Introduction to Lie groups and algebras

Definitions, examples and problems.

Fall 2006

1 Lie groups

1.1 Lie groups, direct products

Notation: by a manifold and submanifold we will always mean C^{∞} -smooth manifolds and submanifolds. We use the following definition of a submanifold. A subset $S \subset M$ of a manifold M is a submanifold if for every point $x \in S$ (not $x \in M$!) there exists a chart U of x in M diffeomorphic to an affine space V such that $U \cap S$ is diffeomorphic to a subspace of V.

Definition 1. A real Lie group is a real manifold G together with a group structure such that the multiplication map

$$G \times G \to G$$
; $(x,y) \mapsto xy$

and the inversion map

$$G \to G; \quad x \to x^{-1}$$

are smooth. A *complex Lie group* is defined analogously: G must be a complex manifold and the multiplication map and the inversion map must be complex differentiable. In the same way, one can define a Lie group over any field K as long as the notion of a manifold over K makes sense.

Note that any complex Lie group of dimension n can be regarded as a real Lie group of dimension 2n.

Examples:

- 1. Any real or complex vector space. The group operation is addition of vectors.
- 2. The multiplicative groups $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
- 3. The circle $\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$ is a real Lie group.
- 4. Any finite or countable group with discrete topology regarded as a 0-dimensional manifold is a Lie group.

- 5. Less trivial examples include linear groups familiar to us from linear algebra, e.g. general linear groups $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$, that are the groups of invertible linear operators in \mathbb{R}^n and \mathbb{C}^n , respectively. A Lie group is called linear if it is a subgroup of some general linear group. Classical examples include:
 - special linear group SL_n = all operators preserving a nondegenerate alternating n-form,
 - orthogonal group O_n = all operators preserving a nondegenerate symmetric 2-form, which in the real case must also be positive definite,
 - symplectic group Sp_{2n} = all operators preserving a nondegenerate skew-symmetric 2-form (which exists only for even-dimensional vector spaces),
 - unitary group $U_n(\mathbb{C})$ = all operators in $GL_n(\mathbb{C})$ preserving a non-degenerate positive definite hermitian form.
 - pseudo orthogonal group $O_{p,q}(\mathbb{R})$ = all operators preserving a nondegenerate symmetric 2-form of signature (p,q) on a (p+q)-dimensional vector space

It is not evident why all these groups are manifolds. We will prove this later. Other classical groups are special orthogonal group $SO_n = O_n \cap SL_n$, special pseudo orthogonal group $SO_{p,q}(\mathbb{R}) = O_{p,q} \cap SL_{p+q}(\mathbb{R})$ and special unitary group $SU_n(\mathbb{C}) = U_n(\mathbb{C}) \cap SL_n(\mathbb{C})$.

One can get new examples of Lie groups by taking *direct products*.

Definition 2. Let G_1 and G_2 be Lie groups. Their direct product $G_1 \times G_2$ is Cartesian product of the manifolds G_1 and G_2 together with group structure of direct product.

Examples:

- 1. A real torus $\mathbb{T}^n = \underbrace{\mathbb{T}^1 \times \ldots \times \mathbb{T}^1}_n$ is a direct product of circles.
- 2. A complex torus $(\mathbb{C}^*)^n$ is a direct product of multiplicative groups \mathbb{C}^* .

Exercises:

- 1. Show that the group G of non-degenerate upper-triangular $n \times n$ matrices over a field K (the field K here is either \mathbb{R} or \mathbb{C}) is a Lie group and find its dimension. Show then that G is diffeomorphic as a manifold but not isomorphic as a group to the direct product of several copies of K^* and K.
- 2. Show that if the multiplication map in Definition 1 is smooth then so is the inversion map (i.e. the second condition in the definition of a Lie group can be dropped).
- 3. Show that the direct product of Lie groups is a Lie group.
- 4. Show that SL_2 , SO_2 and $SU_2(\mathbb{C})$ are Lie groups and find their dimensions.

- 5. Show that all operators from the symplectic group Sp_{2n} have determinant one.
- 6. Which of the classical groups are compact? Connected?
- 7. Show that the tangent bundle of any Lie group is trivial (i.e. it is diffeomorphic to the direct product of the Lie group and a vector space of the same dimension).
- 8. Which of the following manifolds can be endowed with the structure of a Lie group?
 - (a) S^1
- (b) S^2
- (c*) S^3
- $(d^*) \mathbb{RP}^3$
- (e) \mathbb{T}^2
- (f) Klein bottle

- 9. Show that
 - (a) $SO_2(\mathbb{R})$ is isomorphic to \mathbb{T}^1
 - (b) $SO_{1,1}(\mathbb{R})$ is isomorphic to \mathbb{R}^*
 - (c) $SU_2(\mathbb{C})$ is diffeomorphic to S^3
 - (d) $SO_3(\mathbb{R})$ is diffeomorphic to \mathbb{RP}^3

1.2 Subgroups, homomorphisms

Definition 3. A subgroup H of a Lie group G is called a Lie subgroup of G if $H \subset G$ is a submanifold.

Note that not every subgroup of a Lie group is a Lie subgroup. E.g. the cyclic subgroup $H = \langle e^{2\pi i\sqrt{2}} \rangle \subset \mathbb{T}^1$ is everywhere dense in \mathbb{T}^1 and, hence, is not a submanifold.

Examples:

- 1. A subspace of a vector space is a Lie subgroup.
- 2. All n-th roots of unity in \mathbb{C} for a given n form a Lie subgroup of the circle \mathbb{T}^1 .
- 3. The real torus \mathbb{T}^n is a Lie subgroup of the complex torus $(\mathbb{C}^*)^n$ (regarded as a real Lie group) since \mathbb{T}^1 is a subgroup of \mathbb{C}^* .
- 4. The group of non-degenerate upper-triangular square matrices, the group of non-degenerate diagonal square matrices and all classical groups are Lie subgroups of general linear groups.

Definition 4. Let G_1 and G_2 be Lie groups. A map $G_1 \to G_2$ is called a *Lie group homomorphism* if it is a group homomorphism and is also smooth.

Note that the image of a Lie group under a Lie group homomorphism is not always a Lie subgroup. E.g. the map $h: \mathbb{R} \to \mathbb{T}^2$; $h: x \to (e^{2\pi i x}, e^{2\pi i \sqrt{2}x})$ is a Lie group homomorphism but $h(\mathbb{R}) \subset \mathbb{T}^2$ (an irrational winding of the torus) is everywhere dense in \mathbb{T}^2 and, hence, is not a Lie subgroup.

Examples:

- 1. An exponential homomorphism: $\exp : \mathbb{R} \to \mathbb{R}^*$; $\exp : x \mapsto e^x$.
- 2. Another exponential homomorphism: $\exp : \mathbb{R} \to \mathbb{T}; \exp : x \mapsto e^{ix}$.
- 3. Determinant: det : $GL_n(\mathbb{R}) \to \mathbb{R}^*$; det : $A \mapsto \det(A)$.
- 4. For any Lie group G and any element $g \in G$ there is an inner automorphism: $a(g) : G \to G$; $a(g) : x \mapsto gxg^{-1}$.
- 5. A Lie group homomorphism $G \to GL_n$ is called an *n*-dimensional linear representation of a Lie group G.

Exercises:

1. For each pair of real numbers α and β define the subgroup $H \subset \mathbb{T}^2$ as follows:

$$H = \{ (e^{i\alpha x}, e^{i\beta x}), x \in \mathbb{R} \}.$$

Under what conditions on α and β is H a Lie subgroup?

- 2. Show that every Lie subgroup is a Lie group.
- 3. Let H be a subgroup of a Lie group G. Show that if there exists a neighborhood $U_e \subset G$ of the identity element $e \in G$ such that $H \cap U_e$ is a submanifold of G, then H is a Lie subgroup of G.
- 4. Prove that a Lie subgroup is a closed submanifold (note that in Definition 3 we do not require that a Lie subgroup be a closed submanifold). See also Problem 11 in the end of this section.
- 5. For any Lie group G denote by G^0 the connected component of the identity element. Show that G^0 is a normal Lie subgroup of G.
- 6. Find the differential d_e det at the identity of the determinant homomorphism det : $GL_n(\mathbb{R}) \to \mathbb{R}^*$.
- 7. Show that the kernel of a Lie group homomorphism is a Lie subgroup.

1.3 Actions

Definition 5. Let G be a Lie group, and M a manifold. A homomorphism α from G to the group of diffeomorphisms of M is called an *action* of G on M if the map

$$G \times M \to M; \quad (g, x) \to \alpha(g)x$$

is smooth. The orbit Gx of a point $x \in M$ is the image of G under the map

$$\alpha_x: q \to \alpha(q)x$$
.

The stabilizer G_x of x is the preimage of x under the map α_x , i.e.

$$G_x = \{ g \in G : \alpha(g)x = x \}$$

Examples:

1. There are three important types of actions of a Lie group G on itself:

• Left action: $l(g): x \to gx$;

• Right action: $r(g): x \to xg^{-1}$;

• Adjoint action: $a(g): x \to gxg^{-1}$.

2. Any linear representation of a Lie group G on a vector space V provides an action of G on V.

3. The group SO_n acts on the unit sphere in \mathbb{R}^n .

Exercises:

- 1. Find all orbits and stabilizers of the adjoint action of $GL_2(\mathbb{C})$.
- 2. Show that for any point $x \in M$ the map α_x from Definition 5 is smooth of constant rank. Show also that if the rank of α_x is k, then
 - (a) The stabilizer G_x is a Lie subgroup of codimension k in G. The tangent space T_eG_x of G_x at the identity is the kernel of the differential $d_e\alpha_x: T_eG \to T_x(Gx)$.
 - (b) For some sufficiently small neighborhood $U_e \subset G$ of the identity element $e \in G$ the set $\alpha(U_e)x$ is a submanifold of dimension k in M.
 - (c) If the orbit Gx is a submanifold of M, then its dimension is k.

In part (c), could it be that the orbit is not a submanifold?

- 3. Let a compact Lie group act on a manifold M. Show that all orbits are closed submanifolds of M.
- 4. Prove that SL_n is a Lie subgroup of GL_n and find its dimension. Describe explicitly the tangent space T_eSL_n (this is some vector subspace in the space of all linear operators).
- 5. Do the same for O_n . (Hint: consider the action of GL_n on the space of positive definite symmetric bilinear forms.)
- 6. Do the same for the other classical groups.

1.4 Exponential map and one-parameter subgroups

The case of linear Lie groups. Denote by $\mathfrak{g}l_n = T_eGL_n$ the space of all linear operators on an *n*-dimensional vector space.

Definition 6. Define the exponential map $\exp : \mathfrak{gl}_n \to GL_n$ by the formula $\exp(A) = e^A$, where e^A is the matrix exponent:

$$e^A = I + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots$$

Exercises:

- 1. (a) Show that the power series defining the matrix exponent e^A converges for every operator $A \in \mathfrak{g}l_n$.
 - (b) Show that if AB = BA, then $e^A e^B = e^{A+B}$.
 - (c) Show that $det(e^A) = e^{trace(A)}$
- 2. Show that the exponential map is a diffeomorphism of some neighborhood of 0 in $\mathfrak{g}l_n$ to some neighborhood of the identity element e in GL_n .
- 3. Show that the exponential map for $GL_n(\mathbb{C})$ is surjective. Is it injective?
- 4. Show that the exponential map for $SL_2(\mathbb{R})$ is not surjective.
- 5. Verify that for each classical group G the image of the tangent space T_eG under the exponential map lies in G. Show also that the exponential map provides a diffeomorphism between some neighborhood of 0 in T_eG and some neighborhood of e in G.
- 6. Verify that for any $A \in \mathfrak{g}l_n$ the map

$$\varphi: \mathbb{R} \to GL_n; \quad \varphi: t \to e^{At}$$

satisfies the differential equation

$$\frac{d\varphi(t)}{dt} = A\varphi(t); \quad \varphi(0) = e, \frac{d\varphi(t)}{dt}\Big|_{t=0} = A.$$

The product $A\varphi(t)$ here is the composition of linear operators A and $\varphi(t)$.

General case

Definition 7. Let G be a Lie group. A Lie group homomorphism $\varphi : \mathbb{R} \to G$ is called a one-parameter subgroup.

Examples:

- 1. $G = \mathbb{R}^*$; $\varphi : t \to e^t$;
- 2. $G = \mathbb{C}^*$; $\varphi : t \to e^{at}$, where $a \in \mathbb{C}$ is any complex number;
- 3. $G = GL_n$; $\varphi : t \to e^{At}$,
- 4. G is any Lie group; $\varphi_v(t)$ is the solution of the differential equation

$$\frac{d\varphi(t)}{dt} = \varphi(t)v; \quad \varphi(0) = e, \frac{d\varphi(t)}{dt}\Big|_{t=0} = v,$$

where v is any vector in the tangent space T_eG and vg is a shorthand notation for the image of v in T_qG under the differential $d_er(g): T_eG \to T_qG$.

Exercise:

1. Show that the one-parameter subgroup $\varphi_v(t)$ in the last example is well-defined (i.e. the solution of the differential equation exists for all $t \in \mathbb{R}$ and provides a homomorphism $\mathbb{R} \to G$).

Definition 8. Let G be a Lie group. Denote by \mathfrak{g} the tangent space T_eG . Define the exponential map $\exp : \mathfrak{g} \to G$ as follows:

$$\exp(v) = \varphi_v(1).$$

Exercises:

- 1. Show that for the classical groups this definition is equivalent to Definition 6.
- 2. Show that the exponential map is a diffeomorphism of some neighborhood of 0 in \mathfrak{g} to some neighborhood of the identity element e in G.

Problems:

- 1. Classify all connected real Lie groups of dimension 1.
- 2. Classify all connected complex Lie groups of dimension 1.
- 3. Show that a compact Lie group of positive dimension has Euler characteristic zero.
- 4. Find all compact real Lie groups of dimension 2.
- 5. Denote by \widetilde{G} the universal cover of a Lie group G. Show that \widetilde{G} can be endowed with the structure of a Lie group.
- 6. A discrete normal subgroup of a connected Lie group G lies in the center of G.
- 7. The fundamental group of a connected Lie group is Abelian.

- 8. Show that if G is a connected commutative Lie group then the exponential map $\exp: \mathfrak{g} \to G$ is a Lie group homomorphism (the group operation on the vector space \mathfrak{g} is vector addition). Use this to classify all commutative connected Lie groups.
- 9. Show that any connected compact complex Lie group is commutative.
- 10. Is it true that the intersection of two submanifolds is also a submanifold? Prove that the intersection of Lie subgroups is always a Lie subgroup.
- 11. (very difficult!) Prove that if a subgroup of a Lie group is closed then it is a Lie subgroup.