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A GENERALIZED PARIS' LAW FOR FATIGUE CRACK GROWTH

M. Ciavarella[a], N. Pugno [b]*

[a] Viale Japigia, 182, Politecnico di Bari, Bari, ITALY)
[b] C.so Duca degli Abruzzi, 24, Politecnico di Torino, Torino, ITALY

Abstract

An extension of the celebrated Paris' law for crack propagation is given to take into account of deviations from Linear Elastic Fracture Mechanics (LEFM) in a simple manner using the Woehler SN curve of the material, suggesting they are both special cases of a more general "unified law". In particular, using recent proposals by the second author, the stress intensity factor $K(a)$ is replaced with a suitable mean over a micro structural parameter length scale Δa, the "fracture quantum". The latter is not a material constant but rather an indication of a process zone size. In practice, for a Griffith crack, this is seen to correspond to increasing the effective crack length of Δa. Contrary to other attempts to extend Paris' law to include short cracks, we suggest a dependence of this fracture "quantum" on the applied stress range level such that the correct convergence towards the Wöhler-like regime is obtained. Hence, the final law includes both Wöhler's and Paris' material constants, and can be seen as either a generalized Wöhler's SN curve law in the presence of a crack, or as a generalized Paris' law for cracks of any size. The main aim is not explaining quantitatively the behavior of short cracks (for which detailed modeling of local processes is probably required), but rather to provide a consistent unified treatment of the fatigue damage process.

Key words: Fatigue, Notches, Cracks, Initiation, Fatigue life

1. INTRODUCTION

Fatigue life prediction is still very much an empirical art rather than a science. In the specific case of fatigue crack propagation, after the pioneering work of Paris and Erdogan [1-2], there is large use of phenomenological laws relating the crack growth rate, da/dN, to the amplitude of the applied stress intensity factor, ΔK. However, it is often forgotten that Paris's law describes the experimental data under constant amplitude loading and, importantly, under 'small-scale yielding' which in turn requires either sufficiently low loads, and/or "long cracks" [3]. These stringent requirements are not always well specified and are also confused: "short crack" in particular is nearly always related to the fatigue limit and fatigue threshold (see the definition of intrinsic crack from El Haddad et al. [4]) whereas, as we shall see later on, the definition of "short" should also depend on the loads level. A number of modifications of Paris's law have been suggested to deal with various departure from the ideal conditions: threshold limits, closure, short cracks [4,5], among others. The case of short cracks is one of the most well-known since Paris's law can significantly underestimate their rate of growth. Such large number of ad hoc fatigue laws implies that the "physics" of fatigue-crack growth is not completely well captured by stress-intensity factors-based theories, mainly because of failure of the

* Corresponding author: *Tel.*: ; *Fax.*: ; *E-mail:*

‘small-scale yielding’ assumption, and the involvement of other microstructural or grain-scale parameters. However, this should not come with too much surprise, given until fracture mechanics emerged, fatigue was dealt with older, also empirical, fatigue laws, such as Wöhler’s SN curves approach.

Initiation and propagation of cracks are well distinct phenomena, and depend differently on material, geometry and load levels [5,6]. For nominally plain specimen, at low load levels, where we expect fatigue failure at high cycle numbers (High Cycle Fatigue, HCF), practically the whole life is expended in enucleating the crack, rather than propagating: indeed, the latter phase only takes the final few cycles. At high load levels (those giving Low Number of Cycles, LCF), cyclic plastic deformation takes place rapidly leading to failure. These various processes result in the well-know empirical Wöhler curve (or, more in detail, in the Basquin-Coffin-Manson’s law – but we shall neglect for simplicity this aspect). There is no known fundamental reason to write the curve as a power-law, and indeed alternative equations have been suggested, but the power law between two given points is probably the simplest or most used form for the plain specimen namely:

$$N_0 \Delta\sigma_R{}^k = N_\infty \Delta\sigma_0{}^k = N_f \Delta\sigma^k = \overline{C}\,; \quad N_0 < N_f < N_\infty \tag{1}$$

where $\Delta\sigma_R$ is the range of stress at static failure (i.e. twice the ultimate strength), $\Delta\sigma_0$ the fatigue limit and $\Delta\sigma$ is the stress range for having a life N_f; N_0, and N_∞ are the number of cycles at beginning and end of validity of the law. Clearly, eq. (1) also implies:

$$kLogF_R = Log\frac{N_\infty}{N_0} \tag{2}$$

where $F_R = \dfrac{\sigma_R}{\Delta\sigma_0}$; typically $N_\infty=10^7$ and $N_0=10^3$, and for steels considering $F_R=2$ we would have $k=13.3$, while for $F_R=3$, $k=8.4$, in the typical range k=6-14 for Al or ferrous alloys. Turning to the case of cracked specimen, LEFM applies, and fatigue life (often denominated “residual”) is all given by propagation, generally by the celebrated Paris’s law [1,2]. Paris’ law gives the advancement da of fatigue crack per unit cycle dN, v_a, as a function of the amplitude of stress intensity factor ΔK

$$v_a = \frac{\mathrm{d}a}{\mathrm{d}N} = C\Delta K^m\,; \qquad \Delta K_{th}<\Delta K<K_{Ic} \tag{3}$$

where ΔK_{th} is the “fatigue threshold”, and K_{Ic} the “fracture toughness” of the material; C and m are the so-called Paris’ constants. There is therefore no dependence on absolute dimension of the crack. The law is mostly valid in the range 10^{-5}—10^{-3} mm/cycle, intersecting ΔK_{th} and K_{Ic} at $v_a{}^{th}=10^{-6}$ mm/cycle, *and* $v_a{}^c = 10^{-4}$ mm/cycle, respectively, where $v_a{}^{th}$ is a conventional velocity at the threshold, and $v_a{}^c$ at the critical conditions. This means that the constant C is not really arbitrary, since by writing the condition at the intersections, $C = v_a{}^{th} / \Delta K_{th}{}^m = v_a{}^c / K_{Ic}{}^m$. In practice, these limits are not as precise and only standards can help defining reference values. From the linearity in this range 10^{-5}—10^{-3} *mm/cycle* in the log/log plot, Fleck et al. [6] suggest to find the Paris exponent m as:

$$LogF_K = \frac{4}{m}\,; \tag{4}$$

where $F_K = \frac{K_{Ic}}{\Delta K_{th}}$, and their paper (see specifically Fig. 16) seems to confirm this assumption for the exponent m. An obvious link between the two curves (Wöhler and Paris) is obtained when considering the life of a distinctly cracked specimen having an initial crack size a_i. Under the assumptions of constant remote stress and no geometrical effects, for $m>2$ the following is obtained (where the dependence on the final size of the crack a_f has been removed as relatively not influent):

$$a_i^{\frac{2-m}{2}} = \left(\frac{m-2}{2}\right) C \pi^{m/2} \Delta\sigma^m N_f ; \tag{5}$$

This is to be considered as a Wöhler curve of the cracked component and the Wöhler equivalent exponent in these conditions, *k*', to distinguish it from the "material constant" base value *k*, turns out to be exactly equal to the Paris exponent, *k'=m***.** It is interesting however to remark that the SN fatigue curve depends on the initial crack size, a_i.

2. GENERALIZED PARIS' LAW

Quantized Fracture Mechanics (QFM, Pugno and Ruoff [7]), is prone to generalize the Paris' equation, by substituting $K(a)$ with an appropriate mean value, $K^*(a, \Delta a) = \sqrt{\left\langle K^2(a) \right\rangle_a^{a+\Delta a}}$, where Δa is "fracture quantum", a microstructural material constant. In order to consider the effect of this "apparent" additional crack sizes, it is natural in the study of fatigue crack growth to propose the following generalized Paris' law:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\left(\Delta K^*(a, \Delta a, \Delta\sigma)\right)^m ; \tag{6}$$

where in turn $\Delta a(\Delta\sigma)$. By integrating (6), the total number of cycles N_C^{P*} can be found for the fatigue collapse, arising when the crack length has reached its critical final value a_C, can be deduced as:

$$N_C^{P*} = \frac{1}{C} \int_a^{a_C} \frac{\mathrm{d}a}{\left(\Delta K^*(a, \Delta a, \Delta\sigma)\right)^m} ; \tag{7}$$

Accordingly, in the criterion of eq. (7) we can fix Δa to recover, in the limit case of $a \to 0$, the Wöhler's prediction, eq. (1), which we write here as:

$$N_C^W = \frac{\overline{C}}{\Delta\sigma^k} ; \tag{8}$$

Hence, the quantum should be fixed so that :

$$\Delta a : \quad N_C^{P*}(a \to 0) = N_C^W ; \tag{9}$$

Thus, eqs. (6) or (7), with the position of eq. (9), can be consider a generalized Paris law. Note that such a law is of very simple application, and would allow one to study not only the final condition but also the evolution of the fatigue crack growth $N^{P*}(a(N))$, where $a \leq a(N) \leq a_C$.

As a simple example of application, let us consider the Griffith's case (infinite elastic plate with a symmetric crack of length 2*a)*. For this case the stress-intensity factor (mode I) is $K = \sigma\sqrt{\pi a}$ and the full stress field at the crack tip is $\sigma_{yy} = \dfrac{1}{\sqrt{\left(1-\left(a/(a+x)\right)^2\right)}}$ (where x is the distance from the tip). Accordingly, by integration:

$$K^* = \sigma\sqrt{\pi(a+\Delta a/2)}; \tag{10}$$

By applying eq. (7), and integrating between an initial and a final value of crack size (we suppose $a_f=a_c$ and $a_i=a$) it follows:

$$N_C^{P*} = \frac{1}{C\Delta\sigma^m \pi^{m/2}} \frac{\left(a_C + \dfrac{\Delta a}{2}\right)^{1-m/2} - \left(a + \dfrac{\Delta a}{2}\right)^{1-m/2}}{1-m/2}; \tag{11}$$

From eq. (11), Δa can be obtained by solving:

$$\frac{1}{C\Delta\sigma^m \pi^{m/2}} \frac{\left(a_C + \dfrac{\Delta a}{2}\right)^{1-m/2} - \left(\dfrac{\Delta a}{2}\right)^{1-m/2}}{1-m/2} = \frac{\overline{C}}{\Delta\sigma^k}; \tag{12}$$

Assuming $a_C >> \Delta a$, it gives:

$$\Delta a = 2\left(a_C^{1-m/2} - \frac{C\overline{C}\pi^{m/2}(1-m/2)}{\Delta\sigma^{k-m}}\right)^{\frac{1}{1-m/2}}; \tag{13}$$

For m>2 (usual case), $-2<\frac{1}{1-m/2}<0$, one obtains

$$\Delta a = 2\left(\frac{\Delta\sigma^{k-m}}{C\overline{C}\pi^{m/2}(m/2-1)}\right)^{\frac{1}{m/2-1}}; \tag{14}$$

i.e. a power-law of stress range with exponent (k-m)/(m/2-1). For example, for typical values for a metal m=4, k=12, then (12-4)/(2-1)= 8. Hence, $\Delta a(\sigma)$ increases remarkably with the stress range similarly to what found from simpler independent reasoning (Ciavarella [8,9]).

By assembling eqs. (11) and (14), one obtains:

$$N_C^{P*} \approx \frac{1}{C\Delta\sigma^m \pi^{m/2}} \frac{\left(a + \left(\dfrac{\Delta\sigma^{k-m}}{C\overline{C}\pi^{m/2}(m/2-1)}\right)^{\frac{1}{m/2-1}}\right)^{1-m/2}}{m/2-1}; \tag{15}$$

which is the proposed integrated Paris law. Notice that obviously it is not a power-law type, but a more complicated law depending on initial size of the crack and stress range. An obvious limit case is for very large crack, for which one re-obtains the original Paris law. Hence, the notion of “large crack” can be made quantitative, indicating that:

$$a >> \left(\frac{\Delta\sigma^{k-m}}{C\overline{C}\pi^{m/2}\left(m/2-1\right)} \right)^{\frac{1}{m/2-1}}; \tag{16}$$

Conversely, “short crack” for “<<”and returning to the typical cases of metals $2<m<k$, then short crack is obtained if either the numerator (stress range) is large *or* the denominator is small. Viceversa, the usual definition of short crack is (as discussed earlier) limited to the absolute size, probably because the stress range at which this is measured is not too far from the fatigue limit (or just above it).

Eq. (15) is a “non-power-law” which we can consider as an “asymptotic matching” between the two due power-law regimes (Wöhler and Paris) at the extremes. In fact, Eq. (15) in the other limit of small *a* becomes:

$$N_C^{P*} \approx \frac{1}{C\Delta\sigma^m \pi^{m/2}} \frac{\left(\dfrac{\Delta\sigma^{k-m}}{C\overline{C}\pi^{m/2}\left(m/2-1\right)} \right)^{-1}}{m/2-1} = N_C^W; \tag{17}$$

which is the original Wöhler law. Notice that since we impose to obtain the Wöhler law for negligible crack, we are not sure what is the reduction of the fatigue limit for the a_0 crack size (this will be discussed further in the example cases paragraph).

This result can also be interpreted in terms of the generalized Paris law, which reads for a Griffith crack:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\left(\Delta\sigma \sqrt{\pi\left(a + \left(\frac{\Delta\sigma^{k-m}}{C\overline{C}\pi^{m/2}\left(m/2-1\right)} \right)^{\frac{1}{m/2-1}} \right)} \right)^m; \tag{18}$$

For short cracks ($a \to 0$) one gets:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\pi^{m/2}\left(C\overline{C}\pi^{m/2}\left(m/2-1\right)\right)^{\frac{-m/2}{m/2-1}} \Delta\sigma^{(k-m)\frac{m/2}{m/2-1}+m}; \tag{19}$$

i.e. a Paris’ law power law in terms of $\Delta\sigma$ rather than ΔK.

3. EXAMPLES

To make some illustrative examples, we make use of Tab.1. To start with, we notice that the Wohler curve is more generally defined as a Basquin-Coffin-Manson equation in terms of amplitude of strain (where plastic strains are important for strain controlled experiments). The Basquin (elastic strain) curve in particular reads :

$$\Delta\sigma / 2 = \sigma'_f \left(2N\right)^b ; \qquad (20)$$

where σ'_f is called fatigue strength coefficient, b is fatigue strength exponent, and N the number of cycles for initiation. Table 1 gives various material properties taken from the LIFest database of Somat. To convert equation (20) into the notation of equation (1), just notice that $b=-1/k$ and hence

$$\overline{C} = \left(2\sigma'_f\right)^k / 2 ; \qquad (21)$$

Tabella 1: Mechanical properties of some steels

Material	Stress units	UTS	Yield Strength	Fatigue Strength Coeff	Fatigue Strength Exponent	Crack Growth Coeff	CGC Units / cycle	Crack Growth Exponent	Thresh SIF R=0	Fracture Toughness
1045	MPa	621	382	948	-0,09	8,20E-13	m	3,5	7,1	
A27cast	Ksi	70	40	177	-0,07	1,67E-11	in	3,8		270
EN24	MPa	1277	1166	3956	-0,18	4,45E-11	m	2,45	4,5	72,5
RQT501	MPa	590	472	892	-0,09	1,00E-10	m	1,72	5,35	80
RQT701	MPa	825	735	955	-0,06	1,00E-10	m	1,72	5,35	113

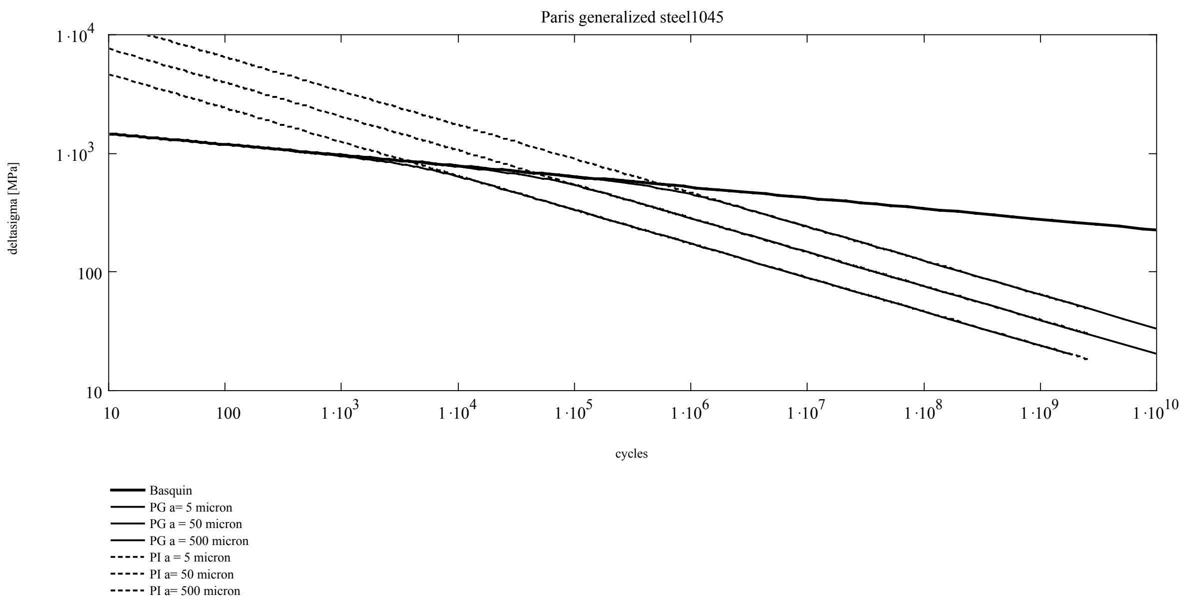

Figure 1: An example of comparison for Basquin curve and Paris integrated (dashed) and generalized (solid) curves for a=5,50,500 microns (1045 steel)

In fig. 1 an example is given of comparisons of the obtained generalized Paris equation (15), with the original integrated Paris law and the Basquin law, equation (1) or (20). Notice that the proposed equation deviates very little from either curves, being very close to the lower of the two curves.

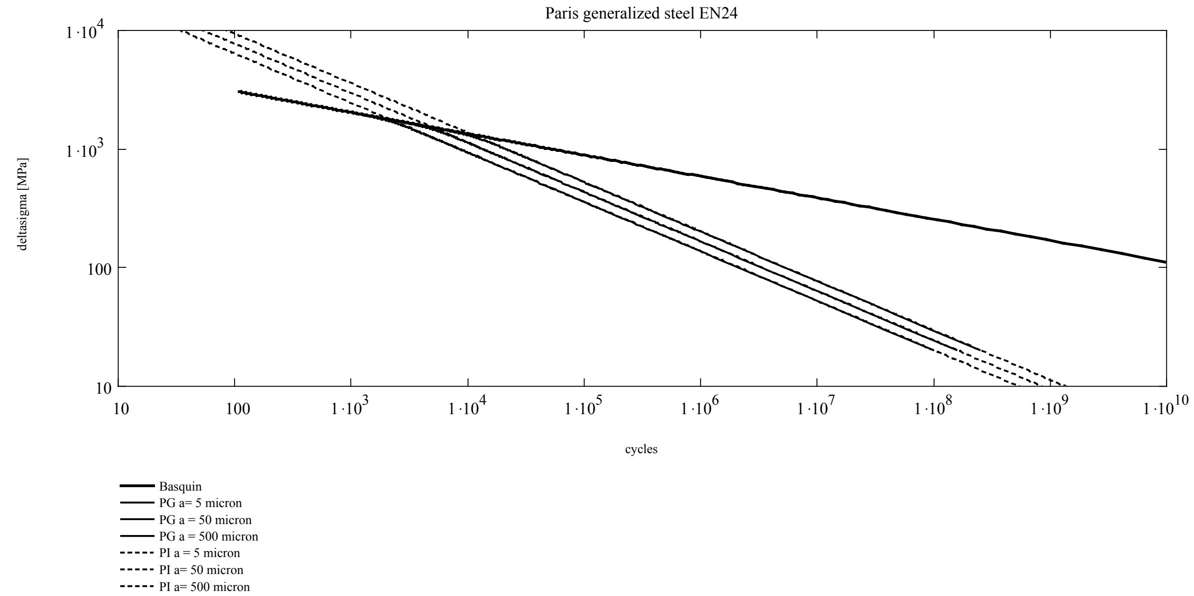

Figure 2: An example of comparison for Basquin curve and Paris integrated (dashed) and generalized (solid) curves for a=5,50,500 microns (EN24 steel)

CONCLUSIONS

We have proposed a new equation generalizing Paris law for fatigue crack propagation by using the ideas of adding a fracture "quantum" to the standard LEFM. This leads to a new law which, upon integration, leads to a general equation for fatigue life. By imposing consistency with the Wöhler's law for the uncracked material, in the limit when the new generalized law is used for short initial cracks, we get the appropriate "quantum crack" size.

In these respects, the proposed model has the advantage of being an "interpolation procedure" between the celebrated Paris and Wöhler's regimes (or perhaps, more elegantly, an "intermediate asymptotics" solution), and hence doesn't have the risk associated to the inevitable "extrapolation" nature of the many other phenomenological but essentially empirical models.

REFERENCES

[1] Paris P., Gomez M. and Anderson W. (1961), A rational analytic theory of fatigue. The trend in engineering 13, 9–14.
[2] Paris P. and Erdogan F. (1963), A critical analysis of crack propagation laws, Journal of Basic Engineering, Transactions of the American Society of Mechanical Engineers, December 1963, 528-534.
[3] Klesnil M. and Lukas P. (1972), Influence of strength and stress history on growth and stabilisation of fatigue cracks. Engineering Fracture Mechanics 4, 77–92.
[4] El Haddad M., Topper T. and Smith K. (1979), Prediction of non-propagating cracks. Engineering Fracture Mechanics 11, 573–584.
[5] Suresh S. (1998), Fatigue of materials, Cambridge University Press, Cambridge.
[6] Fleck N.A., Kang K.J., Asbhy M.F., (1994), Overview 112: The cyclic properties of engineering materials. Acta metal. mater. 42, 365-381.
[7] Pugno N. and Ruoff R. (2004), Quantized Fracture Mechanics, Philosophical Magazine, ***84***, 2829-2845. Pugno N. (2004), Quantized failure criteria and indirect observation for predicting the nanoscale strength of materials: the example of the ultra nano crystalline diamond. arxiv: cond-mat/0411556, 22 nov. 2004.

[8] Ciavarella M., (2002), *Effetti di scala e leggi di potenza in fatica*, Proceedings of the XXXI AIAS Congress, 18-21 September, 2002, Parma, Italy. Also submitted in an extended version to Int J Fract (2005) as "On notch and crack size effects in fatigue, Paris' law and implications for Woehler curves"