PROJECTIVE ARTICULATED DYNAMICS

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by

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PROJECTIVE ARTICULATED DYNAMICS

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SUMMARY

In this thesis, it is proposed to solve for the accelerations and internal forces of multiple connected rigid bodies using recursive formulations and screw theory. Screw theory allows for compact notation of the equations of motion and kinematics. The concept of a single articulated inertia modeling the behavior of an entire subchain is used to simplify the equations such that a recursive solution is possible for systems with no kinematic loops. In planar cases, the graphical interpretation of the equations introduces the connection between projective geometry and dynamics with screw theory. Projective geometry uses subspace decompositions and projections to extract useful information from problems. Similarly in dynamics, to find the reaction forces on all the joints, the splitting of all the internal forces into active and reactive subspaces is required. The splitting of body accelerations into active and reactive parts is also needed in order to understand how each solution affects the next recursion. Together the two decompositions form a symmetric and dualistic set of projections that is used for both single rigid bodies and multiple articulated rigid body chains. These projections are applied to the equations of motion yielding a new set of recursive equations with fewer steps. Each projected articulated equation maintains the two separate parts of active and reactive components. Examining the special cases presented when either of the two parts is zero for either the forces or accelerations indicates the physical interpretation of the parts. A similar non-recursive formulation is introduced to solve systems with kinematic loops. Several planar and spatial examples are included to illustrate solutions to projections, open and closed loop accelerations, and articulated inertias. A MATLAB toolbox is developed, described, and used in some of the examples implementing screw theory and projective articulated dynamics. Suggestions on further development are made that enhance the usability and understanding of multibody dynamics with screw theory.

CHAPTER 1

INTRODUCTION

The study of motion has always inspired scientists throughout history. The exploration of motions has guided pioneers such as Galileo, Newton, Euler, Lagrange and Einstein to define new frontiers in science. One of the many branches of mechanics is rigid body dynamics. It is the study of rigid bodies as they interact with each other and the motions and forces that describe them. These are complex systems with many conceptual and mathematical problems. As a result there are a number of different methods used to describe them, each with its own notation [1]. Most common is the vector method and its association with the Newtonian and Eulerian form of the equations of motion. Alternatively there are variational methods used with the Lagrangian form of the equations of motion for holonimic systems. These two methods are combined in an unified method based on Hamilton's principle. In addition, there are mathematical variations of the above methods including tensor calculus, Gibbs-Appell equations for quasi-coordinates, Hormel equations, Lie groups, quaternions and even graph theory. After three centuries of development, the plethora of methods and formulations for rigid body dynamics may confuse prospective students. Of importance is not the mathematics behind the formulations but the physical insight they may offer. One of these methods which describes dynamics with physically meaningful quantities is screw theory.

1.1 Screws

It is convenient to use compact notation for describing rigid body motions. Three translational and three rotational velocities are encapsulated into a single spatial quantity, the velocity twist. Together with the acceleration twist, they describe the motion of a rigid body.

A single spatial quantity, the wrench, encapsulates the force and moment vectors needed to describe loads on a rigid body. Both twists and wrenches are split into a magnitude and a geometrical screw quantity.

Screws describe helical fields, which are defined by a particular axis and a pitch. In general there are five quantities needed to describe a screw: four quantities are required to locate the screw axis and one quantity specifies the screw pitch. A scalar magnitude and a screw comprise the six quantities in a twist or wrench. Screws serve the same function in rigid body dynamics as unit direction vectors in particle dynamics. For example, the velocity vector of a particle consists of the constitutive part, the magnitude, and the geometrical part, a unit direction vector. The magnitude represents the speed, and the unit vector the direction it is moving. For rigid bodies the velocity twist consists of the geometrical part the screw is moving along and the constitutive part, how fast it is moving.

Spatial inertias are tensor quantities that transform acceleration twists into wrenches and can be represented as 6×6 symmetric positive definite matrices. With spatial quantities, invariants such as kinetic energy or power are easily defined and calculated. Power is the linear form between forces and velocities, and kinetic energy a quadratic form of the velocities.

The components of spatial quantities, such as twists and wrenches, may change as they are expressed in different coordinate systems. However, the information they represent remains unchanged. Invariant screw properties such as the location and direction of the screw axis, or the screw pitch can always be extracted from twists and wrenches. The best way to represent screws is through a unit twist or wrench. This is similar to representing the two independent parameters of a direction using the three components of a unit vector. Although only five quantities are needed to describe a screw, six are often used with the added constraint of the unit magnitude. This concept is also similar to homogeneous coordinates where redundant components are used to simplify the notation. Also similar to homogeneous coordinates, the coordinate transformations are performed by multiplying a single matrix that represents both a translation and a rotation.

Since both twists and wrenches contain screws there is a certain geometrical symmetry between them. The equations used to extract information from twists have symmetrical duals with wrenches. The dual relationships involving velocities, accelerations, forces, and momenta are used to gain physical insight in the equations of dynamics.

1.2 Related Work

Screws were first used to describe statics, kinematics, and dynamics by Ball [2]. More recently, the foundation of modern recursive spatial notation was developed by Featherstone [3]. This is a fundumental source of modern robotic articulated dynamics involving screw theory. Although it deals mostly with open loop rigid body chains, the articulated inertia concept and its prospect of computational efficiency has made the spatial notation very popular in the field of robotics. The initial driving force behind such work was the need for efficient computational methods for solving complex multibody systems.

The linear recursive $O(N)$ methods developed were most efficient for a large number of bodies. Compared to the standard $O(N^3)$ method the new methods were more efficient for problems with seven rigid bodies or more. Most common industrial applications involve robots with six or less rigid bodies, which makes these new methods about on par with the standard methods. However, there are more benefits uncovered by these methods that do not directly address the need for computational efficiency, but are important for the information they convey.

Multibody dynamics methods for constrained tree structures with kinematic loops were later developed by [4, 5, 6, 7, 8, 9]. In particular, Jain et al. [6] used topology and optimal control operators to develop a method similar to Kalman filtering. In addition, Wehage and Balczynski [8, 9] used topology to develop simultaneous stacked form for the equations. The stacked form is converted into recursive form utilizing block matrix factorizations. In addition, projections are used to extract relevant information from the stacked quantities. Another interesting stacked recursive method is by Lathrop [7]. Recursively all the unknowns forces are eliminated and replaced by known quantities in the stacked equations of motion which are solved for the accelerations with a linear recursive method.

The effects of external forces to articulated chains were developed by Lilly [4]. Contacts were modeled as external reaction forces that forced the system to accelerate along particular directions. This model was extended into problems where a common end effector was connected to multiple robots working together, simultaneously. To achieve this, the unknown reaction forces were eliminated using projections. A similar process is used later in this text to model internal forces.

Using recursive spatial notation Phee [10] introduces concepts from graph theory and networks to solve multibody systems. On the other hand, Ploen et al. [11] uses Lie algebra to transform the Newton-Euler equations of motion into recursive equations. In addition, the computational efficiency is examined and improved using a square factorization of the mass matrices. More optimal computational efficiency in recursive methods is developed by Stelzle etr al. [12] in eliminating negligible computations from the transformation matrices and recursive factors.

The geometrical properties of screws are explained by Lipkin and Duffy [13]. An overview of projective geometry, mappings and the principles of duality are found in Pedoe [14]. Although not specific to screws, projective geometry is abstract and applies directly to spatial quantities. Orthogonality, reciprocity, and duality of screws are also explained by Roth [15]. The physical insight of spatial articulated inertias is summarized by Lipkin [16]. Spatial articulated inertias are shown to have six general principal axes, as opposed to the three that single rigid body inertias have. Finally, the articulated recursive spatial algorithms together with duality projections and other geometrical principles are used to completely analyze the loadings and motions of a single constrained rigid body by Alexiou and Lipkin [17]. These projective solutions contain the fundamental building blocks for this text.

1.3 Velocity Twists

To describe the velocity of a rigid body with spatial notation both the linear and angular velocity vectors are needed.

Definition 1 *The velocity twist of a rigid body expressed at a point P is*

$$v_P = \begin{bmatrix} \bar{v}_P \\ \bar{\omega} \end{bmatrix} \tag{1.1}$$

where $\bar{v}_P$ is the linear velocity vector of the body expressed at P and $\bar{\omega}$ the angular velocity of the rigid body.

This vector combination conveys enough information to describe the motion as a simultaneous rotation and translation about an axis is space. The instantaneous path followed by any point on the rigid body through time is a helical curve. Similarly, all the linear velocity vectors of a rigid body describe a helical vector field as seen in Figure 1.1. By definition, the linear velocity vector $\bar{v}_Q$ of any point Q lying on the twist axis is parallel to the angular velocity vector $\bar{\omega}$. The screw pitch ρ is defined by

$$\bar{v}_Q = \rho\bar{\omega} \tag{1.2}$$

The pitch is the scalar factor between the linear velocity vector $\bar{v}_Q$ and the angular velocity vector $\bar{\omega}$. The linear velocity vector at reference point P is

$$\bar{v}_P = \rho\bar{\omega} + \bar{r} \times \bar{\omega} \tag{1.3}$$

where $\bar{r}$ is the position vector from point P to point Q.

The spatial velocity is now split into two parts

$$v_P = \begin{bmatrix} \bar{r} \times \bar{\omega} \\ \bar{\omega} \end{bmatrix} + \begin{bmatrix} \rho\bar{\omega} \\ \bar{0} \end{bmatrix} \tag{1.4}$$

which represent a rotation of $\bar{\omega}$ about an axis located by $\bar{r}$ and a translation $\rho\bar{\omega}$ parallel to the axis of $\bar{\omega}$. The velocity twist v_P magnitude is defined as the magnitute of the angular velocity $|\bar{\omega}|$ if $\bar{\omega} \neq 0$. If $|\bar{\omega}| = 0$, then the velocity twist represents a

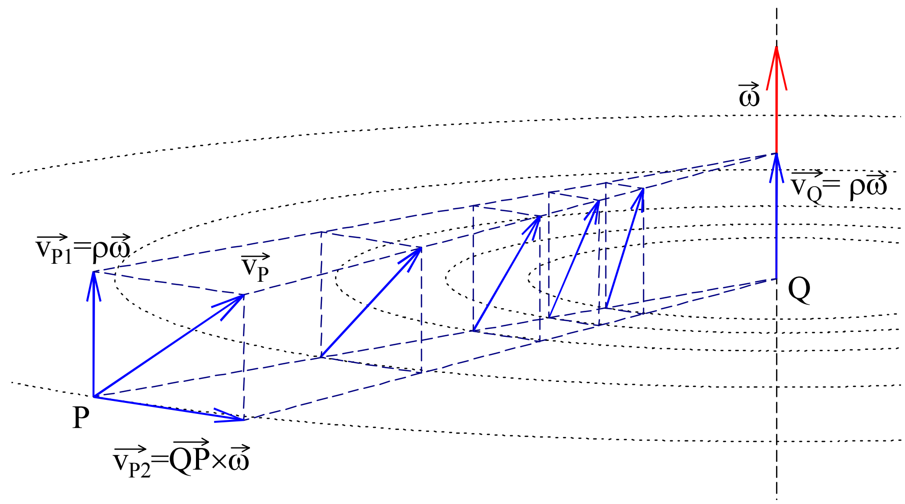

Figure 1.1: The helical field of linear velocity for any point on a rigid body as a result of general motion. The net velocity $\vec{v}_P$ at any point consists of two parts, $\vec{v}_{P1}$ the linear velocity parallel to the screw axis, and $\vec{v}_{P2}$ the tangential velocity that represents the rotational motion of the body about the screw axis.

pure translation along $\bar{v}_P$. The geometrical information in v_P is a screw axis located by $\bar{r}$ in the direction of $\bar{\omega}$ and the pitch, ρ, which has units of length representing the translation along the axis per revolution. A pitch of zero indicates a purely rotational motion and an infinite pitch indicates a purely translational motion. As shown earlier when $|\bar{\omega}| = 0$ and $|v_P| \neq 0$ the screw is degenerate with $\rho = \infty$. Thus, a unit screw with $|\bar{\omega}| = 1$ has a linear velocity along the screw axis equal to the screw pitch.

The first part of (1.4) is associated with a line vector since it contains the orientation and location of the screw axis. The other part is associated with a free vector because its components are independent to the reference point.

To extract the geometrical information from the components of the velocity twist v_P, (1.3) is projected along the screw axis as

$$\bar{\omega}^T \bar{v}_P = \bar{\omega}^T \rho \bar{\omega} + \bar{\omega}^T (\bar{r} \times \bar{\omega}) \tag{1.5a}$$

$$= \rho \bar{\omega}^T \bar{\omega} \tag{1.5b}$$

which is solved for the pitch

$$\rho = \frac{\bar{\omega}^T \bar{v}_P}{\bar{\omega}^T \bar{\omega}} \tag{1.6}$$

Also, to find the closest position of the screw axis, (1.3) is projected on the plane perpendicular to the screw axis as

$$\bar{\omega} \times \bar{v}_P = \bar{\omega} \times \rho \bar{\omega} + \bar{\omega} \times (\bar{r} \times \bar{\omega}) \tag{1.7a}$$

$$= \bar{\omega} \times (\bar{r} \times \bar{\omega}) \tag{1.7b}$$

$$= \left(\bar{\omega}^T \bar{\omega}\right) \bar{r} - \left(\bar{\omega}^T \bar{r}\right) \bar{\omega} \tag{1.7c}$$

If $\bar{r}_\perp$ is assumed to lie on the plane perpendicular to $\bar{\omega}$ and $\bar{\omega}^T \bar{r}_\perp = 0$ then

$$\bar{\omega} \times \bar{v}_P = \left(\bar{\omega}^T \bar{\omega}\right) \bar{r}_\perp \tag{1.8}$$

which is solved for the position vector

$$\bar{r}_\perp = \frac{\bar{\omega} \times \bar{v}_P}{\left(\bar{\omega}^T \bar{\omega}\right)} \tag{1.9}$$

In general there might be a component of $\bar{r}$ along $\bar{\omega}$ and therefore

$$\bar{r} = \bar{r}_\perp + \eta \bar{e} \tag{1.10}$$

$$= \frac{\bar{\omega} \times \bar{v}_P}{\left(\bar{\omega}^T \bar{\omega}\right)} + \eta \bar{e} \tag{1.11}$$

where $\bar{e}$ is the unit direction vector

$$\bar{e} = \frac{\bar{\omega}}{|\bar{\omega}|} = \frac{\bar{\omega}}{\sqrt{\bar{\omega}^T\bar{\omega}}} \tag{1.12}$$

and η is an arbitrary scalar.

1.4 Acceleration Twists

Rigid body accelerations do not form helical vector fields. The reason is that the linear acceleration of any point on a rigid body contains velocity related terms. This is called the material acceleration $\bar{a}^m$ and it is the total derivative of the velocity vector such that

$$\bar{a}^m = \frac{d}{dt}\bar{v} \tag{1.13}$$

Assuming the velocity is a function of position and time, $\bar{v} = \bar{v}\left(\bar{p}, t\right)$ then the material acceleration is split into two parts

$$\bar{a}^m = \frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial \bar{p}}\frac{d\bar{p}}{dt} \tag{1.14a}$$

$$= \bar{a} + \frac{\partial \bar{v}}{\partial \bar{p}}\bar{v} \tag{1.14b}$$

where $\bar{a}$ is called the local acceleration and is defined by

$$\bar{a} = \frac{\partial}{\partial t}\bar{v} \tag{1.15}$$

The change of the velocity vector $\bar{v}$ while holding everything constant and changing the position $\bar{p}$ of the reference point is calculated from the velocity vector expansion

(1.3)

$$\bar{v} = \rho\bar{\omega} + \bar{r} \times \bar{\omega} \tag{1.16}$$

From Figure 1.1 the position vector of the screw axis is $\bar{q}$ and the position of the reference point is $\bar{p}$ then $\bar{r} = \bar{q} - \bar{p}$. A small change in $\bar{p}$ changes $\bar{v}$ as

$$\begin{aligned}
\Delta\bar{v} &= \bar{v}(\bar{p} + \Delta\bar{p}, t) - \bar{v}(\bar{p}, t) && \text{(1.17a)} \\
&= [\rho\bar{\omega} + (\bar{q} - \bar{p} - \Delta\bar{p}) \times \bar{\omega}] - [\rho\bar{\omega} + (\bar{q} - \bar{p}) \times \bar{\omega}] && \text{(1.17b)} \\
&= -\Delta\bar{p} \times \bar{\omega} && \text{(1.17c)} \\
&= \bar{\omega} \times \Delta\bar{p} && \text{(1.17d)}
\end{aligned}$$

Therefore

$$\frac{\partial\bar{v}}{\partial\bar{p}} = \bar{\omega} \times \tag{1.18}$$

To relate the material acceleration with the local acceleration for a rigid body rotating by $\bar{\omega}$ (1.14a) simplifies to

$$\bar{a}^m = \bar{a} + \bar{\omega} \times \bar{v} \tag{1.19}$$

where $\bar{\omega} \times \bar{v}$ is called the convective acceleration.

For particles and rigid bodies, forces through the center of mass and material accelerations are related by a scalar mass such that

$$\bar{F} = m\bar{a}^m \tag{1.20}$$

where $\bar{F}$ is the net force vector, m is the mass, and $\bar{a}^m$ is the material linear acceleration of the center of mass. When local acceleration is used, a convective force must be introduced into the equations of motion to compensate for the difference. Substituting (1.19) into (1.20) the linear equations of motion become

$$\bar{F} = m\bar{a} + \bar{\omega} \times m\bar{v} \tag{1.21}$$

where $\bar{a}$ is the local linear acceleration vector, $\bar{\omega}$ the angular velocity of the body and $\bar{v}$ the linear velocity vector.

The local vector acceleration does form a helical vector field and can be represented by a screw quantity. Together with the angular acceleration vector they form the spatial acceleration.

Definition 2 *The acceleration twist of a rigid body expressed at a point P is*

$$a_P = \begin{bmatrix} \bar{a}_P \\ \bar{\alpha} \end{bmatrix} \tag{1.22}$$

where $\bar{a}_P$ is the local acceleration vector of the body expressed at P and $\bar{\alpha}$ the angular acceleration of the rigid body.

This acceleration is also a screw quantity and can be split into two parts

$$a = \begin{bmatrix} \bar{r} \times \bar{\alpha} \\ \bar{\alpha} \end{bmatrix} + \begin{bmatrix} h\bar{\alpha} \\ \bar{0} \end{bmatrix} \tag{1.23}$$

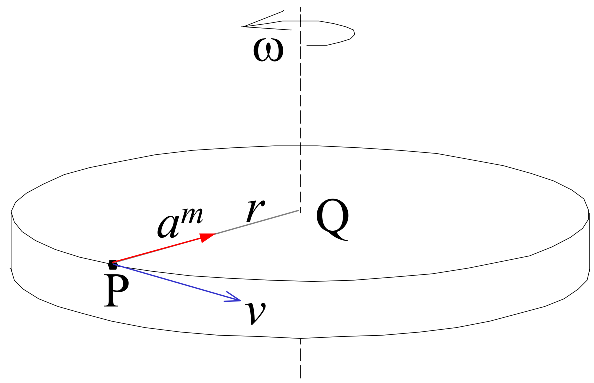

Figure 1.2: Material acceleration of a constant spinning disk has a centripetal component since the rigid body at P follows a circular path.

with $\bar{\alpha}$ the angular acceleration vector, $\bar{r}$ a position vector to the screw axis and h its pitch.

By definition, the spatial acceleration measures a different quantity from the material acceleration. The material acceleration measures the acceleration of a particular point on the rigid body, while spatial acceleration measures the acceleration state of the entire rigid body at a fixed location in space. Consider the case shown in Figure 1.2 where a free disk is spinning with constant angular velocity $\bar{\omega}$ about its center. The acceleration of the particles on the disk under P follow a circular motion and therefore experience a velocity related centripetal acceleration. Said otherwise, the linear velocity vector of that particle changes direction with time and thus it experiences acceleration. On the other hand, in Figure 1.3, the linear velocity vector of the rigid body under P is constant, if the measuring point P is fixed in space and it doesn't move with the rigid body. The spatial acceleration is then zero since the velocity state of the rigid body is constant.

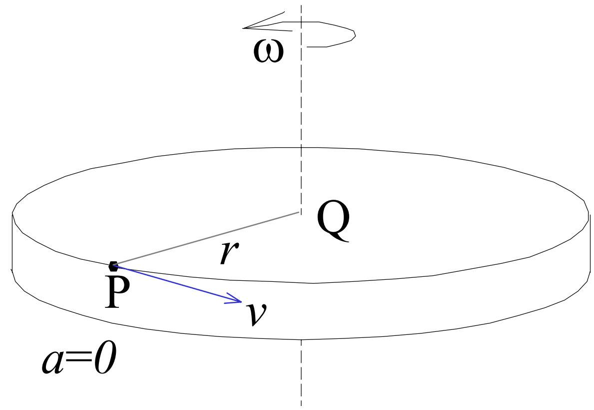

Figure 1.3: Spatial acceleration of a constant spinning disk is zero because the velocity of the rigid body at the fixed spatial location P is constant.

The distinction between measuring a particular point on a body, and measuring the entire body at a particular location is crucial to the understanding of screw theory.

The velocity twist is a function of position, orientation, and time. The change with time is the spatial acceleration

$$a = \frac{\partial v}{\partial t} \tag{1.24}$$

which is the spatial form of (1.15). The change of position and orientation is epxressed through an infinitesimal screw motion

$$\Delta\sigma = \begin{bmatrix} \Delta\bar{p} \\ \Delta\bar{\theta} \end{bmatrix} \tag{1.25}$$

where $\Delta\bar{p}$ is the change of the location of the reference point and $\Delta\bar{\theta}$ the change of the orientation of the reference coordinate frame. If the reference frame is moving along with the body then

$$\Delta\sigma = v\Delta t \tag{1.26}$$

The total derivative of the velocity twist is then split in two parts

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \sigma}\frac{d\sigma}{dt} \tag{1.27a}$$

$$= a + \frac{\partial v}{\partial \sigma}v \tag{1.27b}$$

which is the spatial form of (1.14a). The partial of v with respect to σ is a 6×6 matrix indicating how each of the components of v change as the reference coordinate frame changes with components

$$\frac{\partial v}{\partial \sigma} = \begin{bmatrix} \frac{\partial \bar{v}}{\partial \bar{p}} & \frac{\partial \bar{v}}{\partial \bar{\theta}} \\ \frac{\partial \bar{\omega}}{\partial \bar{p}} & \frac{\partial \bar{\omega}}{\partial \bar{\theta}} \end{bmatrix} \tag{1.28}$$

As seen in (1.18)

$$\frac{\partial \bar{v}}{\partial \bar{p}} = \bar{\omega}\, \times \tag{1.29}$$

Also, since the angular velocity vector does not change with postion

$$\frac{\partial \bar{\omega}}{\partial \bar{p}} = \bar{0} \tag{1.30}$$

For the changes with orientation section 3.3 of [1] defined the change of any arbitrary vector $\bar{A}$ with an infinitesimal rotation $\Delta\bar{\psi}$ as

$$\Delta\bar{A} = \Delta\bar{\psi} \times \bar{A} \tag{1.31}$$

If the vector is fixed and the reference frame changes by $\Delta\bar{\theta}$ then the apparent change in $\bar{A}$ is

$$\Delta\bar{A} = -\Delta\bar{\theta}\times\bar{A} \tag{1.32}$$

$$= \bar{A}\times\Delta\bar{\theta} \tag{1.33}$$

Applying substituting $\bar{v}$ and $\bar{\omega}$ for $\bar{A}$ yields

$$\frac{\partial\bar{v}}{\partial\bar{\theta}} = \bar{v}\times \tag{1.34}$$

and

$$\frac{\partial\bar{\omega}}{\partial\bar{\theta}} = \bar{\omega}\times \tag{1.35}$$

Finally, assembling all the partials together in spatial form

$$\frac{\partial v}{\partial\sigma} = \begin{bmatrix} \bar{\omega}\times & \bar{0} \\ \bar{v}\times & \bar{\omega}\times \end{bmatrix} \tag{1.36}$$

$$= -(v\times)^T \tag{1.37}$$

where $v\times$ is the spatial cross product operator

$$v\times = \begin{bmatrix} \bar{\omega}\times & \bar{v}\times \\ \bar{0} & \bar{\omega}\times \end{bmatrix} \tag{1.38}$$

Equation (1.27a) simplifies to

$$\frac{dv}{dt} = a - (v\times)^T v \tag{1.39}$$

or in component form,

$$\frac{d}{dt}\begin{bmatrix} \bar{v} \\ \\ \bar{\omega} \end{bmatrix} = \begin{bmatrix} \bar{a} + \bar{\omega} \times \bar{v} \\ \\ \bar{\alpha} \end{bmatrix} \tag{1.40}$$

which is the spatial version of equation (1.19).

1.5 Force Wrenches

Spatial loadings need to be defined to formulate the equations of motion. Just as twists describe motion, wrenches describe loads. The combination of force and torque about a point P defines the load wrench f_P as

Definition 3 *The force wrench of a rigid body at a point P is expressed as*

$$f_P = \begin{bmatrix} \bar{f} \\ \\ \bar{\tau}_P \end{bmatrix} \tag{1.41}$$

where $\bar{f}$ is the force vector acting on the body and $\bar{\tau}_P$ is the equipollent moment vector about P.

A force and moment combination represents a helical field of moments around the application axis of the force as shown in Figure 1.4.

Similar to a velocity twist, any force wrench is split into two components which describe a force on a particular line and a pure couple about that line. Using the

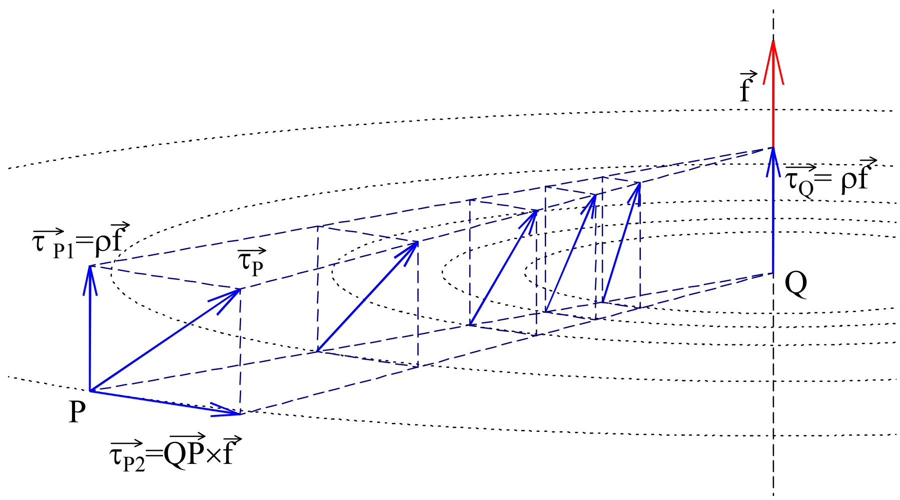

Figure 1.4: The helical field of net moments about any point on a rigid body as a result of general loading. The net moment $\vec{\tau}_P$ has two parts, $\vec{\tau}_{P1}$ which is the moment about the application axis, and $\vec{\tau}_{P2}$ the resulting tangential moment that represents the force $\vec{f}$ applied at a distance.

same concepts of the position vector $\bar{r}$ to the screw axis and the screw pitch ρ, the two parts of a spatial force are

$$f_P = \begin{bmatrix} \bar{f} \\ \bar{r} \times \bar{f} \end{bmatrix} + \begin{bmatrix} \bar{0} \\ \rho \bar{f} \end{bmatrix} \tag{1.42}$$

The first part is associated with a line vector, and the second with a free vector.

The force wrench f_P has magnitude $|\bar{f}|$. The screw axis is the application axis of $\bar{f}$. The rest of the geometrical information contained in f_P is calculated using the equipollent moment equation

$$\bar{\tau}_P = \bar{r} \times \bar{f} + \rho \bar{f} \tag{1.43}$$

The pitch of the screw is found by projecting (1.43) along the screw axis as

$$\bar{f}^T\bar{\tau}_P = \bar{f}^T\left(\bar{r}\times\bar{f}\right) + \bar{f}^T\rho\bar{f} \quad (1.44a)$$

$$= \rho\bar{f}^T\bar{f} \quad (1.44b)$$

which is solved as

$$\rho = \frac{\bar{f}^T\bar{\tau}_P}{\bar{f}^T\bar{f}} \quad (1.45)$$

The closest position of the screw axis is found by projecting (1.43) on the plane perpendicular to the screw axis as

$$\bar{f}\times\bar{\tau}_P = \bar{f}\times\left(\bar{r}\times\bar{f}\right) + \bar{f}\times\rho\bar{f} \quad (1.46a)$$

$$= \bar{f}\times\left(\bar{r}\times\bar{f}\right) \quad (1.46b)$$

$$= \left(\bar{f}^T\bar{f}\right)\bar{r} - \left(\bar{f}^T\bar{r}\right)\bar{f} \quad (1.46c)$$

$$= \left(\bar{f}^T\bar{f}\right)\bar{r}_\perp \quad (1.46d)$$

If $\bar{r}_\perp$ is assumed to lie on the plane perpendicular to $\bar{f}$ and $\bar{f}^T\bar{r}_\perp = 0$ then

$$\bar{f}\times\bar{\tau}_P = \left(\bar{f}^T\bar{f}\right)\bar{r}_\perp \quad (1.47)$$

which is solved for the position vector

$$\bar{r}_\perp = \frac{\bar{f}\times\bar{\tau}_P}{\bar{f}^T\bar{f}} \quad (1.48)$$

If a component of $\bar{r}$ along the screw axis exists then

$$\bar{r} = \bar{r}_\perp + \delta\bar{u} \quad (1.49)$$

$$\frac{\bar{f}\times\bar{\tau}_P}{\bar{f}^T\bar{f}} + \delta\bar{u} \quad (1.50)$$

where $\bar{u}$ the unit direction vector

$$\bar{u} = \frac{\bar{f}}{|\bar{f}|} = \frac{\bar{f}}{\sqrt{\bar{f}^T \bar{f}}} \tag{1.51}$$

and δ an arbitrary scalar.

1.6 Coordinate Transformations

Translating a velocity twist v_A and a force wrench f_A from point A to another point B can be done utilizing position vector $\bar{r}_{AB}$ between points A and B. The linear velocity is $\bar{v}_B = \bar{v}_A - \bar{r}_{AB} \times \bar{\omega}$ and the moment is $\bar{\tau}_B = \bar{\tau}_A - \bar{r}_{AB} \times \bar{f}$. The term $\bar{r}_{AB}\times$ corresponds to the vector cross product which is defined in terms of a 3×3 matrix as

Definition 4 *The skew symmetric vector cross product operator is*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{1.52}$$

Then the spatial form of the translations are

$$v_B = ({}_B X_A)\, v_A \tag{1.53}$$

$$f_B = ({}_B X_A)^{-T}\, f_A \tag{1.54}$$

with the 6×6 spatial transformation

$$ {}_B X_A = \begin{bmatrix} 1 & -\bar{r}_{AB}\times \\ 0 & 1 \end{bmatrix} \tag{1.55} $$

Note that $(\bar{r}\times)^T = -\bar{r}\times$ and $({}_B X_A)^{-1} = ({}_A X_B)$.

Rotations are actually easier to define since both linear and angular parts transform the same way. Using E, an orthogonal 3×3 rotation matrix from coordinate frames B' to B, any $\bar{e}_{B'}$ transforms as $\bar{e}_B = E\bar{e}_{B'}$. Thus,

Definition 5 *A general twist transformation from A to B with translation $\bar{r}_{AB}$ followed by rotation E is described by the transformation matrix*

$$ {}_B X_A = \begin{bmatrix} E & E\,(\bar{r}_{AB}\times)^T \\ 0 & E \end{bmatrix} \tag{1.56} $$

and its inverse

$$ {}_A X_B = \begin{bmatrix} E^T & \bar{r}_{AB} \times E^T \\ 0 & E^T \end{bmatrix} \tag{1.57} $$

Definition 6 *A general wrench transformation from A to B with translation $\bar{r}_{AB}$ followed by rotation E is described by the transformation matrix*

$$ ({}_B X_A)^{-T} = \begin{bmatrix} E & 0 \\ E\,(\bar{r}_{AB}\times)^T & E \end{bmatrix} \tag{1.58} $$

and its inverse

$$({}_AX_B)^{-T} = \begin{bmatrix} E^T & 0 \\ \bar{r}_{AB} \times E^T & E^T \end{bmatrix} \tag{1.59}$$

Spatial accelerations transform the same way velocities do. The transformation of an acceleration twist from coordinate frame A to coordinate frame B is defined as $a_B = ({}_BX_A)\, a_A$ using the same transformation matrix as (1.56).

1.7 Momentum Wrench

A moving rigid body has a linear and an angular momentum associated with it. The spatial momentum as defined in [3] is

$$P = \begin{bmatrix} \bar{p} \\ \bar{r} \times \bar{p} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{h} \end{bmatrix} \tag{1.60}$$

which represents the linear momentum $\bar{p}$ acting on a line through the center of mass, and the intrinsic angular momentum $\bar{h}$ about the center of mass. The intrinsic angular momentum is

$$\bar{h} = \bar{I}\bar{\omega} \tag{1.61}$$

where $\bar{I}$ is the angular inertia about the center of mass and $\bar{\omega}$ is its angular velocity vector. The linear momentum is proportional to the linear velocity vector of the center of mass and thus

$$\bar{p} = m\left(\bar{v} - \bar{r} \times \bar{\omega}\right) \tag{1.62}$$

where $\bar{v}$ is the linear velocity vector of the reference point and $\bar{r}$ is the relative position vector of the center of mass from the reference point.

Assembling the linear and angular parts together the spatial momentum is defined as

$$P = \begin{bmatrix} m\left(\bar{v} - \bar{r} \times \bar{\omega}\right) \\ \bar{I}\bar{\omega} + \bar{r} \times m\left(\bar{v} - \bar{r} \times \bar{\omega}\right) \end{bmatrix} \tag{1.63}$$

The $\bar{r} \times \bar{p}$ term represents the moment of momentum about the reference point.

Momentum wrenches transform the same way force wrenches transform. The momentum transformation from coordinate frame A to coordinate frame B is defined as

$$P_B = \left({}_B X_A\right)^{-T} P_A \tag{1.64}$$

using the transformation matrix $\left({}_B X_A\right)^{-T}$ from (1.58).

1.8 Spatial Inertia

The spatial inertia is defined as the tensor that multiplies a velocity twist to produce a momentum wrench. Therefore, by factoring out the velocity terms from momentum the symmetric 6×6 inertia tensor is defined as

$$P = Iv \tag{1.65}$$

where

$$I = \begin{bmatrix} m\bar{1} & m\left(\bar{r}\times\right)^T \\ m\left(\bar{r}\times\right) & \bar{I} + m\left(\bar{r}\times\right)\left(\bar{r}\times\right)^T \end{bmatrix} \tag{1.66}$$

and

$$v = \begin{bmatrix} \bar{v} \\ \bar{\omega} \end{bmatrix} \tag{1.67}$$

Note that $m\left(\bar{r}\times\right)\left(\bar{r}\times\right)^T$ is the 3×3 angular inertia matrix of a point mass m with relative position vector $\bar{r}$.

The transformation of the spatial inertia from coordinate frame A to coordinate frame B is easily defined from equation (1.65) as

$$I_B = \left({}_AX_B\right)^T I_A \left({}_AX_B\right) \tag{1.68}$$

which can be interpreted from right to left as transforming the spatial velocities from B to A, then evaluating the spatial momentum at A and transforming back the momentum to B.

1.9 Kinetic Energy and Power

It is easy to show that under the spatial transformations the kinetic energy of a rigid body is an invariant quantity. If the kinetic energy is evaluated at a coordinate system A as $K = \frac{1}{2} v_A^T I_A v_A$ then by transforming all the quantities to a coordinate system B the kinetic energy becomes $K = \frac{1}{2} \left({}_A X_B v_B\right)^T \left({}_B X_A\right)^T I_B \left({}_B X_A\right)_A X_B v_B = \frac{1}{2} v_B^T I_B v_B$. Thus the kinetic energy is the same regardless on where it is evaluated.

Power is also an invariant defined as the linear form of a velocity twist with a force wrench. Power evaluated on a coordinate system A as $P = v_A^T f_A$ is transformed to a coordinate system B by $P = \left({}_A X_B v_B\right)^T \left({}_A X_B\right)^{-T} f_B = v_B^T f_B$. If forces and velocities produce zero power then they represent reciprocal screws.

1.10 Summary

Spatial notation using screws provides compact forms for rigid body motions and loads. In addition, the geometrical properties of such quantities are immediately available. With screw theory the motions and loadings of the entire rigid body are considered. On the other hand, with vector analysis the motion or load at a particular point on the body is examined. This conceptual difference is important to the understanding of this spatial notation.

Velocities and accelerations are defined as twists and momenta and forces as wrenches. Transformations between different coordinate frames are established for both twists and wrenches. Spatial inertias are defined by relating the velocity twist to the momentum wrench for a rigid body. The spatial form of the parallel axis theorem is defined to transform inertias between different coordinate frames. Finally, kinetic energy and power are compacted into spatial form.

CHAPTER 2

KINEMATICS

In this chapter joints are modeled using unit twists. A linear chain of rigid bodies models the typical multibody case and recursive equations are developed for the spatial velocities and accelerations. The notation used is shown in Figure 2.1. On each body i there is a local coordinate system i and a joint that connects the body with its parent body $i-1$. The relationships presented below were developed in Chapter 2 of Featherstone[3].

2.1 Velocity Kinematics

Twists are used to describe the motion of rigid bodies. The relative motion twist v^r between two joined bodies is split into two parts

$$v^r = s\dot{q} \tag{2.1}$$

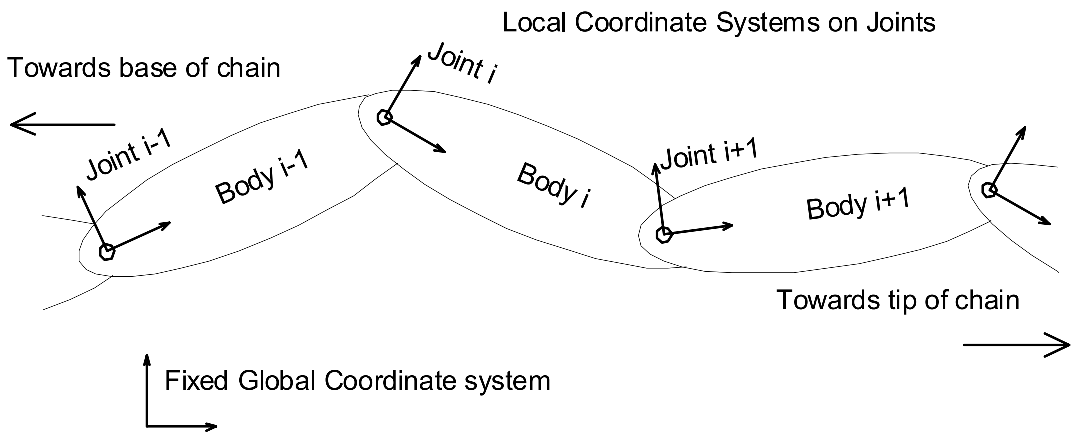

Figure 2.1: A linear chain of rigid bodies represents the basic multibody case. For each body i there is a local coordinate system and a joint that connects that body with its parent body $i-1$.

where s is a unit twist and $\dot{q}$ is the twist magnitude. The unit twist conveys the geometrical information whereas, the constitutive portion is the magnitude. If the joint has a single degree of freedom then it can be modeled by a screw. For example, a revolute joint is modeled by a zero pitch screw and a prismatic joint by an infinite pitch screw. The joint axis is thus represented by a unit twist s. The twist magnitude $\dot{q}$ represents the joint speed.

This notation is extended to multiple degrees of freedom with multiple sequential screws, such that

$$v^r = s_1\dot{q}_1 + s_2\dot{q} + \ldots + s_k\dot{q}_k \tag{2.2}$$

where k is the number of degrees of freedom, s_1 through s_k the unit twists representing each joint, and $\dot{q}_1$ through $\dot{q}_k$ each joint speed. Equation (2.2) is compacted into

$$v^r = s\dot{q} \tag{2.3}$$

where s is the $6 \times k$ joint space matrix

$$s = \begin{bmatrix} s_1 & s_2 & \ldots & s_k \end{bmatrix} \tag{2.4}$$

and $\dot{q}$ the $k \times 1$ joint speed matrix

$$\dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_k \end{bmatrix} \tag{2.5}$$

Each column of s represents a base screw for the subspace of relative motions between joined bodied.

As an example, a slider pin joint is modeled with the slider in the $\bar{x}$ the direction and the pin about the $\bar{z}$ axis. The individual joint axes are

$$s_1 = \begin{bmatrix} \bar{x} \\ \bar{0} \end{bmatrix} \tag{2.6}$$

and

$$s_2 = \begin{bmatrix} \bar{0} \\ \\ \bar{z} \end{bmatrix} \tag{2.7}$$

The 6×2 matrix joint space is

$$\begin{aligned} s &= \begin{bmatrix} s_1 & s_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{x} & \bar{0} \\ \\ \bar{0} & \bar{z} \end{bmatrix} \end{aligned} \tag{2.8}$$

Definition 7 *Let joint i connect body i to body $i-1$, then by*

$$v_i = v_{i-1} + s_i \dot{q}_i \tag{2.9}$$

v_i and v_{i-1} are the velocities twists of bodies i and $i-1$, s_i the joint axis, and $\dot{q}_i$ is the joint speed.

Remark 8 *The Jacobian J of a robotic manipulator is the collection of all the joint axes such that $J = [s_1 \dots s_N]$. This Jacobian can be used to link the equations of motion from the joint space form to the spatial form.*

2.2 Acceleration Kinematics

The acceleration kinematics are defined by substituting (2.9) into (1.24) for each body i

$$
\begin{aligned}
a_i &= \frac{\partial}{\partial t} v_i && (2.10a)\\
&= \frac{\partial}{\partial t}\left(v_{i-1} + s_i \dot{q}_i\right) && (2.10b)\\
&= a_{i-1} + \frac{\partial}{\partial t}\left(s_i \dot{q}_i\right) && (2.10c)\\
&= a_{i-1} + s_i \ddot{q}_i + \frac{\partial s_i}{\partial t}\dot{q}_i && (2.10d)
\end{aligned}
$$

where a_i and a_{i-1} are the spatial accelerations of bodies i and $i-1$, s_i is the joint axis, $\dot{q}_i$ the joint speed, and $\ddot{q}_i$ the joint acceleration. In order to calculate the rate of change of s_i the joint axis is expanded from (1.4) into

$$
s_i = \begin{bmatrix} \rho\bar{e} + \bar{r} \times \bar{e} \\ \bar{e} \end{bmatrix} \tag{2.11}
$$

where ρ is the joint axis pitch, $\bar{e}$ is its direction, and $\bar{r}$ its location. It is known from [3] that the rate of change any vector $\bar{e}$ fixed to a moving rigid body is

$$
\frac{\partial \bar{e}}{\partial t} = \bar{\omega} \times \bar{e} \tag{2.12}
$$

where $\bar{\omega}$ is the angular velocity of the body. Therefore the rate of change of the joint axis is

$$\frac{\partial s_i}{\partial t} = \frac{\partial}{\partial t}\begin{bmatrix} \rho\bar{e} + \bar{r} \times \bar{e} \\ \bar{e} \end{bmatrix} \tag{2.13a}$$

$$= \begin{bmatrix} \frac{\partial}{\partial t}(\rho\bar{e}) + \frac{\partial}{\partial t}(\bar{r} \times \bar{e}) \\ \frac{\partial}{\partial t}\bar{e} \end{bmatrix} \tag{2.13b}$$

$$= \begin{bmatrix} \bar{\omega} \times (\rho\bar{e}) + \bar{\omega} \times (\bar{r} \times \bar{e}) + \frac{\partial \bar{r}}{\partial t} \times \bar{e} \\ \bar{\omega} \times \bar{e} \end{bmatrix} \tag{2.13c}$$

$$= \begin{bmatrix} \bar{\omega}\times & \bar{v}\times \\ \bar{0} & \bar{\omega}\times \end{bmatrix} \begin{bmatrix} \rho\bar{e} + \bar{r} \times \bar{e} \\ \bar{e} \end{bmatrix} \tag{2.13d}$$

which is generalized as

$$\frac{\partial s_i}{\partial t} = v \times s_i \tag{2.14}$$

where v is the velocity twist

$$v = \begin{bmatrix} \bar{v} \\ \bar{\omega} \end{bmatrix} \tag{2.15}$$

and $v\times$ is the spatial cross product operator

$$v\times = \begin{bmatrix} \bar{\omega}\times & \bar{v}\times \\ \bar{0} & \bar{\omega}\times \end{bmatrix} \tag{2.16}$$

The acceleration kinematics are defined recursively from (2.10d) and (2.14) as

Definition 9 *Let joint i connect body i to body $i-1$, then the accelerations are related by*

$$a_i = a_{i-1} + s_i\ddot{q}_i + v_i \times s_i\dot{q}_i \tag{2.17}$$

with a_i and a_{i-1} the spatial accelerations of bodies i and $i-1$, s_i the joint axis, v_i the spatial velocity and $\dot{q}_i$ and $\ddot{q}_i$ then joint speed and acceleration.

Remark 10 *The bias acceleration term $v_i \times s_i\dot{q}_i$ is the convective acceleration of the joint s_i. It is a degenerate screw which is reciprocal to v_i because it always has zero magnitude.*

Sometimes the velocity related terms are grouped into one bias acceleration quantity κ_i such that

$$a_i = a_{i-1} + s_i\ddot{q}_i + \kappa_i \tag{2.18}$$

with

$$\kappa_i = v_i \times s_i\dot{q}_i \tag{2.19}$$

which allows for slightly more compact notation.

Multiple degree of freedom joints are modeled using the joint space s_i as a collection of individual joint axes.

2.3 Summary

The velocity and acceleration kinematics are fundamental tools for recursive modeling of rigid body dynamics. Joints are modeled as the basis screw of the relative motion between bodies. The joint space represents multiple sequential basis screws used to model multiple degree of freedom joints. The velocity difference between two successive bodies is proportional to the screw axis of the joint between them. Differentiating the velocity kinematics yields the acceleration kinematics. The relative acceleration between bodies contains a part along the joint axes as well as bias accelerations in other directions.

CHAPTER 3

EQUATIONS OF MOTION

The Newton-Euler equations of motion follow from Newtons 2^{nd} law. In spatial form, the net load wrench is equal to the rate of change of the momentum wrench. The net load of a joined rigid body contains internal and external loads. Usually, the equations of motion are not solved in recursive form. In [3, 16] the concept of articulated inertia is introduced. This inertia replaces the rigid body inertia in the equations of motion modifying them in such a way that a recursive solution for joint acceleretations is possible. In fact, the calculation of the articulated body inertias is also a recursive algorithm.

3.1 Free Body Diagram

In a serial chain of N rigid bodies there are generally three force wrenches acting on a body i as seen in Figure 3.1: applied force wrench g_i, internal wrench f_i through

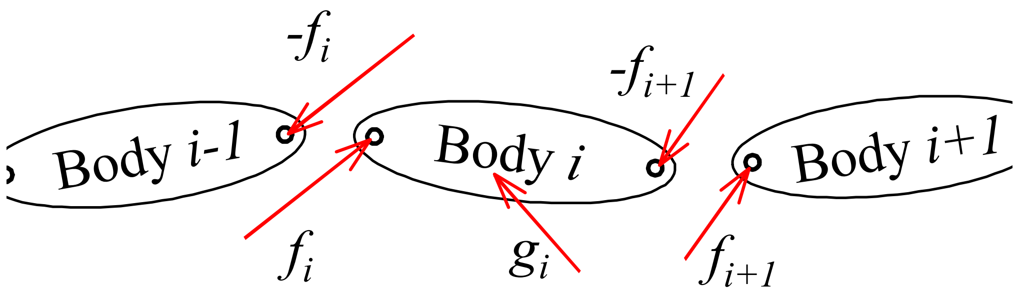

Figure 3.1: Free body diagram of body i in chain of rigid bodies. The forces acting on body i include any applied forces g_i, the internal forces f_i from the previous joint, and the internal reaction forces $-f_{i+1}$ from the next joint.

joint i from the previous body, and the internal wrench $-f_{i+1}$ through the joint $i+1$ from the next body. The net force wrench acting on rigid body i is

$$f_i^{NET} = f_i - f_{i+1} + g_i \tag{3.1}$$

Internal forces maintain the joint constraints between bodies and carry the motive forces of joint actuation. Internal forces act as equal and opposite forces on the two connected bodies. Applied forces are external forces that are applied by the environment. These may include body forces such as gravity or air resistance.

3.2 Joint Torque

Joint torques provide power to the system through the motion of the joints. If a joint with joint velocity $\dot{q}_i$ applies a joint torque τ_i then the power added to the system is

$$W_i = \dot{q}_i^T \tau_i \tag{3.2}$$

This power is the result of the internal force f_i through the joint and the relative velocity between the two connected bodies $v_i - v_{i-1}$. This power is then

$$W_i = (v_i - v_{i-1})^T f_i \tag{3.3}$$

and from (2.9)

$$W_i = (s_i \dot{q}_i)^T f_i \tag{3.4}$$

By equating (3.2) and (3.4) and eliminating $\dot{q}_i$ the joint torque τ_i is

$$\tau_i = s_i^T f_i \tag{3.5}$$

where s_i is the joint axis, and f_i the internal force through the joint.

This definition also shows how joint axes project spatial forces into joint torques. This operation resolves forces to their the net torques about the direction of the joint. This projection is part of a group of projections that transform spatial quantities into components.

3.3 Newton-Euler Equations

The development of the spatial equations of motion follows Chapter 3 of [3]. Newton's 2^{nd} law in spatial form for each body i is

$$\begin{aligned} f_i^{NET} &= \frac{d}{dt}\left(I_i v_i\right) && \text{(3.6a)} \\ &= I_i \frac{d}{dt}\left(v_i\right) + \frac{d}{dt}\left(I_i\right) v_i && \text{(3.6b)} \end{aligned}$$

where f_i^{NET} is the net force wrench acting on i, I_i the spatial inertia and v_i the velocity twist. The rate of change of the spatial inertia is defined in Section 3.3 of [3] as

$$\frac{d}{dt} I_i = I_i \left(v_i \times\right)^T - \left(v_i \times\right)^T I_i \tag{3.7}$$

Substituting (1.27a) and (3.7) into (3.6b) the equations of motion simplify to

$$\begin{aligned} f_i^{NET} &= I_i \left[a_i - \left(v_i \times\right)^T v_i\right] + \left[I_i \left(v_i \times\right)^T - \left(v_i \times\right)^T I_i\right] v_i \\ &= I_i a_i - \left(v_i \times\right)^T I_i v_i \end{aligned} \tag{3.8}$$

Using the net forces (3.1) from the free body diagram the spatial Newton-Euler equations of motion are defined. Note that the spatial cross product $v\times$ in the equations of motion is identical to the spatial cross product in kinematics.

Definition 11 *The equations of motion for a rigid body i are*

$$f_i - f_{i+1} + g_i = I_i a_i - \left(v_i \times\right)^T I_i v_i \tag{3.9}$$

where f_i and f_{i+1} are internal forces on the body through joints i and $i+1$, I_i the spatial inertia, a_i and v_i are spatial acceleration and velocity, and $v_i\times$ is the spatial velocity cross product operator as defined in equation (2.16).

Sometimes the velocity related forces are grouped together so that the equations of motion are expressed as

$$f_i - f_{i+1} = I_i a_i + p_i \tag{3.10}$$

with the bias force

$$p_i = -(v_i\times)^T I_i v_i - g_i \tag{3.11}$$

Note that for the last body in the chain N (3.10) simplifies to

$$f_N = I_N a_N + p_N \tag{3.12}$$

which corresponds to a single free floating rigid body. This type of simplification can be achieved for any body in the chain with the use of the articulated inertias.

3.4 Articulated Inertia Idea

Articulated body inertias were introduced by [3] as means for simplification of the equations of motion. Articulated inertias replace body inertias in the equations of motion such that

$$f_i = A_i a_i + d_i \tag{3.13}$$

where f_i is the internal load from joint i, A_i the articulated inertia of the body, a_i the spatial acceleration and d_i an articulated bias force. Conceptually the articulated equations of motion model a free floating chain as seen in Figure 3.2. To accelerate body i as part of the chain by a_i a force f_i is needed. The articulated inertia A_i contains the single body inertia I_i and parts of the inertias for the rest of the bodies up the chain. The advantage of the articulated equations of motion (3.13) over the standard equations (3.10) is that the internal forces f_{i+1} from the successive body have

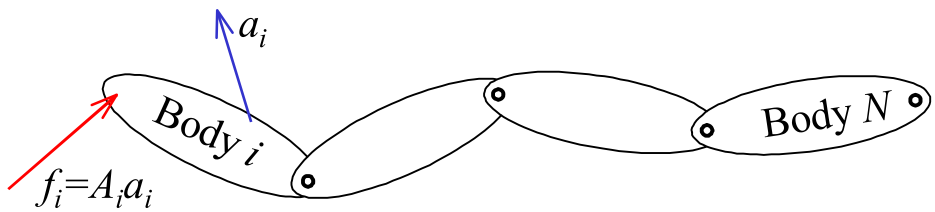

Figure 3.2: The basic idea of articulated inertias is to model all the bodies from body i to the tip of a chain as one single body. This allows a simpler interaction between the internal force f_i and the resulting acceleration a_i.

been eliminated. Therefore there is a one-to-one relationship between the acceleration a_i and the internal force f_i.

A simple example is shown in Figure 3.3. A unit acceleration in the x direction along the joint of the sphere requires a force of magnitude m applied to the cylinder. On the other hand, a unit acceleration in the y direction away from the joint requires a force of magnitude $2m$ applied to the cylinder. The 3×3 articulated inertia submatrix relating linear acceleration and forces is

$$A_{3\times3} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{bmatrix} \tag{3.14}$$

The apparent mass of the system depends on the direction in which an applied force acts. This is achieved by generalizing the inertia matrix to increase the number of independent quantities. Similar effects are observed when moments are applied on the system and the resulting angular accelerations are observed. In general, the

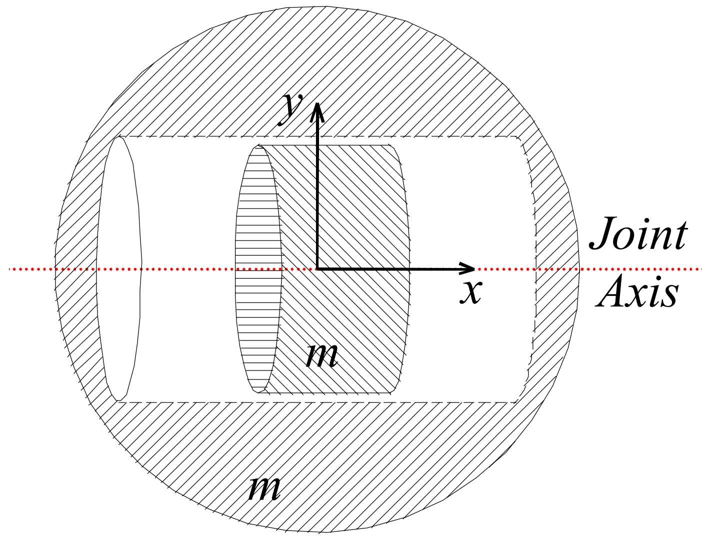

Figure 3.3: Example of articulated body inertia. A sphere of mass m has a cylindrical cutout in the x direction. Inside there is a sliding cylindrical body also of mass m. The direction of an applied force to the cylinder changes the apparent mass of the system. A force in the y direction is results in an apparent mass of $2m$. A force in the x direction changes the apparent mass to m. The directional dependence on apparent mass is encapsulated into the 6×6 articulated inertia matrix.

calculations of articulated inertias can be very complex. The subject of topology and graph theory is often utilized to establish efficient recursions for the articulated inertias. When systems consist of serial chains of rigid bodies the calculations revert to a single recursion down the chain.

3.5 Recursive Articulated Inertias

A solution to calculating articulated inertias is developed in [3]. This is recursive method from the tip of the chain down to the base. It is assumed that all joint axes

s_i, joint velocities $\dot{q}_i$, velocity twists v_i, body inertias I_i, bias accelerations κ_i, joint torques τ_i and bias forces p_i are known from (2.9), (2.19) and (3.11).

For the terminal body the equations of motion (3.10) are

$$f_N = I_N a_N + p_N \tag{3.15}$$

Comparing to the articulated equations of motion (3.13)

$$f_N = A_N a_N + d_N \tag{3.16}$$

it is apparent that the articulated inertia for the terminal body is always

$$A_N = I_N \tag{3.17}$$

and the articulated bias force

$$d_N = p_N \tag{3.18}$$

To calculate recursively the articulated inertia for a body i in the middle of the chain it is assumed that the articulated inertia for body $i+1$ is known such that

$$f_{i+1} = A_{i+1} a_{i+1} + d_{i+1} \tag{3.19}$$

where f_{i+1} is the internal force, A_{i+1} the articulated inertia, a_{i+1} the acceleration of body $i+1$ and d_{i+1} the articulated bias force. Knowing A_{i+1} and d_{i+1} the articulated inertia A_i and bias force d_i is calculated from the equations of motion and the kinematics. The equation of motion for body i (3.10) is

$$f_i = f_{i+1} + I_i a_i + p_i \tag{3.20}$$

The acceleration kinematics (2.17) for body $i+1$ is

$$a_{i+1} = a_i + s_{i+1}\ddot{q}_{i+1} + \kappa_{i+1} \tag{3.21}$$

The joint torque (3.5) for joint $i+1$ is equal to

$$\tau_{i+1} = s_{i+1}^T f_{i+1} \tag{3.22}$$

Substituting (3.21) into (3.19) and then into (3.22) yields

$$\tau_{i+1} = \left(s_{i+1}^T A_{i+1} s_{i+1}\right)\ddot{q}_{i+1} + s_{i+1}^T\left(A_{i+1}\left(a_i + \kappa_{i+1}\right) + d_{i+1}\right) \tag{3.23}$$

which is solved for the joint accelerations $\ddot{q}_{i+1}$ as

$$\ddot{q}_{i+1} = \left(s_{i+1}^T A_{i+1} s_{i+1}\right)^{-1}\left[\tau_{i+1} - s_{i+1}^T\left(A_{i+1}\left(a_i + \kappa_{i+1}\right) + d_{i+1}\right)\right] \tag{3.24}$$

Then the joint accelerations are back substituted into (3.21) relating the acceleration of body i to that of body $i+1$ as

$$a_{i+1} = s_{i+1}\left(s_{i+1}^T A_{i+1} s_{i+1}\right)^{-1}\left[\tau_{i+1} - s_{i+1}^T\left(A_{i+1}\left(a_i + \kappa_{i+1}\right) + d_{i+1}\right)\right] + a_i + \kappa_{i+1} \tag{3.25}$$

The acceleration of body $i+1$ is back substituted into the articulated equations of motion (3.19) as

$$f_{i+1} = \left(s_{i+1}^+\right)^T \tau_{i+1} + \left(1 - s_{i+1}s_{i+1}^+\right)^T\left(A_{i+1}\left(a_i + \kappa_{i+1}\right) + d_{i+1}\right) \tag{3.26}$$

where s_{i+1}^+ is the weighted pseudo-inverse of the joint space s_{i+1}

$$s_{i+i}^+ = \left(s_{i+1}^T A_{i+1} s_{i+1}\right)^{-1} s_{i+1}^T A_{i+1} \tag{3.27}$$

Finally, the internal force f_{i+1} (3.26) is substituted into the equations of motion (3.20) for body i to yield expressions for the articulated quantities

$$f_i = A_i a_i + d_i \tag{3.28}$$

where

$$A_i = I_i + \left(1 - s_{i+1} s_{i+1}^{+}\right)^T A_{i+1} \tag{3.29}$$

and

$$d_i = p_i + \left(s_{i+1}^{+}\right)^T \tau_{i+1} + \left(1 - s_{i+1} s_{i+1}^{+}\right)^T \left(A_{i+1} \kappa_{i+1} + d_{i+1}\right) \tag{3.30}$$

With each recursion a new articulated inertia is calculated moving towards the base of the chain.

Equation (3.24) can be used with the acceleration kinematics (2.17) to solve for all the joint accelerations. A complete algorithm for simulating rigid body chains is shown in the next section.

3.6 Recursive Algorithm

1. Assumptions

   - All the joints are modeled using the joint axes s_i.
   - All the spatial inertias I_i are known.
   - All joint velocities $\dot{q}_i$ are known.

- All the joint torques τ_i are known.
- All applied loads g_i are known.

2. Kinematics

   (a) Initial Values

   - The base of the chain is connected to the ground which has zero velocity.

$$v_0 = [0] \tag{3.31}$$

   (b) For all bodies with $i = 1 \ldots N$

   - The velocity twists are

$$v_i = v_{i-1} + s_i \dot{q}_i \tag{3.32}$$

   - The bias accelerations are

$$\kappa_i = v_i \times s_i \dot{q}_i \tag{3.33}$$

   - The bias forces are

$$p_i = -\left(v_i \times\right)^T I_i v_i - g_i \tag{3.34}$$

3. Articulated Inertias

   (a) Initial Values

- The terminal articulated inertia is

$$A_N = I_N \tag{3.35}$$

- The terminal articulated bias force is

$$d_N = p_N \tag{3.36}$$

(b) For all bodies with $i = N - 1 \dots 1$

- Joint pseudo-inverse is

$$s_{i+i}^{+} = \left(s_{i+1}^{T} A_{i+1} s_{i+1}\right)^{-1} s_{i+1}^{T} A_{i+1} \tag{3.37}$$

- The articulated inertia is

$$A_i = I_i + \left(1 - s_{i+1} s_{i+1}^{+}\right)^{T} A_{i+1} \tag{3.38}$$

- The articulated bias force is

$$d_i = p_i + \left(s_{i+1}^{+}\right)^{T} \tau_{i+1} + \left(1 - s_{i+1} s_{i+1}^{+}\right)^{T} \left(A_{i+1} \kappa_{i+1} + d_{i+1}\right) \tag{3.39}$$

4. Joint Accelerations

(a) Initial Values

- The base of the chain is connected to the ground which has zero acceleration.

$$a_0 = [0] \tag{3.40}$$

(b) For all bodies with $i = 1 \ldots N$

- The joint acceleration is

$$\ddot{q}_i = \left(s_i^T A_i s_i\right)^{-1} \left[\tau_i - s_i^T \left(A_i \left(a_{i-1} + \kappa_i\right) + d_i\right)\right] \tag{3.41}$$

- The acceleration twist is

$$a_i = a_{i-1} + s_i \ddot{q}_i + \kappa_i \tag{3.42}$$

Once all the joint accelerations are determined, they can be integrated to find the values of the joint velocities and positions at subsequent times.

3.7 Questions

There are some important questions evident at this point that are the primary driving forces behind this research:

1. *"What is the physical interpretation of the terms used to calculate articulated inertias?"* Articulated inertias are an essential part of spatial multibody dynamics and are cryptic at this point.

2. *"What are the reaction forces caused by the joint constraints?"* Knowing just the joint accelerations is not enough to completely describe multibody systems. Reaction forces are just as important, and a systematic way of extracting such information is needed. This is accomplished through projections in later chap-

ters, where the relationship between reaction forces and articulated inertias is explored.

3. *"What other structures can be modeled besides linear chains of rigid bodies?"* Linear chains are just a small subset of multibody systems. The greater set contains tree structures and branches, kinematic loops, and constraints to the environment.

4. *"What are the obvious advantages of the spatial notation?"* The initial lure into the spatial notation is the recursive nature and the potential computational efficiency. Other potential benefits are the compactness of the notation and the rich information delivered with each solution.

CHAPTER 4

GRAPHICAL VIEW OF PLANAR DYNAMICS

In this chapter planar dynamics are linked to geometrical constructions. Twists are represented as points in the plane, and wrenches as lines. Representing the kinematics and dynamics graphically requires ways of visualizing the addition of two or more points and the addition of two or more lines. Finally, the idea that forces cause accelerations and accelerations require forces is viewed as a way of mapping points to lines and vice versa. Also some examples of planar joints are presented describing the specific accelerations and forces they require.

4.1 Planar Twists

In mechanics, planar motion of rigid body is generally a rotation about a particular point called a pole. When this pole is located at infinity the motion is a pure translation. This is the projection of a zero pitch twist on a plane. This projection ignores any linear velocity out of the plane and all angular velocities along the plane. The easiest way to perform this projection is through the planar subspace matrix

$$D = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \bar{0} \\ \bar{0} & \bar{0} & \hat{k} \end{bmatrix} \tag{4.1}$$

where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the three unit vectors of a Cartesian coordinate system. If a general spatial twist expressed at the origin as

$$v_{3D} = \begin{bmatrix} \bar{v} \\ \bar{\omega} \end{bmatrix} \tag{4.2}$$

with the component vectors

$$\bar{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{4.3}$$

and

$$\bar{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{4.4}$$

then the planar twist is calculated with

$$v_{2D} = D^T v_{3D} \tag{4.5}$$

which is equal to

$$v_{2D} = \begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} \tag{4.6}$$

This planar velocity twist now conveys important information about the motion of the rigid body. One is the magnitude of the twist which is equal to ω_z. The other is the location of the pole of the twist. This location is the point where the twist has zero linear velocity and it represents the point the rigid body is rotating about. If this location is point P, finding it requires the twist transformations projected on a plane. These transformations for the spatial case are described in (1.56) and in the planar case are defined as

$$X_{2D} = D^T X_{3D} D \tag{4.7}$$

with X_{3D} any spatial transformation and D the planar subspace matrix. Then the planar transformation matrix X_P from the origin to any point $P = (x_P, y_P)$ is defined as

$$X_P = \begin{bmatrix} 1 & 0 & -y_P \\ 0 & 1 & x_P \\ 0 & 0 & 1 \end{bmatrix} \tag{4.8}$$

Now the location of the twist pole is found by solving the following equations

$$\left\{ \begin{array}{c} v_x - y_P \omega_z = 0 \\ v_y + x_P \omega_z = 0 \end{array} \right\} \tag{4.9}$$

yielding the solution for the pole P as

$$(x_P, y_P) = \left(-\frac{v_y}{\omega_z}, \frac{v_x}{\omega_z} \right) \tag{4.10}$$

This point P and the magnitude ω_z completely describe a planar rigid body motion. Since twists describe rigid body motion, planar twists must contain information about point P. In Chapter 5 planar twists are viewed as a set of homogeneous coordinates for P.

4.2 Twist Addition

In looking at the kinematic equations for multibody systems frequently, two or more relative twists need to be added to yield a resultant twist. In a plane this operation is the geometrical addition of two or more points. Such operations are often encountered in projective geometry, but are easy to describe even without the explicit help of such an abstract mathematical tool.

The addition of two planar velocities twists v_1 and v_2 results in a planar twist v_3. Understanding the geometrical properties of the resulting twist is the topic covered in this section.

4.2.1 Addition of two rotations

Figure 4.1 graphically shows the addition of two planar velocity twists. Given two velocities v_1 and v_2, each a rotation about points $Q_1 = (x_{Q1}, y_{Q1})$ and $Q_2 = (x_{Q2}, y_{Q2})$ with magnitudes of ω_1 and ω_2, then their sum $v_3 = v_1 + v_2$ has magnitude

$$\omega_3 = \omega_1 + \omega_2 \tag{4.11}$$

located at a point

$$P = Q_1 + \beta (Q_2 - Q_1) \tag{4.12}$$

with

$$\beta = \frac{\omega_2}{\omega_1 + \omega_2} \tag{4.13}$$

Thus P is a linear combination of Q_1 and Q_2 which means that it lies on the line through these two points. The proportionality ratio β indicates the distance of P from Q_1 relative to the distance of Q_2 from Q_1. In addition, the proportionality ratio β also defines the rotational velocity ω_2 relative to the sum $\omega_1 + \omega_2$.

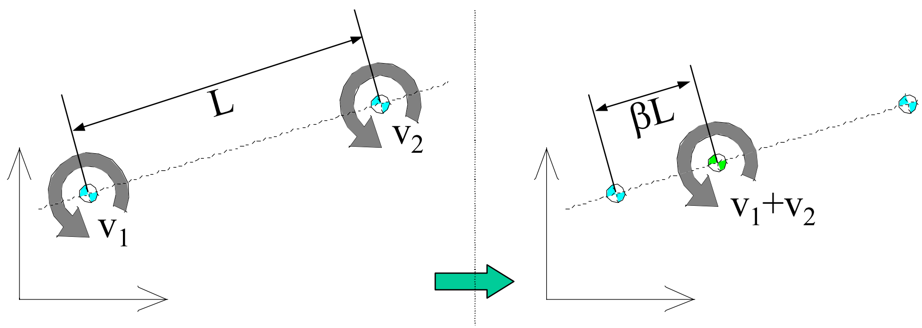

Figure 4.1: The two pure rotations v_1 and v_2 add up to a pure rotation with the pole proportionally along the line that connects the centers of rotation.

This calculation is equivalent to the problem in statics where a load applied on a simply supported beam is countered by two reaction forces at the two ends of the beam as seen on Figure 4.2. The relative strength of each reaction force is proportional to the proximity of the applied force F. The proportionality ratio is

$$\beta = \frac{R_2}{R_1 + R_2} \tag{4.14}$$

and the load is

$$F = R_1 + R_2 \tag{4.15}$$

These are similar to (4.13) and (4.11). In planar dynamics the relative strength of each rotation is proportional to the proximity of the resulting rotation.

The result may not always be a rotation. In the case where $\omega_1 = -\omega_2$ the proportionality ratio becomes $\beta = \infty$ making the result a pure translation to the perpendicular of the line through Q_1 and Q_2. Sometimes a translation is described as a pure rotation about an infinite distant point.

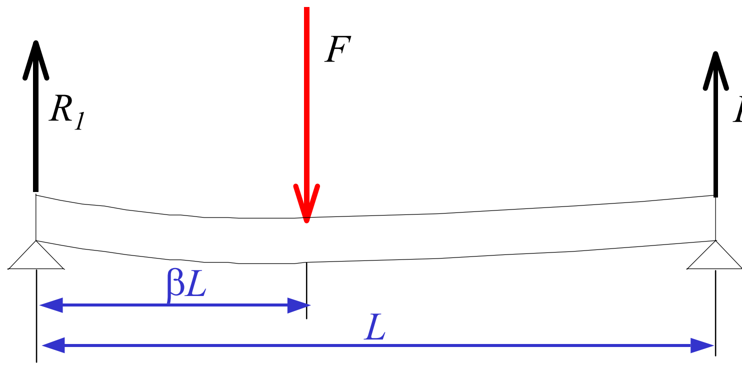

Figure 4.2: Lever rule in statics. The relative magnitide of a reaction force R_1 is proportional to the proximity of the applied force F.

4.2.2 Addition of a rotation with a translation

The addition of a rotation to a translation is shown in Figure 4.3. Given a rotation v_1 about a point $Q = (x_Q, y_Q)$ with magnitude ω, and a translation v_2 with speed u along the direction $e = [\cos\lambda, \sin\lambda]$, then their sum is also a rotation with magnitude ω but located at point P,

$$P = Q + \rho n \tag{4.16}$$

where $n = [-\sin\lambda, \cos\lambda]$ is the direction perpendicular to e, and $\rho = u/\omega$ the distance away from Q.

It is not possible for the result to be a translation. The only special case is when both v_1, v_2 are translations and so $v_3 = v_1 + v_2$ represents the addition of two translations. Such a case results in a translation along the direction of the vector sum of the two translations. An extremely degenerate case arises when the two translations are equal in strength but run in opposite directions resulting in a zero twist. This

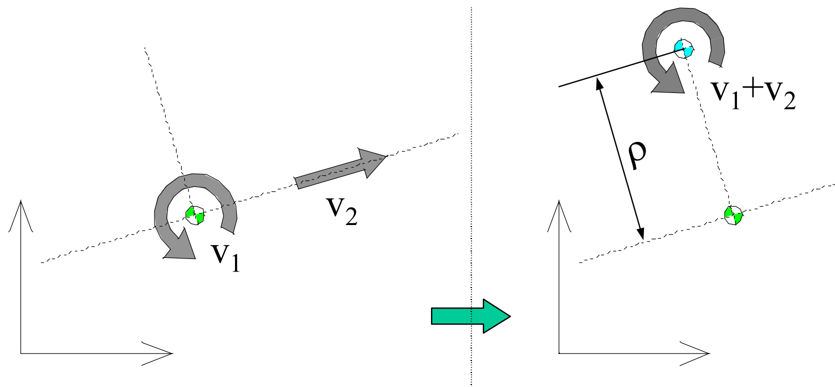

Figure 4.3: A pure rotation v_1 and a translation v_2 add up to a pure rotation with the pole along a line perpendicular to v_2 and through the center of v_1. The distance along this line from the center of v_1 is equal to $\rho = u/\omega$.

represents no motion and is neither a rotation or a translation.

4.3 Planar Wrenches

In planar rigid body dynamics loads represent the application of a net force along a particular line called a polar. A pure couple represents a special case where the polar moves to infinity. Normally the polar line represents the collection of points with zero net moment about them. This is the planar projection of a zero pitch wrench. This projection is also performed by the projection matrix D since it ignores all forces out of the plane, and all moments along the plane. If a general spatial wrench expressed on the origin is

$$f_{3D} = \begin{bmatrix} \bar{f} \\ \bar{\tau} \end{bmatrix} \tag{4.17}$$

with the component vectors

$$\bar{f} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \tag{4.18}$$

and

$$\bar{\tau} = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} \tag{4.19}$$

then the planar wrench is calculated with

$$f_{2D} = D^T f_{3D} \tag{4.20}$$

which is equal to

$$f_{2D} = \begin{bmatrix} f_x \\ f_y \\ \tau_z \end{bmatrix} \tag{4.21}$$

This planar force wrench conveys important information about the loading of the rigid body. One is the magnitude of the net force which is equal to

$$F = \sqrt{f_x^2 + f_y^2} \tag{4.22}$$

The other is the location and direction of the wrench polar. This line represents the location and direction of an equipollent pure force. Finding any point P on the polar requires the planar wrench transformation matrix. This transformation matrix is projected from (1.58) with

$$X_{2D}^{-T} = D^T X_{3D}^{-T} D \tag{4.23}$$

The planar wrench transformation matrix from the origin to any point $P = (x_P, y_P)$ by component is

$$X_P^{-T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y_P & -x_P & 1 \end{bmatrix} \tag{4.24}$$

A wrench f expressed on the origin as

$$f = \begin{bmatrix} f_x \\ f_y \\ \tau_z \end{bmatrix} \tag{4.25}$$

when transformed on a point $P = (x_P, y_P)$ is

$$\begin{aligned} f_P &= X_P^{-T} f \\ &= \begin{bmatrix} f_x \\ f_y \\ y_P f_x - x_P f_y + \tau_z \end{bmatrix} \end{aligned} \tag{4.26}$$

If point P is located somewhere along the axis of application of f then the net moment of f_P has to be equal to zero and thus the equation of the application axis is given by

$$\tau_z + y_P f_x - x_P f_y = 0 \tag{4.27}$$

By normalizing the first two components of f the application axis has a direction

$$e = \begin{bmatrix} \frac{f_x}{F} & \frac{f_y}{F} \end{bmatrix} \tag{4.28}$$

where

$$F = \sqrt{f_x^2 + f_y^2} \tag{4.29}$$

The scalar F represents the magnitude of the wrench.

If point P is the closest point of the axis to the origin then

$$P = nd \tag{4.30}$$

where

$$d = \frac{\tau_z}{F} \tag{4.31}$$

and

$$n = \begin{bmatrix} \frac{f_y}{F} & -\frac{f_x}{F} \end{bmatrix} \tag{4.32}$$

The scalar d represents the shortest distance of the line to the origin and n the unit direction vector perpendicular to the axis.

Now the distance d of the axis to the origin, its direction e, and the magnitude of the wrench F describe the loading of the rigid body completely. In Chapter 5 it is shown that a planar wrench is a set of homogeneous coordinates for the polar line through P.

4.4 Wrench Addition

Looking at the equations of motion and the free body diagrams of multibody systems, often two or more wrenches need to be added to yield a net wrench. This operation on a plane is the geometrical addition of two or more lines. The addition of two planar force wrenches f_1 and f_2 results in a planar wrench f_3. Understanding the magnitude, direction, and position of this wrench is the topic of this section.

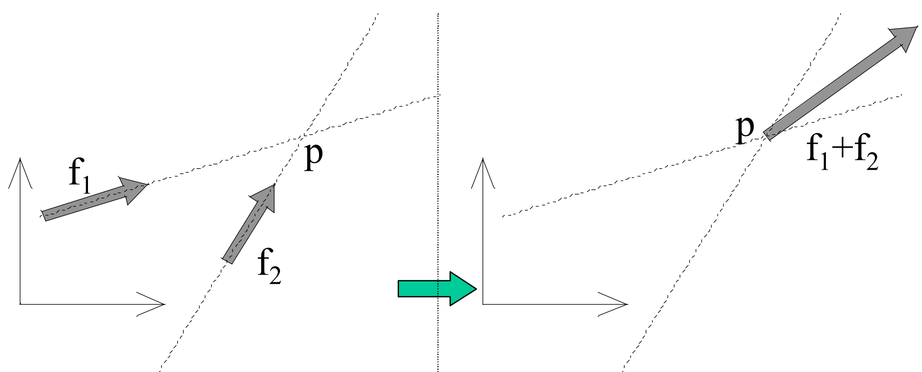

Figure 4.4: Two forces f_1 and f_2 add according to the parallelogram rule, where the net force passes through the point of intersection of the two lines and the magnitudes add vectorially.

4.4.1 Addition of two pure forces

Figure 4.4 graphically show the addition of two general force wrenches. Given two planar forces

$$f_1 = \begin{bmatrix} f_{1x} \\ f_{1y} \\ \tau_1 \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} f_{2x} \\ f_{2y} \\ \tau_2 \end{bmatrix} \tag{4.33}$$

with a directions e_1 and e_2, magnitudes F_1 and F_2, and positions d_1 and d_2 the polar of the resulting wrench $f_3 = f_1 + f_2$ passes through the point P which is the common point of the application axes of f_1 and f_2. Point P is calculated from (4.27) applied to f_1 and f_2 to form the set

$$\left\{ \begin{array}{l} \tau_{1z} + y_P f_{1x} - x_P f_{1y} = 0 \\ \tau_{2z} + y_P f_{2x} - x_P f_{2y} = 0 \end{array} \right\} \tag{4.34}$$

The solution is

$$\begin{aligned} P &= (x_p, y_p) \\ &= \left(\frac{y_1 - y_2}{e_1 - e_2}, \frac{e_2}{e_1 - e_2} y_1 - \frac{e_1}{e_1 - e_2} y_2 \right) \end{aligned} \tag{4.35}$$

where

$$e_1 = \frac{f_{1y}}{f_{1x}} \quad \text{and} \quad e_2 = \frac{f_{2y}}{f_{2x}} \tag{4.36}$$

and

$$y_1 = -\frac{\tau_1}{f_{x1}} \quad \text{and} \quad y_2 = \frac{\tau_2}{f_{x2}} \tag{4.37}$$

The scalars e_1 and e_2 are the slopes of the lines for f_1, f_2 and y_1 and y_2 the interceptions of the lines with the y-axis.

The direction of the pole is

$$e_3 = \frac{1}{F_3}\left[\begin{array}{cc} f_{x1} + f_{x2} & f_{y1} + f_{y2} \end{array}\right] \tag{4.38}$$

where

$$F_3 = \sqrt{\left(f_{x1} + f_{x2}\right)^2 + \left(f_{y1} + f_{y2}\right)^2} \tag{4.39}$$

The scalar F_3 is the magnitude of $f_1 + f_2$.

The minimum position of the resulting line with the origin is $d_3 = \left(\tau_1 + \tau_2\right)/F_3$

4.4.2 Addition of two parallel forces

When the two lines of action are parallel the resulting wrench is also parallel. It lies proportionally between the initial wrenches as seen in Figure 4.5. Looking down the lines of action, it appears similar to the twist addition of two pure rotations, since the axes of rotation are always parallel.

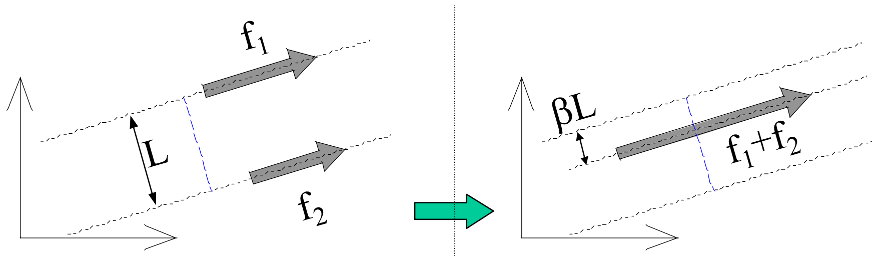

Figure 4.5: The sum of two parallel forces is similar to the addition of two pure rotations. The result is located proportionally between the two componets according to the rule of the lever.

Starting from two parallel forces

$$f_1 = \begin{bmatrix} F_1 \cos\lambda \\ F_1 \sin\lambda \\ \tau_1 \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} F_2 \cos\lambda \\ F_2 \sin\lambda \\ \tau_2 \end{bmatrix} \tag{4.40}$$

their sum $f_3 = f_1 + f_2$ has the same direction

$$e_3 = \begin{bmatrix} \cos\lambda & \sin\lambda \end{bmatrix} \tag{4.41}$$

as f_1 and f_2. The magnitude F_3 is

$$F_3 = F_1 + F_2 \tag{4.42}$$

The distance d_3 to the polar is

$$d_3 = d_1 + \beta\,(d_2 - d_1) \tag{4.43}$$

where β is the proportionality ratio

$$\beta = \frac{F_2}{F_1 + F_2} \tag{4.44}$$

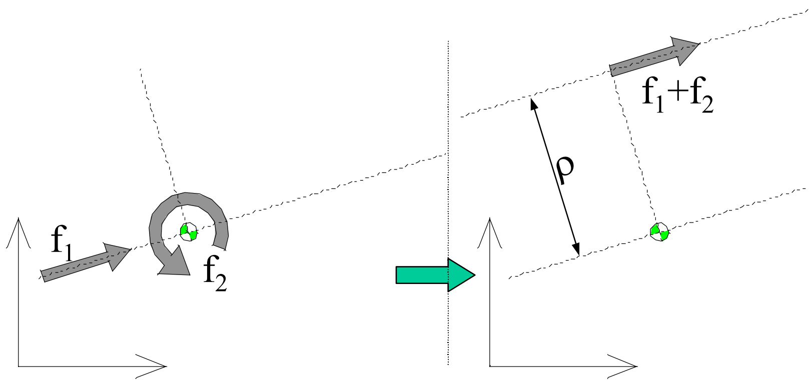

Figure 4.6: A force and a couple create a parallel force located a distance ρ away. This distance increases as the couple becomes stronger.

and d_1, d_2 the polar distances

$$d_1 = \frac{\tau_1}{F_1} \quad \text{and} \quad d_2 = \frac{\tau_2}{F_2} \tag{4.45}$$

The similarities between the addition of two parallel lines, and two points indicate that there is some kind of common behavior which may even apply to the less geometrical features of twists and wrenches.

4.4.3 Addition of a force and a moment

Given a pure force f_1 and a pure couple f_2 the sum $f_3 = f_1 + f_2$ is a pure force. It is parallel to f_1 at a perpendicular distance

$$\rho = \frac{\tau}{F} \tag{4.46}$$

where F is the magnitude of force f_1 and τ the magnitude of the pure couple f_2. The magnitude of f_3 is equal to F.

4.5 Velocity Kinematics

The velocity kinematics in planar form is very easy to visualize and understand. Figure 4.7 graphically shows the planar kinematics equation

$$v_i = v_{i-1} + s_i \dot{q}_i \tag{4.47}$$

as defined in (2.9). Shown here is the most general case involving only rotational joints. The reason rotational joints are the most general case is because any prismatic joint can be modeled as revolute joint located at infinity. In the following figures there are two connected bodies, a connecting joint, and the center of gravity of the body that is being examined. The velocity v_i is the sum of the previous body velocity v_{i-1} and the relative velocity about the joint $s_i \dot{q}_i$. Thus the pole of v_i lies somewhere on the line that connects the poles for v_{i-1} and $s_i \dot{q}_i$. The exact location on the line depends on the relative strength of the two components.

If all the joint velocities $\dot{q}_i$ are known, this method can be applied recursively to find the velocities of all the bodies graphically. Looking at Figure 4.7, the joint is at distance k_i away from the velocity pole of the previous body. Along the same line, the new velocity pole is located a distance $\beta_i k_i$ away, where β_i is the proportionality ratio

$$\beta_i = \frac{\dot{q}_i}{\dot{q}_i + \omega_{i-1}} \tag{4.48}$$

Also, ω_i and ω_{i-1} are the magnitudes of the velocity twists v_i and v_{i-1} where

$$\omega_i = \omega_{i-1} + \dot{q}_i \tag{4.49}$$

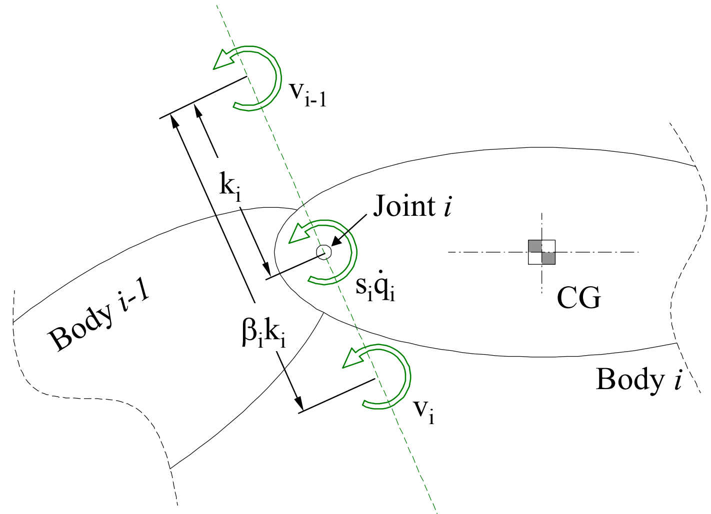

Figure 4.7: Planar velocity kinematics. The velocity of any body is the sum of the velocity of the previous body and the relative velocity between them. The resulting velocity always lies on the connecting line.

To model planar prismatic joints the process is similar. The addition of a rotation with a translation is seen in Figure 4.3. The addition of the rotation v_{i-1} to the translation $s_i\dot{q}_i$ yields an offset rotation by a distance

$$\rho = \frac{\dot{q}_i}{\omega_{i-1}} \tag{4.50}$$

The magnitude of the new velocity remains unchanged as

$$\omega_i = \omega_{i-1} \tag{4.51}$$

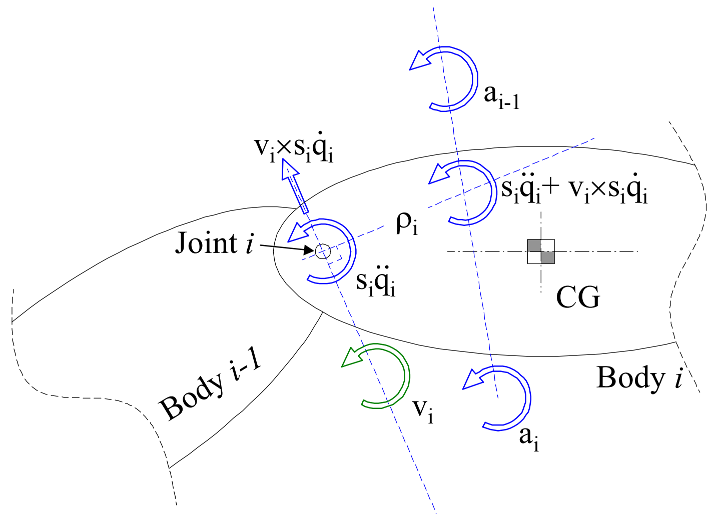

Figure 4.8: The planar acceleration kinematics. Each successive body adds a relative acceleration about the joint and a degenerate bias acceleration on the line between the velocity center and the joint.

4.6 Acceleration Kinematics

The acceleration kinematics in planar form involves a slightly more complicated process than the velocity kinematics. Figure 4.8 shows graphically the acceleration kinematics equation

$$a_i = a_{i-1} + s_i\ddot{q}_i + v_i \times s_i\dot{q}_i \tag{4.52}$$

as defined in (2.17). The acceleration a_i is the sum of three quantities. The first is the acceleration of the previous body a_{i-1}. The second is the relative joint acceleration $s_i\ddot{q}_i$. Both are usually rotations and their poles are easy to find.

The third quantity $v_i \times s_i\dot{q}_i$ is the bias acceleration needed since the joint is moving on a curve around the pole of v_i. This quantity is a translation in the direction of the line that connects v_i and s_i. This is the same line that connects v_i and v_{i-1}. The magnitude of the bias acceleration is

$$|v_i \times s_i \dot{q}_i| = \omega_{i-1} \dot{q}_i k_i \tag{4.53}$$

with k_i the distance shown in Figure 4.7. In Figure 4.8 the combination $s_i \ddot{q}_i + v_i \times s_i \dot{q}_i$ is shown as a rotation at an offset distance of

$$\rho_i = \frac{\omega_{i-1} \dot{q}_i}{\ddot{q}_i} k_i \tag{4.54}$$

away from the joint. The body acceleration a_i is proportionally on the line connecting this bias acceleration and a_{i-1}.

4.7 Equations of Motion

The planar equations of motion are visualized using lines to represent the polars of wrenches, and points to represent the poles of twists. Figure 4.9 shows graphically the Newton-Euler equations of motion

$$f_i^{net} = I_i a_i + p_i \tag{4.55}$$

with

$$p_i = -\left(v_i \times\right)^T I_i v_i \tag{4.56}$$

the bias forces as defined in (3.10). The net force f_i^{net} on the left hand side is equal to the sum of the two forces on the right hand side.

One is the force needed to accelerate the body about a_i and is equal to $I_i a_i$. This mapping of a twist to a wrench through the inertia matrix is an important part of

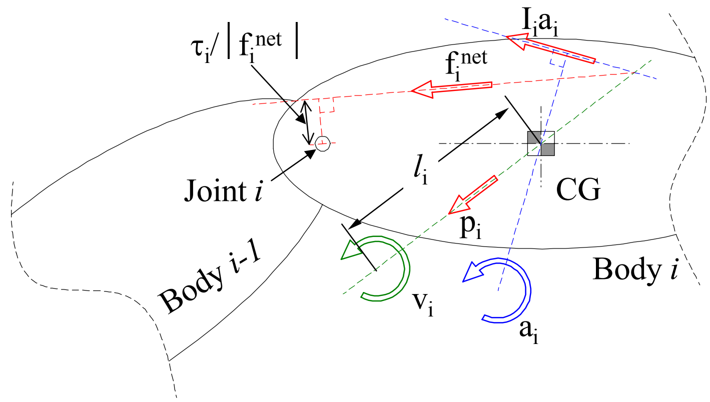

Figure 4.9: Planar Newton-Euler equations of motion. The balance of forces in each body includes the inertial forces resulting from the body acceleration, the centripetal force on a line between the velocity center and the center of mass, and the net internal forces from the joints.

dynamics. In fact, the location of the inertial force $I_i a_i$ is associated with an instant axis of percussion type of calculation and is discussed in the next chapter. In general, the shortest distance between the center of gravity and the axis or percussion is nonzero and depends on the inertial properties of the body.

The other quantity is the centripetal force $p_i = -(v_i \times)^T I_i v_i$. This is the force needed to produce the centripetal acceleration which allows the body to rotate about its instant velocity center v_i. It can be shown that in the planar case this force is through the center of gravity and towards the pole of v_i. This force points towards the instantaneous center of motion keeping the body in orbit around v_i. The magnitude of the centripetal force is $m_i l_i \omega_i^2$ where m_i is the mass of body i and l_i the distance between v_i and the center of gravity.

When the joint is not powered, then the net force f_i^{net} passes through the joint creating zero net moment about the joint. Otherwise a joint torque τ_i appears as the distance ρ_i that the net force has from the joint. This distance is

$$\rho_i = \frac{\tau_i}{F_i^{net}} \tag{4.57}$$

where F_i^{net} is the magnitude of f_i^{net}.

4.8 Examples of Planar Joints

This section contains examples with different joint types. Quantities needed for the solution of planar multibody systems are defined. All of the quantities are expressed in a coordinate system located on the joint and aligned with the direction of the x-axis towards the center of gravity as seen in Figure 4.10. The planar inertia matrix is then

$$I_i = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & m_i C_i \\ 0 & m_i C_i & m_i \left(r_i^2 + C_i^2 \right) \end{bmatrix} \tag{4.58}$$

with r_i the radius of gyration for the body and C_i the distance of the center of gravity from the joint. The general velocity twist of the previous body is

$$v_{i-1} = \begin{bmatrix} v_{x_{i-1}} \\ v_{y_{i-1}} \\ \omega_{z_{i-1}} \end{bmatrix} \tag{4.59}$$

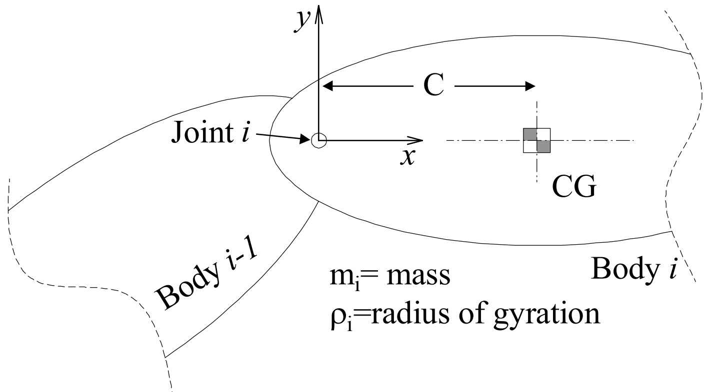

Figure 4.10: Local coordinate system used in planar examples.

and the previous body acceleration twist is

$$a_{i-1} = \begin{bmatrix} a_{x_{i-1}} \\ a_{y_{i-1}} \\ \alpha_{z_{i-1}} \end{bmatrix} \tag{4.60}$$

Before working with the kinematic equations (4.47) and (4.52) and the equations of motion (4.55), the bias accelerations and the bias forces need to be computed. The bias accelerations are defined as

$$\kappa_i = v_i \times s_i \dot{q}_i \tag{4.61}$$

and the centripetal forces

$$p_i = -\left(v_i \times\right)^T I_i v_i \tag{4.62}$$

with the planar spatial cross product operator $v_i\times$ projected as

$$v_i\times = \begin{bmatrix} 0 & -\omega_{z_i} & v_{y_i} \\ \omega_{z_i} & 0 & -v_{x_i} \\ 0 & 0 & 0 \end{bmatrix} \tag{4.63}$$

where the velocity twist is $v_i = \begin{bmatrix} v_{x_i} & v_{y_i} & \omega_{z_i} \end{bmatrix}^T$.

4.8.1 Revolute Joint

A planar pin joint is defined by the joint axis direction

$$s_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.64}$$

and the joint velocity

$$\dot{q}_i = \begin{bmatrix} \dot{\theta}_i \end{bmatrix} \tag{4.65}$$

The velocity kinematics then returns the velocity twist

$$v_i = \begin{bmatrix} v_{x_{i-1}} \\ v_{y_{i-1}} \\ \omega_{z_{i-1}} + \dot{\theta}_i \end{bmatrix} \tag{4.66}$$

which is used to calculate the bias accelerations

$$\kappa_i = \begin{bmatrix} v_{y_{i-1}}\dot{\theta}_i \\ -v_{x_{i-1}}\dot{\theta}_i \\ 0 \end{bmatrix} \tag{4.67}$$

and is a translation from v_i towards the joint s_i. The velocity twist is also used to calculate the bias forces

$$p_i = \begin{bmatrix} -m_i \left[v_{y_{i-1}} \left(\omega_{z_{i-1}} + \dot{\theta}_i \right) + C_i \left(\omega_{z_{i-1}} + \dot{\theta}_i \right)^2 \right] \\ m_i v_{x_{i-1}} \left(\omega_{z_{i-1}} + \dot{\theta}_i \right) \\ C_i m_i v_{x_{i-1}} \left(\omega_{z_{i-1}} + \dot{\theta}_i \right) \end{bmatrix} \tag{4.68}$$

The ratio of the third component to the second is C. The second component is a y-axis force and the third component is the net moment about a point a distance C away in the positive x-axis. The center of gravity is located a distance C away and therefore p_i represents a force through the center of gravity.

This is verified by transforming p_i to the center of gravity with the help of the transformation matrix (4.24).

4.8.2 Prismatic Joint

A planar prismatic joint is defined by the joint axis direction

$$s_i = \begin{bmatrix} \cos\lambda_i \\ \sin\lambda_i \\ 0 \end{bmatrix} \tag{4.69}$$

where λ_i is the orientation angle of the slider. With the joint velocity

$$\dot{q}_i = [u_i] \tag{4.70}$$

the velocity kinematics returns the velocity twist

$$v_i = \begin{bmatrix} v_{x_{i-1}} + u_i \cos\lambda_i \\ v_{y_{i-1}} + u_i \sin\lambda_i \\ \omega_{z_{i-1}} \end{bmatrix} \tag{4.71}$$

This velocity twist is used to calculate the bias acceleration

$$\kappa_i = \begin{bmatrix} -\omega_{z_{i-1}} u_i \sin\lambda_i \\ \omega_{z_{i-1}} u_i \cos\lambda_i \\ 0 \end{bmatrix} \tag{4.72}$$

The bias acceleration is a translation perpendicular to s_i. The velocity twist is also used to calculate the bias forces

$$p_i = \begin{bmatrix} -m_i\omega_{z_{i-1}}\left(v_{y_{i-1}} + u_i \sin\lambda_i\right) - m_i C_i \omega_{z_{i-1}}^2 \\ m_i\omega_{z_{i-1}}\left(v_{x_{i-1}} + u_i \cos\lambda_i\right) \\ m_i\omega_{z_{i-1}} C_i \left(v_{x_{i-1}} + u_i \cos\lambda_i\right) \end{bmatrix} \tag{4.73}$$

which is also a force through the center of gravity.

4.8.3 Revolute and Prismatic Joint

A planar 2DOF pin and prismatic joint is defined by the joint axes directions

$$s_i = \begin{bmatrix} \cos\lambda_i & 0 \\ \sin\lambda_i & 0 \\ 0 & 1 \end{bmatrix} \tag{4.74}$$

where λ_i is the orientation angle of the slider. Using the joint velocities

$$\dot{q}_i = \begin{bmatrix} u_i \\ \dot{\theta}_i \end{bmatrix} \tag{4.75}$$

the velocity twist is

$$v_i = \begin{bmatrix} v_{x_{i-1}} + u_i \cos\lambda_i \\ v_{y_{i-1}} + u_i \sin\lambda_i \\ \omega_{z_{i-1}} + \dot{\theta}_i \end{bmatrix} \tag{4.76}$$

This velocity twist is used to calculate the bias acceleration

$$\kappa_i = \begin{bmatrix} -\left(\omega_{z_{i-1}} u_i \sin\lambda_i - v_{y_{i-1}} \dot{\theta}_i\right) \\ \omega_{z_{i-1}} u_i \cos\lambda_i - v_{x_{i-1}} \dot{\theta}_i \\ 0 \end{bmatrix} \tag{4.77}$$

The bias acceleration is a combination of two translations. The velocity twist is also used to calculate the bias forces

$$p_i = \begin{bmatrix} -m_i \left(v_{y_{i-1}} + u_i \sin\lambda_i\right) \left(\omega_{z_{i-1}} + \dot{\theta}_i\right) - m_i C_i \left(\omega_{z_{i-1}} + \dot{\theta}_i\right)^2 \\ m_i \left(v_{x_{i-1}} + u_i \cos\lambda_i\right) \left(\omega_{z_{i-1}} + \dot{\theta}_i\right) \\ m_i C_i \left(v_{x_{i-1}} + u_i \cos\lambda_i\right) \left(\omega_{z_{i-1}} + \dot{\theta}_i\right) \end{bmatrix} \tag{4.78}$$

which as expected is a force through the center of gravity.

4.8.4 Plane Joint

A planar 2DOF twin prismatic joint is defined by the joint axes directions

$$s_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.79}$$

Using the joint velocities

$$\dot{q}_i = \begin{bmatrix} u_{x_i} \\ u_{y_i} \end{bmatrix} \tag{4.80}$$

the velocity twist is

$$v_i = \begin{bmatrix} v_{x_{i-1}} + u_{x_i} \\ v_{y_{i-1}} + u_{y_i} \\ \omega_{z_{i-1}} \end{bmatrix} \tag{4.81}$$

This velocity twist is used to calculate the bias acceleration

$$\kappa_i = \begin{bmatrix} -u_{y_i}\omega_{z_{i-1}} \\ u_{x_i}\omega_{z_{i-1}} \\ 0 \end{bmatrix} \tag{4.82}$$

The bias acceleration is a translation perpendicular to the relative velocity $s_i \dot{q}_i$. The velocity twist is also used to calculate the bias forces

$$p_i = \begin{bmatrix} -m_i \omega_{z_{i-1}} \left(u_{y_i} + v_{y_{i-1}} \right) - m_i C_i \omega_{z_{i-1}}^2 \\ m_i \omega_{z_{i-1}} \left(u_{x_i} + v_{x_{i-1}} \right) \\ m_i C_i \omega_{z_{i-1}} \left(u_{x_i} + v_{x_{i-1}} \right) \end{bmatrix} \tag{4.83}$$

4.9 Questions

The planar analysis helps to visualize concepts like twists and wrenches. This aids in visualizing some parts of a multibody system. At this point there are no answers to the questions posed in the previous chapter, just some hints, and a few new questions:

1. *"Is it possible to visualize the multibody problem solution?"* The planar explanation of kinematics and dynamics assumes quantities such as joint accelerations and net forces are known. In general these would be considered unknown quantities. But this type of visualization is important because it provides checks and balances in the development of multibody solutions.

2. *"Are twists really points, and wrenches really lines?"* Planar twists and wrenches are expressed as the combination of a geometrical construct and a scalar magnitude. This type of description for points and lines resembles homogeneous coordinate descriptions. The components are different but the idea is similar enough. In the next chapter this observation links planar dynamics

to projective geometry.

3. *"What can geometry tell us about screw algebra?"* In projective geometry there are important ideas that may have significant impact in dynamics. These include mappings and decompositions of geometrical elements, and the fundamental properties of duality.

4. *"How is geometry going to help us answer the questions asked in the end of the previous chapter?"* Geometry provides the tools needed to proceed in a divide and conquer type of approach for dynamics. Ultimately it provides associations and projections for the different quantities involved. Then the flow of information from known to unknown quantities can be assembled piece by piece.

CHAPTER 5

PLANAR PROJECTIVE GEOMETRY

In this chapter homogeneous coordinates are introduced to describe points, lines, and conic curves in a plane. First, the basic ideas of mappings, duality and decompositions are presented. Then a transformation is used to define an alternative set of homogeneous coordinates with the same components as planar screws. This then represents planar twists as points, planar wrenches as lines and planar inertias as conic sections. Then mappings, duality and decompositions are used to examine rigid body velocities, accelerations, forces, and momenta.

5.1 Homogeneous Coordinates

5.1.1 Points in Homogeneous Coordinates

Homogeneous coordinates are used to describe points and lines on a plane. These coordinates contain more information than needed. Planar points and lines require a minimum of two components, but homogeneous coordinates supply three components. Typically a point P is defined as

$$P = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \tag{5.1}$$

which is normalized by dividing by $p_0 \neq 0$ to

$$P = \begin{bmatrix} 1 \\ r\vec{n} \end{bmatrix} \tag{5.2}$$

with

$$\vec{n} = \begin{bmatrix} \cos\lambda \\ \sin\lambda \end{bmatrix} \tag{5.3}$$

and r the distance of the point from the origin. The unit vector $\vec{n}$ is the direction vector from the origin to the point. As seen in Figure 5.1, any point P has homogeneous coordinates

$$P = \begin{bmatrix} p_0 \\ p_0 r \cos\lambda \\ p_0 r \sin\lambda \end{bmatrix} \tag{5.4}$$

for any $p_0 \neq 0$. With homogeneous coordinates a scalar multiple of a point such as $2P$ still designates the same point as P. Normalized homogeneous coordinates with $p_0 = 1$ contain only the geometrical information filtering out the redundancy. These are called the base coordinates for that point. If a point moves to infinity with $r \to \infty$ it can be designated with finite coordinates if $p_0 \to 0$. For example, if P is

$$P = \begin{bmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{bmatrix} \tag{5.5}$$

then it is a point at infinity.

Homogeneous coordinates provide the tools to perform point algebra. The addition of two points

$$P = \begin{bmatrix} 1 \\ p_x \\ p_y \end{bmatrix} \tag{5.6}$$

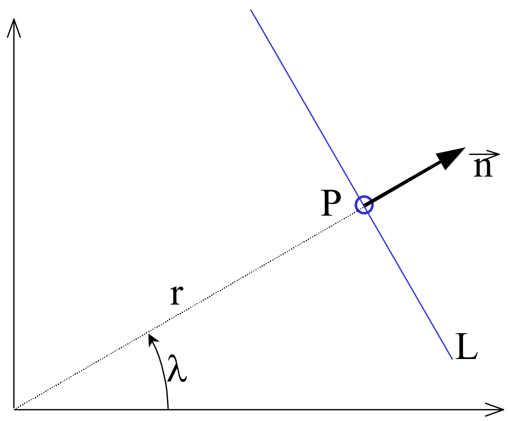

Figure 5.1: Standard homogeneous coordinates. Both points and lines use the unit position vector $\vec{n}$ and the scalar distance r in their definitions.

and

$$Q = \begin{bmatrix} 1 \\ q_x \\ q_y \end{bmatrix} \tag{5.7}$$

results in the mid-point

$$P + Q = \begin{bmatrix} 2 \\ p_x + q_x \\ p_y + q_y \end{bmatrix} \tag{5.8}$$

with base coordinates

$$P + Q \rightarrow \begin{bmatrix} 1 \\ \frac{p_x+q_x}{2} \\ \frac{p_y+q_y}{2} \end{bmatrix} \tag{5.9}$$

Point addition is also used when a linear combination two points is needed such as $P = Q_1 + \beta\,(Q_2 - Q_1)$ with β a scalar proportionality ratio.

5.1.2 Lines in Homogeneous Coordinates

A typical a line L is defined as

$$L = \begin{bmatrix} l_0 \\ l_1 \\ l_2 \end{bmatrix} \tag{5.10}$$

which is normalized by dividing by $l = \sqrt{l_1^2 + l_2^2}$ to

$$L = \begin{bmatrix} -r \\ \vec{n} \end{bmatrix} \tag{5.11}$$

with

$$\vec{n} = \begin{bmatrix} \cos\lambda \\ \sin\lambda \end{bmatrix} \tag{5.12}$$

and r the minimum distance of the line from the origin. The unit vector $\vec{n}$ is the outward perpendicular to the direction of the line from the origin. As seen in Figure 5.1, line L has homogeneous coordinates

$$L = \begin{bmatrix} -rl \\ l\cos\lambda \\ l\sin\lambda \end{bmatrix} \tag{5.13}$$

for any $l \neq 0$. With homogeneous coordinates a scalar multiple of a line such as $2L$ still designates the same line as L. Without loss of generality the homogeneous coordinates for lines can also be defined as

$$L = \begin{bmatrix} r \\ -\vec{n} \end{bmatrix} \tag{5.14a}$$

or

$$L = \begin{bmatrix} rl \\ -l\cos\lambda \\ -l\sin\lambda \end{bmatrix} \tag{5.15}$$

The reason for this change of convention is so that linear forms between points and lines takes on meaningful values as seen in the next section.

Normalized homogeneous coordinates with $l = 1$ contain only the geometrical information filtering out the redundancy. These are called the base coordinates for that line. If a line moves to infinity with $r \to \infty$ it can be designated with finite coordinates if $l \to 0$. For example, if L is

$$L = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5.16}$$

then it is the line at infinity.

Line coordinates are defined such that the equation of that line is given by

$$P^T L = 0 \tag{5.17}$$

where P is any point on the plane. For example if

$$P = \begin{bmatrix} 1 \\ p_x \\ p_y \end{bmatrix} \tag{5.18}$$

then the equation of the line is

$$P^T L = 0 \qquad \Leftrightarrow \qquad r - p_x \cos\lambda - p_y \sin\lambda = 0 \tag{5.19}$$

This is the equation for the line L shown in Figure 5.1, with r the minimum distance of the line to the origin and $(\cos\lambda, \sin\lambda)$ the unit vector normal to the line. Analogous to point algebra, line algebra can be performed to calculate linear combinations of lines.

Two unit independent lines L_1 and L_2 have a linear combination of

$$L = L_1 + \beta(L_2 - L_1) \tag{5.20}$$

where β is a proportionality ratio. This represents a pencil of lines through the intersection point P. The proof is simple. Point P can always be found by solving two scalar equations

$$\left\{ \begin{array}{c} P^T L_1 = 0 \\ P^T L_2 = 0 \end{array} \right\} \tag{5.21}$$

Then point P also belongs to any linear combination L since

$$P^T L = P^T L_1 + P^T \beta(L_2 - L_1) \tag{5.22a}$$

$$= P^T L_1 + \beta(P^T L_2 - P^T L_1) \tag{5.22b}$$

$$= 0 \tag{5.22c}$$

5.1.3 Point and Line Algebra

Between an arbitrary point P and an arbitrary line L the linear form

$$t = P^T L \tag{5.23}$$

is proportional to the minimum distance between them. If the normalized homogeneous coordinates are used for both P and L then t is equal to the minimum distance. This presumes the convention for line coordinates shown in (5.14a) is used. Otherwise the minimum distance would be equal to $-t$. If the point and the line are coincident then $t = 0$. Therefore the expression

$$P^T L = 0 \tag{5.24}$$

is interpreted as either all the points P that lie on the line L, or all the lines L that pass through the point P. This dual interpretation makes the notation powerful and relevant for dynamics. A similar duality exists in spatial screw algebra between twists and wrenches.

Homogeneous coordinates can be used in the calculation of lines that join two points, or points that meet two lines. The sets of equations needed to solve such problems are a pair of linear forms. These can be also viewed as projections.

Finding the line L that joins two points P and Q with

$$P = \begin{bmatrix} 1 \\ p_x \\ p_y \end{bmatrix} \tag{5.25}$$

and

$$Q = \begin{bmatrix} 1 \\ q_x \\ q_y \end{bmatrix} \tag{5.26}$$

requires the two equations

$$\left\{ \begin{array}{c} P^T L = 0 \\ Q^T L = 0 \end{array} \right\} \tag{5.27}$$

By subtracting these two equations the direction normal of the line is

$$\sin\lambda = -\left(p_x - q_x\right)/d \tag{5.28}$$

$$\cos\lambda = \left(p_y - q_y\right)/d \tag{5.29}$$

where

$$d = \sqrt{\left(p_x - q_x\right)^2 + \left(p_y - q_y\right)^2} \tag{5.30}$$

The minimum distance of the line to the origin is

$$r = \left(p_y q_x - p_x q_y\right)/d \tag{5.31}$$

In homogeneous coordinates this line is

$$L = \begin{bmatrix} p_y q_x - p_x q_y \\ -\left(p_y - q_y\right) \\ p_x - q_x \end{bmatrix} \tag{5.32}$$

Dually finding the point P where two lines L_1 and L_2 meet requires the two equations

$$\left\{ \begin{array}{c} P^T L_1 = 0 \\ P^T L_2 = 0 \end{array} \right\} \tag{5.33}$$

These are solved as a system of two equations with two unknowns and result in the point

$$P = \begin{bmatrix} \sin(\lambda_1 - \lambda_2) \\ r_2 \sin\lambda_1 - r_1 \sin\lambda_2 \\ r_1 \cos\lambda_2 - r_2 \cos\lambda_1 \end{bmatrix} \tag{5.34}$$

with the pairs r_1, λ_1 and r_2, λ_2 the defining properties of the two lines respectively.

5.2 Conic Mappings and Duality

5.2.1 Conics and Algebraic Mappings

Besides points and lines, conic sections also have homogeneous coordinates. These curves are used to map points to lines and vice versa in projective geometry. The mappings have both geometrical and algebraic meaning. A conic C has six independent homogeneous coordinates and is represented with a symmetric 3×3 matrix

$$C = \begin{bmatrix} c_0 & c_3 & c_4 \\ c_3 & c_1 & c_5 \\ c_4 & c_5 & c_2 \end{bmatrix} \tag{5.35}$$

This conic maps a point P to a line L through matrix multiplication so that

$$L = CP \tag{5.36}$$

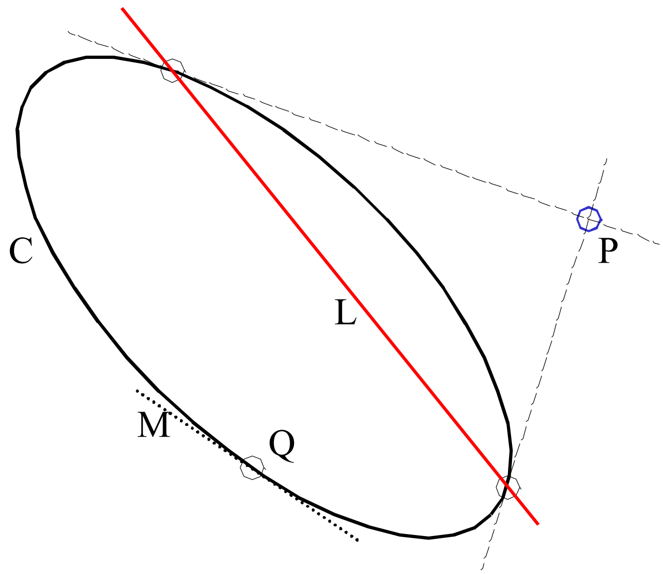

Figure 5.2: Mapping of an exterior point P to a line L through a conic C. A tangential point Q maps to a tangential line M through Q.

and the inverse

$$P = C^{-1}L \tag{5.37}$$

is a unique one-to-one mapping.

This one-to-one mapping is also defined geometrically using the graphical representation of conic sections. Any point maps to a line through a geometric construction. There are three set of points relative to the conic: inside points, coincident points, and outside points. The first two are shown in Figure 5.2 as points P and Q. The third is shown in Figure 5.3 as point P. In order to understand these mappings the conic C must be associated with a curve on the plane. The equation of the curve is given by the collection of points Q that have

$$Q^T C Q = 0 \tag{5.38}$$

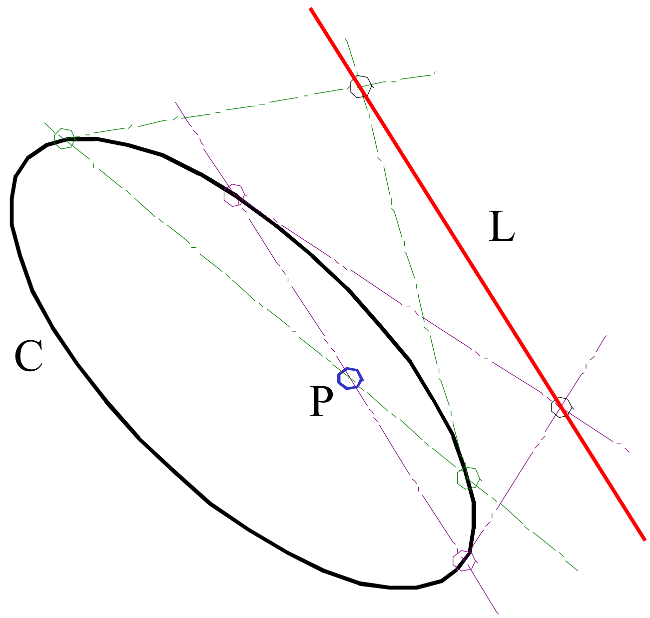

Figure 5.3: Mapping of an interior point P to a line L through a conic C. Any two arbitrary points on L maps to two lines that intersect at P.

Any point Q which is incident to its map $M = CQ$ is incident with the conic C. The curve C is defined by this collection of points Q. The dual to this is defined by the equation

$$M^T C^{-1} M = 0 \tag{5.39}$$

which defines all the lines M that are tangent to the conic at the mapped points $Q = C^{-1}M$. The curve C is then defined by a collection of tangent lines. This means that any point Q which is coincident to the conic C maps to a line L which is tangent to the conic and coincident to Q.

5.2.2 Conics and Geometrical Mappings

A full rank conic C can be a circle, an ellipse, a hyperbola, or a parabola. The mappings and geometric constructions work for either of these types of curves.

Exterior points to the conic map as shown in Figure 5.2. The geometric construction of the map between the point P and the line L is:

1. Draw the two lines that pass through P and are tangent to the conic.
2. Each tangent intersects the conic on one point.
3. Connect the two intersection points to form line L.

This of course may work in reverse to map an interior line to an exterior point. An interior line is one that intersects the conic, in two distinct points.

Interior points to the conic map as shown in Figure 5.3. The geometric construction of the map between the point P and the line L is:

1. Draw two arbitrary lines through P which intersect the conic on two points each.
2. For each set of two points draw the two tangent lines with the conic.
3. Each set of two tangent lines joins on one point.
4. Two points are thus defined from the two sets of two tangent lines.
5. Connect these two points to form line L.

From (5.38) it is obvious that in order to have real solutions for Q, the symmetric matrix C must not be positive definite. If it is positive definite then no real Q can be coincident to the curve C. Therefore it would be impossible to sketch the curve on the real plane. Despite the fact that a positive definite C represents an imaginary conic, it is still capable of mapping real points into real lines and vice versa. This is important because the positive definite inertia matrix maps twists into wrenches. Interestingly, a point P at infinity maps to a line through the center of the conic, and the line at infinity to a point on the center of the conic. This works for both real and imaginary conics. The center of a conic can always be defined as the real point Q that minimizes $Q^T CQ$ (Q is in base coordinates).

The closer a point P moves to the conic curve the closer its mapped line moves to the curve too. In the limit, where the point becomes incident to the curve, the mapped line becomes tangent to the curve and incident to the point.

5.2.3 Duality and Decompositions

The construction for interior points is based on the principle of correlation:

Theorem 12 *Any line L on a point Q maps to point $P = CL$ on the line $M = C^{-1}Q$.*

Proof. If line M passes through P then $P^T M = 0 \Leftrightarrow (CL)^T C^{-1}Q = 0 \Leftrightarrow L^T Q = 0$ which is only true if point Q is on line L. ■

This demonstrates that all algebraic equations between points and lines have dual forms when all the components are mapped with the same conic. For example, the minimum distance between a point P and a line M is ordinarily defined as

$$t = P^T M \tag{5.40}$$

Dual to this is the definition

$$t = L^T Q \tag{5.41}$$

where line $L = C^{-1}P$ and point $Q = CM$. This demonstrates dually that linear forms are invariant to conic mappings and so either set of points and lines is equally sufficient.

This duality provides uniform ways of decomposing points and lines. If a point P on the plane is a linear combination of three base points Q_1, Q_2 and Q_3 so that

$$P = u_1 Q_1 + u_2 Q_2 + u_3 Q_3 \tag{5.42}$$

then any line $L = CP$ on the plane can be decomposed dually by three base lines $M_1 = CQ_1$, $M_2 = CQ_2$ and $M_3 = CQ_3$ so that

$$L = u_1 M_1 + u_2 M_2 + u_3 M_3 \tag{5.43}$$

with the same proportionality ratios.

In Figure 5.4 a special case is shown where Q_2 and Q_3 are coincident to the conic. Also the tangent lines through Q_2 and Q_3 intersect at Q_1. By definition then $Q_2^T C Q_2 = 0$ and $Q_3^T C Q_3 = 0$ but also $Q_1^T C Q_2 = 0$ and $Q_1^T C Q_3 = 0$.

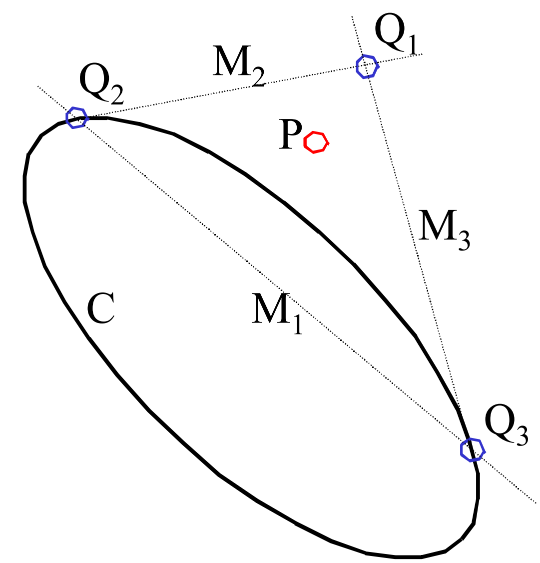

Figure 5.4: Special decomposition of a point P so the components are easily extracted with simple projections.

By premultiplying (5.42) with $Q_1^T C$, $Q_2^T C$ and $Q_3^T C$ the projections

$$Q_1^T CP = Q_1^T CQ_1 u_1 \tag{5.44a}$$

$$Q_2^T CP = Q_2^T CQ_3 u_3 \tag{5.44b}$$

$$Q_3^T CP = Q_3^T CQ_2 u_2 \tag{5.44c}$$

are used to solve for the scalar components

$$u_1 = \left[\left(Q_1^T CQ_1\right)^{-1} Q_1^T C\right] P \tag{5.45a}$$

$$u_2 = \left[\left(Q_3^T CQ_2\right)^{-1} Q_3^T C\right] P \tag{5.45b}$$

$$u_3 = \left[\left(Q_2^T CQ_3\right)^{-1} Q_2^T C\right] P \tag{5.45c}$$

Dual to this is the line decomposition from (5.43) with components

$$u_1 = \left[\left(M_1^T C^{-1} M_1 \right)^{-1} M_1^T C^{-1} \right] L \tag{5.46a}$$

$$u_2 = \left[\left(M_3^T C^{-1} M_2 \right)^{-1} M_3^T C^{-1} \right] L \tag{5.46b}$$

$$u_3 = \left[\left(M_2^T C^{-1} M_3 \right)^{-1} M_2^T C^{-1} \right] L \tag{5.46c}$$

There are other similar ways of decomposing points and lines. All use particular points that lie on mapped lines resulting in some sort of decoupling in the projection equations. In planar dynamics such relationships already exist between joint axes and reaction forces. The fact that reaction forces provide zero power is used to decompose accelerations and forces. In the following sections the link between geometrical points and lines with physical accelerations and forces is made.

5.3 Alternative Homogeneous Coordinates

5.3.1 Transformation To Alternative Coordinates

Standard homogeneous coordinates are used to describe points and lines in the plane. In addition, planar twists and wrenches may also be used to designate points and lines through their poles and polars. Changing from one notation to the other is performed with the help of a 3×3 transformation matrix T,

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{5.47}$$

The transformation matrix is orthonormal since $T^T T = 1$.

5.3.2 Points in Alternative Coordinates

Points in the alternative homogeneous coordinates are defined by transforming (5.4) with T to yield

$$P = \begin{bmatrix} p_0 r \sin\lambda \\ -p_0 r \cos\lambda \\ p_0 \end{bmatrix} \tag{5.48}$$

The normalized coordinates using planar vectors are

$$P = \begin{bmatrix} -r\vec{e} \\ 1 \end{bmatrix} \tag{5.49}$$

where

$$\vec{e} = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \end{bmatrix} \tag{5.50}$$

is the unit vector perpendicular to $\vec{n}$ as seen in Figure 5.5. The transformation matrix performs a $\pi/2$ counter-clockwise rotation of the planar vectors components. The new point coordinates seem a bit awkward, but on closer examination they resemble a planar twist located at P and expressed at the origin. Using the definition of the pole (4.10) and the planar twist in (4.6), a homogeneous point is equal to

$$\begin{bmatrix} p_0 r \sin\lambda \\ -p_0 r \cos\lambda \\ p_0 \end{bmatrix} = \begin{bmatrix} \omega_z y_p \\ -\omega_z x_p \\ \omega_z \end{bmatrix} \tag{5.51}$$

If all the components are converted into polar coordinates with

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} r \sin\lambda \\ r \cos\lambda \end{bmatrix} \tag{5.52}$$

then the twist is equal to the point if

$$p_0 = \omega_z \tag{5.53}$$

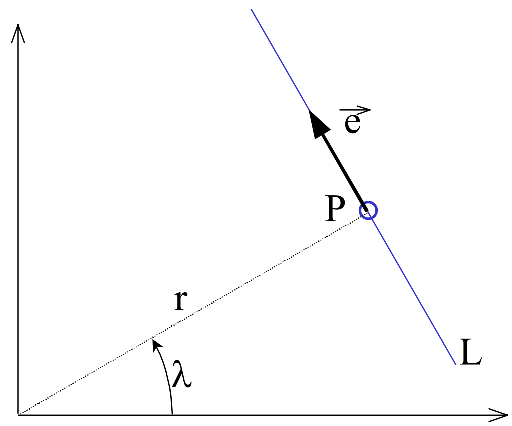

Figure 5.5: Alternate homogeneous coordinates. Both points and lines use the unit direction vector $\vec{e}$ and the scalar distance r in their definitions.

5.3.3 Lines in Alternative Coordinates

Lines in the alternative homogeneous coordinates are defined by transforming (5.14a) with T to yield

$$L = \begin{bmatrix} -l\sin\lambda \\ l\cos\lambda \\ rl \end{bmatrix} \tag{5.54}$$

The normalized coordinates using planar vectors are

$$L = \begin{bmatrix} \vec{e} \\ r \end{bmatrix} \tag{5.55}$$

These coordinates provide a simpler definition for lines since they use the unit vector $\vec{e}$ along the line and its minimum distance to the origin r. Standard homogeneous coordinates use unit vectors perpendicular to the line. They also resemble planar wrenches along L expressed at the origin. Using the equation for the line (4.27) and the planar wrench in (4.21), a homogeneous line is equal to

$$\begin{bmatrix} -l\sin\lambda \\ l\cos\lambda \\ lr \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ x_p f_y - y_p f_x \end{bmatrix} \tag{5.56}$$

If all the quantities are expressed in polar coordinates such that

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = r\vec{n} = \begin{bmatrix} r\cos\lambda \\ r\sin\lambda \end{bmatrix} \tag{5.57}$$

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = F\vec{e} = \begin{bmatrix} -F\sin\lambda \\ F\cos\lambda \end{bmatrix} \tag{5.58}$$

where

$$F = \sqrt{f_x^2 + f_y^2} \tag{5.59}$$

then the homogeneous line is equal to

$$\begin{bmatrix} -l\sin\lambda \\ l\cos\lambda \\ lr \end{bmatrix} = \begin{bmatrix} -F\sin\lambda \\ F\cos\lambda \\ (r\cos\lambda)(F\cos\lambda) + (r\sin\lambda)(F\sin\lambda) \end{bmatrix} \tag{5.60}$$

which is true if $l = F$.

From the viewpoint of geometry, the expression $P^T L = 0$ represents a point P and a line L with zero distance between them. From the viewpoint of dynamics where points are velocities and lines are forces, this equation represents wrenches and twists that produce zero power. These are called reciprocal screws. If s is the unit twist about a joint axis then

$$s^T R = 0 \tag{5.61}$$

defines all possible reaction forces for that joint as the pencil of forces R. Mathematically R is the null-space of s^T and is called the reaction subspace. For example, the prismatic joint

$$s = \begin{bmatrix} \cos\lambda \\ \sin\lambda \\ 0 \end{bmatrix} \tag{5.62}$$

in the direction of $(\cos\lambda, \sin\lambda)$ has the reaction space spanned by

$$R = \begin{bmatrix} -\sin\lambda & 0 \\ \cos\lambda & 0 \\ 0 & 1 \end{bmatrix} \tag{5.63}$$

since $s^T R = 0$. The two basis reactions are a force through the joint, and a pure couple. Any reaction force wrench is a linear combination of these two reactions.

5.4 Inertia Mappings and Decomposition

In projective geometry a conic section C maps points P into lines $L = CP$. In dynamics an inertia matrix I maps acceleration twists a into force wrenches $f = Ia$ as if the body is at rest and $v = 0$. The process is identical, except the inertia matrix I is usually positive definite, and always positive semi-definite. Thus, usually a conic section for I cannot be sketched with real points. However, since inertias are conic mappings they can be used to decompose forces and accelerations.

Inertia mappings define a unique inertial force $f = Ia$ for every acceleration a as seen in Figure 5.6. The acceleration a also defines a point in homogeneous coordinates called the pole of a. The force f defines a line in homogeneous coordinates called the polar of f. The point uniquely maps to the line using the inertia matrix. In dynamics, the polar of f is called the axis of percussion for the pole of a.

A simple planar example demonstrates how to locate the axis of percussion. Consider the body shown in Figure 5.6. A coordinate frame is placed at the acceleration pole a and with the local x-axis along the line that connects a and the center of gravity. If the body has mass m, radius of gyration ρ, and the center of gravity is at distance d then the inertia matrix is

$$I = \begin{bmatrix} m & 0 & 0 \\ 0 & m & dm \\ 0 & dm & m\left(\rho^2 + d^2\right) \end{bmatrix} \tag{5.64}$$

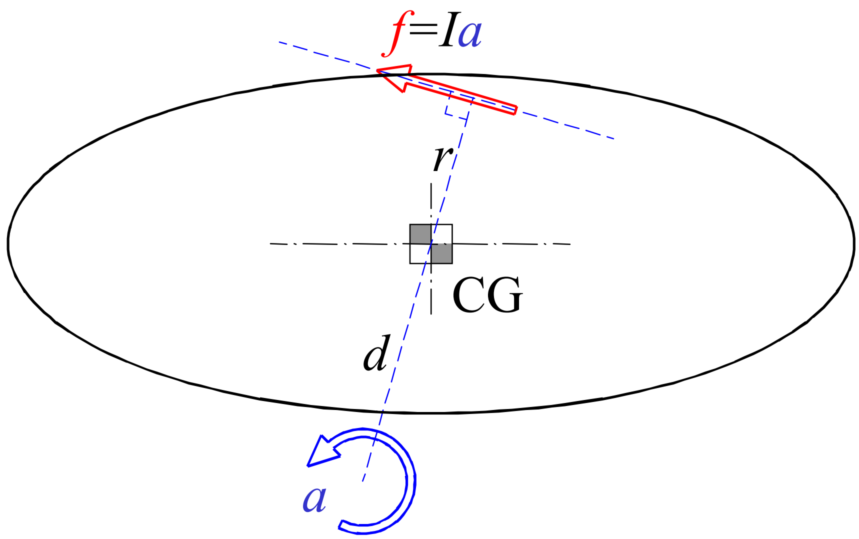

Figure 5.6: Inertia maps accelerations into forces. Accelerations are points and forces are lines on the plane. The line is the axis of percussion of the point.

The force f needed to accelerate the body about a is $f = Ia$ if the body is at rest. If the angular acceleration is α then the acceleration twist is

$$a = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \tag{5.65}$$

and the force required is

$$f = \begin{bmatrix} 0 \\ md\alpha \\ m\left(\rho^2 + d^2\right)\alpha \end{bmatrix} \tag{5.66}$$

Obviously f has no x-axis component and its axis of application is located by normalizing the force wrench. Dividing the wrench by its magnitude $md\alpha$ yields the shortest distance l of the application axis as

$$l = d + \frac{\rho^2}{d} \tag{5.67}$$

The force lies on the so-called "axis of percussion." This axis defines the application line for a force that rotationally accelerates a free body about a specified point. In the world of sports, the effect of the axis of percussion is called the "sweet spot;" where all of the energy of the athlete is transfered most efficiently to the ball. In this case, the rotation caused by f occurs about a. The minimum distance r of the axis of percussion to the center of gravity is

$$\begin{aligned} r &= l - d \\ &= \frac{\rho^2}{d} \end{aligned} \tag{5.68}$$

where ρ is the radius of gyration of the rigid body and d the distance from a to the center of gravity. The farther away a is, the closer f is and vice versa. There are two limiting cases with $d = 0$ or $d = \infty$. In the first case, an acceleration twist about the center of gravity needs a force wrench at infinity which represents a pure couple, and in the second case, an acceleration twist at infinity which represents a pure translation needs a force wrench through the center of gravity.

Figure 5.7 demonstrates mappings in power relations. A powerless force f passes

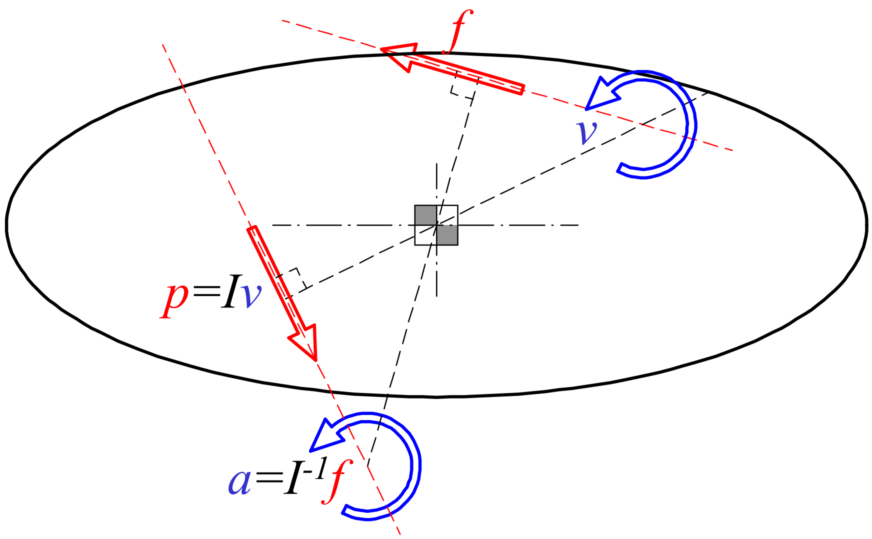

Figure 5.7: Mappings in power relations. A powerless force f passes through the velocity v, or a powerless acceleration a lies on the line of action the momentum p.

through the rigid body velocity v such that

$$v^T f = 0 \tag{5.69}$$

If f represents a force applied on a single rigid body (3.9) relates this force to an acceleration a such that

$$f = Ia - (v\times)^T Iv \tag{5.70}$$

The power relationship expands to

$$v^T f = v^T Ia - v^T (v\times)^T Iv \tag{5.71a}$$

$$= (Iv)^T a \tag{5.71b}$$

$$= p^T a = 0 \tag{5.71c}$$

where

$$p = Iv \tag{5.72}$$

is the momentum wrench.

In fact, in general the power is defined as

$$P = v^T f \tag{5.73}$$

or

$$P = p^T a \tag{5.74}$$

since both calculations are equivalent.

A reaction force f that produces no power has an application axis that passes through the instant center of motion v. Equivalently the resulting acceleration a lies on the application axis of the momentum h. Powerless forces define reaction forces, and powerless accelerations define reactive accelerations. This equivalency is used in the next chapter to decompose forces and accelerations according to reactive and active components. Mappings in power relationships are very important because they intertwine forces, velocities, accelerations and momenta.

5.5 Extension to 3D Spatial Quantities

Projective geometry provides the tools for defining geometrical construction in a general N dimensional space and is not limited to the planar case. In spatial dynamics both twists and wrenches are screw quantities with the dualities, mappings, and projections similar to those in the planar cases. Screws are lines in space with a pitch quantity associated with them. Therefore a minimum of five quantities are needed to describe them. Spatial twists and wrenches are a form of homogeneous coordinates for screws containing one extra redundant coordinate to describe the five dimensional quantity of a screw. Both represent screws because screws are self-dual. On the other hand, planar twists represent points, and planar wrenches lines which also are dual to each other. The principles and concepts developed for the planar cases are easily extended to the spatial case by changing points and lines into screws and increasing the number of homogeneous coordinates needed to describe them from 3 to 6. Reciprocity, duality, and mappings exist between twists and wrenches in both spatial and planar cases.

Spatial inertias map twists into wrenches. Although in the plane they are conics, in the spatial case they are of higher dimension. Spatial inertias still provide unique mappings and are used to dually decompose twists and wrenches according to power relationships. For example, in the previous section the planar power relationships extend to the spatial case simply by redefining the quantities involved as spatial screws. The concept, notation, and mathematics are identical between the two cases.

Often in the following chapters the figures and examples reflect planar cases which are easier to visualize, but the notation and mathematics is also valid for the spatial case. When a quantity such as a joint axis s is depicted in a figure as a point it may represent a planar twist, but it may also represent a spatial twist. Unless all 3 or 6 components are explicitly defined, there is no way of separating the planar and the spatial cases mathematically.

5.6 Questions

1. *"Which part of projective geometry and mappings is most useful in dynamics?"* Homogeneous coordinates in projective geometry are used for decompositions and projections. For example, any point on the plane is a linear combination of three base points. These base points are the subspaces that decompose all the points on a plane. The association between geometry and dynamics described above allows twists and wrenches to be decomposed similarly.

2. *"Which subspaces define components with meaningful information?"* In dynamics it is always meaningful to classify quantities in terms of power. For example, reaction forces are wrenches that provide zero power. The subspaces are defined according to their power relation. With mappings these power relations help define subspaces for any wrench or twist.

CHAPTER 6

SUBSPACES

Power is used to decompose spatial twists and wrenches onto subspaces. The decompositions yield meaningful projections for the components. Applying these projections to the kinematics and the equations of motion result in direct solutions of the unknown quantities. This chapter connects concepts described in previous chapters and provides powerful tools for solving multibody problems. This method of projective dynamics is used in the next chapters for solving multibody chains, trees, and loops. It is also implemented as a MATLAB toolbox using object oriented programming allowing for high level formulation and solution of problems.

6.1 Basic decomposition

Joints in multibody systems are modeled by their joint axis s. This is the subspace of relative motion between the two bodies the joint connects. The joint axis also defines the reaction subspace R as its null-space through the equation

$$s^T R = 0 \tag{6.1}$$

This gives one meaningful subspace for the accelerations and one for the reaction forces. To perform the decomposition, the acceleration and force spaces have to be completed with additional subspaces.

One simple way of completing the planar space is shown in Figure 6.1. The force space $[F]$ is given as

$$[F] = \begin{bmatrix} T & R_1 & R_2 \end{bmatrix} \tag{6.2}$$

where

$$T = Is \tag{6.3}$$

and R_1 and R_2 are any two independent unit reaction forces. The force subspace T represents the axis of percussion for the joint as defined in Section 5.4.

Any force f is decomposed as a linear combination of the subspaces

$$f = T\psi + R_1\mu_1 + R_2\mu_2 \tag{6.4}$$

with ψ, μ_1 and μ_2 the components of the decomposition. Sometimes this is abbreviated to

$$f = T\psi + R\mu \tag{6.5}$$

with ψ the active force component, R the reaction subspace

$$R = \begin{bmatrix} R_1 & R_2 \end{bmatrix} \tag{6.6}$$

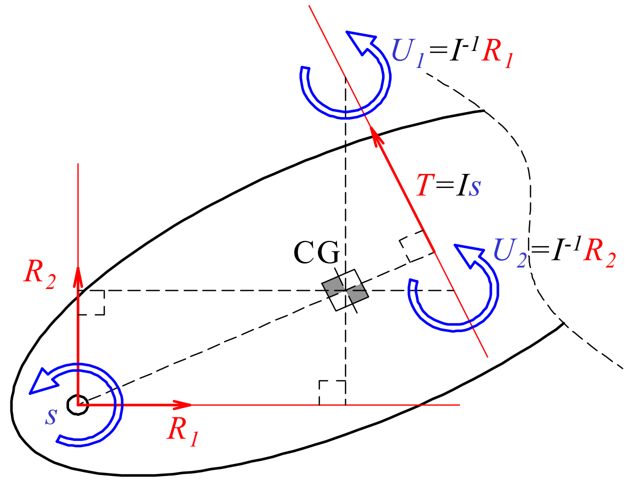

Figure 6.1: Simple planar decomposition of twists and wrenches through three unit forces and their mapped accelerations. Both the joint space s and the reaction subspace R are used.

and μ the reaction force component

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{6.7}$$

This decomposition is meaningful because for a force to accelerate the body around the joint without a reaction force it must lie only in the percussion subspace $f = T\psi$. Sometimes the $T\psi$ component of a force wrench is called the active force component since it does not contribute to the reaction forces.

In the spatial case a joint may have k degrees of freedom and therefore the joint space is

$$s = \begin{bmatrix} s_1 & \dots & s_k \end{bmatrix} \tag{6.8}$$

and the axes of percussion are

$$T = \begin{bmatrix} Is_1 & \dots & Is_k \end{bmatrix} \tag{6.9}$$

Since there are $6 - k$ reaction forces possible the reaction space is

$$R = \begin{bmatrix} R_1 & \dots & R_{6-k} \end{bmatrix} \tag{6.10}$$

such that

$$s^T R = 0 \tag{6.11}$$

The force decomposition is similar to the planar case

$$f = T\psi + R\mu \tag{6.12}$$

but the components are

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_k \end{bmatrix} \tag{6.13}$$

and

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{6-k} \end{bmatrix} \tag{6.14}$$

Since decompositions in the plane (6.5) and space (6.12) have identical notation it is unnecessary to define the two cases separately. Unless the components are defined explicitly the following decompositions and subsequent projections apply to both cases.

The acceleration space $[A]$ is derived as a mapping of the force space $[F]$,

$$\begin{aligned} [A] &= I^{-1}[F] \\ &= \begin{bmatrix} s & U \end{bmatrix} \end{aligned} \tag{6.15}$$

where s is the joint axis and $U = I^{-1}R$ are the complementary accelerations. If the joint is ignored and a reaction force $f = R\mu$ is applied to a stationary body, with $v = 0$, then the resulting acceleration is

$$a = I^{-1}f = U\mu \tag{6.16}$$

In general, acceleration a is expressed as a linear combination of the subspaces

$$a = s\psi + U\mu \tag{6.17}$$

where subspace U is called the reactive acceleration space. The above decomposition is used with any applied wrench f and any body velocity v.

Acceleration and force decompositions are related by $s^T R = 0$ and the dual expression $T^T U = 0$.

One of the problems with the decompositions is the fact that not all subspaces are represented by unit twists or unit wrenches. Although by definition s and R are unit quantities, their maps T and U are not. For example, if s is a single degree of freedom joint, the acceleration component $s\psi$ is easily interpreted as unit twist s multiplied by its magnitude ψ in whatever units is appropriate for the joint. On the other hand, the force quantity $T\psi$ is a non-unit wrench T multiplied by the same ψ in the same acceleration units. The interpretations of T, U, ψ and μ are less direct.

Another problem occurs when projecting the decompositions to extract the components since large coupled matrices need to be inverted. For example, in the planar case shown in Figure 6.1 the projection

$$\begin{aligned} R^T a &= R^T (s\psi + U\mu) \\ &= \left(R^T U\right) \mu \end{aligned} \tag{6.18}$$

is solved for the component μ as

$$\mu = \left(R^T U\right)^{-1} R^T a \tag{6.19}$$

which requires the inversion of

$$\begin{aligned} R^T U &= R^T I^{-1} R \\ &= \begin{bmatrix} R_1^T I^{-1} R_1 & R_1^T I^{-1} R_2 \\ R_2^T I^{-1} R_1 & R_2^T I^{-1} R_2 \end{bmatrix} \end{aligned} \tag{6.20}$$

In Figure 5.4 of the previous chapter the decoupling of the components is possible because two of the points considered are incident to their mapped lines. In addition, the third point considered is lying in the inersection of those two mapped lines. As a result, the projections simplified to the point where each projection is used to calculate one of the components without the need for solving simultaneous systems of equations.

In dynamics, since the inertia matrix is positive defnite, it is impossible that an acceleration a maps to an incident force Ia such that

$$a^T I a = 0 \tag{6.21}$$

As a result, decoupling of the component equations is rare. To avoid this problem U can be chosen such that the matrix $R^T U$ is a diagonal matrix.

If U is chosen such that

$$U^T R = [1]_{(6-k)\times(6-k)} \tag{6.22}$$

and T such that

$$s^T T = [1]_{k\times k} \tag{6.23}$$

then

$$\begin{aligned} R^T a &= R^T (s\psi + U\mu) \\ &= R^T U\mu \\ &= \mu \end{aligned} \tag{6.24}$$

Solving for $\mu = R^T a$ is much simpler than in the example above. Of course, depending on k the degrees-of-freedom of the joint, the result of $U^T R$ and $s^T T$ can either be scalars or identity matrices. In the subsequent development, a number 1 may also designate an identity matrix of appropriate size and a number 0, a zero matrix.

6.2 Decompositions and Pseudo-Inverses

The requirements that $s^T T = 1$ and $R^T U = 1$ are achieved easily with the use of weighted pseudo-inverses. These do not violate the reaction space definition $s^T R = 0$ and its dual $T^T U = 0$.

In Chapter 3 a natural decomposition of the internal forces is derived from (3.26) using the articulated inertia A_i and the joint torque τ_i. This equation is

$$f_i = \left(s_i^+\right)^T \tau_i + \left(1 - s_i s_i^+\right)^T \left(A_i \left(a_{i-1} + \kappa_i\right) + d_i\right) \tag{6.25}$$

where the subspace

$$\left(s_i^+\right)^T = A_i s_i \left(s_i^T A_i s_i\right)^{-1} \tag{6.26}$$

and s_i^+ is called the weighted pseudo-inverse of s_i. The first part of f_i is the active component since

$$\begin{aligned} s_i^T f_i &= s_i^T \left(s_i^+\right)^T \tau_i + s_i^T \left(1 - \left(s_i^+\right)^T s_i^T\right) \left(A_i \left(a_{i-1} + \kappa_i\right) + d_i\right) && (6.27) \\ &= \tau_i + \left(s_i^T - s_i^T\right) \left(A_i \left(a_{i-1} + \kappa_i\right) + d_i\right) && (6.28) \\ &= \tau_i && (6.29) \end{aligned}$$

and thus only τ_i contributes power to the joint.

Therefore if T_i is the weighted pseudo-inverse of a_i,such that

$$T_i = A_i s_i \left(s_i^T A_i s_i\right)^{-1} \tag{6.30}$$

then by definition

$$s_i^T T_i = s_i^T A_i s_i \left(s_i^T A_i s_i\right)^{-1} \quad (6.31)$$

$$= 1 \quad (6.32)$$

Dually, if U_i is the weighted pseudo-inverse of R_i, such that

$$U_i = A_i^{-1} R_i \left(R_i^T A_i^{-1} R_i\right) \quad (6.33)$$

then by definition

$$R_i^T U_i = R_i^T A_i^{-1} R_i \left(R_i^T A_i^{-1} R_i\right) \quad (6.34)$$

$$= 1 \quad (6.35)$$

It is equally valid to define the subspaces using the single rigid body inertia I_i instead of the articulated inertia A_i. The body inertia subspaces are

$$T_i = I_i s_i \left(s_i^T I_i s_i\right)^{-1} \quad (6.36)$$

and

$$U_i = I_i^{-1} R_i \left(R_i^T I_i^{-1} R_i\right)^{-1} \quad (6.37)$$

The choice depends on which equations of motion are been used. In general, when considering multiple connected rigid bodies the articulated inertias are used. Therefore, in the following sections the definition may vary depending on context. If the internal forces are used with (3.10) then I_i is used. If the internal forces are used with (3.13) then A_i is used.

6.2.1 Wrench Decomposition

The force decomposition with pseudo-inverses is

$$f = TQ + R\mu \tag{6.38}$$

where

$$T = As\left(s^T As\right)^{-1} \tag{6.39}$$

and R is the reaction space. Both force components Q and μ have force units as expected.

The power of a body with $v = s\dot{q}$ is

$$\begin{aligned} W &= v^T f && (6.40)\\ &= (s\dot{q})^T (TQ + R\mu) && (6.41)\\ &= \dot{q}^T s^T TQ && (6.42)\\ &= \dot{q}^T Q && (6.43) \end{aligned}$$

6.2.2 Twist Decomposition

The acceleration decomposition with pseudo-inverses is

$$a = s\psi + U\gamma \tag{6.44}$$

where

$$U = A^{-1}R\left(R^T A^{-1} R\right)^{-1} \tag{6.45}$$

and s is the joint space. Both acceleration components ψ and γ have acceleration units as expected.

It is trivial to show that

$$
\begin{aligned}
T^T U &= \left[\left(s^T A s\right)^{-1} s^T A\right]\left[A^{-1} R\left(R^T A^{-1} R\right)^{-1}\right] && (6.46a)\\
&= \left(s^T A s\right)^{-1}\left(s^T R\right)\left(R^T A^{-1} R\right)^{-1} && (6.46b)\\
&= 0 && (6.46c)
\end{aligned}
$$

6.2.3 Planar Inerpretation

Figure 6.2 shows the planar decomposition which differs from the basic decomposition of Figure 6.1 in that the reactive acceleration subspaces U_1 and U_2 are relocated along the axis of percussion T to meet the reaction subspaces R_2 and R_1. This diagonalizes $R^T U$ because as seen in the planar example

$$
R^T U = \begin{bmatrix} R_1^T U_1 & R_1^T U_2 \\ R_2^T U_1 & R_2^T U_2 \end{bmatrix} \qquad (6.47)
$$

$$
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (6.48)
$$

The diagonal terms $R_1^T U_2 = 0$ and $R_2^T U_1 = 0$ represent the fact that U_1 and U_2 lie on R_2 and R_1 respectively.

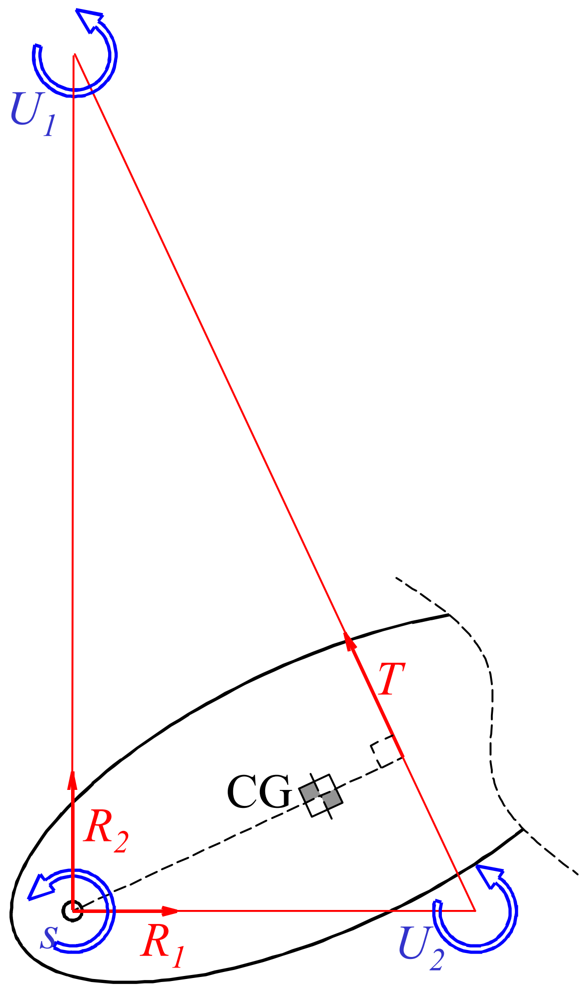

Figure 6.2: Planar decomposition that decouples the projections. The reactive accelerations U_1 and U_2 lie on R_2 and R_1 respectively which diagonilizes the projections. The subspaces use the pseudo-inverses so that their components are in the right units.

6.3 Component Projections

Force components are extracted using projections. Using $s^T R = 0$, $s^T T = 1$ and (6.38)

$$
\begin{aligned}
s^T f &= s^T (TQ + R\mu) && (6.49)\\
&= s^T TQ && (6.50)\\
&= Q && (6.51)
\end{aligned}
$$

In addition, using $T^T U = 0$, $U^T R = 1$ and (6.38)

$$
\begin{aligned}
U^T f &= U^T (TQ + R\mu) && (6.52)\\
&= U^T R\mu && (6.53)\\
&= \mu && (6.54)
\end{aligned}
$$

Acceleration components are also extracted using projections. Using (6.44)

$$
\begin{aligned}
T^T a &= T^T (s\psi + U\gamma) && (6.55)\\
&= T^T s\psi && (6.56)\\
&= \psi && (6.57)
\end{aligned}
$$

and

$$
\begin{aligned}
R^T a &= R^T (s\psi + U\gamma) && (6.58)\\
&= R^T U\gamma && (6.59)\\
&= \gamma && (6.60)
\end{aligned}
$$

6.3.1 Projection Summary

The component projections are

$$Q = s^T f \tag{6.61a}$$

$$\mu = U^T f \tag{6.61b}$$

and

$$\gamma = R^T a \tag{6.62a}$$

$$\psi = T^T a \tag{6.62b}$$

These follow naturally from the definition of the subspaces T and U with pseudo-inverses. These linear forms return either force or acceleration magnitudes. These projections can also be applied to velocities and momenta since their decompositions are similar to accelerations and forces.

6.3.2 Joint Space Inertias

A single pinned rigid body at rest, with $v = 0$, has

$$f = Ia \tag{6.63}$$

Using a general decomposed acceleration (6.44) into the equations of motion (3.10) yields

$$f = Ia \tag{6.64}$$

$$= I\left(s\psi + U\gamma\right) \tag{6.65}$$

Substituting into (6.61a) projects into

$$
\begin{aligned}
Q &= s^T \left(Is\psi + IU\gamma \right) && (6.66) \\
&= \left(s^T Is \right) \psi && (6.67)
\end{aligned}
$$

where $s^T Is$ the joint space inertia of the joint. This relates joint torques Q with active accelerations ψ.

Dually, using a general decomposed force (6.38) into the equations of motion (3.10) yields

$$
\begin{aligned}
a &= I^{-1} f && (6.68) \\
&= I^{-1} \left(TQ + R\mu \right) && (6.69)
\end{aligned}
$$

Substituting into (6.62a) projects into

$$
\begin{aligned}
\gamma &= R^T I^{-1} \left(TQ + R\mu \right) && (6.70) \\
&= \left(R^T I^{-1} R \right) \mu && (6.71)
\end{aligned}
$$

where $R^T I^{-1} R$ the reaction space inverse inertia of the joint. This relates reactive accelerations γ with reaction forces μ.

These are totally decoupled since μ depends only on γ and ψ depends only on Q. For example if a body, is pinned to the ground and the joint accelerates by $\ddot{q} = \psi$ then its acceleration is

$$
a = s\ddot{q} \tag{6.72}
$$

with $\gamma = 0$. Assuming the body is at rest, with $v = 0$, and therefore without bias forces, the reaction forces are

$$\mu = U^T f = U^T I a = 0 \tag{6.73}$$

where U is defined in (6.37). The joint torque needed for such an acceleration is

$$\begin{aligned} Q &= s^T f && (6.74a) \\ &= s^T I a && (6.74b) \\ &= \left(s^T I s\right) \ddot{q} && (6.74c) \end{aligned}$$

6.4 Inertia decomposition

The subspace decompositions are used to decompose a single rigid body inertia I. The subspaces used are accordingly defined in (6.36) and (6.37).

Using the force component projections (6.61a), (6.61b) in the force decomposition (6.38) yields the subspace identity

$$T s^T + R U^T = 1 \tag{6.75}$$

which is used to decompose inertias into two parts

$$I = T s^T I + R U^T I \tag{6.76}$$

Joints are split in two parts with relative motion between them and equal and opposite forces acting on both parts. The forces needed to accelerate each part of the joint are

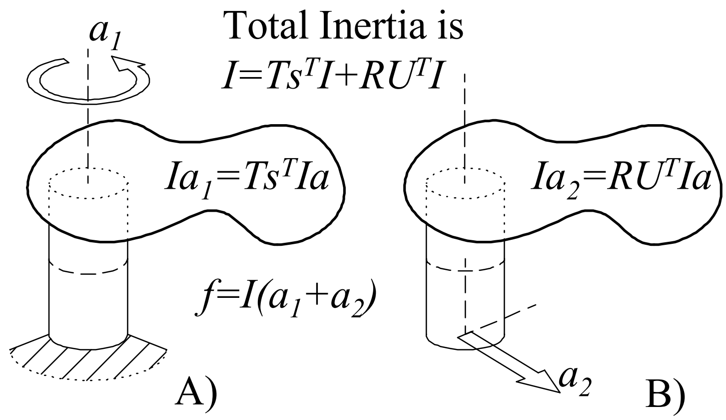

Figure 6.3: Inertias decompose into two parts. A) The inertia that accelerates the body by a_1 which is along the joint. B) The inertia that accelerates the body by a_2 which is along the reactive acceleration space.

seen in Figure 6.3 which describes the inertia decomposition. The first part

$$f_1 = Ia_1 = Ts^T Ia \tag{6.77}$$

represents the active force f_1 needed to accelerate the body by a_1 along the joint when its base is fixed. The second part

$$f_2 = Ia_2 = RU^T Ia \tag{6.78}$$

represents the reactive force f_2 needed to accelerate the base of the joint by a_2 with no acceleration along the joint.

The inverse inertias also decompose into two parts

$$I^{-1} = sT^T I^{-1} + UR^T I^{-1} \tag{6.79}$$

Internal forces are split into two parts. One part contains the reaction forces and the other the active forces. Active forces power the joint while reactions ensure that the

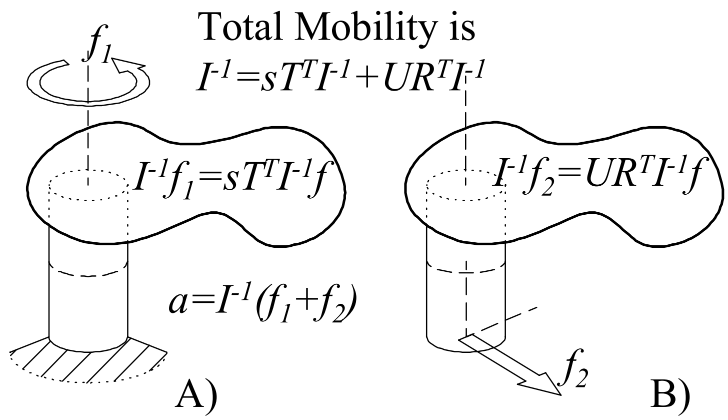

Figure 6.4: Inverse inertias decompose into two parts. A) The inverse inertia that produces force f_1 along the joint. B) The inverse inertia that produces force f_2 along the reaction space.

joint does not accelerate apart. The resulting accelerations from each type of force are seen in Figure 6.4 which describes the inverse inertia decomposition. The active force f_1 results in an acceleration

$$a_1 = I^{-1} f_1 = sT^T I^{-1} f \tag{6.80}$$

which represents the motion of the joint. The reaction force f_2 results in an acceleration

$$a_2 = I^{-1} f_2 = UR^T I^{-1} f \tag{6.81}$$

which represents the motion of the base of the joint.

If a resting body is pinned to the ground and a force f is applied on it then (6.44) is

$$a_1 = s_1 \psi_1 \tag{6.82}$$

and (3.10) is

$$f_1 - f = I_1 a_1 \tag{6.83}$$

If the joint is not powered then $Q_1 = 0$ and the internal force (6.38) is

$$f_1 = R_1 \mu_1 \tag{6.84}$$

The active acceleration projection (6.62b) is

$$\begin{aligned}
\psi_1 &= T_1^T a_1 && (6.85)\\
&= T_1^T I_1^{-1} (f_1 - f) && (6.86)\\
&= T_1^T I_1^{-1} R_1 \mu_1 - T_1^T I_1^{-1} f && (6.87)\\
&= -T_1^T I_1^{-1} f && (6.88)
\end{aligned}$$

since

$$\begin{aligned}
T_1^T I_1^{-1} R_1 &= \left(s_1^T I_1 s_1\right)^{-1} s_1^T I_1 I_1^{-1} R_1 && (6.89)\\
&= \left(s_1^T I_1 s_1\right)^{-1} \left(s_1^T R_1\right) && (6.90)\\
&= 0 && (6.91)
\end{aligned}$$

The body acceleration is then decomposed as

$$\begin{aligned}
a_1 &= s_1 \psi_1 && (6.92)\\
&= -\left(s_1 T_1^T I_1^{-1}\right) f && (6.93)
\end{aligned}$$

which indicates that $\left[s T^T I^{-1}\right]$ is the effective inverse inertia for a constrained body.

If a resting body has acceleration a_1 with no joint torque then (3.10) is

$$f_1 = I_1 a_1 \tag{6.94}$$

and (6.61b) is

$$\begin{aligned} \mu_1 &= U_1^T f_1 && (6.95) \\ &= U_1^T I_1 a_1 && (6.96) \end{aligned}$$

since $Q_1 = 0$ then (6.38) is

$$f_1 = \left(R_1 U_1^T I_1\right) a_1 \tag{6.97}$$

which makes $\left[RU^T I\right]$ the effective inertia of the base of the joint.

The same methodology can be used to decompose articulated inertias using the subspaces defined by (6.30) and (6.33). Then with two joined bodies the effective inertia of one body is added to the articulated inertia of the other body to create the articulated inertia of the system. The generalization of this procedure is in Chapter 7 where the articulated inertia idea is fully expanded for both free floating and constrained chains of rigid bodies.

6.5 Projective Kinematics

The acceleration of a rigid body in a chain of rigid bodies is defined by the acceleration kinematics (4.52) as

$$a_i = a_{i-1} + s_i \ddot{q}_i + \kappa_i \tag{6.98}$$

with a_{i-1} the acceleration of the previous body, s_i the joint axis, v_i the velocity of the body, $\kappa_i = v_i \times s_i \dot{q}_i$ the bias acceleration, $\dot{q}_i$ the joint velocity and $\ddot{q}_i$ the joint acceleration. This acceleration is decomposed into two parts from (6.44)

$$a_i = s_i \psi_i + U_i \gamma_i$$

where s_i is the joint axis and U_i the reactive acceleration space defined by the pseudo-inverse

$$U_i = A_i^{-1} R_i \left(R_i^T A_i^{-1} R_i \right)^{-1} \tag{6.99}$$

Substituting (4.52) into (6.62b) yields

$$\begin{aligned} \psi_i &= T_i^T \left(a_{i-1} + s_i \ddot{q}_i + \kappa_i \right) \\ &= \ddot{q}_i + T_i^T \left(a_{i-1} + \kappa_i \right) \end{aligned} \tag{6.100}$$

This projection relates the absolute acceleration along the joint ψ_i and the relative joint acceleration $\ddot{q}_i$. Given ψ_i the unknown $\ddot{q}_i$ can be calculated easily since all other terms are generally considered to be known. Figure 6.5 describes the difference between the relative and absolute measurements for the joint. Substituting (4.52) into (6.62a) yields

$$\begin{aligned} \gamma_i &= R_i^T \left(a_{i-1} + s_i \ddot{q}_i + \kappa_i \right) \\ &= R_i^T \left(a_{i-1} + \kappa_i \right) \end{aligned} \tag{6.101}$$

This projection calculates the reactive joint accelerations γ_i directly since it doesn't contain any unknowns pertaining to body i. This reactive acceleration is the result

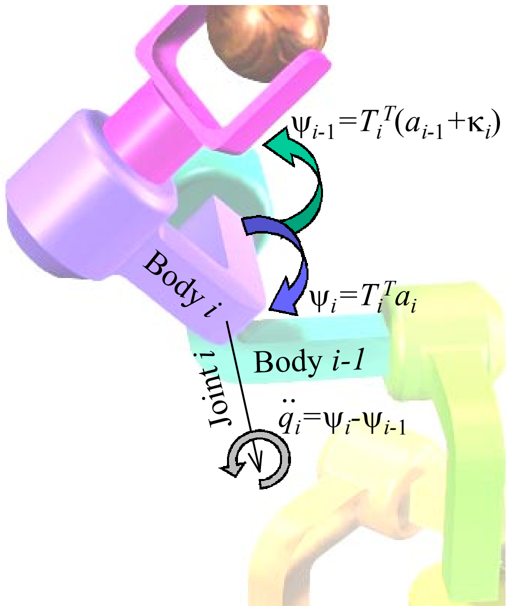

Figure 6.5: The relative joint acceleration $\ddot{q}_i$ is the difference of the two absolute accelerations ψ_i and ψ_{i-1} along joint i. They are both projected from the spatial quantities using the percussion subspace T_i.

of reaction forces. Reaction forces ensure that the base of the joint moves with the previous body and thus the resulting reactive acceleration is directly related to the acceleration of the previous body.

6.6 Projective Dynamics

The equations of motion from (3.9) for a body in a chain of rigid bodies is

$$f_i - f_{i+1} = I_i a_i - (v_i \times)^T I_i v_i \tag{6.102}$$

with f_i and f_{i+1} the internal forces for joints i and $i+1$ respectively, I_i the spatial inertia, a_i the acceleration twist and $-(v_i \times)^T I_i v_i$ the bias forces on the body. The introduction of the articulated inertia idea from (3.13) transforms this equation to

$$f_i = A_i a_i + d_i \tag{6.103}$$

with A_i the articulated inertia of the body and d_i the articulated bias forces. Force wrenches decompose into two parts from (6.38)

$$f_i = T_i Q_i + R_i \mu_i \tag{6.104}$$

where R_i is the reactive force subspace and T_i the active force subspace defined by the pseudo-inverse

$$T_i = A_i s_i \left(s_i^T A_i s_i \right)^{-1} \tag{6.105}$$

The joint torque Q_i is assumed to be a known quantity for each joint. Substituting (6.103) into (6.61b) projects the reaction forces

$$\begin{aligned} \mu_i &= U_i^T \left(A_i a_i + d_i \right) \\ &= U_i^T A_i U_i \gamma_i + U_i^T d_i \end{aligned} \tag{6.106a}$$

Solving equation (6.103) for a_i and substituting into (6.62b) yields

$$\begin{aligned} \psi_i &= T_i^T A_i^{-1} \left(f_i - d_i \right) \\ &= T_i^T A_i^{-1} \left(T_i Q_i - d_i \right) \end{aligned} \tag{6.107}$$

which relates the active acceleration ψ_i directly to the applied torque Q_i.

6.7 Projective Articulated Equations of Motion

Substituting the components γ_i and ψ_i from (6.101) and (6.107) into the acceleration decomposition (6.44) yields an expression for the body acceleration which contains only known quantities. This solution for the body acceleration is

$$a_i = s_i T_i^T A_i^{-1} \left(T_i Q_i - d_i\right) + U_i R_i^T \left(a_{i-1} + \kappa_i\right) \tag{6.108}$$

An equivalent equation with the single rigid body inertias is presented in the next section. The first part is the active acceleration due to the active force $T_i Q_i$ and the bias force d_i projected through the effective mobility $s_i T_i^T A_i^{-1}$. The second part is the reactive acceleration due to the previous body acceleration a_{i-1} and the bias acceleration κ_i projected through the projection matrix $U_i R_i^T$. The reactive acceleration is the acceleration needed to enforce the joint constraint.

Similarly, substituting the component γ_i from (6.101) into μ_i from (6.106a) and then into the force decomposition (6.38) yields an expression for the internal force of the body which contains only known quantities

$$f_i = T_i Q_i + R_i U_i^T \left(A_i \left(a_{i-1} + \kappa_i\right) + d_i\right) \tag{6.109}$$

An equivalent equation with single rigid body inertias is presented in the next section. The first part contains the active forces along the axis of percussion T_i. The second part contains the reaction forces due to the acceleration of the base of the joint a_{i-1} and the bias acceleration κ_i projected through the effective inertia $R_i U_i^T A_i$, and the bias forces d_i projected through $R_i U_i^T$.

Finally the relative joint acceleration is calculated by premultiplying (2.17) with T_i^T and solving for $\ddot{q}_i$

$$\ddot{q}_i = T_i^T \left(a_i - a_{i-1} - \kappa_i \right) \tag{6.110}$$

which is a projection of the relative spatial acceleration through the joint. This joint acceleration is often the end product of most dynamics algorithms. There is always a trade-off between speed and information. The end product for the decomposition methods is not just $\ddot{q}_i$ but the ability to extract any quantity needed, and to observe the flow of information from known to unknown quantities. Equations (6.108) and (6.109) provide all the information needed not only to solve problems in multibody dynamics, but also to observe the relationships between the quantities involved.

These basic equations for multibody dynamics allow for ideas such as articulated inertias or effective inverse inertias to be defined with a few simple steps. In fact, these decomposed solutions for accelerations and forces act as basic building blocks for constructing ideas, methods, algorithms, and solutions to multibody problems. In the next chapter articulated inertias are derived as an example for the decompositions as building blocks.

6.8 Projective Newton-Euler Equations of Motion

Using the Newton-Euler equations of motion (3.10) and the subspace definitions in (6.36) and (6.37) it is possible to solve for the accelerations and forces. The steps are similar to the articulated projections except for the following substitutions

$$A_i \rightarrow I_i \tag{6.111}$$

$$d_i \rightarrow p_i + f_{i+1} \tag{6.112}$$

Equations (6.108) and (6.109) for body inertias are

$$a_i = s_i T_i^T I_i^{-1} \left(T_i Q_i - p_i - f_{i+1}\right) + U_i R_i^T \left(a_{i-1} + \kappa_i\right) \tag{6.113a}$$

$$f_i = T_i Q_i + R_i U_i^T \left(I_i \left(a_{i-1} + \kappa_i\right) + p_i + f_{i+1}\right) \tag{6.113b}$$

These are called the acceleration and force propagators. They are used non recusively to build solutions for constrained rigid body chains which do not require the definition of the articulated inertias. Next chapter contains the definition of articulated inertias and constrained chain equations.

6.9 Planar Example

Before the development of multibody articulated systems an example is presented that illustrates the use of the decompositions as a tool for complex problem solution.

A car has wheelbase L and its center of gravity is located a distance of εL from the front wheels. The car is cornering and accelerating at the same time as seen in Figure 6.6. An applied force f_A acting along the direction of the front wheels is responsible for accelerating the body. This force is a simple way of modeling the torque produced by the engine on the front wheels. The steering angle of the front wheels is such that

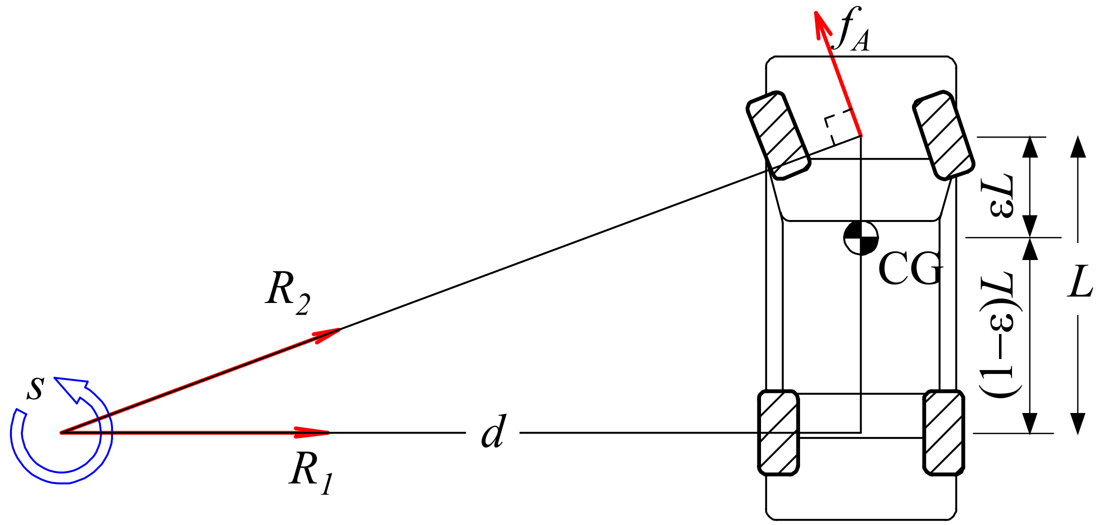

Figure 6.6: Example of a car cornering and accelerating at the same time. The tire forces needed to maintain cornering are calculated with projective dynamics.

the radius of turn is d. The speed of the car measured on the rear wheels is $V = \omega d$ with ω the rotational velocity of the car around the corner.

Using projective dynamics the angular acceleration $\dot{\omega}$ and the magnitudes of the two reaction forces μ_1 and μ_2 are calculated as functions of the angular velocity ω and the applied force f_A. The system is modeled using a revolute joint s on the instant center of rotation. The joint space s and reaction space R are

$$s = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & \frac{d}{\sqrt{d^2+L^2}} \\ 0 & \frac{L}{\sqrt{d^2+L^2}} \\ 0 & 0 \end{bmatrix} \tag{6.114}$$

expressed on the instant center of rotation. The car velocity is calculated directly from (2.9) as

$$v = s\omega = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \tag{6.115}$$

and from (2.19) the bias acceleration κ is

$$\kappa = v \times s\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.116}$$

The inertia matrix of the car, transformed to the joint using (1.68), is

$$I = \begin{bmatrix} m & 0 & -m\left(1-\varepsilon\right)L \\ 0 & m & md \\ -m\left(1-\varepsilon\right)L & md & m\left(\rho^2 + d^2 + \left(1-\varepsilon\right)^2 L^2\right) \end{bmatrix} \tag{6.117}$$

with m the mass and ρ the radius of gyration of the car. The bias force defined in (3.11) as

$$p = -\left(v\times\right)^T I v \tag{6.118}$$

is

$$p = \begin{bmatrix} -md\omega^2 \\ -m(1-\varepsilon)L\omega^2 \\ 0 \end{bmatrix} \tag{6.119}$$

and the applied force wrench is

$$f_A = \begin{bmatrix} -\frac{L}{\sqrt{d^2+L^2}}F_A \\ \frac{d}{\sqrt{d^2+L^2}}F_A \\ \sqrt{d^2+L^2}F_A \end{bmatrix} \tag{6.120}$$

with F_A the magnitude of the force. The equations of motion (3.10) are

$$f + f_A = Ia + p \tag{6.121}$$

with f the internal force for the joint and a the acceleration twist of the car. The internal force is decomposed by (6.38) as

$$f = Ia + p - f_A \tag{6.122}$$

$$= TQ + R\mu \tag{6.123}$$

The active force subspace from (6.30) is

$$T = Is\left(s^T Is\right)^{-1} \tag{6.124}$$

$$T = \begin{bmatrix} -\frac{(1-\varepsilon)L}{\rho^2+d^2+(1-\varepsilon)^2L^2} \\ \frac{d}{\rho^2+d^2+(1-\varepsilon)^2L^2} \\ 1 \end{bmatrix} \tag{6.125}$$

and the reactive acceleration subspace from (6.33) is

$$U = I^{-1}R\left(R^T I^{-1} R\right)^{-1} \tag{6.126}$$

$$U = \begin{bmatrix} 1 & 0 \\ -\frac{d}{L} & \frac{\sqrt{d^2+L^2}}{L} \\ \frac{d^2+(1-\varepsilon)L^2}{L\left(\rho^2+d^2+(1-\varepsilon)^2L^2\right)} & \frac{-d\sqrt{d^2+L^2}}{L\left(\rho^2+d^2+(1-\varepsilon)^2L^2\right)} \end{bmatrix} \tag{6.127}$$

The reactive acceleration component from (6.101) is

$$\gamma = R^T\left(v \times s\omega\right) \tag{6.128}$$

$$\gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.129}$$

which is logical since by definition $a = s\dot{\omega}$.

Since the joint is not powered by a torque then $Q = 0$ and so the angular acceleration from (6.107) is

$$\dot{\omega} = T^T I^{-1} (f_A - p) \tag{6.130}$$

$$\dot{\omega} = \left[\frac{F_A}{m} \frac{\sqrt{d^2 + L^2}}{\rho^2 + d^2 + (1-\varepsilon)^2 L^2} \right] \tag{6.131}$$

The reaction force components from (6.106a) are

$$\mu = U^T I U \gamma + U^T (p - f_A) \tag{6.132}$$

$$\mu = \begin{bmatrix} -\frac{\varepsilon(1-\varepsilon)L^2 - \rho^2}{\rho^2 + d^2 + (1-\varepsilon)^2 L^2} \frac{\sqrt{d^2+L^2}}{L} F_A - \varepsilon d m \omega^2 \\ \frac{\varepsilon(2-\varepsilon)L^2 - \rho^2}{\rho^2 + d^2 + (1-\varepsilon)^2 L^2} \frac{d}{L} F_A - m\omega^2 (1-\varepsilon) \sqrt{d^2 + L^2} \end{bmatrix} \tag{6.133}$$

These represent the actual tire forces needed to maintain the motion of the car. The first component is the force of the rear tires, and the second of the front tires. In general both are negative numbers indicating that the forces act from the tires towards the center of rotation and not away from the center of rotation. In addition, as the applied force F_A increases, the tire forces on the rear tires increase slightly, whereas the tire forces or the front tires decreases slightly.

CHAPTER 7

ARTICULATED INERTIAS

In this chapter articulated inertias are calculated using subspace decomposition. There are two general types of articulated inertia calculations for any body i in a rigid body chain. The first splits the chain on joint i and calculates the acceleration a_i resulting from the internal force f_i as seen in Figure 7.1 where A_i is the articulated inertia for the free floating chain. The second splits the chain at joint $i+1$ and calculates the acceleration a_i resulting from the internal force $-f_{i+1}$ as seen in Figure 7.2 where Λ_i^{-1} is the articulated mobility for a constrained chain. The subspace decomposition allows for meaningful inertia and mobility decomposition.

7.1 Free Floating Chain

From (3.13) the definition of the articulated inertia A_i for a free floating chain is

$$f_i = A_i a_i + d_i \tag{7.1}$$

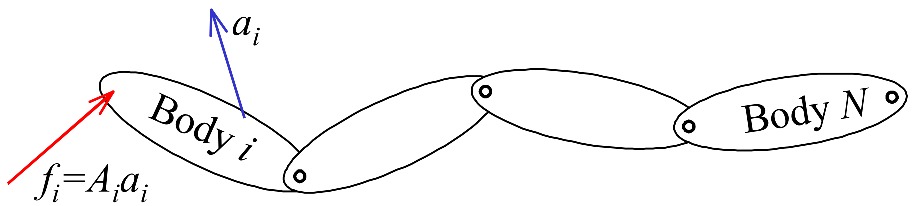

Figure 7.1: Articulated inertia A_i for free floating chain. The acceleration a_i resulting from the inernal force f_i includes the effect from all subsequent bodies.

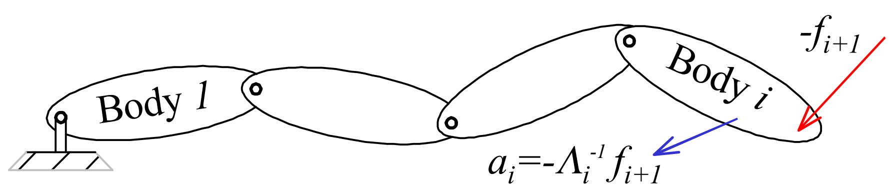

Figure 7.2: Articulated mobility Λ_i^{-1} for constrained chain. The acceleration a_i resulting from the internal force $-f_{i+1}$ includes the effect from all the previous bodies.

which relates the internal force f_i with the spatial acceleration a_i and the articulated bias forces d_i. Articulated inertias are calculated recursively using (3.29) and (3.30). In this section the force decomposition (6.38) is used for an alternative method.

The articulated inertia starts with the terminal body where

$$A_N = I_N \tag{7.2}$$

For $i = N$ (3.10) becomes

$$f_N = I_N a_N + p_N \tag{7.3}$$

Next it is assumed that the articulated inertia for body $i+1$ exists as

$$f_{i+1} = A_{i+1} a_{i+1} + d_{i+1} \tag{7.4}$$

from which the articulated inertia for body i is calculated. From (6.109) the internal force of joint $i+1$ is

$$f_{i+1} = T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T \left(A_{i+1} \left(a_i + \kappa_{i+1} \right) + d_{i+1} \right) \tag{7.5}$$

The internal force f_{i+1} is substituted into (3.10) yielding

$$f_i = A_i a_i + d_i \tag{7.6}$$

with

$$A_i = I_i + R_{i+1} U_{i+1}^T A_{i+1} \tag{7.7a}$$

$$d_i = p_i + T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T \left(A_{i+1} \kappa_{i+1} + d_{i+1} \right) \tag{7.7b}$$

The articulated inertia of a body is the sum of the body inertia and a projection of the articulated inertia for the rest of the chain. This projection is interpreted with the help of the inertia decomposition in (6.76) and Figure 6.3. It represents the forces needed to accelerate the base of joint $i+1$ by a_i.

7.1.1 Articulated Inertia Algorithm

Equations (7.7a) and (7.7b) are used recursively from $i = N$ to $i = 1$ to calculate all the articulated inertias A_i and articulated bias forces d_i with the following algorithm.

1. Given quantities for all bodies in the chain
   - Spatial inertia I_i
   - Joint space s_i
   - Reaction space R_i
   - Spatial velocity v_i
   - Applied forces g_i
   - Joint torques Q_i
   - Joint velocities $\dot{q}_i$
2. Initial values:
   - $\kappa_N = v_N \times s_N \dot{q}_N$
   - $A_N = I_N$

- $d_N = -(v_N)^T \times I_N v_N + g_N$

- $T_N = A_N s_N \left(s_N^T A_N s_N\right)^{-1}$

- $U_N = A_N^{-1} R_N \left(R_N^T A_N^{-1} R_N\right)^{-1}$

3. Recursion from $i = N - 1$ to $i = 1$

- $A_i = I_i + R_{i+1} U_{i+1}^T A_{i+1}$

- $d_i = -(v_i)^T \times I_i v_i + g_i + T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T (A_{i+1} \kappa_{i+1} + d_{i+1})$

- $T_i = A_i s_i \left(s_i^T A_i s_i\right)^{-1}$

- $U_i = A_i^{-1} R_i \left(R_i^T A_i^{-1} R_i\right)^{-1}$

The decompositions can help with eliminating the steps that require the inversion of the 6×6 articulated inertia matrix. Using the identity (6.75) the substitution

$$R_i U_i^T = 1 - T_i s_i^T \tag{7.8}$$

eliminates the need for explicit calculation of U_i. If a joint has only one degree of freedom the calculation of $T_i s_i^T$ is much easier computationally than $R_i U_i^T$.

Although computational efficiency was originally the driving force behind the advances of articulated body inertias, it is not a primary concern in this thesis. The decomposition method presented in the previous chapter is flexible, modular, and complete.

7.2 Articulated Inertia Example

An example of articulated inertias is shown in Figure 7.3. Two floating identical bodies are connected at an angle φ of each other. A horizontal force f_1 accelerates both bodies. The acceleration of the first body is used to define the effective mass in the horizontal direction. If the two bodies are initially at rest with

$$v = 0 \tag{7.9}$$

then the articulated equations of motion for body 1 is

$$f_1 = A_1 a_1 \tag{7.10}$$

with A_1 the articulated inertia and f_1 the applied force. If a horizontal force F_x is needed to accelerate body 1 in the horizontal direction by a_x then the effective mass in that direction is

$$m_x = \frac{F_x}{a_x} \tag{7.11}$$

If F_x is the magnitude of the horizontal force and a_x the magnitude of the linear acceleration in the horizontal direction then the effective mass in that direction is

$$m_x = \{A_1\}_{[1,1]} \tag{7.12}$$

with $\{A_1\}_{[1,1]}$ the first row and first column component of A_1. All the screw quantities described below are expressed on the origin which coincides with the joint coordinate CJ_1.

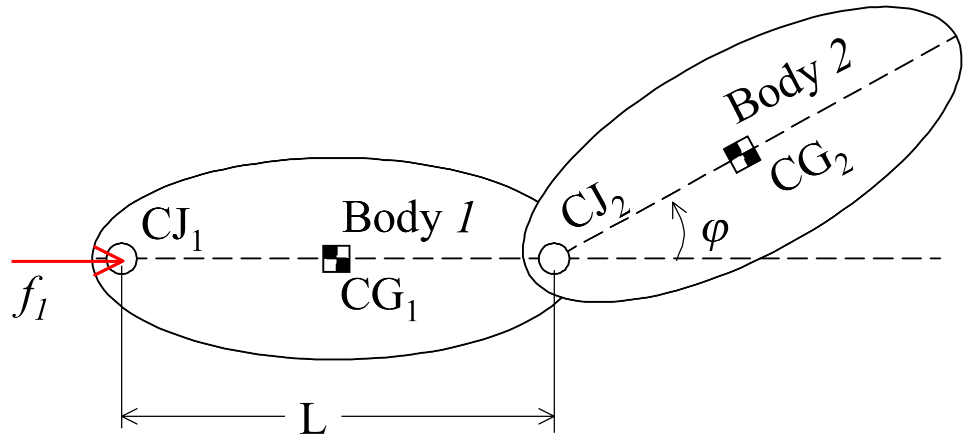

Figure 7.3: Articulated inertia example. A horizontal force f_1 accelerates two connected identical bodies. The effective mass in the horizontal direction is derived from the ratio of the force to the linear acceleration in the horizontal direction. The effective mass is a function of the configuration angle φ and the inertia properties of the two bodies.

The articulated inertias are calculated using the single rigid body inertia for body

1

$$I_1 = \begin{bmatrix} m & 0 & 0 \\ 0 & m & \frac{m}{2}L \\ 0 & \frac{m}{2}L & \frac{m}{4}\left(L^2 + 4\rho^2\right) \end{bmatrix} \tag{7.13}$$

and body 2

$$I_2 = \begin{bmatrix} m & 0 & -\frac{m}{2}L\sin\varphi \\ 0 & m & mL\left(1 + \frac{1}{2}\cos\varphi\right) \\ -\frac{m}{2}L\sin\varphi & mL\left(1 + \frac{1}{2}\cos\varphi\right) & m\rho^2 + \left(\frac{5}{4} + \cos\varphi\right)mL^2 \end{bmatrix} \tag{7.14}$$

with m the mass of the bodies and ρ their radius of gyration. The joint axes are

$$s_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} 0 \\ -L \\ 1 \end{bmatrix} \tag{7.15}$$

The reaction spaces are

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & L \end{bmatrix} \tag{7.16}$$

Going through the articulated inertia analysis, (7.2) is

$$A_2 = I_2 \tag{7.17}$$

and from (6.33) the reactive acceleration space is

$$\begin{aligned} U_2 &= A_2^{-1} R_2 \left(R_2^T A_2^{-1} R_2 \right)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{2L^2 \sin\varphi}{L^2+4\rho^2} & 1 + \frac{2L^2 \cos\varphi}{L^2+4\rho^2} \\ \frac{2L \sin\varphi}{L^2+4\rho^2} & -\frac{2L\cos\varphi}{L^2+4\rho^2} \end{bmatrix} \end{aligned} \tag{7.18}$$

which is used to calculate the articulated inertia for body 1. From (7.7a)

$$A_1 = I_1 + R_2 U_2^T A_2 \tag{7.19}$$

$$= \begin{bmatrix} m + \frac{m\left(L^2\cos^2\varphi+4\rho^2\right)}{L^2+4\rho^2} & \frac{mL^2}{L^2+4\rho^2}\sin\varphi\cos\varphi & \frac{mL^3}{L^2+4\rho^2}\sin\varphi\cos\varphi \\ \frac{mL^2}{L^2+4\rho^2}\sin\varphi\cos\varphi & m + \frac{m\left(L^2\sin^2\varphi+4\rho^2\right)}{L^2+4\rho^2} & m\frac{L}{2} + \frac{mL^3\sin^2\varphi+4mL\rho^2}{L^2+4\rho^2} \\ \frac{mL^3}{L^2+4\rho^2}\sin\varphi\cos\varphi & m\frac{L}{2} + \frac{mL^3\sin^2\varphi+4mL\rho^2}{L^2+4\rho^2} & m\rho^2 + \frac{5}{4}mL^2 - \frac{mL^4\cos^2\varphi}{L^2+4\rho^2} \end{bmatrix}$$

Equation (7.12) returns the effective mass in the horizontal direction as

$$\frac{m_x}{m} = \frac{L^2 + 8\rho^2 + L^2\cos^2\varphi}{L^2 + 4\rho^2} \tag{7.20}$$

Using the size ratio

$$\lambda = \frac{L}{\rho} \tag{7.21}$$

(7.20) becomes

$$\frac{m_x}{m} = \frac{8 + \lambda^2\left(1 + \cos^2\varphi\right)}{\lambda^2 + 4} \tag{7.22}$$

which is plotted in Figure 7.4. The smaller the rotational inertia of the bodies the smaller the radius of gyration ρ is relative to the size L and therefore λ increases. The effective mass ratio m_x/m decreases as the size ratio λ increases. The limiting case where $\lambda \to \infty$ represents bodies that are point masses and the effective mass of the system reaches its minimum of

$$\lim_{\lambda\to\infty} \frac{m_x}{m} = 1 + \cos^2\varphi \tag{7.23}$$

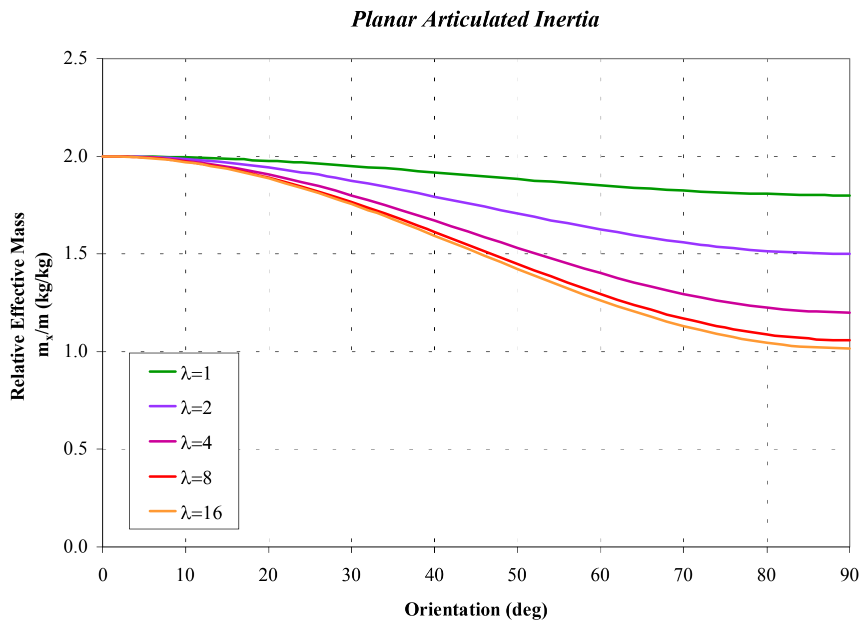

Figure 7.4: Effective mass of planar articulated example as a function of the orientation angle φ. As the size ratio λ increases the effective mass decreases.

On the other hand when the location of the joints approaches the center of mass and $\lambda \to 0$ the effective mass ratio approaches its maximum of

$$\lim_{\lambda \to 0} \frac{m_x}{m} = 2 \tag{7.24}$$

7.3 Articulated Horizontal Chain of Multiple Rigid Bodies

Extending the above example as seen in Figure to multiple horizontal bodies with zero angle between them allows for a tabulated form of the articulated inertia of the

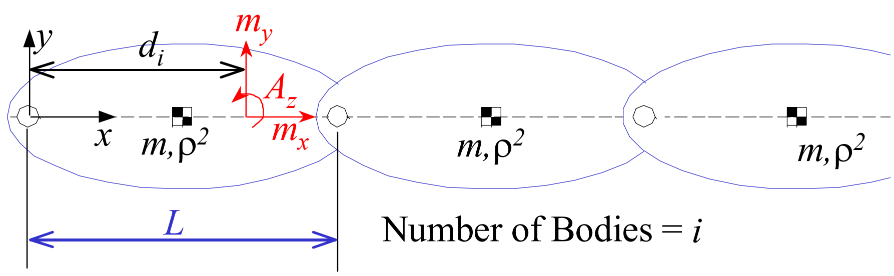

Figure 7.5: Articulated Inertia for a horizontal chain of rigid bodies. With the addition of each successive body the effective center of mass is located by the distance d_i and the diagonal terms of articulated inertia matrix expressed on d_i are m_x, m_y, and A_z.

first body varying the number of bodies.

A horizontal chain of rigid bodies has the following properties. For a chain of i rigid bodies spaced a distance L apart each with a radius of gyration of ρ, the effective center of gravity of the first body is located a distance d_i from the first joint. The articulated inertia of a single rigid body expressed on the center of gravity $d_1 = L/2$ is

$$A_1 = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\rho^2 \end{bmatrix} \tag{7.25}$$

Then the articulated inertia of the i-th body expressed on its effective center of gravity

d_i is

$$A_i = \begin{bmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & A_z \end{bmatrix} \tag{7.26}$$

where m_x, m_y and A_z are shown below as a function of the number of bodies i

i	d_i/L	m_x/m	m_y/m	$A_z/m\rho^2$
1	$\frac{1}{2}$	1	1	1
2	$\frac{1}{2}\frac{\lambda^2+12}{\lambda^2+8}$	2	$\frac{\lambda^2+8}{\lambda^2+4}$	$\frac{2\lambda^2+8}{\lambda^2+8}$
3	$\frac{1}{2}\frac{\lambda^4+40\lambda^2+80}{\lambda^4+32\lambda^2+48}$	3	$\frac{\lambda^4+32\lambda^2+48}{\lambda^4+24\lambda^2+16}$	$\frac{3\lambda^4+40\lambda^2+48}{\lambda^4+32\lambda^2+48}$
4	$\frac{1}{2}\frac{\lambda^6+84\lambda^4+560\lambda^2+448}{\lambda^6+72\lambda^4+400\lambda^2+256}$	4	$\frac{\lambda^6+72\lambda^4+400\lambda^2+256}{\lambda^6+60\lambda^4+240\lambda^2+64}$	$\frac{4\lambda^6+112\lambda^4+448\lambda^2+256}{\lambda^6+72\lambda^4+400\lambda^2+256}$

with $\lambda = L/\rho$.

The behavior of these quantities is better understood when viewed graphically. The relative distance d_i/L is shown in Figure 7.6 as a function of the size ratio λ. The effective mass m_y/m is shown in Figure 7.7 as a function of λ. Finally the effective rotational inertia $A_z/m\rho^2$ is shown in Figure 7.8 as function of λ.

Seen in the graphs for the three and four body systems for specific ranges of the size ratio the quantities are close to the two body systems within about 10%. Interesting is the case where $\lambda = 2.0$ and the behavior of an articulated multibody system is identical to the two body system. For the effective center of gravity for $\lambda > 1.0$ the

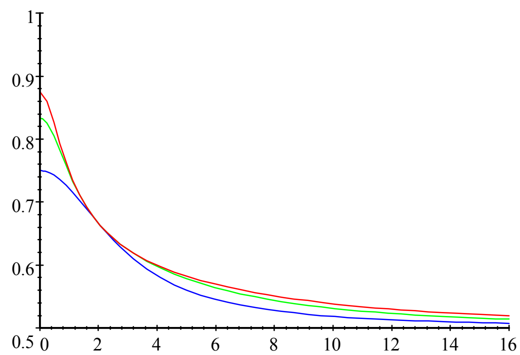

Figure 7.6: The location of the effective center of gravity in a two body system. The x-axis is the size ratio L/ρ and the y-axis is the relative location d/L.

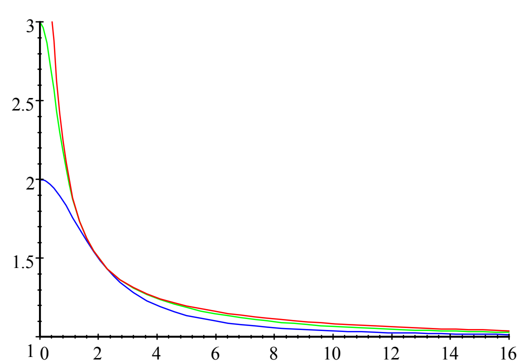

Figure 7.7: The effective mass in the vertical direction in a two body system. The x-axis is the size ratio L/ρ and the y-axis is the relative mass m_y/m.

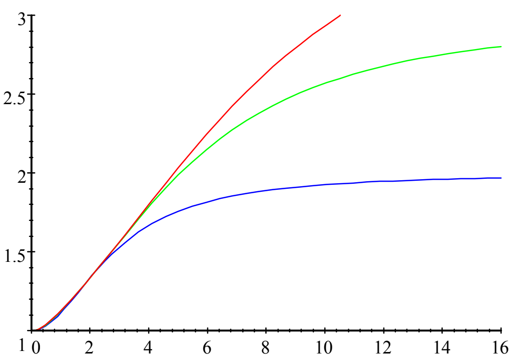

Figure 7.8: The effective rotational inertia in a two body system. The x-axis is the size ratio L/ρ and the y-axis is the relative ineretia $A_z/m\rho^2$.

ratio $d_3/d_2 < 1.1$ and $d_4/d_2 < 1.1$. Even for the rest of the λ the relative locations are not more than 20% off the two body system. For the articulated mass the three and four body systems behave like the two body system for $\lambda > 1.5$ within 10%. The articulated rotational inertia of multibody systems with values $\lambda < 3.0$ exhibit very strong similarity to the two body system well within 5%. But for systems with $\lambda > 3.0$ each successive body contributes significantly to the articulated quantity.

Typically for rectangular shapes $\lambda < 3.46$ and for elliptical shapes $\lambda < 2.24$. But for rhomboidal shapes $\lambda < 4.90$. If the shapes are elliptical disks with their height $h = L/2$ and length L then the radius of gyration is $\rho = L/2$ which yields the size ratio of $\lambda = 2.0$. As shown in the graphs above this is a special case where each successive body contributes nothing to the values of the articulated quantities for more that two bodies. As a result the location of the effective center of gravity for a six body system is identical to the location for a two body system which is equal to

$$\begin{aligned} d &= \frac{1}{2}\frac{\lambda^2+12}{\lambda^2+8}L \\ &= \frac{2}{3}L \end{aligned} \tag{7.27}$$

7.4 Constrained Chain

The definition of the articulated mobility Λ_i^{-1} for a constrained chain is

$$a_i = -\Lambda_i^{-1} f_{i+1} + b_i \tag{7.28}$$

which relates the internal force f_{i+1} with the spatial acceleration a_i and the articulated bias accelerations b_i. Articulated mobilities are defined recursively from the acceleration decomposition (6.44). Articulated mobilities are used extensively by [4] and several different methods for mobility calculation are derived. Unfortunately the recursive method developed contained an error as the acceleration propagator ommited a force related term. A corrected version of this method is shown below. This method is recursive and uses projection matrices.

The acceleration of the first body using the acceleration propagator from (6.113a) is

$$a_1 = s_1 T_1^T I_1^{-1} \left(T_1 Q_1 - p_1 - f_2\right) + U_1 R_1^T \kappa_1 \tag{7.29a}$$

which when compared to the articulated mobility definition

$$a_1 = -\Lambda_1^{-1} f_2 + b_1 \tag{7.30}$$

the articulated quantities are

$$\Lambda_1^{-1} = s_1 T_1^T I_1^{-1} \tag{7.31a}$$

$$b_1 = s_1 T_1^T I_1^{-1} (T_1 Q_1 - p_1) + U_1 R_1^T \kappa_1 \tag{7.31b}$$

To derive the articulated mobility for body i the articulated mobility of body $i-1$

$$a_{i-1} = -\Lambda_{i-1}^{-1} f_i + b_{i-1} \tag{7.32}$$

is used together with the equations of motion (3.10)

$$f_i = I_i a_i + f_{i+1} + p_i \tag{7.33}$$

and the acceleration decomposition (6.113a) as a system of three spatial equations

$$a_i = \eta_i - s_i T_i^T I_i^{-1} f_{i+1} + U_i R_i^T (a_{i-1} + \kappa_i) \tag{7.34a}$$

with η_i the active projected acceleration

$$\eta_i = s_i T_i^T I_i^{-1} (T_i Q_i - p_i) \tag{7.35}$$

The articulated mobility relates a known force f_{i+1} to the unknown acceleration a_i. Other unknowns in this systems are the internal force f_i and the previous body acceleration a_{i-1}.

Substituting (7.33) into (7.32) and then into (7.34a) yields

$$\left(1 + U_i R_i^T \Lambda_{i-1}^{-1} I_i\right) a_i = -\left(s_i T_i^T I_i^{-1} - U_i R_i^T \Lambda_{i-1}^{-1}\right) f_{i+1} + \\ + \eta_i - U_i R_i^T \left(\Lambda_{i-1}^{-1} p_i - b_{i-1} - \kappa_i\right) \tag{7.36}$$

which is solved as

$$a_i = -\Lambda_i^{-1} f_{i+1} + b_i \tag{7.37}$$

with

$$\Lambda_i^{-1} = \left(1 + U_i R_i^T \Lambda_{i-1}^{-1} I_i\right)^{-1} \left(s_i T_i^T I_i^{-1} - U_i R_i^T \Lambda_{i-1}^{-1}\right) \tag{7.38a}$$

$$b_i = \left(1 + U_i R_i^T \Lambda_{i-1}^{-1} I_i\right)^{-1} \left(\eta_i - U_i R_i^T \left(\Lambda_{i-1}^{-1} p_i - b_{i-1} - \kappa_i\right)\right) \tag{7.38b}$$

These recursive equations are used from the base to the tip of the chain to calculate the overall articulated mobility Λ^{-1} of the system

$$\Lambda^{-1} = \Lambda_N^{-1} \tag{7.39}$$

If the system has six or more degrees of freedom then Λ^{-1} is not singular and is the inverse of the operational space inertia as defined by [4].

7.4.1 Articulated Mobility Algorithm

Equations (7.38a) and (7.38b) are used recursively from $i = 1$ to N to calculate all the articulated mobilities Λ_i^{-1} and articulated bias accelerations b_i with the following algorithm.

1. Given variables for all bodies in the chain

    - Spatial inertia I_i
    - Joint space s_i

- Reaction space R_i
- Spatial velocity v_i
- Applied forces g_i
- Joint torques Q_i
- Joint velocities $\dot{q}_i$

2. Initial values:

- $\kappa_1 = v_1 \times s_1 \dot{q}_1$
- $p_1 = -\left(v_1 \times\right)^T I_1 v_1$
- $T_1 = I_1 s_1 (s_1^T I_1 s)^{-1}$
- $U_1 = I_i^{-1} R_1 (R_1^T I_1^{-1} R_1)^{-1}$
- $\Lambda_1^{-1} = s_1 T_1^T I_1^{-1}$
- $b_1 = s_1 T_1^T I_1^{-1} \left(T_1 Q_1 - p_1\right) + U_1 R_1^T \kappa_1$

3. Recursion from $i = 2$ to $i = N$

- $\kappa_i = v_i \times s_i \dot{q}_i$
- $p_i = -\left(v_i \times\right)^T I_i v_i$
- $T_i = I_i s_i (s_i^T I_i s)^{-1}$
- $U_i = I_i^{-1} R_i (R_i^T I_i^{-1} R_i)^{-1}$
- $\eta_i = s_i T_i^T I_i^{-1} \left(T_i Q_i - p_i\right)$

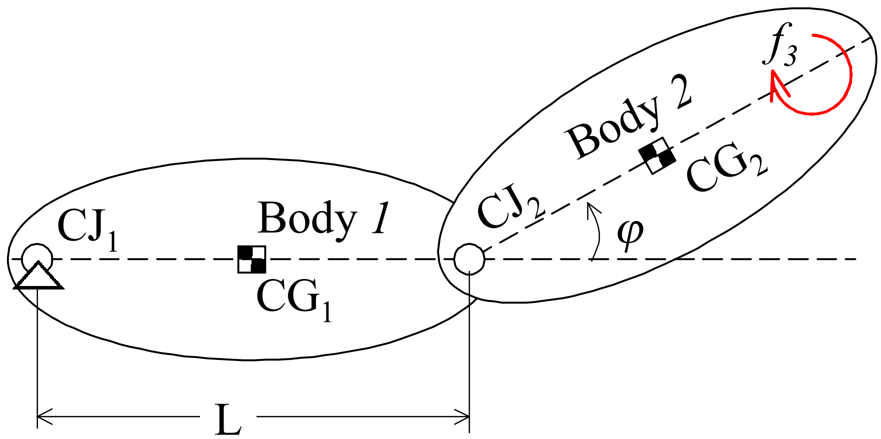

Figure 7.9: Articulated mobility example. The first joint CJ_1 is pinned to the ground and a pure couple is applied to body 2.

- $\Lambda_i^{-1} = \left(1 + U_i R_i^T \Lambda_{i-1}^{-1} I_i\right)^{-1} \left(s_i T_i^T I_i^{-1} - U_i R_i^T \Lambda_{i-1}^{-1}\right)$
- $b_i = \left(1 + U_i R_i^T \Lambda_{i-1}^{-1} I_i\right)^{-1} \left(\eta_i - U_i R_i^T \left(\Lambda_{i-1}^{-1} p_i - b_{i-1} - \kappa_i\right)\right)$

7.5 Example of Articulated Mobility

The same example as in Section 7.2 is used but with the first joint CJ_1 connected to the ground and a pure couple f_3 is applied to body 2 as seen in Figure 7.9.

Equations (6.36) and (6.37) are used to find the subspaces

$$T_1 = \begin{bmatrix} 0 \\ \frac{2L}{L^2+4\rho^2} \\ 1 \end{bmatrix} \quad \text{and} \quad U_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{2L}{L^2+4\rho^2} \end{bmatrix} \tag{7.40}$$

and

$$T_2 = \begin{bmatrix} -\frac{2L\sin\varphi}{L^2+4\rho^2} \\ \frac{2L\cos\varphi}{L^2+4\rho^2} \\ 1 \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} 1 & 0 \\ -\frac{2L^2\sin\varphi}{L^2+4\rho^2} & 1+\frac{2L^2\cos\varphi}{L^2+4\rho^2} \\ \frac{2L\sin\varphi}{L^2+4\rho^2} & -\frac{2L\cos\varphi}{L^2+4\rho^2} \end{bmatrix} \tag{7.41}$$

Equation (7.31a) is used to find the articulated mobility of body 1

$$\Lambda_1^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{4}{m(L^2+4\rho^2)} \end{bmatrix} \tag{7.42}$$

and from (7.38a) the articulated mobility of body 2 is

$$\Lambda_2^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2L^2\left(2L^2(\cos\varphi+3)+4\rho^2\right)}{m\left(L^4\left(\frac{1}{4}+\sin^2\varphi\right)+6L^2\rho^2+4\rho^2\right)} & -\frac{L\left(6L^2\left(\cos\varphi+\frac{7}{6}\right)+12\rho^2\right)}{m\left(L^4\left(\frac{1}{4}+\sin^2\varphi\right)+6L^2\rho^2+4\rho^2\right)} \\ 0 & -\frac{L\left(2L^2\left(\cos\varphi+\frac{5}{2}\right)+4\rho^2\right)}{m\left(L^4\left(\frac{1}{4}+\sin^2\varphi\right)+6L^2\rho^2+4\rho^2\right)} & \frac{4L^2\left(\cos\varphi+\frac{5}{4}\right)+4\rho^2}{m\left(L^4\left(\frac{1}{4}+\sin^2\varphi\right)+6L^2\rho^2+4\rho^2\right)} \end{bmatrix} \tag{7.43}$$

From (7.37) the application of a pure couple τ

$$f_3 = \begin{bmatrix} 0 \\ 0 \\ \tau \end{bmatrix} \tag{7.44}$$

on body 2 accelerates the body by

$$a_2 = -\Lambda_2^{-1} f_3 \tag{7.45}$$

$$= \begin{bmatrix} 0 \\ \frac{\tau L\left(6L^2\left(\cos\varphi+\frac{7}{6}\right)+12\rho^2\right)}{m\left(L^4\left(\frac{1}{4}+\sin^2\varphi\right)+6L^2\rho^2+4\rho^2\right)} \\ -\frac{\tau\left(4L^2\left(\cos\varphi+\frac{5}{4}\right)+4\rho^2\right)}{m\left(L^4\left(\frac{1}{4}+\sin^2\varphi\right)+6L^2\rho^2+4\rho^2\right)} \end{bmatrix} \tag{7.46}$$

which represents a twist about a point on the $y = 0$ axis located at

$$\frac{x}{L} = \frac{-\{a_2\}_{[2]}}{L\{a_2\}_{[3]}} \tag{7.47}$$

$$= \frac{\frac{3}{2}\eta^2\cos\varphi + \frac{7}{4}\eta^2 + 3}{\eta^2\cos\varphi + \frac{5}{4}\eta^2 + 1} \tag{7.48}$$

with the ratio

$$\eta = L/\rho \tag{7.49}$$

indicating the relative size compared to the radius of gyration. The position of the twist is shown in the polar Figure 7.10 as a function of the configuration angle φ.

It is important to point out that the position of the acceleration twist resulting from a pure couple is important because it provides valuable information about the inertial properties of the system. In the planar case, if a pure couple f applied on any connected body in a chain results in the acceleration a, then the point defined by a is the effective center of mass of that body. As a result any force reciprocal to a results in purely translational motion. This is a result of duality and inertia mappings.

7.6 Chain Splitting

Using articulated inertias the calculation of the acceleration of a body i requires the acceleration of the previous body $i-1$. On the other hand with articulated mobilities calculating the acceleration of body i requires the internal forces f_{i+1} acting on the body. It is possible to combine these two methods to establish direct solutions for acceleration a_i without first calculating a_{i-1} or f_{i+1}.

Conceptually this is done by splitting the linear chain into two problems. One is the articulated inertia problem for the sub-chain from body $i+1$ up to the tip n as seen in Figure 7.11. The internal force f_{i+1} accelerates this sub-chain and the motion a_{i+1} of the first body in the sub-chain is described by the equation

$$f_{i+1} = A_{i+1} a_{i+1} + d_{i+1} \tag{7.50}$$

where A_{i+1} and d_{i+1} are defined recursively from (7.7a) and (7.7b). The internal force solution for the articulated inertia problem is

$$f_{i+1} = T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^{T} \left(A_{i+1} \left(a_i + \kappa_{i+1} \right) + d_{i+1} \right) \tag{7.51}$$

as defined using decompositions in (6.109). This relates the previous body acceleration a_i to the internal force f_{i+1}.

The other problem is the articulated mobility problem for the sub-chain for body i down to the ground. The internal force f_{i+1} accelerates this sub-chain and the motion a_i of the tip of the sub-chain is described by the equation

$$a_i = b_i - \Lambda_i^{-1} f_{i+1} \tag{7.52}$$

where Λ_i^{-1} and b_i are defined recursively from (7.38a) and (7.38b).

Combining these two problems by substituting (7.51) into (7.52) and moving all the terms related to a_i on the left hand side returns

$$\left(1 + \Lambda_i^{-1} R_{i+1} U_{i+1}^T A_{i+1}\right) a_i = b_i - \Lambda_i^{-1} \left[T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T \left(A_{i+1} \kappa_{i+1} + d_{i+1}\right)\right] \tag{7.53}$$

which is solved for

$$a_i = \left(1 + \Lambda_i^{-1} R_{i+1} U_{i+1}^T A_{i+1}\right)^{-1} \left[b_i - \Lambda_i^{-1} \left[T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T \left(A_{i+1} \kappa_{i+1} + d_{i+1}\right)\right]\right] \tag{7.54}$$

Now given any end conditions on an open linear chain such as a base acceleration a_0 or an externally applied force on the tip f_{n+1} through the recursions of b_i and d_{i+1} a direct solution for the accelerations is computed.

Similarly, to solve for the internal forces, (7.52) is substituted into (7.51) eliminating the accelerations

$$\left(1 + R_{i+1} U_{i+1}^T A_{i+1} \Lambda_i^{-1}\right) f_{i+1} = T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T \left(A_{i+1} \left(b_i + \kappa_{i+1}\right) + d_{i+1}\right) \tag{7.55}$$

This equation is solved, using the definition of the subspace $U_i = A_i^{-1} R_i \left(R_i^T A_i^{-1} R_i\right)^{-1}$ and symmetry, as

$$f_{i+1} = \left(1 + \Lambda_i^{-1} R_{i+1} U_{i+1}^T A_{i+1}\right)^{-T} \left[T_{i+1} Q_{i+1} + R_{i+1} U_{i+1}^T \left(A_{i+1} \left(b_i + \kappa_{i+1}\right) + d_{i+1}\right)\right] \tag{7.56}$$

This method eliminates the recursion up the chain with the kinematics to solve for all the accelerations. In addition, it provides a direct force propagator from the tip of the chain down to body i and an acceleration propagator from the base up to body i at the same time. This method answers the question of what total effect an externally applied load has on any body in the chain with one calculation.

There is a simple algorithm to this method.

1. Calculate A_{i+1} and d_{i+1} using the articulated inertia recursion algorithm.

2. Calculate subspaces T_{i+1} and U_{i+1} from the same algorithm.

3. Calculate Λ_i^{-1} and b_i using the articulated mobility recursive algorithm.

4. Use (7.54) for the acceleration of body i.

5. Use (7.56) for the internal force on joint $i + 1$.

7.7 Questions

The articulated inertias and articulated mobilities are tools for understanding the behavior of multibody systems. The power of this type of analysis is that it provides useful insight on system behavior. Planar cases are easier to understand and often the observed patterns can be extended to three dimensions.

As these ideas unfold, further questions arise.

1. *"Can programming techniques such as object oriented structures be used to construct a spatial toolbox for numerical or symbolic problem solutions?"* Programs such as MATLAB® are useful in describing and solving problems. Version 5.0

and above support object oriented techniques for user toolboxes. New datatypes that extend simple matrices to screw specific quantities can handle screw algebra transparently for the user. Then a few simple commands can describe complex multibody systems without the need for explicit component by component definitions yielding a high-level language for dynamics.

2. *"Is it possible to solve multibody systems in a non-recursive way?"* Sometimes just putting everything in big matrices helps show patterns in the solutions. In addition, matrices are a convenient way of representing multibody systems which are not just linear chains of bodies. Modified recursive representations can handle multibody structures but stacked formulations are easier to implement.

3. *"Is it possible to use decompositions and articulated quantities to model contacts with the environment?"* When parts of a multibody system are in contact with the environment the reaction forces are not known and cannot be considered just applied forces. Using decompositions and the articulated mobilities, a solution may be found that modifies the unconstrained accelerations and computes the constrained accelerations.

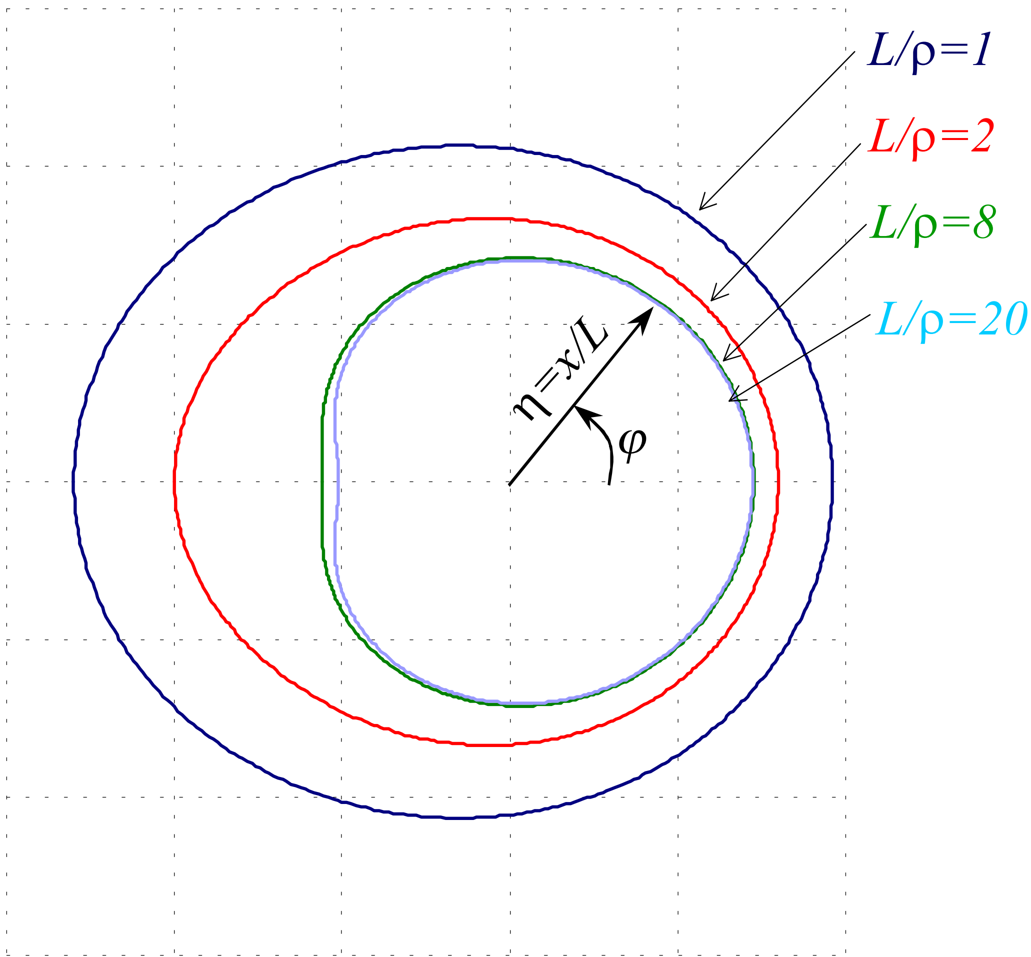

Figure 7.10: In the articulated mobility example the location x of the instant center of acceleration for body 2 is illustrated using a polar plot. The angle φ is the relative orientation between the two bodies and the ratio η the relative location of x with respect to the size L.

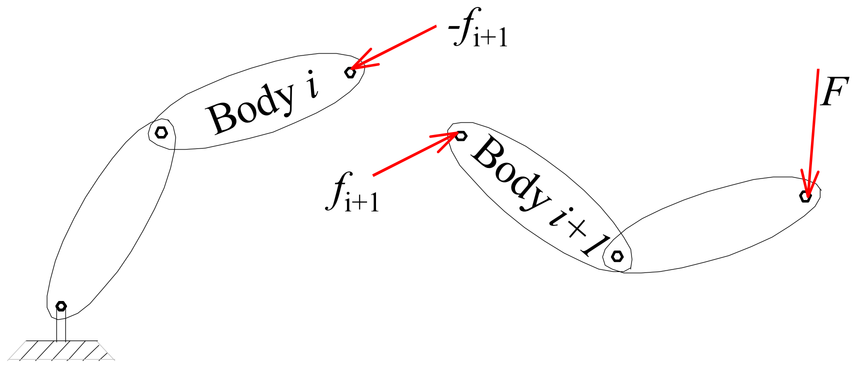

Figure 7.11: Chain splitting reveals the effect of externally applied forces F and base constraints to any internal force f_i. The system is split into a constrained chain from the base to body i and a free floating chain from body $i+1$ to the tip.

CHAPTER 8

MATLAB SPATIAL TOOLBOX (SPAT)

MATLAB® v5.0 and above allow for object oriented programing. This style of programing is ideally suited for a dynamics toolbox using screw theory. Object oriented programing is based on programming operators that manipulate user defined objects. The spatial toolbox (SPAT) defines two new objects. One holds coordinate frame information and the other spatial quantities. Operators are defined to perform the screw algebra and coordinate frame transformations needed for high level modeling of rigid body dynamics. This chapter is a reference for the functions that define this toolbox. In addition, there are some small examples to aid in the understanding of some of the functions. In the end there is a simulation example modeled using this toolbox.

8.1 Constants and Functions

8.1.1 Elementary Unit Vectors

Four vector constants are globally declared, `i_`, `j_`, `k_`, and `o_` that represent the three unit vectors, and the zero vector. They can be used to construct vectors component by component.

Example 13 *The mathematical expression* $3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ *is typed in MATLAB as*

$$\texttt{3} * \texttt{i\_} - \texttt{2} * \texttt{j\_} + \texttt{k\_}$$

$$= 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \tag{8.1}$$

8.1.2 Rotation Matrices

The function `E=elrot(vector,angle)` is declared returning the 3×3 elementary rotational matrix that corresponds to a rotation by an angle about a vector. The vector does not have to be a unit vector, since the function normalizes it before its used.

Example 14 *A rotation about the direction $2\hat{\imath} - \hat{\jmath}$ by an angle $\pi/3$ is typed as*

$$\texttt{elrot}(2 * \texttt{i\_} - \texttt{j\_}, \texttt{pi}/3)$$

$$= \begin{bmatrix} 0.9 & -0.2 & -0.3873 \\ -0.2 & 0.6 & -0.7746 \\ 0.3873 & 0.7746 & 0.5 \end{bmatrix} \tag{8.2}$$

8.1.3 Vector Cross Product

The function `b=cr(a)` is declared returning the 3×3 skew symmetric matrix of the cross product operator.

Definition 15 *If a vector has components $\bar{r}^T = \begin{bmatrix} x & y & z \end{bmatrix}$ then the operator cross product operator $\bar{r}\times$ is defined as*

$$\bar{r}\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{8.3}$$

This notation is used to evaluate cross products such as $\bar{c} = \bar{a} \times \bar{b}$ with matrix multiplication. In MATLAB this operation is typed in as `c=cr(a)*b`.

8.1.4 Spatial Cross Product

The function `vx=cr(v)` is also used when the argument `v` is a 6×1 quantity such as the value of a screw.

Definition 16 *If a screw has components* $v^T = \begin{bmatrix} \bar{v}^T & \bar{\omega}^T \end{bmatrix}$ *then the cross product operator is defined as*

$$v\times = \begin{bmatrix} \bar{\omega}\times & \bar{v}\times \\ 0 & \bar{\omega}\times \end{bmatrix} \tag{8.4}$$

with $\bar{\omega}\times$ *and* $\bar{v}\times$ *the vector cross product operators. The result is a singular* 6×6 *matrix with zero trace.*

In multibody dynamics a different type of spatial cross product is used in bias forces but that can be evaluated by `vx=-cr(v)'`.

8.2 Coordinate Systems

In general, a coordinate system has the following three quantities associated with it:

1) A character string representing a descriptive name for that coordinate system

2) A vector representing the global position of the coordinate origin and

3) A rotation matrix representing the orientation of the coordinate system.

New coordinate systems without specific definition of position or orientation assumes the default global coordinates when initialized. The global position is a 3×1 zero vector and the global orientation is a 3×3 identity matrix.

Remark 17 *The orientation matrix* E *is a local to global rotational transformation. In essence, each column of* E *is the global representation of the local* $\hat{\imath}$, $\hat{\jmath}$ *and* $\hat{k}$

directions. Hence for any local vector $\bar{a}$ its global representation $\bar{a}_o$ is

$$\bar{a}_o = E\bar{a} \tag{8.5}$$

*The orientation matrix E of a coordinate system is best defined with the **`elrot()`** function as shown in the coordinate system example later.*

8.2.1 Coordinate Constructor

A coordinate system is declared with the `A=coord(arguments)` constructor. It can be called with any number of arguments. Below is a list of some possible constructor calls

- `A=coord` returns the global coordinate system which has its origin at `o_` and its unit vectors are aligned with `i_` , `j_` ,and `k_`. The name, position, and orientation of the global coordinate system are used since neither is specified as a parameter.

- `B=coord(A)` makes a copy of coord `A`.

- `A=coord(n,r)` names the coord the character string `n` and sets its origin as the vector `r`.

  Example 18 *The definition `A=coord('foo',2*i_)` returns a coord `A` named `'foo'` with its origin at $2\hat{\imath}$ and aligned with the global orientations.*

- `A=coord(n,r,E)` is the same as above, but now it reorients coord `A` according to the rotation matrix `E`.

  Example 19 *The definition example* `A=coord('foo',2*i_,elrot(k_,pi/2))` *returns a coord* `A` *named* `'foo'` *with it's origin at* $2\hat{\imath}$*, but rotated in the* $\hat{k}$ *direction by* $\pi/2$*. This results in the local* $\hat{\imath}$ *direction corresponding with the global* $\hat{J}$ *and the local* $\hat{\jmath}$ *direction with the global* $-\hat{I}$*.*

- `Z=coord(A,B,C,..)` with `A,B,C,..` being other coord objects, assumes that each successive coord is declared local to the previous one. The result is the global representation of the last coord. The first coord is local to the global coord, and therefore an absolute coordinate. The rest are relative coordinates.

Example 20 *The coordinate system attached on the end of a hook joint is defined as shown in Figure 8.1:*

Starting from the default coordinate system XYZ *the tip coordinate system* xyz *is defined by a series of rotations and translations. First the translations along the local* z *directions are defined with the relative coordinate* `A=coord(h*k_)`*. Then a rotation* ϕ *along the local* y *coordinate is defined by* `E1=elrot(j_,phi)` *and a rotation* ψ *along the local* x *coordinate by* `E2=elrot(i_,psi)`*. The tip coordinate is defined with the series* `A`*,*`E1`*,*`E2` *and* `A`*. Assembling the parts with* `T=coord('Tip',A,E1,E2,A)` *returns*

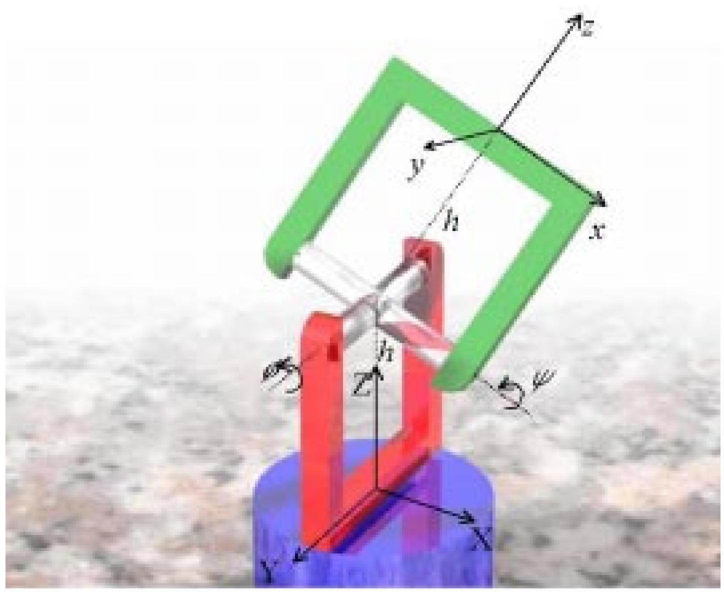

Figure 8.1: Example of coordinate definition on a hook joint. Starting from the default coordinate system XYZ a translation along z of h, a rotation along y of ϕ, a rotation along x of ψ and a translation along z of h define the tip coordinate system xyz.

the following information:

$$\textit{Origin at } \bar{r} = \begin{bmatrix} h \sin\phi \cos\psi \\ -h \sin\psi \\ h + h\cos\phi\cos\psi \end{bmatrix} \tag{8.6}$$

$$\textit{Orientation as } E = \begin{bmatrix} \cos\phi & \sin\phi\sin\psi & \sin\phi\cos\psi \\ 0 & \cos\psi & -\sin\psi \\ -\sin\phi & \cos\phi\sin\psi & \cos\phi\cos\psi \end{bmatrix} \tag{8.7}$$

8.2.2 Member Functions and Operators

The member functions and operators provide ways of accessing the information stored inside the coord objects. In addition, they provide functionality by establishing operators for expressions such as addition, and equality tests.

- `name(A)` is used to return the character string of a coord. To change the name of a coord the constructor `A=coord(n,A)` is called redefining coord `A` with the new name string `n`.

- `pos(A)` returns the position vector of the origin of the coordinate system.

- `orient(A)` returns the orientation matrix of the coordinate system. This matrix provides the local to global transformation for the coordinate system.

- `A+B` adds the relative coordinate `B` to `A` and returns the result as an absolute coordinate. Addition here is not associative since `A+B` is not the same conceptually as `B+A`. Essentially `A+B` returns the function `coord(A,B)`.

- `B-A` returns the relative position and orientation of `B` with respect to `A`. This is useful in extracting the relative position and orientation of one coordinate system with respect to another coordinate system.

- `A==B` or `A~=B` tests for equality or inequality between coords. Two coords are considered equal when both their positions and orientations are equal. Two equal coords may have different names.

- `xform(A,B)` returns a 6×6 transformation matrix between two coordinate systems for transforming screws that represent twists.

Definition 21 *The twist transformation between two coordinate systems is A and B declared as follows*

$$\text{xform}(A, B) = \begin{bmatrix} E_B^T E_A & E_B^T(\bar{r}_A - \bar{r}_B) \times E_A \\ 0_{3\times 3} & E_B^T E_A \end{bmatrix} \tag{8.8}$$

with $\bar{r}_A$ and E_A the position and orientation of coord A, and $\bar{r}_B$ and E_B the position and orientation of coord B. This transformation is derived from the two transformation steps of moving from coord A to the global coordinate system and then to coord B.

- `yform(A,B)` same as `xform` but returns transformation matrix for a wrench.

Definition 22 *The wrench transformation between two coordinate systems is A and B declared as follows*

$$\text{yform}(A, B) = \begin{bmatrix} E_B^T E_A & 0_{3\times 3} \\ E_B^T(\bar{r}_A - \bar{r}_B) \times E_A & E_B^T E_A \end{bmatrix} \tag{8.9}$$

The following identity is used in the transformation of inertias and mobilities.

$$\text{yform}(A, B) \equiv [\text{xform}(B, A)]^T \tag{8.10}$$

8.3 Screw and Inertia Quantities

In general each screw quantity has the following four properties associated with it. One is a character string representing a descriptive name for the screw, another is a number from 0 to 3 representing different screw types, a coord object represents the coordinate system the screw is expressed in, and finally a matrix with the screw value. The value of a screw usually contains six rows and may contain one or more columns.

8.3.1 Screw Constructor

A spatial screw or inertia quantity is declared by the `s=screw(stype,arguments)` constructor. The `stype` argument is a number ranging from 0 to 3 depending on what type of screw quantity is being declared. An `stype` of 0 represents a twist, a `stype` of 1 represents a wrench, a `stype` of 2 represents an inertia, and a `stype` of 3 represents an inverse inertia or mobility. This type of screw type definition is rarely used as more specific constructors are defined. The rest of the arguments can vary in number, and may be of different types. Here is a list of some general constructor forms that do not require a specific `stype` declaration:

- `s=screw` returns an empty screw. There is no name and coordinates system associated with this default screw.
- `s=screw(u)` makes a copy of the screw `u`. It assumes the name, and coordinate

system of its argument.

- `s=screw(u,A)` also makes a copy of the screw `u` but transforms its contents to a coordinate system `A`. This notation can be used in addition to the declarations below, adding as the last argument a coordinate system.

8.3.2 Twist Function

The function call `s=twist(arguments)` is equivalent to the constructor call `s=screw(0,arguments)`.

A twist is used to describe rigid body motions.

- `s=twist(A,[v;w])` creates a screw type 0 at coord `A` with contents the 6×1 quantity derived from the two vectors `v` and `w`. Twists assume the first vector represents the linear motion of a body, and the second vector the angular part. The linear motion of a body changes as the position of the coordinate system moves, in contrast with the angular part which only depends on the orientation of the coordinate system.

Definition 23 *Any vector whose components does not change when expressed at different positions is called an* isophoric *vector. On the other hand, any vector whose components vary with the position it is represented in is called an* anisophoric *vector. These terms are derived from Greek and describe either constant or not constant directions.*

Screws like twists and wrenches always contain one isophoric and one anisophoric vector.

Definition 24 *In this toolbox twists carry the anisotropic vector first and then the isophoric one. This arrangement is called having* axis *screw style. On the other hand, wrenches carry the isophoric vector first and then the anisophoric making them have* ray *screw style.*

This convention for screw arrangements is adopted so that the linear part is always the first vector and the angular is the second. This is useful in power and kinetic energy calculations.

8.3.3 Wrench Function

The function call `s=wrench(arguments)` is equivalent to the constructor call `s=screw(1,arguments)`.

A wrench is used to describe forces and torques on a rigid body.

- `s=wrench(A,[f;t])` creates a screw type 1 at coord `A` with contents the 6×1 quantity derived from the two vectors `f` and `t`. Wrenches assume the first vector represents the linear load of a body, and the second vector the angular part. In contrast to twists, the linear part is the isophoric vector and the angular part the anisophoric vector. As a result wrenches transform differently from twists.

Remark 25 *To manually transform the wrench quantity `[f;t]` from coord `A` to `B` the calculation `yform(A,B)*[f;t]` is needed. Similarly to transform the* *twist quantity `[v;w]` from coord `A` to `B` the calculation `xform(A,B)*[v;w]` is needed.*

8.3.4 Spatial Inertia Function

The function call `s=spi(arguments)` is equivalent to the constructor call `s=screw(2,arguments)`.

A spatial inertia transforms twists into wrenches. It is usually constructed on a coord that represents the center of gravity and the principal inertial axis, and then transformed to any coordinate system the user wishes.

- `I=spi(A,IC)` creates a screw type 2 at coord `A` with contents the 6×6 quantity `IC`.

  Example 26 *If `A` is aligned properly then `IC` should be a diagonal matrix and is declared using the MATLAB `diag()` function. As an example if m is the rigid body mass, and `Ix`,`Iy`, and `Iz` are the moments of inertia about the local $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ axis then the spatial inertia at `A` is `I=spi(A,diag([m,m,m,Ix,Iy,Iz]))`.*

  Remark 27 *Sometimes the spatial inertia is defined on one coordinate system and transformed to another coordinate system using the constructor call `I=spi(A,IC,B)`. This assigns the value `IC` on `A` and transforms the inertia to `B`. Spatial inertia transformations are easily obtained from the transformations* *of twists and wrenches. The spatial inertia transformation from* `A` *to* `B` *is declared as*

$$I_B = X^T I_A X \tag{8.11}$$

with the matrix X being

$$X = \text{xform}(B, A) \tag{8.12}$$

Equation (8.11) is interpreted from right to left as:

1. Transform any twists from `B` to `A` with $\text{xform}(B, A)$
2. Multiply by I_A the spatial inertia at `A` to produce a wrench at `A`.
3. Transform that wrench back to `B` from `A` with $\text{yform}\,(A, B) = \text{xform}^T(B, A)$

8.3.5 Spatial Mobility Function

The function call `s=spm(arguments)` is equivalent to the constructor call `s=screw(3,arguments)`.

A spatial mobility is functionally the inverse of the spatial inertia since it transforms wrenches into twists. The single rigid body mobility is the mathematical inverse of the single rigid body inertia, but other forms of the mobility are sometimes singular.

- `M=spm(A,MC)` creates a screw type 3 at coord `A` with contents the 6×6 quantity `MC`.

Example 28 *If* `A` *is aligned properly then the mobility of a single rigid body* `MC` *is the diagonal matrix which is the inverse of the inertia matrix* `IC`*, and thus* `MC=inv(IC)`*.*

Remark 29 *A spatial mobility declared in one coordinate system and then transformed to another is declared with* `M=spm(A,MC,B)`*. The rules of transformation are derived from the transformation of twists and wrenches. It is shown that the spatial mobility transformation rule is*

$$M_B = X M_A X^T \tag{8.13}$$

with $X = \text{xform}(A, B)$*.*

8.3.6 Other Constructor Options

Using a number `st` for declaring the screw type, there are several common options for the screw constructor as shown below:

- `s=screw(st,n,A,[v;w])` this declares a screw of type `st` with name `n` at coord `A` with contents `[v;w]`. This declaration is general and very versatile.

Example 30 *Defining a wrench representing a force of magnitude* 10 *along the* $\hat{\jmath}$ *direction passing through a point located at* $2\hat{\imath}$ *in addition to a torque of magnitude* 5 *along the* $-\hat{k}$ *direction is accomplished with the following declaration:*

*f=screw(1,'myforces',coord('A',2*i_),[10*j_;-5*k_])returns*

$$f_A = \begin{bmatrix} 0 \\ 10 \\ 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} \tag{8.14}$$

- `s=screw(u,A,[v;w])` this copies the screw `u` transforms it to coord `A` and appends the quantity `[v;w]` returning the result. This is a very powerful way to extend the content of existing screws.

Example 31 *Extending the force system declared above by appending a separate force of magnitude* 2 *along the* $-\hat{\imath}$ *direction and passing through a point at* $\hat{\imath} + 5\hat{\jmath}$ *with zero torque is accomplished with the following declaration:*

`f=screw(f,coord('B',i_+5*j_),[-2*i_;o_])` and returns

$$f_B = \begin{bmatrix} 0 & -2 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 5 & 0 \end{bmatrix} \tag{8.15}$$

Finally this force system is transformed to the global coordinate system with: `f=screw(f,coord('global'))` and returns

$$f_{global} = \begin{bmatrix} 0 & -2 \\ 10 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 15 & 10 \end{bmatrix} \tag{8.16}$$

8.3.7 Member Functions and Operators

The member functions provide ways of extracting useful information from screw objects. Also the operators add functionality by allowing algebraic expressions to be evaluated allowing manipulation of equations involving screw quantities. For multi-body systems screws need to be added, subtracted, multiplied and inverted to solve dynamics problems. Below is a list of all the possible functions and operators for screw objects:

- `cs(s)` returns a `coord` object with the coordinate system that screw `s` is declared in.
- `val(s)` returns the value of the screw `s` as a MATLAB matrix.
- `double(s)` similar to `val(s)` but first it transforms the screw to the global coordinates and then returns its value.
- `s'` returns the transpose of any screw quantity. The result is a screw of the same type but with its contents transposed.
- `s|A` transforms the contents of any screw quantity `s` into the coord `A`. The rules of transformation are observed for twists, wrenches, spatial inertias, and spatial mobilities. All the operators described below use this form to transforms all their arguments into the same coordinate system automatically.
- `s&A` transplants any screw quantity `s` into the coord `A`. It is useful when the same screw quantity is applied to many different coordinate systems.

Example 32 *An example is a system of many similar rigid bodies with the same spatial inertia declared as* `I`*. Then it can be applied to the first body as*

`I1=I&cg1` with `cg1` the coord representing the center of gravity. Similarly for body 2 the spatial inertia would be `I2=I&cg2` and so on so forth.

- `s=a+b` returns the addition of two similar type screws `a` and `b`. This operator first checks to see if the two screws are declared on the same coordinate system, and if not then screw `a` is transformed to the coordinate system of screw `b`. Then it adds the two screws component by component. The same procedure applied to subtraction operator with syntax `s=a-b`.

- `s=[a,b]` appends the screw quantities in `a` with those in `b`. It follows the transformation convention of always transforms the first argument to the coordinate system of the second.

Example 33 *This operation is very useful in calculating the Jacobian of a multibody chain, by appending all the individual joint axes (twists) by typing `J=[s1,s2,..,sN]`. Even if all the individual joint axes are declared on their own coordinate system, the resulting Jacobian is a twist in the coordinate system that `sN` is declared in. Then by calling the function `J|coord` the Jacobian is resolved into the global coordinate system.*

- `s=[a;b]` is similar to the previous operator but it appends the rows of the two screws. It is intended to be used only when the arguments are transposed and appending the columns does not have the desired effect.

Remark 34 *Although included for compatibility reasons, this operator is to be avoided and if needed use the syntax* `s=[a',b']'`.

- `inv(I)` inverts the value of the spatial inertia or mobility. The inverse of an spatial inertia is a spatial mobility, and the inverse of a spatial mobility is a spatial inertia. Note that sometimes inertias or mobilities may be singular, and cannot be inverted. This function can be called with twist and wrench screw types, and the result is of the opposite screw type.

- `c=a*b` is screw multiplication and it has a different meaning depending in the argument types. Any of the following rules apply regardless of which argument type comes first and which second. Although mathematically `a*b` is different from `b*a` the resulting type is the same.

  1. $\text{screw} = \text{number} * \text{screw}$

     This multiplication rule states that the product of a screw quantity with a MATLAB matrix results in a screw quantity of the same type. The same rule applies to both pre- and post-multiplication forms.

     Example 35 `vr=s*qp` *is used when a twist joint axis* `s`*, is multiplied by a vector of generalized velocities* `qp`*. The result is the twist of relative velocities* `vr` *between the two ends of the joint.*

  2. $\text{number} = \text{twist} * \text{wrench}$

This multiplication rule states that the product of any twist with a wrench is a MATLAB matrix. This rule applies even if the order was reversed.

Example 36 ***`Q=s'*f`*** *Here the joint axis twist* ***`s'`*** *is multiplied with the internal force wrench* ***`f`*** *to produce the vector of generalized torques* ***`Q`***.

3. wrench = inertia ∗ twist

   This multiplication rule states that the product of a spatial inertia with a twist is a wrench.

   Example 37 ***`f=I*a`*** *Here the product of the spatial inertia* ***`I`*** *with the rigid body spatial acceleration* ***`a`*** *results in the internal force wrench* ***`f`***.

4. twist = mobility ∗ wrench

   This multiplication rule states that the product of a spatial mobility with a wrench is a twist.

   Example 38 ***`a=M*f`*** *Here the product of the spatial mobility* ***`M`*** *with the internal force* ***`f`*** *results in the rigid body spatial acceleration* ***`a`***.

5. number = mobility ∗ inertia

   This multiplication rule states that when a spatial inertia is multiplied by a mobility the result is a MATLAB matrix.

   Example 39 ***`M*I=1`*** *Here a single rigid body mobility* ***`M`*** *is multiplied by the body's spatial inertia* ***`I`*** *to produce a* 6×6 *identity matrix.*

- `k_bias(v,s,qp)` this is a special function that returns the bias acceleration twist which is equal to $\kappa = v \times s\dot{q}$ with $v\times$ the spatial form of the cross product operator for twists.

- `p_bias(v,I)` this is a special function that returns the bias force wrench which is equal to $p = v \times Iv$ with $v\times$ the spatial form of the cross product operator for wrenches.

8.3.8 Screw Information Functions

Some functions are being redefined for screw quantities while others apply only to screws.

- `name(s)` returns the name of a screw quantity.

- `size(s)` returns the size of the contents of the screw. The number of columns is the number of individual screws appended together in `s`, while the number of rows should be equal to 6 for every screw quantity. The transpose operator switches rows with columns and through this function, the toolbox figures out if an argument to a function is a regular or a transposed screw.

- `isovec(s)` returns the isophoric vector of a screw. The definition for isophoric vector is on page 178 .

- `anisovec(s)` returns the anisophoric vector of a screw.

- `dirvec(s)` returns a unit vector along the screw axis.
- `posvec(s)` returns the location of the screw axis relative to the reference point. A screw `s` expressed on the coordinate frame `coord(posvec(s))` has both vector parts of the screw parallel to each other.
- `pitch(s)` returns the pitch of a screw. The pitch is the ratio of the anisophoric vector component along the axis of the screw to the magnitude of the isophoric vector.
- `subs(s,vars,expr)` performs the substitution of `vars` with `expr` in `s`. This is intended to be used when the screw has symbolic content, and if all the symbolic variables are being substituted with numbers it converts the screw contents to numbers also.

  Example 40 *For example, if the screw `s` uses a symbolic variable `tau` from its declaration on some coordinate system `A` as `s=wrench(A,[o_;tau*k_])`, then the call `subs(s,'tau',0.5)` replaces `tau` with the number `0.5` and returns the numerically evaluated result, making the contents of that wrench equal to `[o_;0.5*k_]`.*

- `simplify(s)` returns a symbolically simplified version of the screw. Best used after complex symbolic operations involving multiple screw multiplications with trigonometric content.

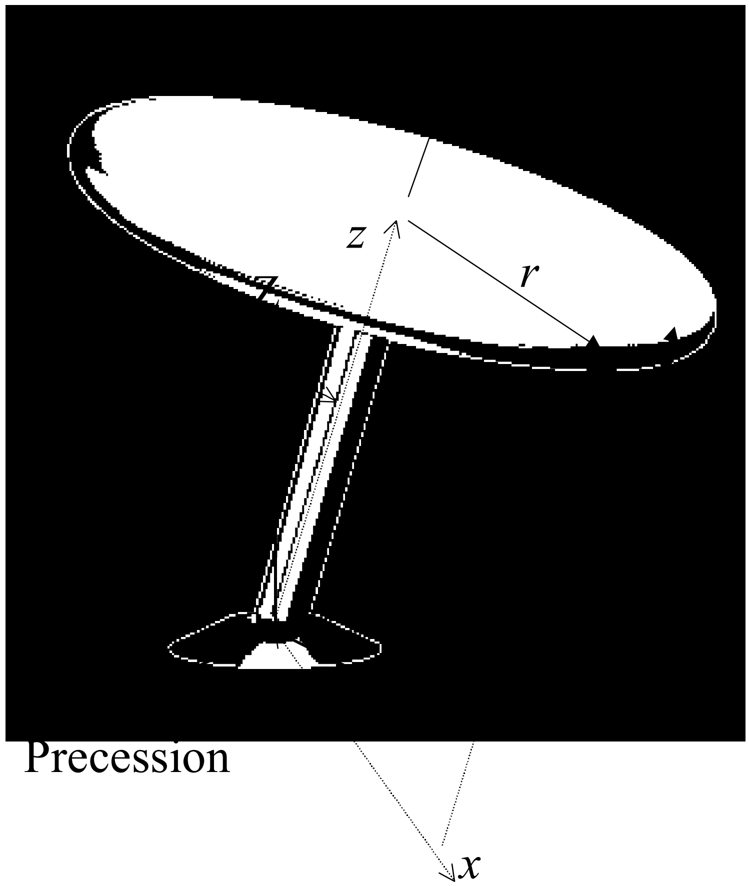

Figure 8.2: Geometry of a spining top. The orientation is given by the precession angle φ, the nutation angle θ and the spin angle ψ. The top consists of a weighless support rod with length l and a disk of radius r, thickness h and mass m.

8.4 Example Simulation

A spinning top shown in Figure 8.2 is used as a simulation example. The top consists of a disk with radius r and thickness h. It is located a distance l from the base. The top has mass $m = 5kg$, radius $r = 6cm$, thickness $h = 1cm$ and height $l = 4cm$. The spherical joint on the base has a coefficient of friction of $\nu = 0.004Nms/rad$.

Initially the top has nutation angle $\theta = \pi/20rad$ and spin rate of $\dot{\psi} = 200rad/s$.

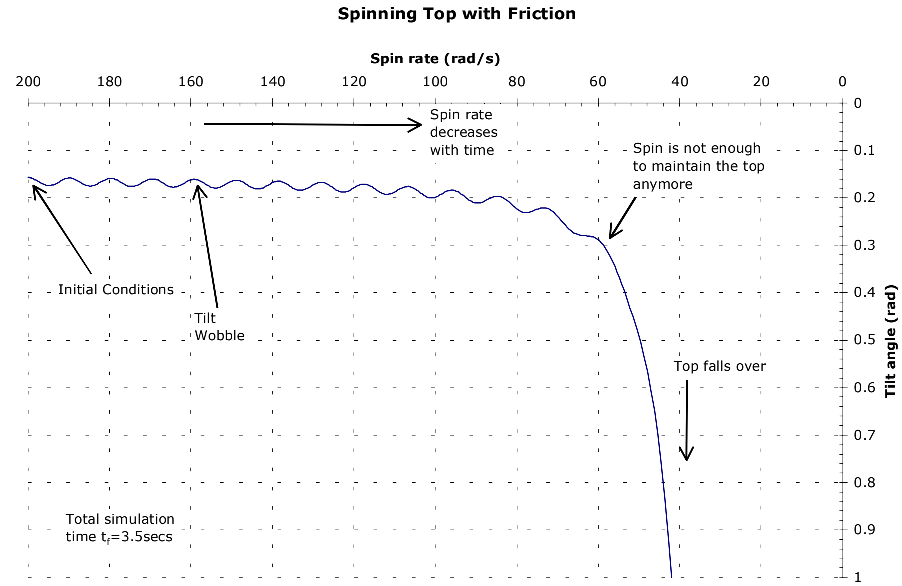

Figure 8.3: Solution of a spining top with friction. Initialy the spin rate is $\dot{\theta} = 200 rad/s$ and the tilt angle $\psi = \pi/20$. When friction reduces the spin rate below $60 rad/s$ the tilt angle increases until the top falls over on its side.

The spin ψ and precession φ are initially zero. In this example a MATLAB program is used to simulate the spinning top. The source code for finding the joint accelerations in this example is shown in the next section. The simulation is run for approximately $t_f = 3.5s$ until the top slows down enough to fall over.

Figure 8.3 shows the notation angle θ as a function of the spin rate $\dot{\psi}$. Because of friction the spin rate continuously decreases. Until the spin rate falls under $100 rad/s$, the tilt angle wobbles around the initial value of $\pi/20$. Between the spin rates of $100 rad/s$ and $60 rad/s$ the tilt angle slowly drifts away and when the spin rate drops below $60 rad/s$ the tilt angle rapidly increases indicating that the top is falling over.

Figure 8.4 shows the reaction force components as a function of time. The reaction

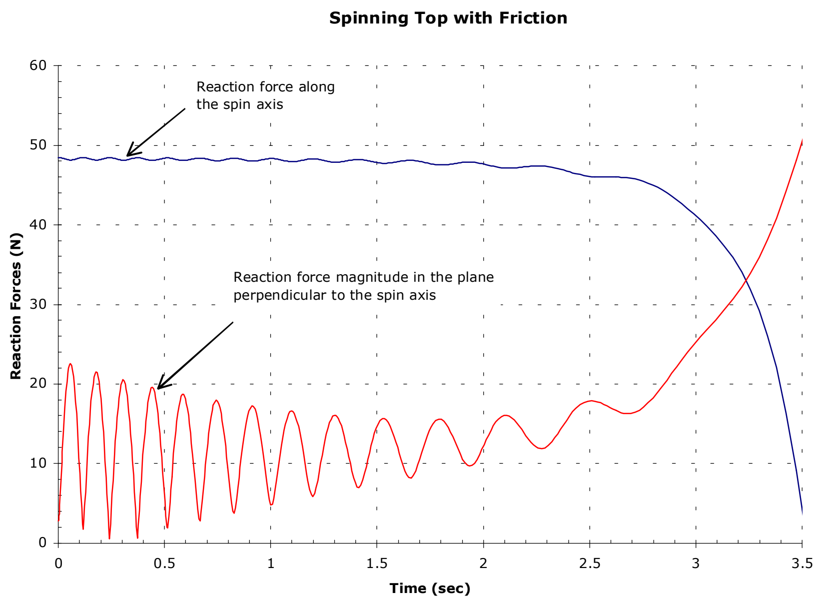

Figure 8.4: Reaction forces on the joint for the spinning top as it slows down.

forces along the spinning axis of the top are constant and equal approximately to $\mu_z = mg \sin\psi$. When the top slows down and start to fall over these reaction forces also drop as the tilt angle ψ increases. The other line in the figure shows the magnitude of the reaction forces in the plane perpendicular to the spin axis. These fluctuate as the top wobbles and increase as the top falls over.

8.4.1 Source Code for Example

Joint Accelerations

The following MATLAB function takes two vectors of joint angles `q` and joint velocities `qp` and returns the joint accelerations `qpp`.

```
function qpp=topsim(q,qp)
%                               John Alexiou, 1999.
%
```

```
%  Simulation of a spinning top with viscus friction in the joint.
%  Arguments q and qp are 3x1 vectors containing the joint angles
%  and joint velocities as defined in the workspace.
%
%  Uses SPAT toolbox.  All units are in SI.

% Definition of a global structure 'tops' containing the geometry
% defined at the workspace.
global tops

% Extraction of geometry form tops
m = tops.m;       %mass of disk
l = tops.l;       %height of disk from joint
r = tops.r;       %radius of disk
h = tops.h;       %thickness of disk
cf = tops.cf;     %coefficient of friction on joint Q=-cf*qp
g = 9.81;         %gravity acceleration

phi = q(1);       %precession angle from q
th = q(2);       %nutation angle from q
psi = q(3);       %spin angle from q

%Coordinate System definitions
%coord() is the constructor for coordinate systems
%elrot() returns 3x3 elementary rotation matrix
%i_,j_ and k_ are three unit vectors and o_ is the zero vector

%CJ is world coord rotated by phi about x-dir and th about y-dir
CJ = coord('Joint',elrot(k_,phi),elrot(j_,th));

%CG is coord CJ translated by l in local z-dir
CG = coord('CG',CJ,l*k_);

%6x6 Diagonal Inertia Matrix
IC = diag([m,m,m,m/12*(3*r^2+h^2),m/12*(3*r^2+h^2),m/2*r^2]);

%spi() is the spatial inertia constructor
%I at CG has value IC and it is xformed to CJ
I = spi('Inertia Matrix',CG,IC,CJ);

%Mobility is the inverse of Inertia
M = inv(I);
```

```
%Joint torques define the frictional forces
Q = -cf*qp;

%Weight of the disk is applied on the location of CG
%wrench() is the wrench constructor
%pos() returns a 3x1 position vector of a coord system
%[k_;o_] is the value of a unit wrench in the global z-dir
%fg is defined on pos(CG) and xformed to CJ
fg = wrench('Weight',coord(pos(CG)),-m*g*[k_;o_],CJ);

%The spherical 3DOF joint axis is defined on CJ
%twist() is the twist constructor
s = twist('Joint Space',CJ,[zeros(3);eye(3)]);

%The reaction space of the joint is defined on CJ
R = wrench('Reaction Space',CJ,[eye(3);zeros(3)]);

%The percussion space is defined by the pseudo-inverse of s
T = I*s*inv(s'*I*s);

%The reactive acceleration space is defined by the pseudo-inverse
of R
U = M*R*inv(R'*M*R);

%The velocity twist is defined from the joint velocities qp
v = s*qp;

%The cross product operator for velocity is 6x6 matrix
%val() returns the value of spatial quantity
%cr() returns either the vector or spatial cross product operator
vx = cr(val(v));

%Bias accelerations use vx
k = vx*s*qp; %bias acceleration
%Bias forces use -vx' the minus transpose of vx
p = (-vx')*I*v; %bias force

%Reactive accelerations components are
gamma = R'*k;

%Joint accelerations are equal to the active accelerations components
```

```
qpp = T'*M*(T*Q+fg-p);

%Reaction force components are
mu = U'*(I*U*gamma+p-fg);
```

State Space Rate of Change

The following MATLAB code is used multiple times by the `ode45()` Matlab function which numerically integrates multiple differential equations. It takes the state space vector of the system and returns the rate of change of the state space.

```
function yp=topode(t,y)
%
% Argument t is simulation time and
% y is the state space vector of the top.
% Returns yp the rate of chage of y.

% The first three components of y defines the joint angles.
q = y(1:3);

% The last three components of y defines the joint velocities.
qp = y(4:6);

% The joint accelerations are calculated from topqpp()
qpp = topqpp(q,qp);

% The rate of change of y contains the joint velocities
% in the first three components and the joint accelerations
% in the last three components.
yp = [qp;qpp];
```

Simulation Initialization

The following MATLAB function initializes all the variables and calls the `ode45()` for the simulation time.

```
function ys=top
%                  Solve a top problem
% top initializes all variables and calls ode45() for solution
% which is returned by this function back to the workspace.

% Initialize system information in tops structure
global tops
```

```
tops.m = 5.00; %mass
tops.l = 0.04; %distance of CG from pivot
tops.r = 0.06; %radius of disk
tops.h = 0.01; %height of disk
tops.cf = 0.004; %friction coeff

% Ask user about initial conditions and simulation time
th = input('Initial Nutation [pi/2]=');
wz = input('Initial spin [180]=');
tf = input('Simulation time [2]=');

% Create initial state space variable
q = [0;th;0];
qp = [0;0;wz];
y0 = [q;qp];

% Call numerical integrator function ode45()
[t,y] = ode45('topode',[0 tf],y0);

% Return the solution of joint angles and velocities.
ys = [t,y];
```

8.5 Summary

Coordinate frames contain two defining quantities. One is the position vector of the origin of the frame, and the other the 3×3 orthonormal matrix describing the orientation of the local $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ vectors. Twists and wrenches contain $6 \times k$ subspace matrices with each column representing a different screw. They also contain the coordinate frame they are expressed in. Spatial inertias and mobilities are 6×6 quantities that map twists into wrenches and vice versa. A coordinate frame is also associated with them.

If quantities are defined right, then the toolbox can be used to express spatial relationships with high level algebra. Addition, linear forms, projections and scalings are performed transparently to the user without ever the need for component by component manual calculation.

Removing the comments from the spinning top source listing makes it apparent that the number of equations needed to model the system is really small.

CHAPTER 9

TREES AND LOOPS

An alternative to recursive methods is the stacked form. The stacked form uses block matrices to describe the motions and loads of the entire system. When rigid body systems form tree like structures the stacked form incorporates the topology and hierarchy of each body. The matrix that associates connected bodies is usually called the connectivity or adjacency matrix. According to graph theory, rigid bodies are treated as nodes, and joints as chords that connect various nodes. The equations of motion in stacked form can incorporate any rigid body chain or tree structure. In fact, general contacts between bodies, or with the environment that form kinematical loops are treated with a different connectivity matrix called the contact matrix. The accelerations are solved for by correcting the open loop solution with the necessary contact forces. Articulated inertias and articulated mobilities are well suited for associating contact forces and resulting accelerations.

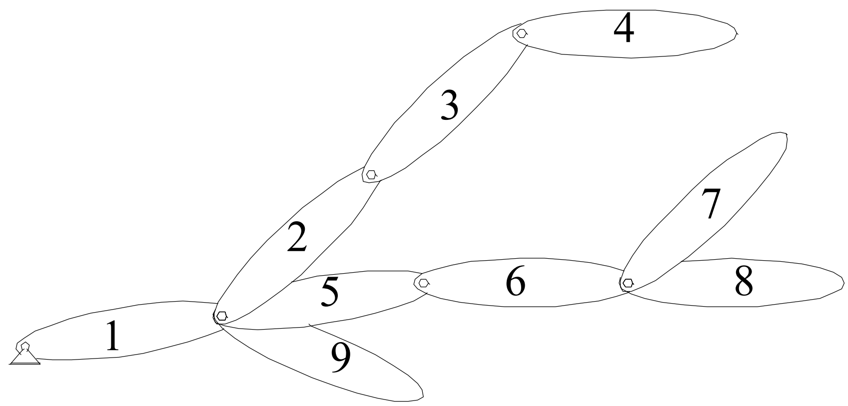

Figure 9.1: Example of rigid body tree structure. Body 1 is the root and bodies 4, 7, 8 and 9 are tips of each branch.

9.1 Stacked Form and Rigid Body Trees

9.1.1 Adjacency Matrix

A linear system of n rigid bodies is a special case of a tree system. A tree system has multiple branches but only one root body. Each body in the system has one parent body and multiple child bodies. For example, body 6 in Figure 9.1 has two children, bodies 7 and 8, and body 5 as a parent. Body 1 is the root because all other bodies are children to it. Body 8 is the tip of the chain that goes through bodies 1, 5, 6 and 8.

To mathematically describe such a tree structure of n rigid bodies, a $n \times n$ adjacency matrix C is defined such that:

Definition 41 *If body j is the immediate parent of body i then $C_{ij} = [1]_{6\times6}$, otherwise it is $C_{ij} = [0]_{6\times6}$.*

For the example shown in Figure 9.1 the 9×9 adjacency matrix is

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9.1}$$

where each row i places a 1 in the j-th column indicating that body j is the parent of body i. Therefore the row corresponding to the root of the system is a row of zeros. With this notation a 0 or a 1 may actually be a zero or a identity matrix depending on the context the adjacency matrix is used in.

The adjacency matrix of a serial chain is a block diagonal matrix with ones in the cells under the diagonal such as

$$C = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \tag{9.2}$$

This means that each body i has a parent the body $i-1$.

9.1.2 Stacked Form

The adjacency matrix is used to transform equations from a recursive to a stacked form. In the recursive form equations are applied sequentially from one body to the next moving either from root to tip or vice versa. In the stacked form, similar recursive equations are joined together to form large block matrix equations. As an example, the stacked form of the velocity kinematics equation (2.9) is

$$v = Cv + S\dot{q} \tag{9.3}$$

with C the adjacency matrix and the following definitions for the remaining quantities.

The stacked velocity twists

$$v = \{v_i\} \tag{9.4}$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{9.5}$$

the stacked joint velocities

$$\dot{q} = \{\dot{q}_i\} \tag{9.6}$$

$$= \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{9.7}$$

and the stacked joint axes

$$S = \{\{s_i\}\} \tag{9.8}$$

$$= \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix} \tag{9.9}$$

With this notation the expression Cv represents a stacked form of all the parent body velocities. Together with the stacked form of all the relative velocities $S\dot{q}$ they add up to the velocity v of all the bodies.

Specifically, with the stacked form a quantity defined by single brackets such as $\blacklozenge = \{\blacklozenge_i\}$ represents a $n \times 1$ block matrix, and with double brackets such as $\blacklozenge = \{\{\blacklozenge_i\}\}$ a $n \times n$ block diagonal matrix.

Now all the indexed quantities associated with each rigid body have non-indexed stacked equivalents. In general, the following definitions apply

1. Joint velocities $\dot{q} = \{\dot{q}_i\}$
2. Joint accelerations $\ddot{q} = \{\ddot{q}_i\}$
3. Body velocities $v = \{v_i\}$
4. Body accelerations $a = \{a_i\}$
5. Bias accelerations $\kappa = \{\kappa_i\}$
6. Forces $f = \{f_i\}$
7. Bias forces $p = \{p_i\}$
8. Joint axes $S = \{\{s_i\}\}$ and reaction spaces $R = \{\{R_i\}\}$
9. Body inertias $I = \{\{I_i\}\}$
10. Percussion space $T = \{\{T_i\}\}$ and quake space $U = \{\{U_i\}\}$

11. Joint torques $Q = \{Q_i\}$ and reaction forces $\mu = \{\mu_i\}$

12. Active accelerations $\alpha = \{\{\alpha_i\}\}$ and reactive accelerations $\gamma = \{\{\gamma_i\}\}$

13. Articulated inertias $A = \{\{A_i\}\}$ and articulated mobilities $\Lambda^{-1} = \left\{\left\{\Lambda_i^{-1}\right\}\right\}$

14. Articulated bias forces $d = \{d_i\}$ and articulated bias accelerations $b = \{b_i\}$

The velocity and acceleration kinematics for any tree system are expressed with a single equation relating block quantities. The acceleration kinematics

$$a = Ca + S\ddot{q} + \kappa \tag{9.10}$$

can be solved for the accelerations as

$$a = \Delta^{-1}\left(S\ddot{q} + \kappa\right) \tag{9.11}$$

with

$$\Delta^{-1} = \left(1 - C\right)^{-1} \tag{9.12}$$

The lower triangular matrix Δ^{-1} is called the trunk matrix. Each row i of Δ^{-1} has ones in the j-th column if body j is between body i and the root. The trunk matrix of the system shown in Figure 9.1 is

$$\Delta^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9.13}$$

Interpreting the trunk matrix requires looking at it row by row. For example the 8-th row has ones in columns 1, 5, 6 and 8. This means that these bodies form a continuous chain with body 1 the root and body 8 the tip. So each row indicates which bodies lie in the path to the root. For a serial chain Δ is a lower triangular matrix with ones from the diagonal and down.

Similar to the adjacency matrix C indicating which bodies are parents, its transpose C^T indicates which bodies are children. It is used in the equations of motion since the internal forces from all the children need to be added up to the internal forces of a body to form the net forces acting on each body. The stacked form of the equations of motion (3.10) is

$$f = C^T f + Ia + p \tag{9.14}$$

or

$$\Delta^T f = Ia + p \tag{9.15}$$

The definition of articulated inertias A from (3.13) is

$$f = Aa + d \tag{9.16}$$

which is used to decompose forces

$$f = TQ + R\mu \tag{9.17}$$

and accelerations into

$$a = S\psi + U\gamma \tag{9.18}$$

with the force and acceleration subspaces are defined from (6.30) and (6.33) as

$$T = AS\left(S^T AS\right)^{-1} \tag{9.19}$$

$$U = A^{-1}R\left(R^T A^{-1}R\right)^{-1} \tag{9.20}$$

and the projected components from (6.61a) and (6.61b)

$$Q = S^T f \tag{9.21}$$

$$\mu = U^T f \tag{9.22}$$

and from (6.62b) and (6.62a)

$$\psi = T^T a \tag{9.23}$$

$$\gamma = R^T a \tag{9.24}$$

Substituting (9.16) into (9.22) and then into (9.17), the internal forces are expressed as

$$f = TQ + RU^T Aa + RU^T d \tag{9.25}$$

These forces are substituted into the right hand side of (9.14) and the left hand side is compared to (9.16) yielding the expression

$$Aa + d = \left(C^T RU^T A + I\right) a + \left(p + C^T TQ + C^T RU^T d\right) \tag{9.26}$$

which is split into

$$A = C^T RU^T A + I \tag{9.27}$$

and

$$d = p + C^T TQ + C^T RU^T d \tag{9.28}$$

The first expression is

$$\left(1 - C^T RU^T\right) A = I \tag{9.29}$$

which defines the articulation matrix L as

$$A = LI \tag{9.30}$$

with

$$L = \left(1 - C^T R U^T\right)^{-1} \tag{9.31}$$

The second expression is then

$$d = Lp + LC^T TQ \tag{9.32}$$

There is a possible computational shortcut in calculating L since C is an adjacency matrix and $C^n = 0$. When the Taylor series expansion is applied to (9.31) the series is truncated to contain only n terms resulting in

$$\begin{aligned} L &= \left(1 - C^T R U^T\right)^{-1} && (9.33) \\ &= 1 + C^T R U^T + \left(C^T R U^T\right)^2 + \ldots + \left(C^T R U^T\right)^{n-1} && (9.34) \end{aligned}$$

Substituting (9.18) into (9.16) and then into (9.21) and solving for the active acceleration components

$$\psi = T^T A^{-1} \left(TQ - d\right) \tag{9.35}$$

Also substituting (9.10) in (9.24) results into the reactive acceleration components

$$\gamma = R^T \left(Ca + \kappa\right) \tag{9.36}$$

These acceleration components are assembled into (9.18) as

$$\begin{aligned} a &= ST^T A^{-1} \left(TQ - d\right) + UR^T \left(Ca + \kappa\right) && (9.37) \\ \left(1 - UR^T C\right) a &= ST^T A^{-1} \left(TQ - d\right) + UR^T \kappa && (9.38) \end{aligned}$$

which is solved for the accelerations as

$$a = L^T \left[S T^T A^{-1} \left(TQ - d \right) + U R^T \kappa \right] \tag{9.39}$$

where

$$L^T = \left(1 - U R^T C \right)^{-1} \tag{9.40}$$

Substituting (9.10) into (9.16) and then into (9.22) gives the reaction force component

$$\mu = U^T \left(A \left(Ca + \kappa \right) + d \right) \tag{9.41}$$

Using (9.17) the internal force solution is

$$f = TQ + RU^T \left(A \left(Ca + \kappa \right) + d \right)$$

which depends on the accelerations a.

Rearranging (9.10) into

$$S\ddot{q} = \Delta a - \kappa \tag{9.42}$$

and pre-multiplying by the percussion space T^T projects the body accelerations into the relative joint accelerations

$$\ddot{q} = T^T \left(\Delta a - \kappa \right) \tag{9.43}$$

9.2 Direct Solution

Applying equations (3.10) and (7.37) in stacked form to any system results in a direct solution of accelerations similar to that in Section 7.6. Equation (9.14) shows the stacked form of (3.10) as

$$f = C^T f + Ia + p \tag{9.44a}$$

$$f = \left(1 - C^T\right)(Ia + p) \tag{9.44b}$$

where C is the adjacency matrix, f are the internal forces, a are the accelerations, I are the body inertias and p are the bias forces. The stacked form of (7.37) is

$$a = b - \Lambda^{-1} C^T f \tag{9.45}$$

where b are the articulated bias accelerations and Λ^{-1} the articulated mobilities.

Substituting (9.44b) into (9.45) yields

$$a = b - \Lambda^{-1} C^T \left(1 - C^T\right)(Ia + p) \tag{9.46}$$

$$\left(1 + \Lambda^{-1} C^T \left(1 - C^T\right) I\right) a = b - \Lambda^{-1} C^T \left(1 - C^T\right) p \tag{9.47}$$

which is solved for the body accelerations as

$$a = \left(1 + \Lambda^{-1} C^T \left(1 - C^T\right) I\right)^{-1} \left(b - \Lambda^{-1} C^T \left(1 - C^T\right) p\right) \tag{9.48}$$

This is a direct solution for the accelerations that can be used after all the articulated mobilities are known. It may replace the last recursion needed to solve for all the accelerations starting from the base towards the tips.

9.3 Stacked Form and Contacts Between Bodies

Many methods for solving constrained multibody systems reply on a similar idea. First the system is split into an open loop structure, and the open loop accelerations are calculated. Then the forces needed to close the loops are calculated. Finally, these contact forces are use to calculate the close loop accelerations. This section follows the methodology developed in [4] but in a stacked form. The adaptation of the contacts in stacked form utilizes the contact matrix used in [9].

9.3.1 Contact Matrix

A constrained system has m contacts between its n bodies. Each contact is described by a separate row in the $m \times n$ contact matrix G as defined below.

Definition 42 *The k-th contact between bodies i and j is modeled by* $G_{ki} = 1$ *and* $G_{kj} = -1$.

An example is shown in Figure 9.2 with a contact matrix

$$G = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{9.49}$$

The branch matrix Υ is defined as

$$\Upsilon = G\Delta^{-1} \tag{9.50}$$

where Δ^{-1} is the trunk matrix. For the example shown above Υ is equal to

$$\Upsilon = \begin{bmatrix} 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \end{bmatrix} \tag{9.51}$$

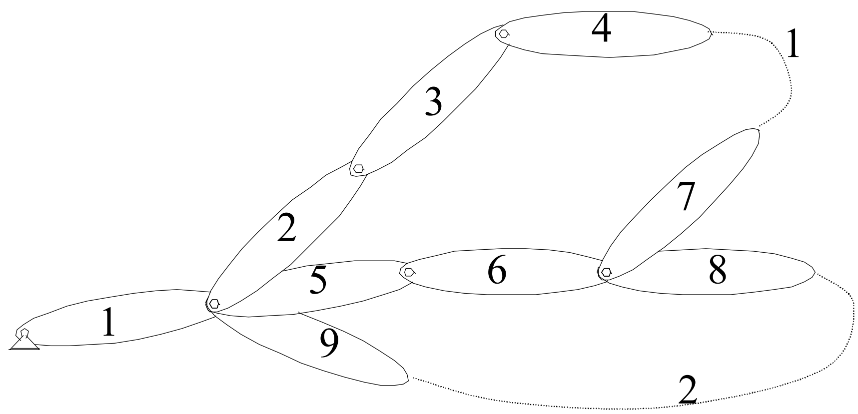

Figure 9.2: A constrained tree system, with two contacts. Body 4 and body 7 make the first contact, and body 8 and body 9 the second.

This shows the two sides of the kinematic loop. Starting from the common ancestor one branch of the loop is indicated by ones and the other by minus ones. For example in Figure 9.2 the first constraint is a kinematic loop where all the bodies have body 1 as a common ancestor. From body 1 the two branches of the loop are bodies 2, 3, 4 and 5, 6, 7.

9.3.2 Stacked Form

In stacked form the velocity kinematics (9.3) are defined as

$$\Delta v = S\dot{q} \tag{9.52}$$

and the acceleration kinematics (9.10) as

$$\Delta a = S\ddot{q} + v \times S\dot{q} \tag{9.53}$$

where $\Delta = 1 - C$.

Similarly the kinematics of the contact are defined as

$$Gv = z\dot{\theta} \tag{9.54}$$

and

$$Ga = z\ddot{\theta} + v \times z\dot{\theta} \tag{9.55}$$

where G is the contact matrix, z is the relative motion spaces between the contacts, and $\dot{\theta}$, $\ddot{\theta}$ the contact velocities and accelerations. These equations are velocity and acceleration constrain equations where the net relative velocity around a kinematic loop is zero. This is similar to circuit rule that the net voltage change around a loop is zero.

Using the contact kinematics requires knowledge of $\dot{\theta}$. It can either be calculated by integrating $\ddot{\theta}$ if the contact accelerations are known or by projecting the velocity twists v directly. Integration usually introduces errors and explicit calculation of $\ddot{\theta}$ may be difficult. Alternatively, from (9.54) multiplying both sides with $z^T J$ yields

$$z^T J z\dot{\theta} = z^T JGv \tag{9.56}$$

where

$$J = \Upsilon I \Upsilon^T \tag{9.57}$$

and I is the stacked inertia of the system. Equation (9.56) is solved for

$$\dot{\theta} = \left(z^T J z\right)^{-1} z^T JGv \tag{9.58}$$

The contact forces F acting on each of the contacting bodies must lie in the reaction space of the contact since contacts are powerless. Therefore

$$F = B\xi \tag{9.59}$$

where

$$z^T B = 0 \tag{9.60}$$

and ξ are the contact reaction force magnitudes.

The stacked equations of motion with contacts are derived from (9.14) as

$$\Delta^T f = Ia + p + G^T F \tag{9.61}$$

where $G^T F$ are the net contact forces acting on each body.

9.4 Overview of Constrained Equations

There are three main steps in solving general tree systems with contacts. First the contacts are broken and an articulated inertia analysis is performed on the resulting open loop tree structure to solve for the open loop accelerations. Then an articulated mobility analysis is performed to establish the effect of contact forces on the body accelerations. Finally the contact forces are solved for using projections and these forces are propagated down through the tree to solve for the closed loop accelerations. The body accelerations are projected into joint accelerations which are then integrated for the next time step.

9.4.1 Open Loop Accelerations

If u are the open loop tree accelerations, then the equations of motion for the system (9.14) are

$$\Delta^T f = Iu + p \tag{9.62}$$

The recursive articulated inertia method in (7.7a) and (7.7b) is used to compute the articulated quantities A and d. Also the decomposition matrices T, U and L are calculated from (9.19), (9.20) and (9.40). Then the open loop accelerations are solved using (9.39) as

$$u = L^T \left[ST^T A^{-1} (TQ - d) + UR^T \kappa\right] \tag{9.63}$$

It would probably be more efficient to use the recursive method in equation (6.108) moving from base towards the tips with

$$u_i = s_i T_i^T A_i^{-1} (T_i Q_i - d_i) + U_i R_i^T (u_j + \kappa_i) \tag{9.64}$$

where u_j is the open loop acceleration of the parent body.

9.4.2 Contact Forces

Before the contact forces are calculated it is important to establish their effect to the rest of the chain. From the definition of the articulated mobility (7.37) the accelerations are

$$a = b - \Lambda^{-1} G^T F \tag{9.65}$$

If the contacts are ignored so $F = 0$ then the open loop accelerations u are

$$u = b \tag{9.66}$$

Thus the closed loop accelerations as a function of contact forces, are

$$a = u - \Lambda^{-1} G^T F \tag{9.67}$$

where a are the closed loop accelerations, u the open loop accelerations, Λ^{-1} the articulated mobilities and F the contact forces. Without any contact forces $a = u$. Otherwise F corrects the open loop accelerations u such that the contact constraints are enforced.

Substituting (9.59) into (9.67) and then into (9.55) yields

$$z\ddot{\theta} + v \times z\dot{\theta} = Gu - G\Lambda^{-1} G^T B\xi \tag{9.68}$$

$$B^T \left(z\ddot{\theta} + v \times z\dot{\theta} \right) = B^T Gu - B^T G\Lambda^{-1} G^T B\xi \tag{9.69}$$

which is solved for

$$\xi = \left(B^T G\Lambda^{-1} G^T B \right)^{-1} B^T \left(Gu - v \times z\dot{\theta} \right) \tag{9.70}$$

since the reaction forces are powerless and $B^T z = 0$. Finally (9.70) is back substituted into (9.59) to yield the contact forces

$$F = \Gamma \left(Gu - v \times z\dot{\theta} \right) \tag{9.71}$$

where

$$\Gamma = B \left(B^T G\Lambda^{-1} G^T B \right)^{-1} B^T \tag{9.72}$$

It is possible Γ may not exist if $\left(B^T G \Lambda^{-1} G^T B\right)$ is singular. In that case the kinematic loops introduce arbitrary forces through the system. This is similar to singular configurations in serial robots where arbitrary motions exist between the links.

9.4.3 Closed Loop Solution

Once the contact forces F are known they can either be substituted into (9.67)

$$a = u - \Lambda^{-1} G^T F \tag{9.73}$$

Then the joint accelerations are calculated by either the recursive projection (6.110)

$$\ddot{q}_i = T_i^T \left(a_i - a_{i-1} - \kappa_i\right) \tag{9.74}$$

or the stacked form (9.43)

$$\ddot{q} = T^T \left(\Delta a - \kappa\right) \tag{9.75}$$

The joint accelerations are then be integrated to calculate the joint velocities and orientations for the next time frame. The contact joint velocities used are calculated from (9.58)

$$\dot{\theta} = \left(z^T J z\right)^{-1} z^T J G v \tag{9.76}$$

with

$$J = \Upsilon I \Upsilon^T \tag{9.77}$$

the branch composite inertia matrix.

9.5 Constrained Projective Dynamics Algorithm

1. Preparatory Work

   - The contact matrix G is defined according to which bodies are in contact.
   - The contact motion space z
   - The contact reaction space B such that $z^T B = 0$

2. Open Loop Accelerations

   - Articulated inertia analysis
   - Articulated mobility analysis
   - Open loop accelerations are $u = L^T \left[ST^T A^{-1} (TQ - d) + UR^T \kappa\right]$

3. Kinematics

   - The branch composite inertia matrix is $J = \Upsilon I \Upsilon^T$
   - The contact velocities are projected with $\dot{\theta} = \left(z^T J z\right)^{-1} z^T J G v$
   - The contact bias accelerations are $\nu = v \times z\dot{\theta}$

4. Contact Forces

   - The contact reactive inertia is $\Gamma = B \left(B^T G \Lambda^{-1} G^T B\right)^{-1} B^T$
   - The contact forces are $F = \Gamma (Gu - \nu)$

5. Closed Loop Accelerations

- Closed loop accelerations are $a = u - \Lambda^{-1} G^T F$

9.6 Planar Constrained Example

A slider crank mechanism consists of three bodies as seen in Figure 9.3. The three bodies have masses $m_1 = 0.40$, $m_2 = 0.15$ and $m_3 = 0.30$ and the lengths of the links are $r = 2.29$, $l = 5.00$, and $t = 0.5$. The center of gravity of body 1 is located a distance $r/4$ from the joint. For the second body a distance $l/2$ and for the third body a distance t. The angular inertias on their center of gravities are $I_{z1} = 0.1464 m_1 r^2$, $I_{z2} = 0.0866 m_2 l^2$, and $I_{z3} = 0.666 m_3 t^2$.

The initial conditions are

$$q = \begin{bmatrix} \psi \\ \theta \\ \varphi \end{bmatrix} = \begin{bmatrix} 0.400 \\ 1.603 \\ -0.4355 \end{bmatrix} \tag{9.78}$$

and

$$\dot{q} = \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 20.00 \\ 23.9343 \\ -3.9343 \end{bmatrix} \tag{9.79}$$

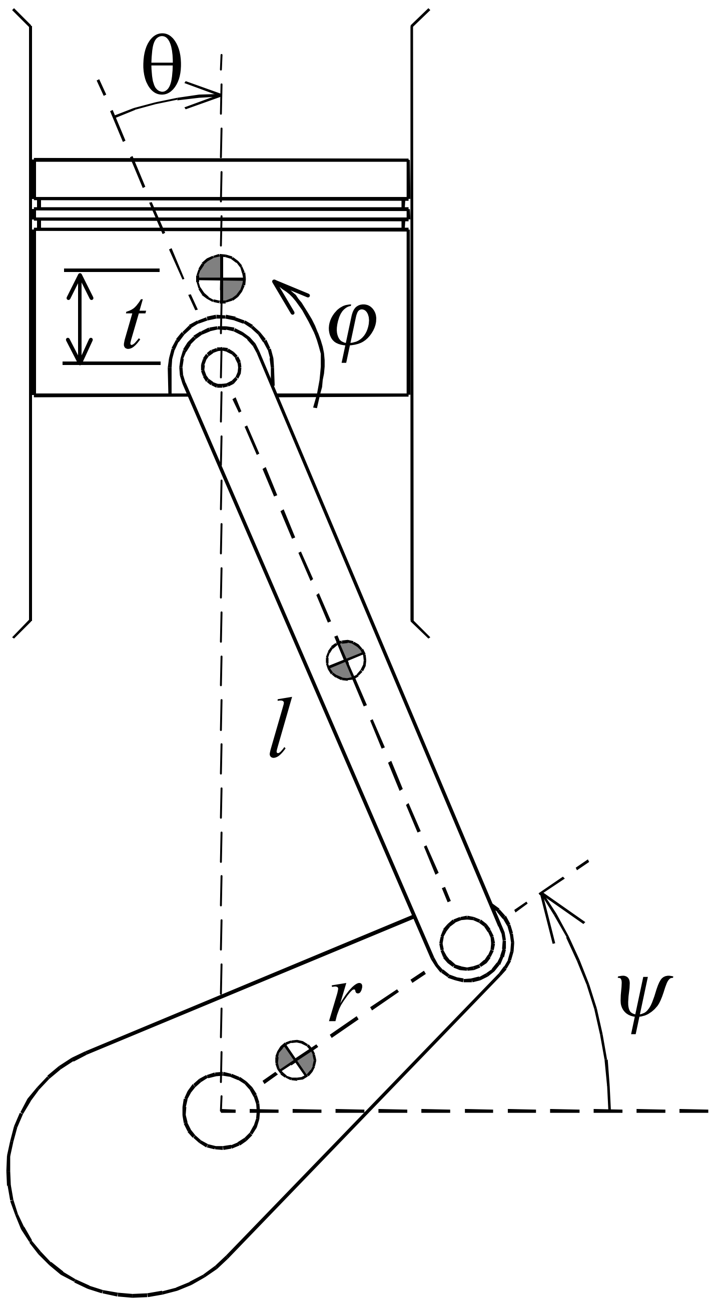

Figure 9.3: Example of constrained projective dynamics with a slider crank mechanism. The three relative sequential rotations ψ, θ, and φ are constrained such that $\psi + \theta + \varphi = \pi/2$.

which results in the planar velocity twists

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 20.0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -21.344 \\ 50.483 \\ -3.934 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 50.483 \\ 0 \end{bmatrix} \tag{9.80}$$

Using MATLAB® and SPAT the articulated mobility of body 3 is

$$\Lambda_3^{-1} = \begin{bmatrix} -668.13 & -42.89 & -146.22 \\ 173.01 & 3.087 & 36.20 \\ -100.98 & -6.853 & -22.30 \end{bmatrix} \tag{9.81}$$

and open loop planar accelerations are

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 5.3123 \end{bmatrix} \quad u_2 = \begin{bmatrix} -1199.3 \\ 21.7 \\ -207.4 \end{bmatrix} \quad u_3 = \begin{bmatrix} 933.409 \\ 21.7561 \\ 149.1452 \end{bmatrix} \tag{9.82}$$

Since the piston constraint is not enforced the open loop acceleration u_3 is not a pure translation in the vertical direction. The contact motion space is

$$z_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{9.83}$$

The contact force (9.59) is defined as

$$F_3 = B_3 \xi_3 \tag{9.84}$$

where B_3 is the reaction subspace of the contact which contains two unit force wrenches. One is a force along the horizontal direction and through the center of gravity of the third body and the other a unit torque. Since the center of gravity is at a height of

$$h = r\sin(\psi) + l\sin(\psi+\theta) + t\sin(\psi+\theta+\varphi) \tag{9.85}$$

$$= 5.925 \tag{9.86}$$

relative to the origin the subspace B_3 is

$$B_3 = \begin{bmatrix} 1.000 & 0 \\ 0 & 0 \\ -5.925 & 1 \end{bmatrix} \tag{9.87}$$

The contact reaction space B_3 restricts motion in the horizontal direction and rotation for the third body. The constraint closing equation (9.67) is

$$a_3 = u_3 - \Lambda_3^{-1} F_3 \tag{9.88}$$

and the contact force components (9.70) are

$$\xi_3 = \left(B_3^T \Lambda_3^{-1} B_3\right)^{-1} B_3^T u_3 \tag{9.89}$$

$$= \begin{bmatrix} 7.459 \\ 3.729 \end{bmatrix} \tag{9.90}$$

since $v_3 \times z_3\dot{\theta}_3 = 0$ by definition of z_3 and the current v_3. The total contact force is

$$F_3 = B_3\xi_3 \tag{9.91}$$

$$= \begin{bmatrix} 7.459 \\ 0 \\ -40.464 \end{bmatrix} \tag{9.92}$$

The closed loop acceleration is

$$a_3 = u_3 - \Lambda_3^{-1}F_3 \tag{9.93}$$

$$= \begin{bmatrix} 0 \\ 196.17 \\ 0 \end{bmatrix} \tag{9.94}$$

which obviously is a pure translation in the vertical direction. Therefore the contact force F_3 has forced body three to slide along the vertical direction only.

9.7 Summary

An alternative to the recursive formulation is the stacked form. Quantities are grouped together in large block matrices. The topology and hierarchy of the system is described with the adjacency matrix. Expressing the kinematics and equations of motion in stacked form, stacked decompositions and projections are possible. The stacked form is well suited for constrained systems as the complexity of closed kinematic loops is described with the contact matrix. The methodology for closed loop accelerations involves three main steps. First the contacts are ignored and the system is treated as an open loop system. Then the contact forces are calculated as reaction forces, and finally entered into the equations of motion to yield the closed loop accelerations.

CHAPTER 10

CONCLUSION AND REMARKS

10.1 Contributions

The subject of this thesis is the theoretical development and understanding of the dynamic behavior of multibody systems. It uses concepts and principles from various areas of mathematics, physics, and engineering. In the past twenty years, research in multibody dynamics has had a renewed interest by various researchers. Although, some of this thesis is a structured summary of previously developed methods, listed below are some original contributions:

1. Visualization for the equations of motion and kinematics of planar multibody systems.

2. Summary of planar joints and related bias forces and accelerations.

3. Alternative planar homogeneous coordinates and their equivalency to planar twist and wrenches.

4. A generalization from planar to spatial representation that preserves the equivalency and links projective geometry to dynamics.
5. Dualistic subspace decomposition with respect to power relationships.
6. Recursive articulated projective dynamics. The dualistic projections of kinematics and equations of motion provide projected solutions for forces and accelerations.
7. Inertia and mobility decomposition and projection through the joint subspaces.
8. Interpretation of articulated inertia recursion through inertia decomposition.
9. Correction of the acceleration propagator and articulated mobility recursive algorithm.
10. Direct solution of multibody dynamics based on chain splitting.
11. Object oriented MATLAB® toolbox for spatial dynamics.
12. Stacked form of dualistic decomposition, articulated projective dynamics and direct solutions.
13. Integration of articulated projective dynamics into constrained multibody systems with kinematic loops.

10.2 Future Work

10.2.1 Greater Duality and Symmetry

Duality starts as a property of projective geometry. It is then propagated into the subspace decompositions and therefore into the solution of problems in dynamics. On the other hand, the duality and symmetry between systems such as serial and parallel robots seem to originate not from the methodology used but from the systems themselves. Recognizing and linking together all the possible manifestations of duality may be possible in such a way that a deeper understanding of the laws of motion can be reached. In general, the eigen structure of principal axes always has a dual representation in both translational and rotational modes. Also, power is seen to have duality as it may be used to decompose both forces and accelerations. Typically, power helps in the definition of workless forces in a moving system. Dually it assists in the definition of workless accelerative motions in the same moving system. With power expressions, forces and velocities are related the same way as accelerations and momenta. All these important variables are always be intertwined by duality.

10.2.2 Convergence in Chains

The articulated inertia components become increasingly complex with each iteration. On the other hand, numerically the result may not differ significantly from the values on previous iterations. This was demonstrated when the articulated inertia was calculated for a planar horizontal chain of rigid bodies. The expressions needed to calculate the components contained polynomials of incresing power. Some of the components when evaluated had negligible changes for each iteration.

Examining the conditions under which complex recursions can be omitted and developing a strategy may greatly improove the computational efficiency in complex systems.

It may even be possible to calculate the articulated inertias for some simpler cases just by inspection. Some components may change in a linear fashion with each iteration which provide fast estimates for these components with out any recursions. Also, there may be simple algebraic formulas that estimate some of the other values based on the fact that three iterations are almost as precise as six iterations.

10.2.3 Visualization of Articulated Inertias

Together with duality, some other concepts of projective geometry propagate through dynamics. In the plane there is a similarity between planar inertias and conic sections. This similarity must propagate into the spatial case, as well. Just as conic sections map points into lines, hyperconics map screws into screws in the spatial case. Both map the basis elements of their respective spaces and therefore neither their function nor their behavior change as the dimensionality of the space increases.

To help in the deeper understanding of the laws of motion, it may be necessary to find appropriate ways of visualizing hyperconics such as inertias. Even in the planar cases regular conics are visualized by conic curves such as an ellipse or a parabola. But planar inertias are imaginary conics and therefore do not have any real points on the plane. They still map real points into real lines just as regular conics do. And here is an interesting point. If an imaginary conic is expressed in such coordinates as to be represented by a positive definite diagonal matrix, then changing the sign of the homogeneous part this matrix represents a real conic. The lines that this real conic maps are symmetrically located from the lines that the imaginary conic maps. This is a point symmetry around the center of the conic. So by graphing this similar real conic and considering its symmetric lines or points, an idea of the behavior of the imaginary conic is established.

Of course this trick works in the planar case, but visualizing a spatial hyperconic, real or imaginary, is a task considerably more difficult. A curve on a plane is easily visualized as one of the sections of a spatial surface through the plane. But how is a three dimensional surface visualized as a section of a six dimensional hypersurface?

10.2.4 Special Cases

The kinematics and equations of motion in planar cases are easily viewed graphically by representing twists as points and wrenches as lines. With the addition of the subspaces and projections it is possible to visualize some special cases and understand better the relationships between all the quantities involved. For example when a force wrench is said to be workless it means that its screw is reciprocal to the velocity twist.

By making points or lines coincident or moving them to infinity special cases may be examined. Then the kinematics, dynamics, and subspace decompositions provide information on how the remaining variables preserve these special relationships.

The step from planar to spatial in visualization and the study of special cases provides tools in the design of multibody systems. From robot arms, suspension systems, to sports, and even space exploration the process of designing mechanisms for particular uses would be greatly improved by geometrical constructions and graphical visualization tools.

BIBLIOGRAPHY

[1] J. H. Ginsberg. *Advanced Engineering Dynamics*, 2nd Edition,Cambridge University Press, 1995.

[2] R. S. Ball. *A Treatise on the Theory of Screws*, Cambridge University Press, 1900.

[3] R. Featherstone. *Robot Dynamics Algorithms*, Kluwer Academic Publishers, Boston, 1987.

[4] K. W. Lilly. *Efficient Dynamic Simulation of Robotic Mechanisms*, Kluwer Academic Publishers, 1993.

[5] D. Orin, and S. McMillan. "*Efficient Computation of Articulated-Body Inertias Using Successive Axial Screws*," IEEE Transactions on Systems, Man, and Cybernetics, Vol 24, pp. 1194-1206, 1995.

[6] A. Jain, G. Rodriguez, and K. Kreutz-Delgato. "*Spatial Operator Algebra for Multibody System Dynamics*," The Journal of the Astronautical Sciences, Vol. 40, pp. 27-50, 1992.

[7] R. Lathrop. "*Constrained (Closed-Loop) Robot Simulation*," Robotics and Automation, IEEE International Conference, Vol. 2, pp. 689-694, 1986.

[8] R. A. Wehage, and M. J. Balczynski. "*Constrained Multibody Dynamics*," Computer Aided Analysis of Rigid and Flexible Mechanical Systems, Proceedings of the NATO - Advanced Study Institute Vol. I, pp. 461-487, 1993.

[9] R. A. Wehage. "*Workshop on: Computational Methods in Multibody Vehicle Dynamics*," Godollo University of Agriculture, 1993.

[10] J. Phee. "*On the Use of Linear Graph Theory in Multibody System Dynamics*," Computer Aided Analysis of Rigid and Flexible Mechanical Systems, Proceedings of the NATO - Advanced Study Institute, Vol II, pp. 131-150, 1993.

[11] S. R. Ploen, J. E. Bobrow, and F. C. Park. "*Geometric Algorithms for Operational Space Dynamics and Control*," International Conference on Robotics and Automation, IEEE Proceedings, pp. 1606-1611, 1997.

[12] W. Stelzle, A. Kecskeméthy, and M. Hiller. "*A Comparative Study of Recursive Methods*," Archive of Applied Mechanisms 66, pp. 9-19, 1995.

[13] H. Lipkin, and J. Duffy. "*On Ball's Mapping of the Two-System of Screws,*" Mechanism and Machine Theory Vol. 21, No. 6, pp. 499-507, 1986.

[14] D. Pedoe. *Geometry, a Comprehensive Course*, Dover Publications, Inc., 1970.

[15] B. Roth. "*Screws, Motors and Wrenches That Cannot Be Bought in a Hardware Store,*" Robotics Research, The First International Symposium, pp. 679-93, 1984.

[16] H. Lipkin. "*Extension of Euler's Principal Axes to Articulated Inertia*," 9th World Congress on the Theory of Machines and Mechanisms, pp. 1475-1479, 1992.

[17] J. Alexiou, and H. Lipkin. "*Dual Dynamic Formulation for Two Connected Bodies*," 5th National Applied Mechanisms and Robotics Conference, Vol. 2, pp. AMR97-065-001 - AMR97-065-007, 1997.