z}$ where p divides n.

Note that the notion of orbifold extends the classical notion of manifold. Thus we say that the orbifold $\mathcal{O}$ is a manifold if all the Γ_i's are trivial. Sometimes it will be necessary to distinguish between the orbifold $\mathcal{O}$ and its *underlying space* (i.e. the topological space obtained by forgetting the orbifold structure.) When we want to make the distinction clear, we will denote this underlying space by $|\mathcal{O}|$. In our case, $|\mathcal{O}|$ is a topological manifold. We say that $\mathcal{O}$ is connected (resp. compact) if $|\mathcal{O}|$ is connected (resp. compact).

Let M be a manifold and Γ a discrete group acting properly on M by diffeomorphisms. Then the quotient space M/Γ has a natural orbifold structure, associated to the branched covering projection $M \to M/\Gamma$. An orbifold is called *good* if it is obtained in this way, and *bad* otherwise. It is *very good* if it is the quotient of a manifold by a *finite* group. In our case, the orbifold $\mathcal{O}$ is very good.

An orbifold is called *spherical* (resp. *discal*, resp. *annular*, resp. *toric*) if it is a quotient of a sphere (resp. a disc, resp. an annulus, resp. a torus) by an orthogonal action. One defines similarly *Euclidean* and *hyperbolic* orbifolds, extending the definitions of the previous chapter.

The *local group* of $\mathcal{O}$ at a point $x \in \mathcal{O}$ is the group Γ_x defined as follows: let $\psi : \tilde{U} \to U \ni x$ be a chart. Then Γ_x is the stabilizer of any point of $\psi^{-1}(x)$ under the action of Γ. It is well defined up to isomorphism. If Γ_x is trivial, we say that x is *regular*, otherwise it is *singular*. The *singular locus* is the set Σ of singular points of $\mathcal{O}$. Note that $\Sigma = \emptyset$ iff $\mathcal{O}$ is a manifold. Since every smooth action of a finite group on a manifold is locally conjugated to an orthogonal action, local groups are isomorphic to subgroups of $SO(n)$. This fact can be used to study the structure of the singular locus.

In dimension 3, the singular locus is in general a trivalent graph. In our case, since the local groups are all cyclic, the singular locus is a link, i.e. a finite collection of embedded, disjoint circles. It is the image $\Sigma = p(\operatorname{Fix}\varphi)$ of the fixed point set of φ under the natural projection $p\colon M \to \mathcal{O}$. the orbifold $\mathcal{O}$ is said to be a *cyclic* orbifold (or of cyclic type).

The (topological) pair $(|\mathcal{O}|, \Sigma)$ is called the topological type of the orbifold $\mathcal{O}$.

In the remaining of these notes all 2-orbifolds and 3-orbifolds are assumed to be connected and orientable unless mentioned otherwise. In general, 2-suborbifolds of 3-orbifolds are assumed to be either properly embedded or suborbifolds of the boundary. We extend now to the setting of orbifolds some basic notions of low dimensional manifolds.

Let $F_0, F_1 \subset \mathcal{O}$ be 2-suborbifolds (either properly embedded or contained in $\partial\mathcal{O}$). We say that F_0, F_1 are *parallel* if they cobound in $\mathcal{O}$ a suborbifold $F \times [0,1] \subset \mathcal{O}$, called a *product region* such that $F \times \{0\} = F_0$, $F \times \{1\} = F_1$ and $\partial F \times [0,1] \subset \partial\mathcal{O}$. A properly embedded 2-suborbifold $F \subset \mathcal{O}$ is *∂-parallel* (boundary-parallel) if F is parallel to a suborbifold of ∂M.

A 2-suborbifold $F \subset \mathcal{O}$ is *compressible* if either F is spherical and bounds a discal 3-suborbifold, or there exists a discal 2-suborbifold $D \subset \mathcal{O}$, called a *compression disk,* such that $\partial D = D \cap F \subset \operatorname{int}\mathcal{O}$ does not bound a discal 2-suborbifold in F. Otherwise F is said *incompressible.* Note that the term “compression disk” is a slight abuse of language since it might be a disk with a singular point.

It is obvious that discal 2-suborbifolds, bad 2-suborbifolds and nonspherical turnovers (i.e. spheres with 3 singular points) are always incompressible, since they do not have essential curves. A spherical turnover is compressible if and only if it surrounds a vertex in the singular locus.

A 2-suborbifold $F \subset \mathcal{O}$ is *∂-compressible* if there exists a discal 2-suborbifold $D \subset \mathcal{O}$, called a *∂-compression disk,* such that ∂D is the union of two arcs α, β with $\partial\alpha = \partial\beta = \alpha \cap \beta$, $\alpha \subset F$, $\beta \subset \partial\mathcal{O}$, and α does not cobound a discal suborbifold of F with an arc in ∂F. Otherwise F is said *∂-incompressible.*

A 2-suborbifold $F \subset \mathcal{O}$ is *essential* if it is incompressible, ∂-incompressible and not ∂-parallel.

Definition 22.
- *A* 3*-orbifold* $\mathcal{O}$ *is* irreducible *if* $\mathcal{O}$ *contains no bad* 2*-suborbifold, and every (orientable) spherical* 2*-suborbifold of* $\mathcal{O}$ *bounds a spherical 3-suborbifold.*
- *A* 3*-orbifold is* atoroidal *if it is irreductible and contains no essential toric* 2*-suborbifold.*

The Theorem 23 gives a condition for a compact 3-orbifold to be decomposed into finitely many irreducible orbifolds.

Theorem 23. *Let $\mathcal{O}$ be a compact* 3*-orbifold without bad* 2*-suborbifold. There exists a finite collection $\mathcal{S}$ of disjoint, embedded and nonparallel spherical 2-suborbifolds in $\mathcal{O}$ such that for every component of $\mathcal{O}\backslash\mathcal{S}$ the* 3*-orbifold, obtained from this component by gluing a discal* 3*-orbifold along each spherical boundary component, is irreducible.*

We denote by $\mathcal{O}\backslash\mathcal{S}$ the orbifold obtained from $\mathcal{O}$ by removing a disjoint union of open product neighborhoods of the components of $\mathcal{S}$. Thus the study of compact 3-orbifolds without bad 2-suborbifold reduces to the case of irreducible 3-orbifolds. For these orbifolds there is a toric decomposition, analogous to the JSJ-decomposition for 3-manifolds:

Theorem 24. *Let $\mathcal{O}$ be a compact, irreductible* 3*-orbifold. There exists a system $\mathfrak{T}$ of essential, pairwise nonparallel toric* 2*-suborbifolds of $\mathcal{O}$ such that every component of $\mathcal{O}\backslash\mathfrak{T}$ is Seifert fibered or topologically atoroidal.*

Definition 25. *An orbifold $\mathcal{O}$ is* Seifert fibered *if it has a partition into 1-dimensional closed orbifolds (i.e. circles and mirrored intervals) such that each fibre has a saturated neighbourhood. In particular such an orbifold fibers over a* 2*-orbifold with generic fiber a circle or a mirrored interval.*

It follows from Theorem 23 that we need only to prove Theorem 13 with the extra hypothesis $\mathcal{O} = M/\langle\varphi\rangle$ is *irreducible.* A priori this does not follow directly from the fact that M is irreducible, since we do not know a priori that a cyclic quotient $B^3/\mathbb{Z}n$ of a 3-ball is a discal 3-orbifold.

Let $\mathcal{O}$ be a compact, orientable 3-orbifold which admits a finite, regular, manifold covering M. Any statement about $\mathcal{O}$ can be translated into an "equivariant" statement for M with respect to the covering group of transformations. For instance, the existence of a compression disc D for a given 2-suborbifold $F \subset \mathcal{O}$ is equivalent to the existence of an equivariant compression disk $\widetilde{D}$ for the preimage $\widetilde{F}$ of F in M. Here *equivariant* means that for every covering transformation g, either $g(\widetilde{D}) = D$ or $g(\widetilde{D}) \cap D = \emptyset$. The following equivariant version of the Loop Theorem has been obtained by Meeks and Yau [MY1, MY2], see also [DD], motivated by the Smith Conjecture and other questions about group actions on 3-manifolds. Their proof were based on minimal surfaces, i.e. surfaces that locally minimize area.

Theorem 26 (Equivariant Dehn Lemma)**.** *Let M be a compact, orientable 3-manifold with a smooth action of a finite group Γ. Let F be an equivariant subsurface of ∂M. If F is compressible, then it admits an equivariant compression disk.*

Using the equivariant Dehn Lemma we have:

Proposition 27. *Let M be a compact orientable and irreducible 3-manifold and let $p\colon M \to \mathcal{O}$ be a finite regular covering of a compact orientable 3-orbifold $\mathcal{O}$. Then any incompressible 2-submanifold $F \hookrightarrow \mathcal{O}$ lifts to an inconpressible surface in M.*

Corollary 28. *If M is a small* 3*-manifold and $\mathcal{O} = M/\langle\varphi\rangle$ is irreducible, then $\mathcal{O}$ does not contain any incompressible 2-suborbifod (i.e. is small).*

So, we are reduced to prove the following theorem:

Theorem 29. *Let $\mathcal{O}$ be an orientable, closed, small 3-orbifold* of cyclic type *then $\mathcal{O}$ is geometric (i.e. Hyperbolic, Spherical, Euclidean or Seifert fibred).*

The following exercise shows that an orientable, closed, small orbifold of cyclic type can be obtained as the quotient of a closed 3-manifold by an action of a finite group of diffeomorphisms.

Exercise 30. *Let $\mathcal{O}$ be an orientable, closed, small 3-orbifold of cyclic type, show that there is a finite manifold covering $p\colon M \to \mathcal{O}$. (Hint: The underlying space of $\mathcal{O}$ is a rational homology sphere, by smallness of $\mathcal{O}$. Then, using the exact sequence*

$$0 \to H_2(\mathcal{N}(\Sigma), \partial\mathcal{N}(\Sigma); \mathbb{Z}) \to H_1(|\mathcal{O}| \setminus \Sigma; \mathbb{Z}) \to H_1(|\mathcal{O}|, \mathbb{Z}) \to 0\,,$$

show that the meridians are linearly independent in $H_1(\mathcal{O} \setminus \Sigma, \mathbb{Q})$.)

4. Reduction to a hyperbolic singular locus

Proposition 31. *Either $\mathcal{O} \setminus \Sigma$ has a complete hyperbolic structure with finite volume or $\mathcal{O} \setminus \Sigma$ is Seifert fibered. In the last case $\mathcal{O}$ is Seifert fibered and Σ is an union of fibres.*

Proof. This follows from Thurston's hyperbolization theorem for Haken 3-manifolds, once we have proved that $\mathcal{O}\setminus \overset{\circ}{\mathcal{N}}(\Sigma)$ is irreducible and topologically atoroidal. This follows immediatly from the fact that $\mathcal{O}$ is small:

- If $S^2 \hookrightarrow \mathcal{O}\setminus \overset{\circ}{\mathcal{N}}(\Sigma)$ bounds a discal 3-orbifold in $\mathcal{O}$ then it must bound a ball in $\mathcal{O} \setminus \Sigma$.
- If $T^2 \hookrightarrow \mathcal{O} \setminus \Sigma$ is compressible in $\mathcal{O}$, then either it is compressible in $\mathcal{O} \setminus \Sigma$ or there is a compression disk Δ such that $\Delta \cap \Sigma = 1$ point. By cutting T^2 along Δ and gluing back two copies of Δ, we get a spherical 2-suborbifold that bounds a discal 3-orbifold in $\mathcal{O}$, hence T^2 is parallel to $\partial\mathcal{N}(\Sigma)$.

When $\mathcal{O}\setminus \overset{\circ}{\mathcal{N}}(\Sigma)$ is Seifert fibred, the fibre is not the boundary of a meridian disc of $\mathcal{N}(\Sigma)$, otherwise $\mathcal{O}$ would be reducible. Thus we can extend the Seifert fibration to $\mathcal{N}(\Sigma)$ with the components of Σ as fibers.

Otherwise, by Thurston's hyperbolization theorem, $\mathcal{O} \setminus \Sigma$ admits a complete hyperbolic structure with finite volume. $\square$

Exercise 32. *Show that if $\mathcal{O}\setminus \overset{\circ}{\mathcal{N}}(\Sigma)$ is Seifert fibered and the boundary of a meridian disc of $\mathcal{N}(\Sigma)$ is isotopic to the fiber, then $\mathcal{O}$ is reducible. (Hint: produce a spherical or bad 2-suborbifold by gluing a vertical annulus with a discal orbifold bounded by the meridian.)*

5. Deformations of Hyperbolic cone 3-manifolds

From now on we assume that $\mathcal{O} \setminus \Sigma$ is hyperbolic. In this section we introduce cone manifolds, which are the basic tool for the proof. Then we study sequences of cone manifolds s and we explain the proof of the orbifold theorem modulo some results to be proven later. Those are explained in the remaining sections.

5.1. **Cone manifolds.** Before defining cone manifolds we recall briefly the notion of *path metric space.* In a metric space X one defines the *length* of a path ξ as the supremum of the lengths of piecewise geodesic paths inscribed in ξ. Then X is a *path metric space* if for all $x, y \in X$, the distance between x and y is the infimum of the length of paths joining x to y. For instance, a Riemannian manifold is (by definition) a path metric space. If a topological space X results from an isometric gluing construction on one or more path metric spaces, then there is an obvious way to measure lengths of paths in X, and one can *define* a metric on X by taking the infimum of lengths of paths joining two points. We call this *the* path metric space obtained by the gluing construction.

Definition 33. *A* 3*-dimensional* cone manifold C *of constant curvature* $K \leq 0$ *is a complete path metric space whose underlying space is a smooth* 3*-manifold* $|C|$ *and such that every* $x \in C$ *has a neighborhood* U_x *that embeds isometrically in one of the model spaces* $H^3_K(\alpha)$ *defined below.*

To define $H^3_K(\alpha)$, we first let H^3_K denote the complete, simply connected Riemannian manifold of constant sectional curvature $K \leq 0$. Thus $H^3_{-1} \cong H^3$ and $H^3_0 \cong E^3$. For $\alpha \in (0, 2\pi)$, consider in H^3_K a solid angular sector S_α obtained by taking the intersection of two half-spaces, such that the dihedral angle at its infinite edge is α. Then $H^3_K(\alpha)$ is the path metric space obtained by gluing together the faces of S_α by an isometric rotation around the edge. Let Σ be the image of the edge in $H^3_K(\alpha)$. The induced metric on $H^3_K(\alpha) \setminus \Sigma$ is an incomplete Riemannian metric of constant curvature K, whose completion is precisely $H^3_K(\alpha)$. Points of Σ have no neighborhood isometric to a ball in a Riemannian manifold. In cylindrical or Fermi coordinates, the metric on $H^3_K(\alpha) \setminus \Sigma$ is:

$$ds^2_K = \begin{cases} dr^2 + \left(\frac{\alpha}{2\pi}\frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}}\right)^2 d\theta^2 + \cosh^2(\sqrt{-K}r)dh^2 & \text{for } K < 0 \\ dr^2 + \left(\frac{\alpha}{2\pi}r\right)^2 d\theta^2 + dh^2 & \text{for } K = 0 \end{cases}$$

where $r \in (0, +\infty)$ is the distance from Σ, $\theta \in [0, 2\pi)$ is the rescaled angle parameter around Σ and $h \in \mathbb{R}$ is the distance along Σ. More generally, if C is a cone manifold and $x \in C$, we say x is *regular* if it has a neighborhood isometric to a subset of H^3_K. Otherwise it is *singular*. The set of singular points is denoted by Σ_C and called the *singular locus*. It is a 1-dimensional totally geodesic submanifold.

Remark 34. *These definitions are very similar to those for orbifolds, but there is a fundamental differeeen the two concepts: orbifolds are topological objects which may carry metrics, whereas cone manifolds are by definition metric spaces.*

To every singular point is associated a *cone angle*, which is the only real number $\alpha \in [0, 2\pi]$ such that x has a neighborhood isometric to a subset of $H^3_K(\alpha)$. The induced metric on $|C| \setminus \Sigma_C$ is a Riemannian metric of constant curvature $K \leq 0$ whose completion is precisely the cone manifold C with cone type singularities along Σ_C. It is easy to see that the cone angle is constant along components of Σ_C, so we can talk about the cone angle of a component of Σ_C. Here are some useful definitions:

- The topological pair (C, Σ_C) is called the topological type of the cone manifold C.

- The developping map of C is the developping map of the induced metric on $C \setminus \Sigma_C$:
$$D\colon \widetilde{C \setminus \Sigma_C} \to H^3_K\,.$$
It is not a covering map because the metric is incomplete. ($\widetilde{C \setminus \Sigma_C}$ is the universal covering of $C \setminus \Sigma_C$).
- The associated holonomy representation $\rho\colon \pi_1(C\setminus\Sigma_C)\to \mathrm{Isom}^+(H^3_K)$ is called the holonomy of C. It is defined by:
$$D \circ \gamma = \rho(\gamma) \circ D,$$
where γ acts as a covering translation of the universal covering. The image $\rho(\pi_1(C \setminus \Sigma_C))$ need not be discrete.
- When $\mu \in \pi_1(C \setminus \Sigma_C)$ is a meridian around a component of Σ, then $\rho(\mu)$ is an *elliptic element*= rotation of angle α along a geodesic. We have the equality:
$$Tr\rho(\mu) = \pm 2 \cos \frac{\alpha}{2}\,.$$

In the case where $K = -1$, $\mathrm{Isom}^+(H^3_K) = PSL(\mathbb{C})$.

Note that with our definition, the singular locus of a 3-dimensional cone manifold is a 1-submanifold. One can give more general definitions where the cone manifold may have arbitrary dimension, the singular locus may have a more complex topology, or cone angles may be greater than 2π. (Compare [BLP2].)

Remark 35. *The orbifold $\mathcal{O}$ has a metric of constant curvature iff there exists a cone manifold C with $(|C|, \Sigma_C) \cong (|\mathcal{O}|, \Sigma)$ and with cone angles $\frac{2\pi}{m_1}, \ldots, \frac{2\pi}{m_k}$, where $m_1, \ldots, m_k$ are the branching orders of the components of Σ. This motivates the next definition.*

Definition 36. *The angles $\frac{2\pi}{m_1}, \ldots, \frac{2\pi}{m_k}$ are called* the orbifold angles.

5.2. **Deforming cone manifolds.** Before proceeding, we have to define formally hyperbolic cone structures on an orbifold.

Definition 37. *Let $\mathcal{O}$ be a compact, orientable 3-orbifold of cyclic type. Let $\Sigma_1, \ldots, \Sigma_k$ be the components of the singular locus Σ. Let $(\alpha_1, \ldots, \alpha_k)$ be a k-uple of real numbers belonging to the interval $(0, \pi)$. A* hyperbolic cone structure *on $\mathcal{O}$ with* cone angles $(\alpha_1, \ldots, \alpha_k)$ *is a pair (C, h) where C is a hyperbolic cone manifold and h is a homeomorphism of pairs $(|\mathcal{O}|, \Sigma) \to (|C|, \Sigma_C)$ such that for all i, the cone angle along $h(\Sigma_i)$ is α_i. By convention, we define a hyperbolic cone structure on $\mathcal{O}$ with angles $(0, \ldots, 0)$ to be the complete hyperbolic structure of finite volume on the 3-manifold $\mathcal{O} \setminus \Sigma$.*

From now on, we fix an order on the components of the singular locus of our small orbifold $\mathcal{O}$, let m_i denote the branching index of the i-th component, and set:

$$\mathcal{I} := \left\{ t \in [0,1] \;\middle|\; \begin{array}{l} \text{There exists a hyperbolic cone structure} \\ \text{on } \mathcal{O} \text{ with cone angles } (\frac{2\pi t}{m_1}, \dots, \frac{2\pi t}{m_k}). \end{array} \right\}$$

Our hypothesis that $\mathcal{O} \setminus \Sigma$ is hyperbolic translates into the fact that $0 \in \mathcal{I}$. The first step is to deform this structure to get hyperbolic cone structures with small cone angles on $\mathcal{O}$. A hyperbolic cone structure on $\mathcal{O}$ induces a non-complete hyperbolic structure on $\mathcal{O} \setminus \Sigma$. In particular it has a holonomy representation $\pi_1(\mathcal{O} \setminus \Sigma) \to PSL_2(\mathbb{C})$. The variety of representations $\mathfrak{R} = \mathrm{Hom}(\pi_1(\mathcal{O} \setminus \Sigma), PSL_2(\mathbb{C}))$ is a finite dimensional affine algebraic set, possibly reducible, since $\pi_1(\mathcal{O} \setminus \Sigma)$ is finitly generated (cf. [CS], [GM]). The group $PSL_2(\mathbb{C})$ acts on $\mathfrak{R}$ by conjugation, and we are interested in the quotient. The topological quotient is not Hausdorff, and therefore we consider the algebraic quotient:

$$\mathfrak{X} = \mathrm{Hom}\left(\pi_1(\mathcal{O} \setminus \Sigma), PSL_2(\mathbb{C})\right) // PSL_2(\mathbb{C})$$

which is again an affine algebraic set. Note that the irreducible representations form a Zariski open subset of $\mathfrak{R}$. More precisely, $\mathfrak{R}^{irr}$ is the inverse image of a Zariski open subset $\mathfrak{X}^{irr} \subseteq \mathfrak{X}$, and $\mathfrak{X}^{irr}$ is the topological quotient of $\mathfrak{R}^{irr}$. Notice that the holonomy representation ρ_0 of the complete hyperbolic structure on $\mathcal{O} \setminus \Sigma$ is irreducible.

Given $\mu \in \pi_1(\mathcal{O} \setminus \Sigma)$, we define the function $\tau_\gamma : X(M) \to \mathbb{C}$ as the function induced by

$$\begin{array}{rcl} \mathfrak{R} & \to & \mathbb{C} \\ \rho & \mapsto & \mathrm{trace}\left(\rho(\mu)^2\right). \end{array}$$

Let $a = \ell + i\theta$ be the complex translation length of $\rho(\mu)$. This means that $\rho(\mu)$ is a translation of length ℓ along it axis plus a rotation of angle θ around the same axis. Then:

$$tr\rho(\mu) = \pm 2 \cosh \frac{a}{2}\,.$$

Let $\mu_1, \dots, \mu_k$ be a family of meridian curves, one for each component of Σ.

Theorem 38 (Local parametrization)**.** *The map*

$$\tau_\mu = (\tau_{\mu_1}, \dots, \tau_{\mu_k})\colon \mathfrak{X} \to \mathbb{C}^k$$

is locally bianalytic at $[\rho_0]$.

This result is the main step in the proof of Thurston's hyperbolic Dehn filling theorem (see [BoP, App. B] or [Ka] for the proof). It implies in particular the following special case of Thurston's Generalized Hyperbolic Dehn Filling Theorem.

Corollary 39 (Generalized hyperbolic Dehn filling)**.** *There exists $\varepsilon > 0$, so that $[0,\varepsilon) \subset \mathcal{I}$.*

Proof. We have $\tau([\rho_0]) = (2,\dots,2)$. Consider the path

$$\begin{array}{rcl} \gamma\colon [0,\varepsilon) & \to & \mathbb{C}^k \\ t & \mapsto & \left(2\cos\frac{2\pi t}{m_1},\dots,2\cos\frac{2\pi t}{m_k}\right) \end{array} \tag{1}$$

where $\varepsilon > 0$ is sufficiently small. The composition $\tau_\mu^{-1}\circ\gamma$ gives a path of conjugacy classes of representations. It can be lifted to a path $t\mapsto\rho_t$, because there are slices to the action of $PSL_2(\mathbb{C})$ on the representation variety. The representations ρ_t are the holonomies of incomplete hyperbolic structures on $\mathcal{O}\setminus\Sigma$. By construction, the holonomies of the meridians are rotations with angles $2\pi t/m_1,\dots,2\pi t/m_q$. By a classical argument due to Ereshmann and Thurston, the deformation of holonomies is, locally near $t=0$, induced by a deformation of hyperbolic cone structures on $\mathcal{O}$ with cone angles $2\pi t/m_j$. □

Lemma 40. *There exists a unique irreducible curve $\mathcal{D}\subset\mathbb{C}^k$ such that $\gamma([0,1])\subset\mathcal{D}$.*

Proof. For $n\in\mathbb{N}$, we consider the Chebyshev-like polynomial

$$p_n(x) = 2\cos(n\arccos(x/2)).$$

It has the following property:

$$\operatorname{trace}(A^n) = p_n(\operatorname{trace}(A)), \qquad \forall A\in SL_2(\mathbb{C}),\ \forall n\in\mathbb{N}.$$

An easy computation shows that $p_n'(2)=n$, and therefore

$$\{z\in\mathbb{C}^k \mid p_{m_1}(z_1)=\dots=p_{m_k}(z_k)\}$$

is an algebraic curve with $(2,\dots,2)$ as a smooth point. We take $\mathcal{D}$ to be the unique irreducible component containing $(2,\dots,2)$. Then $\gamma([0,\varepsilon))\subset\mathcal{D}$ for small $\varepsilon>0$. Since γ is an analytic curve, it remains in $\mathcal{D}$. □

We define the algebraic curve $\mathcal{C}\subset\mathfrak{X}$ to be the irreducible component of $\tau_\mu^{-1}(\mathcal{D})$ that contains $[\rho_0]$. By construction, $[\rho_t]\in\mathcal{C}$ for small $t\geq 0$.

Theorem 41 (Openness)**.** *$\mathcal{I}$ is open to the right.*

Proof. Openness of $\mathcal{I}$ at $t = 0$ is a consequence of Thurston's hyperbolic Dehn filling, so we only prove openness at $t > 0$.

Consider the path

$$\begin{array}{rcl} \gamma : [t, t+\varepsilon) & \to & \mathcal{D} \subset \mathbb{C}^k \\ s & \mapsto & (2\cos(s2\pi/m_1), \ldots, 2\cos(s2\pi/m_k)) \end{array}$$

defined for some $\varepsilon > 0$. By construction, the image of γ is contained in the curve $\mathcal{D} \subset \mathbb{C}^k$ of Lemma 40. Since $\tau_\mu : \mathcal{C} \to \mathcal{D}$ is non constant, it is open, and therefore γ can be lifted to $\mathcal{C}$. We can lift it further to a path

$$\begin{array}{rcl} \tilde{\gamma} : [t, t+\varepsilon) & \to & \mathfrak{R} \\ s & \mapsto & \rho_s \,. \end{array}$$

To justify this second lift, notice that the holonomy ρ_t is irreducible (because the corresponding cone structure has finite volume) and therefore the $PSL_2(\mathbb{C})$-action is locally free. By construction, $\rho_s(\mu_i)$ is a rotation of angle $\frac{2\pi s}{m_i}$. According to Ereshmann and Thurston, a small deformations of holonomies can be realized by a deformation of cone structures. Therefore, the cone structure on $\mathcal{O}$ with holonomy ρ_t can be deformed to a continuous family of cone structures on $\mathcal{O}$ with holonomies ρ_s, and thus $[t, t+\varepsilon) \subset \mathcal{I}$ for $\varepsilon > 0$ sufficiently small. $\square$

The following theorem of Hodgson and Kerckhoff gives a more precise result about the possible deformations of these hyperbolic cone structures:

Theorem 42 ([HK])**.** *The space of hyperbolic structures on $\mathcal{O}$ with cone angles $< 2\pi$ is open, and it is locally parametrized by the cone angles $(\alpha_1, \ldots, \alpha_k)$.* $\square$

Theorem 42 contains a local ridigity statement: there are no deformations of hyperbolic cone structures with cone angles fixed. A global rigidity result as been obtained by Kojima:

Theorem 43 ([Ko])**.** *Two hyperbolic structures on $\mathcal{O}$ with the same cone angles are isometric, provided that their cone angles are $< \pi$.* $\square$

This theorem implies that for each $t \in \mathcal{I}$, the hyperbolic cone structure on $\mathcal{O}$ with angles $(\frac{2\pi t}{m_1}, \ldots, \frac{2\pi t}{m_q})$ is unique. We denote it by $C(t)$. A crucial consequence of the openness of the deformation space $\mathcal{I}$ is:

Corollary 44. *Let $t_\infty = \sup\mathcal{I}$. Then $t_\infty \in \mathcal{I}$ if and only if $t_\infty = 1 \in \mathcal{I}$. In this case $\mathcal{O}$ is hyperbolic.*

To prove the orbifold theorem it remains to show:

Theorem 45. *When* $t_\infty = \sup \mathcal{I} \notin \mathcal{I}$*, then either* $\mathcal{O}$ *is spherical, Euclidean or Seifert fibred.*

To prove theorem 45 we analyze the accidents that can occur at the *limit of hyperbolicity* $t_\infty = \sup \mathcal{I}$. This is done by looking at sequences of cone hyperbolic 3-manifolds with increasing cone angles:

$$C_n = C(t_n)\,, \quad t_n \in \mathcal{I}\,, \quad t_n \to t_\infty \notin \mathcal{I}\,.$$

So we need first some preliminaries about convergence of sequences of cone manifolds. The following examples have to be kept in mind while reading the proof of the Orbifold Theorem.

Example 46. *We start with a* 2*-dimensional example. Consider an equilateral triangle in* H^2*,* E^2 *or* S^2 *with angle* $\alpha \in (0, \pi]$*. In the hyperbolic case (resp. Euclidean, resp. spherical), one has* $\alpha < \pi/3$ *(resp.* $\alpha = \pi/3$*, resp.* $\alpha > \pi/3$*). Let* $S(\alpha, \alpha, \alpha)$ *be the double of this triangle, i.e the length space obtained by gluing two copies of the triangle by an isometry. Then* $S(\alpha, \alpha, \alpha)$ *is a cone* 2*-manifold with underlying space* S^2 *and three cone points of cone angle* 2α*. When* $n = \pi/\alpha$ *is an integer, it is a hyperbolic (resp. Euclidean, resp. spherical) structure on an orbifold. We see on this example how cone manifolds can be seen as "interpolating continuously between geometric orbifolds". It is worth noting that when* α *goes to* $\pi/3$ *from below, the diameter of the hyperbolic cone manifold* $S(\alpha, \alpha, \alpha)$ *goes to* 0*. Thus there is a limit angle which corresponds to a degeneration of the hyperbolic structure.*

Example 47. *This kind of behavior happens in dimension* 3*. In* [HLM]*, it is proved that for every* $\alpha \in (0, \pi)$ *there is a cone manifold of constant curvature with underlying space* S^3*, singular locus the figure eight knot and angle* α*. The structure is explicitly constructed; it is hyperbolic for* $\alpha < 2\pi/3$*, Euclidean for* $\alpha = 2\pi/3$ *and spherical for* $2\pi/3 < \alpha \leq \pi$*. Again by looking at angles of the form* $2\pi/n$*, one gets geometric structure on certain orbifolds. Since orbifold coverings are branched covering, one also gets geometric structures on branched coverings of the figure eight knot.*

6. Sequences of cone manifolds

6.1. **Geometric convergence.**

Definition 48. *Let* X *and* Y *be two metric spaces. A map* $f\colon X \to Y$ *is* $(1+\varepsilon)$*-bi-Lipschitz, for* $\varepsilon > 0$*, if* $\forall (x, y) \in X$*:*

$$\frac{1}{1+\varepsilon} \leq d(f(x), f(y)) \leq (1+\varepsilon)\, d(x, y)\,.$$

Remark 49. *A $(1+\varepsilon)$-bi-Lipschitz map is always an embedding.*

Definition 50. *A sequence of pointed cone manifolds (C_n, x_x)* converges geometrically *to a cone-manifold (C_∞, x_∞) if for every $R > 0$ and $\varepsilon > 0$, there exists an integer n_0 such that, for $n > n_0$, there is a $(1+\varepsilon)$-bi-Lipschitz map $f_n \colon B_R(x_\infty) \to C_n$ satisfying:*

(1) $d(f_n(x_\infty), x_n) < \varepsilon$,
(2) $B_{R-\varepsilon}(x_n) \subset f_n(B_R(x_\infty))$,
(3) $f_n(B_R(x_\infty) \cap \Sigma_\infty) = \Sigma_n \cap f_n(B_R(x_\infty))$.

Remark 51. *The geometric convergence is the convergence in the Gromov's* pointed bi-Lipschitz topology. *If $(C_n, x_n) \to (C_\infty, x_\infty)$ geometrically and if the limit C_∞ is compact, then for n large enough the pairs $(|C_n|, \Sigma_n)$ and $(|C_\infty|, \Sigma_\infty)$ are homeomorphic.*

A *standard ball* is a metric ball in a model space $H^3_K(\alpha)$ which either does not intersect the singular axis or is centered on it.

Definition 52. *The* cone injectivity radius *of a point p in a cone manifold is defined as:*

$$\textit{cone-inj}\,(x) = \sup\{r > 0 \mid B_r(p) \text{ is contained in a standard ball } \}$$

Definition 53. *A sequence of cone manifolds C_n* collapses *if*

$$\sup_{x \in C_N} \textit{cone-inj}\,(x) \to 0 \qquad \text{as } n \to \infty.$$

If $\mathrm{Diam}(C_n)$ goes to 0, then obviously the sequence collapses. For instance, our first 2-dimensional example $S(\alpha_n, \alpha_n, \alpha_n)$ collapses when $\alpha_n \to \pi/3$. The converse is false. For instance, one obtains a collapsing sequence of flat metrics on the 2-torus by starting with a product metric and pinching one factor to a point. In this example the diameter is eventually constant.

If a sequence C_n does not collapse, then by definition there is a sequence $x_n \in C_n$ such that for some subsequence, the numbers cone-inj (x_n) are uniformly bounded away from zero. Thus the following theorem is relevant to non-collapsing sequences. It is a version of Gromov's compactness theorem for Riemannian manifolds with pinched sectionnal curvature, in the setting of cone 3-manifolds with sectional curvature in $[-1, 0]$ and cone angles $\leq \pi$.

Theorem 54 (Compactness Theorem)**.** *Let (C_n, x_n) be a sequence of pointed cone manifolds. Suppose that there exist constants $a, \omega > 0$ such that for all n we have:*

(1) *cone-inj* $(x_n) > a$;
(2) *C_n has constant curvature $K_n \in [-1, 0]$;*
(3) *all cone angles of C_n lie in $[\omega, \pi]$.*

Then there is a subsequence (C_{n_k}, x_{n_k}) *which geometrically converges to a cone 3-manifold* (C_∞, x_∞) *with curvature* $K_\infty = \lim_{n\to+\infty} K_n$ *and cone angles that are limit of the cone angles of* C_n*.*

Theorem 54 is proved by using Gromov's compactness criterion to say that there is a subsequence that converges to a metric space for the Gromov-Hausdorff topology. To show that the limit is a cone manifold, one proves a uniform lower bound for the injectivity radius of points in C_n that are at bounded distance from x_n. This is where the upper bound of cone angle is used, via convexity of the Dirichlet domain.

Proposition 55 (Uniform decay of injectivity radius)**.** *Given* $R > 0$*,* $a > 0$ *and* $\omega \in (0, \pi)$*, there is a uniform constant* $b(R, a, w) > 0$ *such that for any pointed cone 3-manifold* (C, x) *with* $\mathrm{inj}(x) \geq a$ *and cone angles* $\in [\omega, \pi]$*, then* $\mathrm{inj}(y) \geq b > 0$, $\forall y \in B(x, R)$*.*

This proposition implies in particular that there is a uniform lower bound for the radius of a tubular neighbourhood of the singular locus Σ_n of C_n. When the sequence C_n is not collapsing, the singular locus cannot cut itself, but it may go to infinity.

6.2. **Analyzing $C(t_n)$ as $t_n \to t_\infty$.** We can now start the proof of the orbifold theorem. Let (t_n) be an increasing sequence in $\mathcal{I}$ with limit $t_\infty = \sup \mathcal{I}$. Assume that $C(t_n)$ does not collapse. Then Theorem 54 implies that $C(t_n)$ subconverges geometrically to a hyperbolic cone manifold C_∞.

Theorem 56 (Stability)**.** *If* $C(t_n)$ *does not collapse, then* C_∞ *is compact.*

We will prove this theorem in Section 7. Assuming it, the bi-Lipschitz convergence implies that the limit C_∞ is a hyperbolic cone structure on $\mathcal{O}$. Since $\mathcal{I}$ is open, it follows that $t_\infty = 1 \in \mathcal{I}$ and hence $\mathcal{O}$ is hyperbolic and we are done. So we are left with the harder case where $C(t_n)$ collapses. Our main tool is the following theorem, whose proof is postponed in Section 8. A cone manifold C is called *δ-thin* if $\sup_{x\in C}$ cone-inj $(x) \leq \delta$ for some $\delta > 0$.

Theorem 57 (Fibration)**.** *Let* C *be a cone manifold structure on* $\mathcal{O}$ *of constant curvature in* $[-1, 0]$ *and with cone angles less than or equal to the orbifold angles of* $\mathcal{O}$*. For* $\omega > 0$ *there exists* $\delta(\omega) > 0$ *such that, if* C *has cone angles* $\geq \omega$*, Diam*$(C) \geq 1$ *and* C *is* $\delta(\omega)$*-thin, then* $\mathcal{O}$ *is Seifert fibered.*

Now the proof of the orbifold theorem goes as follows:

Case 1: Diam$(C(t_n))$ is bounded away from zero. In this case the fibration theorem (Thm. 57) shows that $\mathcal{O}$ is Seifert fibered, hence geometric.

Case 2: $\mathbf{Diam}(\boldsymbol{C(t_n)}) \to \mathbf{0}$. Consider the **rescaled** sequence

$$\overline{C}(t_n) = \frac{1}{\mathrm{Diam}(C(t_n))} C(t_n)$$

of cone 3-manifolds with constant curvature $K_n = -\mathrm{Diam}(C(t_n))^2 \in [-1, 0)$ and diameter equal to 1. If $\overline{C}(t_n)$ collapses, then for n sufficiently large the fibration theorem (Thm. 57) applies again. If it does not, then by Theorem 54, a subsequence converges to a compact Euclidean cone 3-manifold C_∞ with diameter one. Hence C_∞ corresponds to a closed Euclidean cone structure on $\mathcal{O}$. If $t_\infty = 1$, this proves that $\mathcal{O}$ is Euclidean. Hence we assume that $t_\infty < 1$. Our goal is to prove the following:

Proposition 58. *If $t_\infty < 1$ then $\mathcal{O}$ is spherical.*

Proof. Recall that $\mathcal{O}$ has a finite, regular, covering which is a manifold. The Euclidean cone metric C_∞ lifts to a Euclidean cone metric on this finite manifold covering M. The singular locus $\widetilde{\Sigma}$ of this metric is a link in M and every cone angle equals $2\pi t_\infty < 2\pi$. Moreover, this metric is invariant by the group of deck transformations of the covering.

One can deform this Euclidean cone metric on M by using a radial deformation in a tubular neighbourhood of the singular locus $\widetilde{\Sigma}$, to get an equivariant non flat smooth Riemannian metric with a curvature ≥ 0 (cf. [BoP] or [Zh]). Hamilton's flow [Ha1, Ha2] allows to deform this Riemannian metric equivariantly to one of constant curvature $+1$. Since it is equivariant, this gives a spherical structure on $\mathcal{O}$.

We explain now quickly how to deform the metric.

The singular Euclidean metric in a tubular neighbourhood of $\widetilde{\Sigma}$ is:

$$ds^2 = dr^2 + t^2 r^2 d\theta^2 + dh^2\,, \qquad t < 1\,.$$

We replace it by a metric of the form:

$$ds^2 = dr^2 + f^2(v) d\theta^2 + dh^2$$

where $f\colon [0, r_0 - \varepsilon) \to [0, +\infty)$ is a $\mathcal{C}^\infty$ function with the following properties:

(1) $f(r) = r$, $\forall r \in [0, \varepsilon/2)$
(2) $\forall r \in [r_0/2, r_0 - \varepsilon)\,,\ f(r) = t(r + \varepsilon)$
(3) $f''(r) \leq 0\,,\ \forall r \in [0, r_0 - \varepsilon)$, (i.e. it is concave).

From the properties of f we deduce the following properties on the metric:

- Property 1) implies that the new metric is non singular.
- Property 2) implies that the new metric fits with the original one at $\partial \mathcal{N}_{r_0}(\Sigma)$.

- Property 3) implies that the sectional curvature is ≥ 0 and even > 0 at some points.

□

Remark 59. *Using Ricci flow, Hamilton proved first that a metric of > 0 Ricci curvature may be deformed to a spherical one with $K \cong +1$.*

In a second paper he proved that if a metric has Ricci curvature ≥ 0, then, appart from two exceptions, there is a deformation to a metric with Ricci curvature > 0. The exceptions are:

(1) The original metric is flat (Ricci tensor $\cong 0$).
(2) After the deformation the Ricci tensor has rank one everywhere and the metric is modeled on $S^2 \times E^1$.

The first case is eliminated because the starting metric is not flat. The second case is ruled out because $\mathcal{O}$ is irreducible.

This finishes the proof of the Orbifold Theorem, modulo the Stability theorem 56 and the fibration theorem (Thm. 57).

7. The stability theorem

We have an increasing sequence t_n in $\mathcal{I}$ that converges to t_∞ and the corresponding cone manifolds $C(t_n)$ converge geometrically to a hyperbolic cone manifold C_∞. We have to prove that C_∞ is compact.

Set $X := \mathcal{O} \setminus \Sigma$, $C_n^{smooth} := C(t_n) \setminus \Sigma_{C(t_n)}$, and $C_\infty^{smooth} = C_\infty \setminus \Sigma_{C_\infty}$. By definition of a hyperbolic cone structure on $\mathcal{O}$, there is for each n a homeomorphism $g_n : (|\mathcal{O}|, \Sigma_{\mathcal{O}}) \to (|C(t_n)|, \Sigma_{C(t_n)})$ such that for all i, the cone angle along $g_n(\sigma_i)$ is $\frac{2\pi t_n}{m_i}$. Using these homeomorphisms, we can identify $\pi_1 X$ with $\pi_1 C_n^{smooth}$, and thus consider the holonomy ρ_n of C_n^{smooth} as a representation of $\pi_1 X$. We write χ_n for the character of this holonomy in the the character variety $\mathfrak{X} = \mathrm{Hom}(\pi_1(X), SL_2(\mathbb{C}))//SL_2(\mathbb{C})$.

Lemma 60. *The singular locus of C_∞ is compact.*

Proof. Seeking a contradiction, assume that C_∞ has a non-compact singular locus. Using the bi-Lipschitz convergence, it is easy to see that the number of singular components of C_∞ is not greater than that of $C(t_n)$; in particular it is finite. Thus C_∞ has a non-compact singular component. Again the bi-Lipschitz convergence shows that there exists $i \in \{1, \ldots, k\}$, say 1, such that the length of $g_n(\Sigma_1)$ goes to infinity. According to the lemma 40, there exists an algebraic curve $\mathcal{C}$ in the character variety $\mathfrak{X}$ containing χ_n for all n. By passing to a subsequence, we can assume that (χ_n) converges in $\overline{\mathcal{C}}$. The following fact tells us that the limit must be an ideal point. There is an element $\lambda \in \pi_1 X$ such that the real part of the complex length of $\rho_n(\lambda)$ goes to infinity. In fact $\lambda \in H_1(T_1, \mathbb{Z})$, where $T_1 \subset \partial(\mathcal{O} \setminus \overset{\circ}{\mathcal{N}}(\Sigma))$ corresponds to the singular component Σ_1. Moreover:

- $|\text{trace}(\rho_n(\lambda))| = 2\cosh\left(\frac{\text{complex length}}{2}\right) \to +\infty$
- $|\text{trace}(\rho_n(\mu_i))| = \left|2\cos\frac{\pi t_n}{m_i}\right| \leq 2$, for any meridian μ_i of a component $\sigma_i \subset \Sigma$.

Then the Culler-Shalen's theory of valuations [CS], associated to ideal points of the curve $\mathcal{C} \subset \mathfrak{X}$, gives rise to an incompressible, ∂-incompressible surface in $\mathcal{O} \setminus \overset{\circ}{\mathcal{N}}(\Sigma)$ with boundary an union of meridians.

This gives an incompressible 2-suborbifold in $\mathcal{O}$ contradicting the smallness of $\mathcal{O}$. Hence the singular locus of C_∞ is compact. □

Lemma 61. *For every $t \in \mathcal{I}$, vol$(C(t)) \leq \text{vol}(C(0))$.*

Proof. The proof uses Schläfli's formula. For a smooth deformation of a polyhedron P_t in hyperbolic space, the variation of volume is:

$$d\,\text{vol}\,(P_t) = -\frac{1}{2}\sum_e \text{ length}\,(e) d\alpha_e$$

where the sum is taken over all edges e of P_t and α_e denotes the dihedral angle of e. For our family of cone manifolds $C(t)$, we can take a totally geodesic triangulation that varies smoothly with t (cf. [Por]). By adding up all contributions of volume, we realize that only edges corresponding to singularities are relevant, and we have:

$$d\,\text{vol}\,(C(t)) = -\frac{1}{2}\sum_{i=1}^{k} \text{ length}\,(\Sigma_i)\ d(2\pi/m_i t) = -\pi \sum_{i=1}^{k} \text{ length}\,(\Sigma_i)\frac{1}{m_i}\ d\,t,$$

where $\Sigma_1, \ldots, \Sigma_k$ denote the components of Σ_C. Hence the volume of $C(t)$ decreases with t. □

Assume that C_∞ is not compact. By Lemma 61, the volume of $C(t_n)$ is bounded above, thus C_∞ has finite volume. Since its singular locus is compact, the ends of C_∞ are smooth and one can apply a local version of the Margulis Lemma. In particular one can prove (cf.[BLP2]):

Proposition 62. *The manifold C_∞ has a finite number of ends, which are smooth cusps.*

Lemma 63. *The manifold C_∞^{smooth} is hyperbolic.*

Proof. The incomplete hyperbolic metric can be deformed around the singularity to have a complete metric of strictly negative curvature, (cf.[Ko]). The metric is unchanged along the complete smooth cusps of C_∞^{smooth}. This implies that C_∞^{smooth} is irreducible and atoroidal, since strictly negative curvature forbides essential spheres or essential tori (using Cartan-Hadamard Theorem and minimal surfaces). Then the result follows from Thurston's hyperbolization theorem. □

Let Y be a compact core of C_∞^{smooth}. By convergence, there exists a $(1+\varepsilon_n)$-bi-Lipschitz embedding $f_n : Y \to C_n^{smooth}$ with $\varepsilon_n \to 0$.

Lemma 64. *$C(t_n) \setminus f_n(Y)$ is a union of smooth or singular solid tori.*

Proof. The boundary ∂Y is a union of tori $T_1^2, \ldots, T_r^2$. Since $C(t_n)^{smooth}$ is hyperbolic, each $f_n(T_i^2)$ is either compressible or end parallel in $C(t_n)^{smooth}$. Assume first that $f_n(T_i^2)$ is end parallel, i.e. $f_n(T_i^2)$ bounds an end neighborhood U of C_n^{smooth} homeomorphic to $T^2 \times [0, +\infty)$. If $f_n(T_i^2)$ separates $f_n(Y)$ from U, then the component of $C(t_n) \setminus f_n(Y)$ corresponding to T_i^2 is a singular solid torus. Now it is impossible that $f_n(Y) \subset U$ for infinitely many n, because $\rho_n \circ f_{n*}$ converges to the holonomy of the incomplete structure on C_∞^{smooth}, which is non-abelian. When $f_n(T_i^2)$ is compressible, then one of the following occurs:

(1) $f_n(T_i^2)$ bounds a solid torus disjoint from $f_n(Y)$.
(2) $f_n(T_i^2)$ bounds a solid torus that contains $f_n(Y)$.
(3) $f_n(T_i^2)$ is contained in a ball.

As before, an argument with convergence of holonomy representations eliminates cases (2) and (3), because the holonomy of C_∞^{smooth} is non-abelian and the holonomy of T_i^2 is nontrivial. □

For each n, let $\lambda_1^n, \ldots, \lambda_k^n$ be curves on ∂Y such that:

(1) there is one curve λ_i^n for each component of ∂Y corresponding to a cusp of C_∞.
(2) $f_n(\lambda_i^n)$ is a meridian of a possibly singular solid torus lying in $C(t_n) \setminus f_n(Y)$.

Lemma 65. *For each i, $\lambda_i^n \to \infty$.*

Proof. Otherwise, for some i the curve $\lambda_i^n = \lambda_i$ is independent of n. Thus $\rho_n(f_{n*}(\lambda_i))$ converges to the holonomy of λ_i in C_∞^{smooth}, which is parabolic. This gives a contradiction with the fact that $\rho_n(f_{n*}(\lambda_i))$ is either trivial or a rotation of angle $\frac{2\pi}{m_i} t_n$ (that converges to $\frac{2\pi}{m_i} t_\infty$). □

For each n we consider the Dehn filling of Y along $\lambda_1^n, \ldots, \lambda_k^n$. This manifold is the underlying space of $C(t_n)$ minus open regular neighborhoods of some singular components (the ones that correspond to the singular components of C_∞). Thus we may assume that topologically this Dehn filling is independent of n. Now using Lemma 65, Thurston's hyperbolic Dehn filling theorem, and volume estimates, one can show that those Dehn fillings are different. This contradiction finishes the proof of Theorem 56. □

8. The fibration theorem

First we introduce Gromov's simplicial volume which a crucial ingredient used to prove the fibration theorem. It has been introduced by M. Gromov [Gro] and it is connected to volumes of hyperbolic manifolds.

8.1. **Gromov's simplicial volume.** Let M be a topological space. Our first goal is to define a semi-norm on $H_k(M,\mathbb{R})$. A real k-chain on M is a linear combination $c = \Sigma_i a_i \sigma_i$, where the a_i's are real numbers and the σ_i's are continuous maps from the standard k-simplex to M. We set $\|c\| := \Sigma_i |a_i|$. The semi-norm $\|z\|$ of an element $z \in H_k(M,\mathbb{R})$ can now be defined as the infimum of the norms of cycles representing z.

If M is a closed n-manifold, then it has a fundamental class $[M] \in H_n(M,\mathbb{R})$. We define the *simplicial volume* of M, by $\|M\| := \|[M]\|$. More generally:

Definition 66. *Let M be a compact orientable n-manifold,*

$$\|M\| = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| \;\middle|\; \begin{array}{c} \sum_{i=1}^{n} \lambda_i\sigma_i \textit{ is a cycle representing a fundamental} \\ \textit{class in } H_3(M,\partial M;\mathbb{R}),\ \textit{where } \sigma_i : \Delta^3 \to M \\ \textit{is a singular simplex and } \lambda_i \in \mathbb{R},\ i = 1,\dots,n. \end{array} \right\}$$

A basic idea we will exploit is that nonvanishing of simplicial volume is associated to some kind of "hyperbolic" behavior. Let us illustrate this on examples.

Proposition 67. *Let M be a closed manifold. If there exists a self-map $f : M \to M$ with $|\deg(f)| \geq 2$, then $\|M\| = 0$.*

Corollary 68. *Spheres and tori of any dimension have zero simplicial volume.*

Exercise 69. *Prove Proposition 67. (Hint: recall that the degree of a map $f\colon M \to N$ can be defined by $f_*[M] = \deg(f) \cdot [N]$.)*

By contrast, hyperbolic manifold s have nonzero simplicial volume. More precisely their simplicial volume is equal to the hyperbolic volume up to a constant factor:

Theorem 70. *For $n \geq 2$, let v_n be the supremum of volumes of geodesic simplices in H^n. Then for all hyperbolic n-manifolds we have:*

$$\|M\| = \frac{\operatorname{vol} M}{v_n}.$$

Remark 71. *Proving $\geq$ is not too hard. It uses the idea of "straightening" cycles, together with the fact that for any "straight" cycle c representing $[M]$, the volume of M is equal to the weighted sum of the volumes of the simplices of c provided they are counted "algebraically", i.e. taking into account orientations and multiplicities. The other direction is more involved. See* [BeP, C4] *for a detailed proof.*

Here are some important properties of the simplicial volume:

Properties

- $||M_1 \sharp M_2|| = ||M_1|| + ||M_2||$.
- For 3-dimensional manifolds, $||M_1 \bigcup_{T^2} M_2|| \leq ||M_1|| + ||M_2||$ with equality if the boundary torus T^2 is incompressible in both M_1 and M_2.(cf. [Gro] (see also [Ku]).

Since the simplicial volume is additive with respect to gluing manifolds along incompressible tor, the following corollary is a consequence of the JSJ-decomposition of a Haken 3-manifold.

Corollary 72. *Let M be a compact, orientable, Haken 3-manifold. Then $\|M\| \neq 0$ if and only if M has at least one hyperbolic piece in its JSJ-decomposition.*

Simplicial volume will be used in the next section to analyze collapses. We shall use Corollary 72 and the following vanishing theorem of Gromov (see [Gro] and [Iva]). We say that a covering of a manifold has dimension k if its nerve has dimension k (i.e. $k + 1$ is the maximal number of sets of the covering that contain a given point of the manifold).

We say that a subset S in a manifold M is *amenable* if the image of $\pi_1 S \to \pi_1 M$ is amenable. Notice that virtually abelian groups are amenable.

Theorem 73 (Gromov's Vanishing Theorem)**.** *If M is a n-manifold with a $(n-1)$-dimensional covering by amenable sets, then $\|M\| = 0$.*

8.2. **Strategy of the proof of the fibration theorem.** Throughout this section, we assume that C is a cone manifold structure on $\mathcal{O}$ of constant curvature in $[-1, 0]$, with cone angles between ω and the orbifold angles of $\mathcal{O}$. We also assume that C is δ-thin (i.e. each point has cone injectivity radius $< \delta$). We shall show the existence of a constant $\delta_0(\omega) > 0$ such that if $\delta < \delta_0(\omega)$ then $\mathcal{O}$ is Seifert fibered.

The strategy of the proof consists in choosing a Seifert fibered suborbifold $W_0 \subset \mathcal{O}$ which is a quotient of a solid torus or a thickened torus such that:

(i) $\mathcal{O}_0 := \mathcal{O} - int(W_0)$ is irreducible;
(ii) any manifold covering of $\mathcal{O}_0$ has trivial simplicial volume.

Then Corollary 72 implies that any finite regular manifold covering of $\mathcal{O}_0$ is a graph manifold, and therefore by Meeks and Scott [MS] $\mathcal{O}_0$ itself is a graph orbifold. Hence $\mathcal{O}$ is Seifert fibered by atoroidality.

8.3. **Local Euclidean structures.** To understand the local geometry of thin cone manifolds of nonpositive curvature, we need some facts about non-compact Euclidean cone manifolds. We first give some examples.

Example 74. *The following are non-compact Euclidean cone* 3*-manifolds.*

1. *The model spaces H^3_0 and $H^3_0(\alpha)$.*
2. *Quotients of H^3_0 (resp. $H^3_0(\alpha)$) by an infinite cyclic group generated by a screw motion (resp. a screw motion respecting the singular axis.) The underlying space is $S^1 \times \mathbb{R}^2$, and the singular locus is empty (resp. a core circle).*
3. *The product of $\mathbb{R}$ with a closed Euclidean cone* 2*-manifold.*
4. *A slightly more complicated example is obtained by taking the quotient of the previous one by a metric involution τ that reverses the orientation of both factors. For instance T^2 admits an involution such that the quotient is topologically an annulus. This gives $S^1 \times \mathbb{R}^2$ with singular locus two circles of angle π.*

Definition 75. *A* soul *S of a non-compact Euclidean cone manifold E is a totally geodesic compact submanifold with boundary either empty or singular with cone angle π, such that E is isometric to the normal bundle on E (with infinite radius).*

In example (i) above the soul is a point. In example (ii) it is a circle. We leave it as an exercise to determine the soul in examples (iii) and (iv).

Proposition 76. *Every non-compact Euclidean cone manifold with cone angles $\leq \pi$ has a soul.*

This proposition can be used to classify Euclidean cone 3-manifolds. See [BLP2, BoP, CHK] for a complete list.

Next lemma is the orbifold analogue of [CG, part 2, Proposition 3.4] in the case of Riemannian manifolds with bounded curvature, which gives a local description of collapsing manifolds.

Lemma 77. *For every $\varepsilon > 0$ and $D > 1$, there exists $\delta_0 = \delta_0(\varepsilon, D, \omega) > 0$ such that, if C is a cone* 3*-manifold satisfying all hypotheses of Theorem 57, in particular is δ-thin with $\delta < \delta_0$, then for each $x \in C$ there is a neighborhood $U_x \subset C$ of x, a number $\nu_x \in (0,1)$ and a $(1+\varepsilon)$-bi-Lipschitz homeomorphism f_x between U_x and the normal cone fiber bundle of radius*

ν_x of the soul S of a non-compact Euclidean cone 3-manifold. In addition $\dim S = 1$ *or* 2*, and*

$$\max\big(d(f_x(x), S)\,,\ Diam(S)\big) \leq \nu_x/D.$$

Proof. The proof is by contradiction. If the assertion were false, then there would exist $\varepsilon > 0$, $D > 1$ and a sequence of cone manifolds C_n with diameter ≥ 1, curvature in $[-1, 0]$ such that C_n is $\frac{1}{n}$-thin, and there would exist points $x_n \in C_n$ for which the conclusion of the lemma does not hold with the constants ε, D.

Set $\lambda_n =$ cone-inj (x_n). By the compactness theorem (Thm. 54), a subsequence of $(\frac{1}{\lambda_n} C_n, x_n)$ converges to a non-compact Euclidean 3-manifold (E, x_∞). Since cone-inj $(x_\infty) = 1$, the soul of E has dimension one or two. Using the properties of geometric convergence, one can prove that the conclusion of the lemma holds for x_n provided that n is large (see [BLP2] and [BoP] for details). □

The neighborhoods U_x in this lemma are called (ε, D)-*Margulis' neighborhoods*, and the Euclidean cone manifolds with soul S are called local models. The local models E are described according to the dimension of the soul S (cf. [BoP]):

- When S is two dimensional and orientable, then E is isometric to the product $S \times \mathbb{R}$. The possible 2-dimensional cone manifolds S are a torus T^2, a pillow $S^2(\pi, \pi, \pi, \pi)$ and a turnover $S^2(\alpha, \beta, \gamma)$, with $\alpha + \beta + \gamma = 2\pi$.
- When S is two dimensional but non orientable (possibly with mirror boundary), then $E = \tilde{S} \times \mathbb{R}/\tau$, where $\tilde{S}$ is the orientable covering of S and τ is an involution that preserves the product structure and reverses the orientation of each factor. It is a twisted line bundle over S.
- When $\dim(S) = 1$, then either $S = S^1$ or S is an interval with mirror boundary (a quotient of S^1). In the former case, E is either a solid torus or a singular solid torus. In the latter, E is a solid pillow.

We apply this lemma to each point of C with some constants $D > 1$, $\varepsilon > 0$ to be specified later. Consider the thickening

$$W_x := f^{-1}(\overline{\mathcal{N}_{\lambda\nu_x}(S)})$$

of the soul of U_x where $0 < \lambda < \frac{1}{D}$. We will also view W_x as a suborbifold of $\mathcal{O}$.

The topology of W_x is easily described from the list of all possible non-compact Euclidean cone manifolds. Moreover, not all the models can occur: W_x contains no turnover, because $\mathcal{O}$ is small and of cyclic type.

From the classification of noncompact Euclidean cone 3-manifolds, one can deduce:

Lemma 78. *Each W_x admits a Seifert fibration (in the orbifold sense). In particular, ∂W_x is a union of tori and pillows.*

Proof. To prove this, notice that $\mathcal{N}_{\nu_x/D}(S)$ is a Euclidean structure on W_x, with cone angles $\leq$ than the orbifold angles. Then the lemma is proved by looking at all the possible cases for S, using the fact that S is never a turnover nor a quotient of it. □

Lemma 79. *If $\varepsilon = \varepsilon(\omega) > 0$ is small enough, $D = D(\omega)$ is large enough and if $W_x \cap \Sigma \neq \emptyset$, then $\mathcal{O} \setminus int(W_x)$ is irreducible.*

Proof. We have to show that W_x is not contained in a discal suborbifold. Since the pairs $(\mathcal{O}, W_x)$ and $(\mathcal{O}, \bar{U}_x)$ are homeomorphic, this amounts to showing that U_x is not contained in a discal suborbifold, and we can use the metric properties of U_x.

Suppose that U_x meets the singularity and is contained in the discal suborbifold Δ. Topologically, Δ is a singular ball with one axis a. Hence U_x cannot contain an entire singular component. By looking at the possible local models one can show that U_x contains at least two singular segments of length $> \nu_x$ whose midpoints m_1 and m_2 have distance $< \frac{\nu_x}{D}(1+\varepsilon)$. By developing the smooth part of D into model space, and composing the developing map with the projection onto the axis fixed by the holonomy representation, we find a 1-Lipschitz function on D whose restriction to the axis a is linear with slope 1. It follows that a is distance minimizing inside Δ and hence $d_\Delta(m_1, m_2) > \nu_x$, a contradiction. This finishes the proof of irreducibility. □

8.4. **Covering by virtually abelian subsets.** We assign a special role to one of the subsets W_x along which we will cut $\mathcal{O}$ later on. We choose $x_0 \in C$ with $W_{x_0} \cap \Sigma \neq \emptyset$ and:

$$\nu_{x_0} \geq \frac{1}{1+\varepsilon} \sup\{\nu_x | W_x \cap \Sigma \neq \emptyset\}.$$

We denote $W_0 = W_{x_0}$, $\mathcal{O}_0 = \mathcal{O} \setminus int(W_0)$, $\nu_0 = \nu_{x_0}$. In view of Lemma 79, $\mathcal{O}_0$ is irreducible.

Definition 80. *We say that a subset $S \subset \mathcal{O}$ is* virtually abelian in *$\mathcal{O}_0 = \mathcal{O} \setminus int(W_0)$ if the image of $\pi_1(S \setminus int(W_0)) \to \pi_1(\mathcal{O}_0)$ is virtually abelian. Moreover, for $x \in C$ we define:*

$$vab(x) = \sup\{r > 0 \mid B_r(x) \textit{ is virtually abelian in } \mathcal{O}_0\}$$

and $r(x) = \inf(\frac{vab(x)}{8}, 1)$.

Lemma 81. *Let $x, y \in C$. If $B_{r(x)}(x) \cap B_{r(y)}(y) \neq \emptyset$, then*

(a) $3/4 \leq r(x)/r(y) \leq 4/3$;
(b) $B_{r(x)}(x) \subset B_{4r(y)}(y)$.

Proof. Assume $r(x) \geq r(y)$. Either $r(y) = 1$ or $r(y) = vab(y)/8$. If $r(y) = 1$, then $r(x) = 1$ and assertion (a) is clear. If $r(y) = vab(y)/8$, by using the inclusion $B_{6r(x)}(y) \subset B_{8r(x)}(x)$ and the fact that $8r(x) \leq vab(x)$, it follows that $B_{6r(x)}(y)$ is virtually abelian in $\mathcal{O}_0$. Hence $r(x) \leq vab(y)/6 \leq 4r(y)/3$ and (a) is proved.

Assertion (b) follows easily from (c) and the inclusion $B_{r(x)}(x) \subset B_{2r(x)+r(y)}(y)$. □

Lemma 82. *For D sufficiently large holds $W_0 \subset B_{\frac{1}{9}r(x_0)}(x_0)$.*

Proof. This follows from the fact that $vab(x_0) \geq \frac{1}{1+\varepsilon}\nu_{x_0}(1 - 1/D)$ and W_0 is contained in the ball of radius $2(1+\varepsilon)\nu/D$ centered at x_0. □

We have already fixed a point $x_0 \in W_0$. We consider then a finite sequences $\{x_0, x_1, \ldots, x_p\}$ starting with x_0, such that:

(2) the balls $B_{\frac{1}{4}r(x_0)}(x_0), \ldots, B_{\frac{1}{4}r(x_p)}(x_p)$ are pairwise disjoint.

A sequence satisfying (2) is finite, by Lemma 81 and by compactness. Moreover we have the following property:

Lemma 83. *If the sequence $\{x_0, x_1, \ldots, x_p\}$ is maximal for property (2), then the balls $B_{\frac{2}{3}r(x_0)}(x_0), \ldots, B_{\frac{2}{3}r(x_p)}(x_p)$ cover C.*

Proof. Let $x \in C$. By maximality, the ball $B_{\frac{1}{4}r(x)}(x)$ intersects $B_{\frac{1}{4}r(x_i)}(x_i)$ for some $i \in \{1, \ldots, p\}$. From property (a) of Lemma 81, $r(x) \leq \frac{4}{3}r(x_i)$ and thus $x \in B\big(x_i, \frac{r(x_i)+r(x)}{4}\big) \subset B_{\frac{2}{3}r(x_i)}(x_i)$. □

Given a sequence $\{x_0, x_1, \ldots, x_p\}$, maximal for property (2) and starting with $x_0 \in W_0$, we consider the covering of C by the following open sets:

- $V_0 = B_{r(x_0)}(x_0)$, and
- $V_i = B_{r(x_i)}(x_i) \setminus W_0$, for $i = 1, \ldots, p$.

Lemma 83 guarantees that the open sets $V_0, \ldots, V_p$ cover C.

We put $r_i = r(x_i)$ and $B_i = B_{r(x_i)}(x_i)$.

We want to replace the covering $\{V_0, \ldots, V_p\}$ by a covering of $\mathcal{O}$ still consisting of virtualy abelian subsets, but that has dimension 2 and dimension 0 in W_0. To obtain the new covering we prove the following proposition:

Proposition 84. *If D is large enough and $\varepsilon > 0$ is sufficiently small, then there exists a 2-dimensional complex $K^{(2)}$ and a continuous map $f : C \to K^{(2)}$ such that:*

(1) $f(W_0)$ is a vertex of $K^{(2)}$.
(2) The inverse image of the open star of each vertex is virtually abelian in $\mathcal{O}_0$.

Notice that by taking the inverse images of the open stars of vertices of $K^{(2)}$, this proposition tells us that $\mathcal{O}_0$ has a covering by virtually abelian subsets. This covering has dimension 2 (i.e. each point belongs to at most 3 open sets). Moreover, only one open set intersects W_0.

We first construct a Lipschitz map $f : C \to K$, where K is the nerve of the covering (i.e. 0-cells or K correspond to open sets, 1-cells to pairs V_i and V_j with $V_i \cap V_j \neq \emptyset$, and so on), such that:

(i) If $x \in C$ belongs to only an open set V_i of the covering, then $f(x)$ is the corresponding vertex v_i of K.
(ii) For every vertex v_i of K, $f^{-1}(star v_i) \subset \bigcup_{V_j \cap V_i \neq \emptyset} V_j$.
(iii) The Lipschitz constant of f restricted to $\bigcup_{V_j \cap V_i \neq \emptyset} V_j$ is $\leq \frac{\xi}{r_i}$ for some uniform constant $\xi > 0$ depending only on the dimension.

Lemma 85. *There is a uniform bound N on the dimension of the nerve of the covering $\{V_0, \dots, V_p\}$.*

Proof. It suffices to bound the number of balls B_i that can intersect a given ball B_k. For every ball B_i intersecting B_k holds $d(x_i, x_k) \leq r_i + r_k < 3r_k$. These points x_i are separated from each other: $d(x_{i_1}, x_{i_2}) \geq \frac{r(x_{i_1})+r(x_{i_2})}{4} \geq \frac{1}{4} r(x_i)$. Since $r_k \leq 1$ it follows that there is a uniform lower bound on the number of points x_i. $\square$

We consider the nerve K of the covering. It is a simplicial complex of dimension $k \leq N$ that can be canonically embedded in $\mathbb{R}^{p+1}$, where every vertex correspond to a vector $(0, \dots, 1, \dots 0)$ and where simplices of positive dimension are defined by linear extension. $|| \quad ||$ is the usual norm on $\mathbb{R}^{p+1}$.

Consider a smooth function $\tau\colon \mathbb{R} \to [0,1]$ such that $\tau((-\infty, 0]) = 0$, $\tau([1/3, +\infty)) = 1$, and $|\tau'(t)| \leq 4$ for every $t \in \mathbb{R}$.

We define
$$\begin{array}{rcl} \tau_i\colon \overline{V}_i & \to & \mathbb{R} \\ x & \mapsto & \tau(d(x, \partial V_i)/r_i)\,. \end{array}$$
Since $\tau_i \cong 0$ on ∂V_i, we extend it by 0 on C.

We need the following property of our covering:

Lemma 86. *Every $x \in C$ belongs to an open set V_i such that $d(x, \partial \overline{V}_i) \geq r_i/3$.*

Proof. Let $x \in C$, then $x \in B_{\frac{2}{3}r_i}(x_i)$ for some $i \in \{0,\dots,p\}$; we fix this index i. If $i = 0$ or if $B_i \cap W_0 = \emptyset$, then $V_i = B_i$ and the lemma holds. Hence we may assume that $i > 0$ and $B_i \cap W_0 \neq \emptyset$. Moreover, we can suppose $d(x,x_0) > \frac{2}{3}r_0$. In this case $V_i = B(x_i,r_i) \setminus W_0$ and we claim that $d(x,W_0) > \frac{1}{3}r_i$.

To prove this claim, we use the inequality:

$$d(x,W_0) \geq d(x,x_0) - \mathrm{Diam}(W_0) \geq \tfrac{2}{3}r_0 - \tfrac{1}{9}r_0 = \tfrac{5}{9}r_0,$$

which is true because $d(x,x_0) > \frac{2}{3}r_0$ and $\mathrm{Diam}(W_0) \leq \frac{1}{9}r_0$.

Since $B(x_0,r_0) \cap B(x_i,r_i) \neq \emptyset$, $r_0 \geq \frac{3}{4}r_i$. Hence $d(x,W_0) \geq \frac{5}{12}r_i > \frac{1}{3}r_i$, and the claim is proved. $\square$

A consequence of this lemma is that:

$$\sum_{i=0}^{p} \tau_i(x) \geq 1 \quad \forall x \in C.$$

Hence the following application $f\colon C \to K^{(k)}$ is well defined:

$$f(x) = \frac{1}{\sum_{i=0}^{p} \tau_i(x)} (\tau_0(x),\dots,\tau_p(x)).$$

Next proposition summarizes the properties of f.

Proposition 87. *$f\colon C \to K^{(k)}$ is a Lipschitz map such that:*

1. *For every vertex $v \in K^{(0)}$, there exists $i(v)$ such that $f^{-1}(Star(v)) \subset \bigcup_{V_j \cap V_{i(v)} \neq \emptyset} V_j$ is virtually abelian in $\mathcal{O}_0$.*
2. *If $x \in C$ belongs to only one open set of the covering, then $f(x) \in K^{(0)}$. In particular $f(W_0) \subset K^{(0)}$.*
3. *For all x and $y \in \bigcup_{V_0 \cap V_i \neq \emptyset} V_j$, $||f(x) - f(y)|| \leq \frac{\xi}{r_i} d(x,y)$.*

Proof. Properties (1) and (2) follow from the construction and the properties of the balls B_i.

By definition the function τ_i are $\frac{4}{r_i}$-Lipschitz. Let $x \in V_i$. Then at most $N+1$ functions τ_j are non-zero in x, and all of them have Lipschitz constant $\leq \frac{4}{3} \cdot \frac{4}{r_i}$. Now property (3) follows since the functions

$$(x_0,\dots,x_N) \mapsto \frac{x_k}{\sum_{i=0}^{N} x_i}$$

are Lipschitz on $\{x \in \mathbb{R}^{N+1} \mid x_i \geq 0 \; \forall i \wedge \sum_{i=0}^{N} x_i \geq 1\}$.

More details of the proof can be found in [BoP, 5.3]. $\square$

We now homotope f into the 3-skeleton $K^{(3)}$ by an inductive procedure while controlling the local Lipschitz constant.

Lemma 88. *For $d \geq 4$ and $\xi > 0$, there exists $\xi'(d, \xi) > 0$ such that the following is true: a continuous map $g : C \to K^{(d)}$ which verifies properties (1), (2) and (3) of proposition 87 can be homotoped to a map $\tilde{g} \colon C \to K^{(d-1)}$ with the same properties, but with the Lipschitz constant ξ being replaced by ξ'.*

Proof. It suffices to find a constant $\theta(d, \xi) > 0$ such that every d-dimensional simplex $\sigma \subset K$ contains a point z at distance $\geq \theta$ from both $\partial\sigma$ and the image $g(C)$. To deform g into the $(d-1)$-skeleton we compose it on σ with the central projection from z. This will increase the Lipschitz constant by a factor bounded in terms of d, and it reduces the inverse images of open stars of vertices.

If θ does not satisfy the desired property for some d-simplex σ, then $g(C) \cap int(\sigma)$ must contain a subset of at least $\lambda_1(d) \cdot \frac{1}{\theta^d}$ points with pairwise distances $\geq \theta$. Let $A \subset C$ be a set of inverse images, one for each point. Let v_{V_k} be a vertex of σ. Then $A \subset V_k \subseteq B_k$. Since f is $\frac{\xi}{r_k}$-Lipschitz on V_k, the points of A are separated by distance $\frac{1}{\xi} r_k \theta$. Since $r_k \leq 1$, volume comparison (using Bishop-Gromov inequality) implies that A contains at most $\lambda_2 \cdot (\frac{\xi}{\theta})^3$ points. The inequality $\lambda_1(d) \cdot \frac{1}{\theta^d} \leq \lambda_2 \cdot (\frac{\xi}{\theta})^3$ yields a positive lower bound $\theta_0(d, \xi)$ for θ. Hence any constant $\theta < \theta_0$ has the desired property. □

Now we can further homotope f into the 2-skeleton.

Proposition 89. *For suitable constants $\varepsilon > 0$ and $D > 1$, the map $f \colon C \to K^{(k)}$ can be homotoped to a map $\tilde{f} : C \to K^{(2)}$ that satisfies properties (1) and (2) of proposition 87.*

Proof. Using Lemma 88 repeatedly, we can homotope f to a map $\hat{f} : C \to K^{(3)}$ which satisfies properties (1), (2) and (3) of proposition 87. Then it suffices to show that no 3-simplex $\sigma \subset K^3$ is contained in the image of $\hat{f}$ and therefore $\hat{f}$ can be deformed into the 2-skeleton by a retraction in each 3-simplex σ.

To show that the image $f(C)$ misses a point from every 3-simplex σ of K^3 we show the following estimate:

Lemma 90. *For sufficiently small $\varepsilon > 0$ there exists a constant $b = b(\varepsilon) > 0$ such that for all i:*

$$\mathrm{vol}(V_i) \leq b \frac{1}{D} r_i^3.$$

Proof. We first show that W_0 does not enter to far into the other sets U_{x_i}.

Claim 91. *There exists a constant $c(\varepsilon) > 0$ such that, for $D > 1$ is sufficiently large, $d(x_i, W_0) \geq c\,\nu_{x_i}$ for all $i \neq 0$. In particular $r_i \geq c\,\nu_{x_i}$ for all i.*

Proof. Using Lemma 81, we obtain:

$$d(x_i, W_0) \geq d(x_i, x_0) - \mathrm{Diam}(W_0 \cup \{x_0\}) \geq$$
$$\geq \frac{1}{4} r_0 - \frac{1}{9} r_0 > \frac{1}{8} r_0 \geq \frac{1}{64(1+\varepsilon)}\,\nu_{x_0}\,.$$

For the last estimate, we use that $vab(x_0) \geq \frac{1}{1+\varepsilon}\,\nu_{x_0}$ and, since $\nu_{x_0} \leq 1$, $r_0 = \inf\left(\frac{va(x_0)}{8}, 1\right) \geq \frac{1}{8(1+\varepsilon)}\,\nu_{x_0}$.

We now assume that W_0 intersects U_{x_i} because otherwise there is nothing to show. If $W_0 \subset U_{x_i}$ then, according to our choice of W_0, $\nu_{x_0} \geq \frac{1}{1+\varepsilon}\,\nu_{x_i}$, and the assertion holds with $c < (8(1+\varepsilon))^{-2}$.

We are left with the case that $W_0 \not\subset U_{x_i}$ but intersects the ball of radius, say, $\frac{\nu_{x_i}}{4}$ around x_i. Then we can bound the ratio $\frac{\mathrm{Diam}(W_0)}{\nu_{x_i}}$ from below by:

$$(1+\varepsilon)\frac{\nu_{x_i}}{D} + \frac{\nu_{x_i}}{4} + \mathrm{Diam}(W_0) \geq \frac{\nu_{x_i}}{1+\varepsilon}\,.$$

By definition of W_0 we have $\mathrm{Diam}(W_0) \leq (1+\varepsilon)\frac{2}{D}\nu_{x_0}$ Combining these estimates, we obtain a lower bound for $\frac{d(x_i, W_0)}{\nu_{x_i}}$, as claimed. $\square$

Thus, by Bishop-Gromov inequality

$$\mathrm{vol}(V_i) \leq \mathrm{vol}\,B(x_i, r_i) \leq \mathrm{vol}\,B(x_i, c\,\nu_{x_i})\frac{\mathrm{vol}_{-1}(r_i)}{\mathrm{vol}_{-1}(c\,\nu_{x_i})}\,.$$

Since $\mathrm{vol}_{-1}(t) = \pi(\sinh(t) - t)$, it follows that there is a constant $c_1 > 0$ such that:

$$\mathrm{vol}\,B(x_i, r_i) \leq \mathrm{vol}\,B(x_i, c\,\nu_{x_i}) c_1 \frac{r_i^3}{(\nu_{x_i})^3}\,.$$

On the other hand, since $d(f_{x_i}(x_i), S_i) \leq \nu_i/D$, if we choose $c < \frac{1}{3D}$, we get that: $f_{x_i}\big(B(x_i, c\,\nu_{x_i})\big) \subset \mathcal{N}_{\nu_i}(S_i)$.

This inclusion together with the previous inequality gives:

$$\mathrm{vol}\,B(x_i, c\,\nu_{x_i}) \leq (1+\varepsilon)^3\,\mathrm{vol}\,\mathcal{N}_{\nu_{x_i}}(S_i) \leq 2^3\,\mathrm{vol}\,\mathcal{N}_{\nu_i}(S_i) \leq 2^3\left(\frac{2\pi}{D}\right)\nu_{x_i}^3$$

Therefore: $\mathrm{vol}(V_i) \leq \mathrm{vol}(B_i) \leq \frac{b}{D} r_i^3$, with $b = 2^3 2\pi c_1$. $\square$

We finish now the proof of proposition 89.

Since

$$\mathrm{vol}(\sigma \cap \hat{f}(C)) \leq \sum_{V_j \cap V_{i(v)} \neq \emptyset} \mathrm{vol}(f(V_j))\,.$$

By the Lipschitz property 3) of $\hat{f}$:

$$\mathrm{vol}(\sigma \cap \hat{f}(C)) \leq \sum_{V_j \cap V_{i(v)} \neq \emptyset} \left(\frac{\xi}{r_i(v)}\right)^3 \mathrm{vol}(f(V_j))\,.$$

Since $r_j/r_{i(v)} \leq \frac{4}{3}$, using lemma 90 we obtain:

$$\mathrm{vol}(\sigma \cap \hat{f}(C)) \leq \sum_{V_j \cap V_{i(v)} \neq \emptyset} \left(\frac{4}{3}\,\frac{\xi}{r_j}\right)^3 \mathrm{vol}(V_j) \leq (N+1)\left(\frac{4}{3}\,\xi\right)^3 \frac{b}{D}\,.$$

Hence:

$$\mathrm{vol}(\sigma \cap \hat{f}(C)) \leq \frac{a}{D},$$

for some uniform constant $a = (N+1)\left(\frac{4}{3}\,\xi\right)^3 b > 0$. By choosing $D > 1$ sufficiently large we get

$$\mathrm{vol}(\sigma \cap \hat{f}(C)) < \mathrm{vol}(\Delta^3)\,.$$

□

Remark 92. *If we forget about the singular locus Σ and consider orbifolds $\mathcal{O}$ such that $||\,|\mathcal{O}|\,|| \neq 0$, then the proof shows that there is no collapse and therefore $\mathcal{O}$ is hyperbolic.*

8.5. **Vanishing of simplicial volume.** The orbifold $\mathcal{O}$ has a finite, regular covering which is a manifold M. There is an induced finite, regular covering $p\colon M_0 \to \mathcal{O}_0$, where M_0 is an irreducible manifold, by the equivariant sphere Lemma, and whose boundary ∂M is an union of tori. Therefore M_0 is Haken and has a JSJ-splitting into Seifert fibered and hyperbolic submanifolds.

Proposition 93. *All components in the JSJ-splitting of M_0 are Seifert fibered.*

Proof. We may assume that the boundary of M_0 is incompressible because otherwise M_0 is a solid torus and the assertion holds. We construct a closed manifold $\overline{M_0}$ by Dehn filling on M_0 as follows. Let $Y \subset M_0$ be a component of the JSJ-splitting which meets the boundary, $Y \cap \partial M_0 \neq \emptyset$. When Y is hyperbolic we choose, using the Hyperbolic Dehn Filling Theorem, the Dehn fillings at the tori of $Y \cap \partial M_0$ in such a way that the resulting manifold $\overline{Y}$ remains hyperbolic. When Y is Seifert fibered, we fill so that $\overline{Y}$ is Seifert fibered and the components of $\partial Y - \partial M_0$ remain incompressible (i.e. the surgery slope meets the fiber in at least two points). This can be done because the base of the Seifert fibration of Y is neither an annulus nor a disk with zero or one cone point. The manifold $\overline{M_0}$ has a JSJ-splitting along the same tori as M and with the same number of hyperbolic (and also Seifert fibered) components.

It suffices to show that $\overline{M_0}$ has zero simplicial volume, because then Corollary 72 imply that $\overline{M_0}$ contains no hyperbolic component in its JSJ-splitting. To this purpose we will apply Gromov's vanishing theorem (Thm. 73).

We compose the map of Proposition 84 $\tilde{f} : \mathcal{O} \to K^{(2)}$ with the projection p and extend the resulting map $M_0 \to K^{(2)}$ to a map $h : \overline{M_0} \to K^{(2)}$ by mapping the filling solid tori to the vertex v_{V_0}. Notice that h is continuous because $\tilde{f}(\partial \mathcal{O}_0) = \{v_{V_0}\}$. The inverse images under h of open stars of vertices are virtually abelian as subsets of $\overline{M_0}$. These subsets yield an open covering of $\overline{M_0}$ with covering dimension ≤ 2. By Gromov's theorem, the simplicial volume of $\overline{M_0}$ vanishes. □

Conclusion of the proof of Theorem 57.

Since M_0 is a graph manifold and M results from M_0 by gluing in a Seifert manifold, M is itself a graph manifold (cf. [Wa1]). Since $p\colon M \to \mathcal{O}$ is a regular covering, it follows from Meeks and Scott work [MS] that there is a graph structure on M which is invariant by the group of covering translations. Hence $\mathcal{O}$ is a graph orbifold. As $\mathcal{O}$ is atoroidal, it must be Seifert fibered. □

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