Global Positioning Systems

October 31, 2002

1 The Positioning Problem

A Global Positioning System (GPS) is a device that uses satellite readings to determine a receivers's *geocentric coordinates* (x, y, z); that is, the coordinates relative to the center of the earth $(0, 0, 0)$. These measurements are accurate to at least 80 meters, usually better.

A GPS device will read information from a number of satellites, at least four satellites are necessary as we shall see. The ith satellite will send out four pieces of information: its geocentric coordinates (x_i, y_i, z_i) and the time t_i of transmission. The distances x_i, y_i, z_i are measured in meters m from the center of the earth, and the time t_i is measured in nanoseconds nsec. In GPS, time t_i is the fourth dimension. Furthermore, these values are highly accurate. The receiver gets this information at a time T_i, measured on the receiver's clock. The distance between the receiver and the satellite can then be computed by

$$\rho_i = c(T_i - t_i) \text{ meters,} \qquad \text{where } c \text{ is the speed of light in m/nsec.}$$

We will keep track of the information that we get from satellite i in a single vector

$$\vec{\mathbf{s}}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ \rho_i \end{bmatrix} \qquad \text{(data we get from satellite } i\text{).}$$

Finding the geocentric position of the receiver is a fundamental problem in GPS. Part of the problem comes from the fact that receiver clocks are inexpensive and not perfectly in sync with the satellite clock. This can cause a major error when multiplied by the speed of light in the calculation $\rho_i = c(T_i - t_i)$. Since there are 10^9 nanoseconds (nsec) per second, the speed of light is approximately

$$\begin{aligned} c &= 299792458 \text{ m/s} \\ &= 0.299792458 \text{ m/nsec.} \end{aligned}$$

Thus, each nanosecond error in T_i makes ρ_i deviate from the true range by .29 meters. For this reason, ρ_i is called a *pseudorange*. There are other factors, such as atmospheric variations in the speed of radio messages, that complicate these calculations but we will not address them here.

Fortunately, the clocks on all of the satellites are in sync with each other, so the receiver's clock will be out of sync with each satellite by the same amount. We let Δt denote the error in the receiver clock. Then the error in the pseudorange is $b = c\,\Delta t$ m, and each satelite's pseudorange ρ_i is off by the same distance b. If (x, y, z) is the actual position of the receiver, then we put the unknown receiver quantities together in a vector

$$\vec{\mathbf{u}} = \begin{bmatrix} x \\ y \\ z \\ b \end{bmatrix} \qquad \text{(unknown data about the receiver).}$$

Our variables are related by fact that the actual distance between the satellite and the receiver plus the error equals the pseudorange. Thus, for each i,

$$\sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2} + b = \rho_i. \tag{1}$$

If we bring b to the other side and square both sides, we get

$$(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2 = (\rho_i - b)^2. \tag{2}$$

We'd like to solve this equation for the four unknowns x, y, z, b. If there are four satellites, then there will be four equations and four unknowns, and we should be able to find these values. However, this equation is *nonlinear*. It is possible to solve these using methods from calculus. In particular, Newton's method can iteratively approximate a solution.

2 A Least-Squares Solution

We want to turn it into a linear algebra problem. Here is a clever method due to Bancroft [1] that does some algbraic manipulations to reduce the equations to a least-squares problem. Multiplying things out in equation (2), we get

$$x_i^2 - 2x_i x + x^2 + y_i^2 - 2y_i y + y^2 + z_i^2 - 2z_i z + z^2 = \rho_i^2 - 2\rho_i b + b^2. \tag{3}$$

Now rearrange terms to get

$$(x_i^2 + y_i^2 + z_i^2 - \rho_i^2) - 2(x_i x + y_i y + z_i z - \rho_i b) + (x^2 + y^2 + z^2 - b^2) = 0. \tag{4}$$

Motivated by the form of the values within the parentheses, we define a modified dot product, called a *Lorentz inner product*,

$$\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 - u_4 v_4.$$

Then our equation becomes

$$\langle \vec{\mathbf{s}}_i, \vec{\mathbf{s}}_i \rangle - 2\langle \vec{\mathbf{s}}_i, \vec{\mathbf{u}} \rangle + \langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle = 0, \tag{5}$$

or equivalently

$$\frac{1}{2}\langle \vec{\mathbf{s}}_i, \vec{\mathbf{s}}_i \rangle - \langle \vec{\mathbf{s}}_i, \vec{\mathbf{u}} \rangle + \frac{1}{2}\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle = 0, \tag{6}$$

and this holds for each of the satellites.

In order to simultaneously analyze the equations for each satellite we organize our data as follows

$$\mathbf{B} = \begin{bmatrix} x_1 & y_1 & z_1 & -\rho_1 \\ x_2 & y_2 & z_2 & -\rho_2 \\ x_3 & y_3 & z_3 & -\rho_3 \\ x_4 & y_4 & z_4 & -\rho_4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad \vec{\mathbf{a}} = \frac{1}{2}\begin{bmatrix} \langle \vec{\mathbf{s}}_1, \vec{\mathbf{s}}_1 \rangle \\ \langle \vec{\mathbf{s}}_2, \vec{\mathbf{s}}_2 \rangle \\ \langle \vec{\mathbf{s}}_3, \vec{\mathbf{s}}_3 \rangle \\ \langle \vec{\mathbf{s}}_4, \vec{\mathbf{s}}_4 \rangle \\ \vdots \end{bmatrix}, \quad \vec{\mathbf{e}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}, \quad \text{and} \quad \Lambda = \frac{1}{2}\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle. \tag{$*$}$$

If we have n satellites, then $\mathbf{B}$ is an $n \times 4$ matrix, $\vec{\mathbf{a}}$ and $\vec{\mathbf{e}}$ are $n \times 1$ vectors, and Λ is a scalar. We now can simultaneously write our n equations from (6) as

$$\vec{\mathbf{a}} - \mathbf{B}\vec{\mathbf{u}} + \Lambda\vec{\mathbf{e}} = \mathbf{0}$$

or

$$\mathbf{B}\vec{\mathbf{u}} = (\vec{\mathbf{a}} + \Lambda\vec{\mathbf{e}})$$

If we have more than 4 satellites, then a *least-squares solution*, solves the normal equation,

$$\mathbf{B}^{\mathrm{T}}\mathbf{B}\vec{\mathbf{u}} = \mathbf{B}^{\mathrm{T}}(\vec{\mathbf{a}} + \Lambda\vec{\mathbf{e}}).$$

Such a solution is of the form

$$\vec{\mathbf{u}}^* = \mathbf{B}^+(\vec{\mathbf{a}} + \Lambda\vec{\mathbf{e}}), \qquad \text{where} \qquad \mathbf{B}^+ = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}. \tag{7}$$

However, *our solution* $\vec{\mathbf{u}}^*$ *involves* Λ *which involves our unknown* $\vec{\mathbf{u}}$. This is a problem that we cleverly work around with some algebraic manipulations. Substituting $\vec{\mathbf{u}}^*$ into the definition $(*)$ of the scalar Λ and using the linearity of the Lorentz innerproduct gives

$$\Lambda = \frac{1}{2}\left\langle \mathbf{B}^+(\vec{\mathbf{a}} + \Lambda\vec{\mathbf{e}}), \mathbf{B}^+(\vec{\mathbf{a}} + \Lambda\vec{\mathbf{e}})\right\rangle = \frac{1}{2}\left\langle \mathbf{B}^+\vec{\mathbf{a}}, \mathbf{B}^+\vec{\mathbf{a}}\right\rangle + \Lambda\left\langle \mathbf{B}^+\vec{\mathbf{a}}, \mathbf{B}^+\vec{\mathbf{e}}\right\rangle + \frac{1}{2}\Lambda^2\left\langle \mathbf{B}^+\vec{\mathbf{e}}, \mathbf{B}^+\vec{\mathbf{e}}\right\rangle$$

which can be rewritten as

$$\Lambda^2\left\langle \mathbf{B}^+\vec{\mathbf{e}}, \mathbf{B}^+\vec{\mathbf{e}}\right\rangle + \Lambda 2(\left\langle \mathbf{B}^+\vec{\mathbf{a}}, \mathbf{B}^+\vec{\mathbf{e}}\right\rangle - 1) + \left\langle \mathbf{B}^+\vec{\mathbf{a}}, \mathbf{B}^+\vec{\mathbf{a}}\right\rangle = 0. \tag{Q}$$

This is a quadratic equation in Λ with coefficients $\langle \mathbf{B}^+\vec{\mathbf{e}}, \mathbf{B}^+\vec{\mathbf{e}}\rangle$, $2(\langle \mathbf{B}^+\vec{\mathbf{a}}, \mathbf{B}^+\vec{\mathbf{e}}\rangle - 1)$, and $\langle \mathbf{B}^+\vec{\mathbf{a}}, \mathbf{B}^+\vec{\mathbf{a}}\rangle$. All three of these values can be computed (they do not involve any of our unknowns), so we can solve for two possible values of Λ using the quadratic equation (yes, the one from high school). If we get the two solutions to this equation Λ_1 and Λ_2, then we can solve for two possible solutions $\vec{\mathbf{u}}_1^*$ and $\vec{\mathbf{u}}_2^*$ in equation (7). One of these solutions will make sense, it will be on the surface of earth (which has a radius of approximately 6378 km), and one will not.

To summarize, here is the Bancroft Algorithm for finding $\vec{\mathbf{u}}^*$:

1. Organize your data into $\mathbf{B}$, $\vec{\mathbf{a}}$, and $\vec{\mathbf{e}}$.
2. Solve the quadratic equation (Q) for Λ_1 and Λ_2.
3. Solve for the least-squares solutions $\vec{\mathbf{u}}_1^*$ and $\vec{\mathbf{u}}_2^*$, and pick the one that makes sense.

3 Bibliography

[1] Bancroft, S. An algebraic solution of the GPS equations, *IEEE Transactions on Aerospace and Electronic Systems* **21** (1985) 56–59.

[2] Strang, G. and Borre, K., *Linear Algebra, Geodesy, and GPS*, Wellesley-Cambridge, Wellesley, MA 1997.

4 Your Assignment

Your assignment is to demonstrate the Bancroft Algorithm in a Mathematica Notebook. Your solution should come with documentation that explains both how to use the computer code and how it works. It also should explain what we mean by a "least-squares" solution. Assume that your audience knows the language of matrices and vectors but does not know about least-squares solutions. You will be evaluated both on your ability to implement the algebra and on your writing.

DUE: Friday, November 8, by 3:30 p.m.

GROUPS:

1. Jacob Luby, Marc de Kruijf, Adugna Tolosa
2. Arjun Bhattacherjee, Martin Lacayo-Emery, Michael Decker
3. Paulo Casaes, Devin Reyes, Camilo Paris
4. Jemimah Hendry, Duoduo Cai, Daniel Sword
5. Bibek Pandey, Mariya Sadakova, Artemis Hadjivarnava
6. John Michet, Heather Lendway, Swapnil Prabha
7. Vincent Sseremba, Jan Humaj, Tak Iwanaga
8. Karla Hillstrom, Huseyin Ozhan, Larry Lo

5 Sample Data

Here is some actual satellite data to from 6 satellites for you to use. Geocentric coordinates are given in meters and time is given in nanoseconds.

i (Satellite number)	x_k (meters)	y_k (meters)	z_k (meters)	$T_i - t_i$ (nanoseconds)
1	14177553.47	−18814768.09	12243866.38	70446329.64
2	15097199.81	−4636088.67	21326706.55	75142197.81
3	23460342.33	−9433518.58	8174941.25	78968497.20
4	−8206488.95	−18217989.14	17605231.99	69887173.01
5	1399988.07	−17563734.90	19705591.18	67231182.38
6	6995655.48	−23537808.26	−9927906.48	80796265.09

Speed of light:

$$\begin{aligned} c &= 299792458 \text{ m/s} \\ &= 0.299792458 \text{ m/nsec.} \end{aligned}$$

Radius of the earth

$$\begin{aligned} r &= 6378 \text{ km} \\ &= 6378000 \text{ m} \end{aligned}$$