Higher Transcendental Functions

CALIFORNIA INSTITUTE OF TECHNOLOGY

BATEMAN MANUSCRIPT PROJECT

A. ERDÉLYI, *Editor*

W. MAGNUS, F. OBERHETTINGER, F. G. TRICOMI, *Research Associates*

Higher Transcendental Functions, 3 volumes.

Tables of Integral Transforms, 2 volumes.

HIGHER TRANSCENDENTAL FUNCTIONS

Volume II

Based, in part, on notes left by

Harry Bateman

Late Professor of Mathematics, Theoretical Physics, and Aeronautics at the California Institute of Technology

and compiled by the

Staff of the Bateman Manuscript Project

Prepared at the California Institute of Technology under Contract No. N6onr-244 Task Order XIV with the Office of Naval Research

Project Designation Number: NR 043-045

NEW YORK TORONTO LONDON
McGRAW-HILL BOOK COMPANY, INC.
1953

HIGHER TRANSCENDENTAL FUNCTIONS, vol. II.

PRINTED IN THE UNITED STATES OF AMERICA

Library of Congress Catalog Card Number: 53-5555

7 8 9 10 11 12 13 HL 9 8 7 6

This work is dedicated to the
memory of

HARRY BATEMAN

as a tribute to the imagination which led him to undertake a project of this magnitude, and the scholarly dedication which inspired him to carry it so far toward completion.

STAFF OF THE BATEMAN MANUSCRIPT PROJECT

Director

Arthur Erdélyi

Research Associates

Wilhelm Magnus (1948-50)
Fritz Oberhettinger (1948-51)
Francesco G. Tricomi (1948-51)

Research Assistants

David Bertin (1951-52)
W. B. Fulks (1949-50)
A. R. Harvey (1948-49)
D. L. Thomsen, Jr. (1950-51)
Maria A. Weber (1949-51)
E. L. Whitney (1948-49)

Vari-typist

Rosemarie Stampfel

FOREWORD

The purpose and the history of these volumes were described in the prefatory material to vol. I. The present second volume contains chapters on Bessel functions and other particular confluent hypergeometric functions, on orthogonal polynomials and related matters, and on elliptic functions and integrals. The method of compilation was similar to that of the first volume. Of the chapters presented here, Magnus participated actively in the preparation of Chapters IX and XI, Oberhettinger of Chapter VII, and Tricomi of Chapters VIII, IX, X, and XIII. Since the final version of several of the later chapters in this volume was prepared after the author of the first draft left Pasadena, the editorial work was much more onerous, and in several cases the revised version differs considerably from the first draft.

For Bessel functions we drew heavily on Watson's *Treatise* for a (comparatively) brief summary of the topics to be found there, while results obtained since the publication of Watson's book are presented in more detail. Functions of the parabolic cylinder are described fairly fully, those of the paraboloid of revolution only very briefly: a recent book by H. Buchholz (*Die konfluente hypergeometrische Funktion*, Springer-Verlag, 1953) gives full information on the latter functions. In the case of functions defined by integrals (error functions, exponential integral, and the like) we adopted (by no means unanimously) notations which are a compromise between the notations which seem the best ones from the mathematical point of view and those most convenient for the user of existing mathematical tables. In the chapters on orthogonal polynomials we summarized briefly some aspects of the general theory, using extensively Szegö's book: mainly we presented the properties of the classical orthogonal polynomials, although we found it useful to include some of the less well-known polynomials, polynomials of a discrete variable, hyperspherical harmonics, and some biorthogonal systems of polynomials of several variables. The chapter on elliptic functions and integrals is comparatively brief but we hope that it will be found to contain most of the material frequently required when dealing with these functions. In particular, we have included more material on elliptic integrals of the third kind than is often found in presentations as brief as ours, and attempted to include practically everything that may be required in dealing with Lamé functions or ellipsoidal wave functions. We hope that the tabular arrangement of many of the formulas of Chapter XIII will contribute to the usefulness of this chapter.

As in the first volume, a list of references has been given at the end of each chapter. The length of this list varies with the subject in hand. In the case of elliptic functions and integrals we listed merely some of the newer books, and those memoirs or older books to which we explicitly refer. In cases where bibliographies are available, we give very few references to work covered in the bibliographies, and more numerous references to books and papers which have appeared since the publication of the bibliographies.

At the end of the volume there is an *Index of notations* and a *Subject index*. Notations introduced in vol. I are often used here without further explanation. Their definition may be located by means of the *Index of notations* appended to vol. I. The system of references is the same as in vol. I. In the text, references to literature state the name of the author followed by the year of publication, more details being given in the list of references at the end of the chapter. Equations within the same section are referred to simply by number, equations in other sections by the number of the equation. Chapters are numbered consecutively, Chapters I to VI being in vol. I, Chapters VII to XIII in the present volume. Thus 3.7 (27) refers to equation (27) in section 3.7 and will be found on p. 159 of vol. I, while 9.7 (12) is on p. 144 of the present volume.

Since the editor had less assistance in the preparation of this volume than in the preparation of vol. I, errors and mistakes are more likely to be prevalent here. Suggestions for improvement and corrections will be gratefully received.

A. ERDÉLYI

CONTENTS

CHAPTER VII

BESSEL FUNCTIONS

FIRST PART: THEORY

SECOND PART: FORMULAS

CHAPTER VIII

FUNCTIONS OF THE PARABOLIC CYLINDER AND OF THE PARABOLOID OF REVOLUTION

CHAPTER IX

THE INCOMPLETE GAMMA FUNCTIONS AND RELATED FUNCTIONS

CHAPTER X

ORTHOGONAL POLYNOMIALS

CHAPTER XI

SPHERICAL AND HYPERSPHERICAL HARMONIC POLYNOMIALS

CHAPTER XII

ORTHOGONAL POLYNOMIALS IN SEVERAL VARIABLES

ORTHOGONAL POLYNOMIALS IN THE TRIANGLE

ORTHOGONAL POLYNOMIALS IN CIRCLE AND SPHERE

HERMITE POLYNOMIALS OF SEVERAL VARIABLES

CHAPTER XIII

ELLIPTIC FUNCTIONS AND INTEGRALS

PART ONE: ELLIPTIC INTEGRALS

PART TWO: ELLIPTIC FUNCTIONS

ERRATA

HIGHER TRANSCENDENTAL FUNCTIONS, VOL. II.

P. 21, equation (34): Insert the factor i on the left-hand side.

P. 28, line 2: Read 7.13(34) instead of 7.1(34).

P. 29, line 14: Read 7.13.2 instead of 7.3.2.

P. 41, equation (82): The right-hand side should read
$\pi^{-1}\sin(\nu\pi)\,[s_{0,\nu}(z)-\nu s_{-1,\nu}(z)]$.

P. 49, equation (17): Read $\pi^{-1/2}$ instead of $\pi^{1/2}$.

P. 74, equation (66): Read $\int_a^\infty$ instead of $\int_0^\infty$ on the first line.

P. 91, equation (21): Insert z^{-1} on the right-hand side, and read $(2z)$ instead of (z).

P. 95, equation (53): Read $(t^2-y^2)^{1/2}$ instead of $(t^2-b^2)^{1/2}$, and $(y^2-t^2)^{1/2}$ instead of $(b^2-t^2)^{1/2}$.

P. 124, equation (2): Insert $(\cos t)^\nu$ on the right-hand side.

P. 148, equation (18): Read $\sum_{n=1}^{\infty}$ instead of $\sum_{n=0}^{\infty}$.

P. 149, equation (2): Read $\int_x^\infty$ instead of $\int_0^\infty$.

P. 214, equation (11): Read $(2m+3/2)$ instead of $(2m+3/4)$.

P. 226, equation (6): Read e^z instead of e^{-z}.

P. 312, line 11 up: Read Byrd instead of Bird.

CHAPTER VII

BESSEL FUNCTIONS

FIRST PART: THEORY

7.1. Introduction

Bessel functions are probably the most frequently used higher transcendental functions. Broadly speaking they occur in connection with partial differential equations, usually when the variables are separated, or else in connection with certain definite integrals. We shall briefly describe both types of applications and will start with the latter.

In 1770, Lagrange investigated the elliptic motion of a planet about the sun. Let a, b be the semi-major and semi-minor axes, of the elliptic orbit; write $\epsilon = a^{-1}(a^2 - b^2)^{1/2}$ for the eccentricity; also let r, M, E, be respectively, the radius vector, mean anomaly, and eccentric anomaly. The equations obtained by Lagrange are

(1) $M = E - \epsilon \sin E,$

(2) $r = a(1 - \epsilon \cos E) = a dM/dE.$

They give rise to the expansions

(3) $\sin E = \sum_{n=1}^{\infty} A_n \sin(nM), \qquad \cos E = B_0 + \sum_{n=1}^{\infty} B_n \cos(nM)$

in which Bessel, in 1819, expressed the coefficients in the form of integrals. For instance

$$A_n = (\tfrac{1}{2}\pi n)^{-1} \int_0^{\pi} \cos E \cos(nE - n\epsilon \sin E)\, dE.$$

By easy manipulations the integral occurring here can be expressed in terms of Bessel coefficients [compare 7.3(2) and the recurrence relations of 7.2(56)], and the first expansion (3) becomes

(4) $\sin E = (\tfrac{1}{2}\epsilon)^{-1} \sum_{n=1}^{\infty} \sin(nM)\, J_n(n\epsilon)/n.$

Similarly, the second expansion (3) can be transformed into

(5) $\cos E = -\tfrac{1}{2}\epsilon + 2 \sum_{n=1}^{\infty} \cos(nM)\, J_n'(n\epsilon)/n.$

Later, in 1824, Bessel made the integral, see 7.3(2), the basis for the examination of the functions which now bear his name.

Bessel functions occur most frequently in connection with differential equations. In Watson's monumental *Treatise* (Watson, 1944), which is the standard work on Bessel functions, the history of these functions is traced back to James Bernoulli (about 1700). Since Euler (1764) and Poisson (1823) Bessel functions are associated most commonly with the partial differential equations of the potential, wave motion, or diffusion, in cylindrical or spherical polar coordinates. However, Bessel functions occasionally occur in connection with other differential equations or systems of coordinates.

Let x, y, z be Cartesian coordinates, ρ, ϕ, z, cylindrical coordinates, and r, θ, ϕ, spherical polar coordinates, determined by the equations

(6) $x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z,$

(7) $x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta.$

In these coordinates we have

(8) $\Delta F = F_{xx} + F_{yy} + F_{zz} = F_{\rho\rho} + \rho^{-1} F_{\rho} + \rho^{-2} F_{\phi\phi} + F_{zz},$

(9) $$\Delta F = F_{rr} + 2\frac{F_r}{r} + \frac{F_{\theta\theta}}{r^2} + \operatorname{ctn}\theta\,\frac{F_\theta}{r^2} + \frac{F_{\phi\phi}}{r^2 \sin^2\phi}.$$

If solutions of the wave equation $\Delta F = k^2 F = 0$ in the form $f(\rho)\,g(\phi)\,h(z)$ or $f(r)\,g(\theta)\,h(\phi)$ are sought, one obtains, in the respective cases, the ordinary differential equations for f,

(10) $$\frac{d^2 f}{d\rho^2} + \rho^{-1}\frac{df}{d\rho} + (k^2 - a^2 - \nu^2\rho^{-2})\,f = 0,$$

(11) $$r^{-1}\frac{d^2(rf)}{dr^2} + [k^2 - \nu(\nu+1)\,r^{-2}]\,f = 0,$$

in which a and ν are separation constants. The general solutions of these equations are respectively:

(12) $f(\rho) = Z_\nu[\rho(k^2 - a^2)^{1/2}]$

(13) $f(r) = r^{-1/2} Z_{\nu+1/2}(kr),$

where Z_ν denotes any Bessel function, or a linear combination with constant coefficients of Bessel functions of order ν.

The wave equation, and its solutions in various systems of coordinates, can be used to give a physically plausible approach to the theory of Bessel functions (Weyrich, 1937). Spherical waves of frequency ν, wave length λ, and wave number $k = 2\pi/\lambda$, originating at a source (ξ, η, ζ), may be described by the wave function

$$R^{-1} e^{-i2\pi(\nu t - R/\lambda)} = R^{-1} e^{-i2\pi\nu t + ikR}$$

where R is the distance between the points (ξ, η, ζ) and (x, y, z). If the z-axis is covered with sources of uniform density and phase, the resulting wave motion may be obtained by superposition in the form

$$(14)\quad u = e^{-i2\pi\nu t} \int_{-\infty}^{\infty} [\rho^2 + (z-\zeta)^2]^{-\frac{1}{2}} \exp\{ik[\rho^2 + (z-\zeta)^2]^{\frac{1}{2}}\}\, d\zeta$$

where $\rho^2 = x^2 + y^2$, and by Huyghens' principle this function represents a cylindrical wave. With $\zeta = z + \rho \sinh \tau$, equation (14) may be written as

$$(15)\quad u = e^{-i2\pi\nu t} \int_{-\infty}^{\infty} e^{ik\rho\cosh\tau}\, d\tau,$$

thus leading to Sommerfeld's integral representation of the Bessel functions of the third kind.

Notations: In this chapter we adhere to the notations used in Watson's *Bessel functions*. It may be worth while to mention a few notations which occur in the literature but are not used here.

In Gray-Mathews, (1922, p. 25 and 23, respectively), two functions $F_\nu(z)$ and $G_\nu(z)$ are introduced by

$$(16)\quad F_\nu(z) = z^{-\frac{1}{2}\nu} J_\nu(2z^{\frac{1}{2}}),$$

$$(17)\quad G_\nu(z) = \tfrac{1}{2} i\pi H_\nu^{(1)}(z).$$

Jahnke-Emde (1945, p. 128) has

$$(18)\quad \Lambda_\nu(z) = \Gamma(\nu+1)\,(\tfrac{1}{2}z)^{-\nu} J_\nu(z).$$

In Whittaker-Watson (1946, p. 373), the modified Hankel function $K_\nu(z)$ is defined by

$$(19)\quad K_\nu(z) = \tfrac{1}{2}\pi\,[I_{-\nu}(z) - I_\nu(z)]\,\mathrm{ctn}(\nu\pi).$$

This differs from our notation sec. 7.2(13).

A function closely related to Neumann's function $Y_\nu(z)$, 7.2(4), is denoted by $\mathbf{Y}_\nu(z)$ (Watson, 1944, p. 63) or by $\overline{Y}_\nu(z)$ (Gray-Mathews, 1922, p. 24):

$$(20)\quad \mathbf{Y}_\nu(z) = \overline{Y}_\nu(z) = \pi Y_\nu(z)\, e^{i\nu\pi} \sec(\nu\pi).$$

For other notations of the "related" functions see sec. 7.5.6.

7.2. Bessel's differential equation

7.2.1. Bessel functions of general order

Bessel functions are solutions of Bessel's differential equation

(1) $\nabla_\nu w \equiv z^2 \dfrac{d^2 w}{dz^2} + z \dfrac{dw}{dz} + (z^2 - \nu^2)\, w = z \dfrac{d}{dz}\left(z \dfrac{dw}{dz}\right)$
$+ (z^2 - \nu^2)\, w = 0\,;$

ν, z are unrestricted, but for the present we assume that ν is not an integer. (For integer values of ν see sec. 7.2.4.) The differential equation (1) is a limiting case of the hypergeometric differential equation (cf. Klein, 1933, p. 156); it has a singularity of the regular type at $z = 0$ and an irregular singularity at $z = \infty$; all other points are ordinary points of the differential equation. The standard method of obtaining solutions of a linear differential equation in the neighborhood of a regular singularity (Whittaker-Watson, 1927, 10.3) leads to the solution

(2) $J_\nu(z) = \sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2} z)^{2m+\nu} / [m!\,\Gamma(m+\nu+1)]$

and $J_{-\nu}(z)$. The first solution, $J_\nu(z)$, is called the Bessel function of the first kind; z is the *variable*, ν the *order* of the Bessel function. It is easily seen that the series for $z^{-\nu} J_\nu(z)$ converges absolutely, and uniformly in any bounded domain of z and ν. Equation (2) may be written as

(3) $J_\nu(z) = (\tfrac{1}{2} z)^\nu\, {}_0F_1(\nu+1; -\tfrac{1}{4} z^2)/\Gamma(\nu+1)$
$= (\tfrac{1}{2} z)^\nu e^{-iz}\, {}_1F_1(\nu + \tfrac{1}{2}; 2\nu+1; 2iz)/\Gamma(\nu+1)$

by Kummer's relation, 6.3 (7).

The linear combinations

(4) $Y_\nu(z) = (\sin \nu\pi)^{-1} [J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)]$,

(5) $H_\nu^{(1)}(z) = J_\nu(z) + i Y_\nu(z) = [i \sin(\nu\pi)]^{-1} [J_{-\nu}(z) - J_\nu(z)\, e^{-i\nu\pi}]$,

(6) $H_\nu^{(2)}(z) = J_\nu(z) - i Y_\nu(z) = (i \sin \nu\pi)^{-1} [J_\nu(z)\, e^{i\nu\pi} - J_{-\nu}(z)]$

are likewise solutions of (1). Y_ν is called the Bessel function of the second kind or Neumann's function. $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Bessel functions of the third kind, also called the first and second Hankel functions. From (5) and (6) we have

(7) $J_\nu(z) = \tfrac{1}{2} [H_\nu^{(1)}(z) + H_\nu^{(2)}(z)]$,

(8) $Y_\nu(z) = \tfrac{1}{2} [H_\nu^{(1)}(z) - H_\nu^{(2)}(z)]/i$.

From the definition it is seen that

(9) $H_{-\nu}^{(1)}(z) = e^{i\nu\pi} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z)$.

Also, if $\bar{z}$ denotes the complex number conjugate to z, and similarly for other quantities, we have

$$(10)\quad \overline{J_\nu(z)} = J_{\bar{\nu}}(\bar{z}), \qquad \overline{Y_\nu(z)} = Y_{\bar{\nu}}(\bar{z}),$$

$$\overline{H_\nu^{(1)}(z)} = H_{\bar{\nu}}^{(2)}(\bar{z}), \qquad \overline{H_\nu^{(2)}(z)} = H_{\bar{\nu}}^{(1)}(\bar{z}).$$

In particular, J_ν and Y_ν are real if the order, ν, is real and the variable z is positive. All the four Bessel functions are single-valued in the z-plane cut along the negative real axis from 0 to $-\infty$. For general ν, they all have branch points at $z = 0$. The Bessel function of the first kind is clearly an entire function of ν, and later it will be seen that, with a suitable definition for integer $\nu = n$, the Bessel functions of the second and third kind are also entire functions of ν.

7.2.2. Modified Bessel functions of general order

If z is replaced by iz, Bessel's differential equation (1) becomes

$$(11)\quad z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2)\, w = 0.$$

If ν is not an integer (for integer values of ν see sec. 7.2.5), $J_\nu(iz)$ and $J_{-\nu}(iz)$ are two linearly independent solutions of (11), but more often the function

$$(12)\quad I_\nu(z) = e^{-i\frac{1}{2}\nu\pi} J_\nu(ze^{i\frac{1}{2}\pi}) = \sum_{m=0}^{\infty} (\tfrac{1}{2}z)^{2m+\nu}/[m!\Gamma(m+\nu+1)]$$

$$= \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, {}_0F_1(\nu+1; \tfrac{1}{4}z^2) = \frac{(\frac{1}{2}z)^\nu e^{-z}}{\Gamma(\nu+1)}\, {}_1F_1(\nu+\tfrac{1}{2}; 2\nu+1; 2z)$$

$$= 2^{-2\nu-\frac{1}{2}} z^{-\frac{1}{2}} M_{0,\nu}(2z)/\Gamma(\nu+1)$$

[compare 6.9(11)] and $I_{-\nu}(z)$ are used. They are known as the modified Bessel functions of the first kind and are real when ν is real and z is positive.

The function

$$(13)\quad K_\nu(z) = \tfrac{1}{2}\pi(\sin\nu\pi)^{-1}\,[I_{-\nu}(z) - I_\nu(z)] = (\tfrac{1}{2}\pi/z)^{\frac{1}{2}}\, W_{0,\nu}(2z)$$

[compare 6.9(14)] is likewise a solution of (11). It is known as the modified Bessel function of the third kind or Basset's function (although the present definition is due to Macdonald).

Clearly we have

$$(14)\quad K_{-\nu}(z) = K_\nu(z),$$

and from (12), (5) and (6) it follows that

$$(15)\quad K_\nu(z) = \tfrac{1}{2}i\pi e^{i\frac{1}{2}\nu\pi} H_\nu^{(1)}(ze^{i\frac{1}{2}\pi}) = -\tfrac{1}{2}i\pi e^{-i\frac{1}{2}\nu\pi} H_\nu^{(2)}(ze^{-i\frac{1}{2}\pi}),$$

so that

(16) $K_\nu(ze^{i\frac{1}{2}\pi}) = \frac{1}{2} i\pi e^{i\frac{1}{2}\nu\pi} H_\nu^{(1)}(ze^{i\pi}) = -\frac{1}{2} i\pi e^{-i\frac{1}{2}\nu\pi} H_\nu^{(2)}(z)$,

(17) $H_\nu^{(1)}(z) = -\dfrac{2i}{\pi} e^{-i\frac{1}{2}\nu\pi} K_\nu(ze^{-i\frac{1}{2}\pi})$.

$K_\nu(z)$ is real when ν is real and z is positive.

7.2.3. Kelvin's function and related functions

Kelvin's functions ber(x) and bei(x) (x real) are defined by the equation

(18) $\operatorname{ber}(x) + i \operatorname{bei}(x) = J_0(xe^{i\frac{3}{4}\pi}) = I_0(xe^{i\frac{1}{4}\pi})$.

Extensions of this definition to Bessel functions of any order and complex z are given by the relations

(19) $\operatorname{ber}_\nu(z) \pm i \operatorname{bei}_\nu(z) = J_\nu(ze^{\pm i\frac{3}{4}\pi})$,

(20) $\operatorname{ker}_\nu(z) \pm i \operatorname{kei}_\nu(z) = e^{\mp i\frac{1}{2}\nu\pi} K_\nu(ze^{\pm i\frac{1}{4}\pi})$.

Instead of (20) we may use

(21) $\operatorname{her}_\nu(z) + i \operatorname{hei}_\nu(z) = H_\nu^{(1)}(ze^{i\frac{3}{4}\pi})$

(22) $\operatorname{her}_\nu(z) - i \operatorname{hei}_\nu(z) = H_\nu^{(2)}(ze^{-i\frac{3}{4}\pi})$

so that

(23) $2\operatorname{ker}_\nu(z) = -\pi \operatorname{hei}_\nu(z)$; $\quad 2\operatorname{kei}_\nu(z) = \pi \operatorname{her}_\nu(z)$.

The functions $\operatorname{ber}_\nu(z)$, $\operatorname{bei}_\nu(z)$, $\operatorname{ker}_\nu(z)$, $\operatorname{kei}_\nu(z)$, $\operatorname{her}_\nu(z)$, $\operatorname{hei}_\nu(z)$ are real when ν is real and z is real and positive. (For details see McLachlan, 1934, pp. 119, 168.)

7.2.4. Bessel functions of integer order

Bessel functions of the first kind of integer order are known as *Bessel coefficients*. If n is a positive integer, the first $n - 1$ terms in the infinite series defining $J_{-n}(z)$ vanish because of the poles of the gamma function in the denominator. The remaining gamma functions may be rewritten as factorials, and we have

$$J_{-n}(z) = \sum_{m=n}^{\infty} (-1)^m (\tfrac{1}{2}z)^{2m-n}/[m!\,(m-n)!],$$

or, with $m = n + l$, $l = 0, 1, 2, \ldots$,

(24) $J_{-n}(z) = (-1)^n J_n(z)$.

This relation holds for all integers n.

Bessel coefficients are generated by the expansion of $\exp[\tfrac{1}{2}z(t-t^{-1})]$ in powers of t. To prove this we note that

$$e^{\frac{1}{2}zt}\, e^{-\frac{1}{2}z/t} = \sum_{l=0}^{\infty} (\tfrac{1}{2}zt)^l/l!\ \sum_{m=0}^{\infty} (-\tfrac{1}{2}zt^{-1})^m/m!\,,$$

and the coefficient of t^n in this expansion is exactly $J_n(z)$. This leads to the generating function

$$\exp[\tfrac{1}{2}z(t-t^{-1})] = \sum_{n=-\infty}^{\infty} t^n J_n(z),$$

or, replacing z by az and t by t/a to the more general expression

$$(25)\quad \exp[\tfrac{1}{2}z(t-a^2t^{-1})] = \sum_{n=-\infty}^{\infty} (t/a)^n J_n(az)$$

$$= J_0(az) + \sum_{n=1}^{\infty} J_n(az)\,[(t/a)^n + (-t/a)^{-n}]$$

for the Bessel coefficients. With $a = 1$ and $t = e^{i\phi}$ we obtain the formula of Jacobi-Anger

$$(26)\quad e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z)$$

$$= J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\phi) + 2i\sum_{n=1}^{\infty} J_{2n-1}(z)\sin[(2n-1)\phi]$$

and with $t = ie^{i\phi}$

$$(27)\quad e^{iz\cos\phi} = \sum_{n=-\infty}^{\infty} i^n e^{in\phi} J_n(z) = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\phi).$$

If ν is an integer, the right-hand sides of (4), (5), (6) appear in indeterminate form. However the limits of these right-hand sides as $\nu \to n$ (integer) exist and may be taken as the definition of Bessel functions of the second and third kinds of integer order. Clearly it will be sufficient to evaluate

$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z) \qquad n = 0, 1, 2, \ldots,$$

By L'Hospital's rule applied to (4) we obtain

$$(28)\quad Y_n(z) = \pi^{-1}\left[\frac{\partial J_\nu}{\partial\nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial\nu}\right]_{\nu=n}.$$

From (2) and 1.7 (1)

$$(29)\quad \frac{\partial J_\nu}{\partial\nu} = J_\nu(z)\log(\tfrac{1}{2}z) - \sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2}z)^{\nu+2m} \frac{\psi(\nu+m+1)}{m!\,\Gamma(\nu+m+1)},$$

$$(30)\quad \frac{\partial J_{-\nu}}{\partial \nu} = -J_{-\nu}(z)\log(\tfrac{1}{2}z) + \sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2}z)^{\nu+2m} \frac{\psi(-\nu+m+1)}{m!\,\Gamma(-\nu+m+1)},$$

and from formulas 1.17(11) and 1.17(12) for $m \le n-1$,

$$\lim_{\nu \to n} \psi(-\nu+m+1)/\Gamma(-\nu+m+1) = (-1)^{n-m}(n-m-1)!\,,$$

so that from (30) and (24)

$$\left(\frac{\partial J_{-\nu}}{\partial \nu}\right)_{\nu=n} = (-1)^n \Bigg[-J_n(z)\log(\tfrac{1}{2}z) + \sum_{m=0}^{n-1} (\tfrac{1}{2}z)^{2m-n}(n-m-1)!/m!$$

$$+ \sum_{m=n}^{\infty} (-1)^{m-n}(\tfrac{1}{2}z)^{2m-n} \frac{\psi(m+1-n)}{\Gamma(m+1-n)\,m!}\Bigg].$$

(For special values of ν in (29) see Mitra, 1925, Airey, 1935 a, and also Müller, 1940.) With a new index of summation $l = m-n$, the infinite sum in this expression can be written as

$$\sum_{l=0}^{\infty} (-1)^l (\tfrac{1}{2}z)^{2l+n}\psi(l+1)/[l!\,(l+n)!],$$

and so we obtain

$$(31)\quad \pi Y_n(z) = 2J_n(z)\log(\tfrac{1}{2}z) - \sum_{m=0}^{n-1} (\tfrac{1}{2}z)^{2m-n}(n-m-1)!/m!$$

$$- \sum_{l=0}^{\infty} (-1)^l (\tfrac{1}{2}z)^{n+2l} \frac{\psi(n+l+1)+\psi(l+1)}{l!\,(n+l)!} \qquad n = 1, 2, 3, \ldots,$$

which may be written as

$$(32)\quad \pi Y_n(z) = 2[\gamma + \log(\tfrac{1}{2}z)]\,J_n(z) - \sum_{m=0}^{n-1} (\tfrac{1}{2}z)^{2m-n}(n-m-1)!/m!$$

$$- \sum_{m=0}^{\infty} \left[(-1)^m \frac{(\tfrac{1}{2}z)^{n+2m}}{m!\,(n+m)!}\,(h_{m+n}+h_m)\right] \qquad n = 1, 2, 3, \ldots,$$

where we have used 1.7(9) and put

$$h_m = 1^{-1} + 2^{-1} + \cdots + m^{-1} \qquad m = 1, 2, 3, \ldots, \quad h_0 = 0.$$

If $\nu = 0$, it follows from (30) that the finite sum in (32) is to be omitted. Therefore, we have

$$(33)\quad \pi Y_0(z) = 2[\gamma + \log(\tfrac{1}{2}z)]\,J_0(z) - 2\sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2}z)^{2m}(m!)^{-2}h_m,$$

with the same meaning of h_m as in (32). It is to be noticed that according to (28)

$$(34)\quad Y_{-n}(z) = \lim_{\mu \to n} [\cos(\mu\pi)]^{-1} \left[\frac{J_{\mu}(z)\cos(\mu\pi) - J_{-\mu}(z)}{\sin(\mu\pi)} \right]$$
$$= (-1)^n Y_n(z) \qquad n = 1, 2, 3, \ldots .$$

With this definition of $Y_n(z)$ and $Y_{-n}(z)$ and with the corresponding definition of Bessel functions of the third kind, all Bessel functions become entire functions of ν.

7.2.5. Modified Bessel functions of integer order

From (24) and (12) we have

$$(35)\quad I_{-n}(z) = I_n(z) \qquad n = 1, 2, 3, \ldots .$$

We therefore take $I_n(z)$ and $K_n(z)$ as a fundamental system of solutions of (11) where

$$(36)\quad K_n(z) = \lim_{\nu \to n} K_{\nu}(z) = (-1)^n \tfrac{1}{2} \left[\frac{\partial I_{-\nu}}{\partial \nu} - \frac{\partial I_{\nu}}{\partial \nu} \right]_{\nu = n} .$$

In a similar manner as in sec. 7.2.4 we obtain

$$(37)\quad K_n(z) = (-1)^{n+1} I_n(z) \log(\tfrac{1}{2}z) + \tfrac{1}{2} \sum_{m=0}^{n-1} (-1)^m (\tfrac{1}{2}z)^{2m-n} \frac{(n-m-1)!}{m!}$$
$$+ \tfrac{1}{2}(-1)^n \sum_{m=0}^{\infty} (\tfrac{1}{2}z)^{n+2m} [\psi(n+m+1) + \psi(m+1)]/[m!(n+m)!]$$
$$n = 1, 2, 3, \ldots .$$

In case $n = 0$ we have

$$(38)\quad K_0(z) = -I_0(z)\log(\tfrac{1}{2}z) + \sum_{m=0}^{\infty} (\tfrac{1}{2}z)^{2m} \psi(m+1)/[(m!)^2].$$

With the definition of $K_{\nu}(z)$ completed in this manner, we have an entire function of ν.

7.2.6. Spherical Bessel functions

The Bessel functions and modified Bessel functions reduce to combinations of elementary functions if and only if ν is half of an odd integer (Watson, 1944, 4.7 to 4.75). We shall express here $K_{n+\frac{1}{2}}(z)$ for $n = 0, 1, 2, \ldots$, in terms of elementary functions. The corresponding expressions for the other Bessel functions follow from (16), (17), (7), and (8), and are recorded in sec. 7.11. When $n = 0, 1, 2, \ldots$, and $\nu = n + \tfrac{1}{2}$ we have from 7.3 (16)

$$(39)\quad K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \frac{e^{-z}}{n!} \int_0^{\infty} e^{-t}(1 + t/2z)^n t^n \, dt.$$

Now the binomial expansion of $(1+\frac{1}{2}t/z)^n$ terminates and at once leads to the representation of $K_{n+\frac{1}{2}}(z)$ in finite terms in the form

$$(40)\quad K_{n+\frac{1}{2}}(z)=\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\sum_{m=0}^{n}(2z)^{-m}\frac{\Gamma(n+m+1)}{m!\,\Gamma(n+1-m)}.$$

Using Hankel's symbol

$$(\nu,m)=\frac{2^{-2m}}{m!}\{(4\nu^2-1)(4\nu^2-3^2)\cdots[4\nu^2-(2m-1)^2]\}$$

$$=\Gamma(\tfrac{1}{2}+\nu+m)/[m!\,\Gamma(\tfrac{1}{2}+\nu-m)],$$

[compare 1.20(3)], this can be written as

$$(41)\quad K_{n+\frac{1}{2}}(z)=(\tfrac{1}{2}\pi/z)^{\frac{1}{2}}e^{-z}\sum_{m=0}^{n}(n+\tfrac{1}{2},m)(2z)^{-m}.$$

Hence, for instance if $n=0$, we have

$$(42)\quad K_{\frac{1}{2}}(z)=(\tfrac{1}{2}\pi/z)^{\frac{1}{2}}e^{-z}.$$

From (42) and also 7.11(22) we obtain the representation

$$(43)\quad K_{n+\frac{1}{2}}(z)=(-1)^n\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} z^{n+1}\left(\frac{d}{zdz}\right)^n\frac{e^{-z}}{z}.$$

For the other types of Bessel functions see formulas 7.11(1) to 7.11(13).

Bessel functions whose order is half of an odd integer often occur in connection with spherical waves, and in this context Sommerfeld's notation,

$$(44)\quad \psi_m(z)=(\tfrac{1}{2}\pi/z)^{\frac{1}{2}}J_{m+\frac{1}{2}}(z),$$

$$(45)\quad \zeta_m^{(1)}(z)=(\tfrac{1}{2}\pi/z)^{\frac{1}{2}}H_{m+\frac{1}{2}}^{(1)}(z),$$

$$(46)\quad \zeta_m^{(2)}(z)=(\tfrac{1}{2}\pi/z)^{\frac{1}{2}}H_{m+\frac{1}{2}}^{(2)}(z),$$

is often used. Sometimes $\psi_m(z)$ denotes a slightly different function (Watson, 1944, 3.41). For a class of polynomials connected with the spherical Bessel functions compare Krall and Frink (1949) and Burchnall, (1951).

7.2.7. Products of Bessel functions

In order to obtain an expression for the product $J_\mu(az)\,J_\nu(\beta z)$ of two Bessel functions in the form of a series of ascending powers of z we use (2) and Cauchy's rule for the multiplication of power series. Thus the coefficient of $(-1)^m(\frac{1}{2}az)^\mu(\frac{1}{2}\beta z)^\nu(\frac{1}{2}az)^{2m}$ is found to be

$$\sum_{n=0}^{m}(\beta/a)^{2n}/[n!\,\Gamma(\nu+n+1)\,(m-n)!\,\Gamma(\mu+m-n+1)].$$

This may be expressed as a terminating hypergeometric series by means of formulas 1.2(3), 1.20(5), and 2.1(2) and leads to the expansion

(47) $$\Gamma(\nu+1)\, J_\nu(\beta z)\, J_\mu(az) = (\tfrac{1}{2}az)^\mu (\tfrac{1}{2}\beta z)^\nu \times \sum_{m=0}^{\infty} \frac{(-1)^m (\tfrac{1}{2}az)^{2m}}{m!\,\Gamma(\mu+m+1)}\, {}_2F_1(-m, -\mu-m; \nu+1; \beta^2 a^{-2}).$$

This expansion simplifies when $\beta = a$, because then the hypergeometric series may be summed by Gauss's formula 2.1(14), so that

(48) $$J_\nu(z)\, J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\tfrac{1}{2}z)^{\nu+\mu+2m}\, \Gamma(\nu+\mu+2m+1)}{m!\,\Gamma(\mu+m+1)\Gamma(\nu+m+1)\Gamma(\nu+\mu+m+1)}.$$

In the notation of generalized hypergeometric series

(49) $$\Gamma(\nu+1)\,\Gamma(\mu+1)\, J_\nu(z)\, J_\mu(z) = (\tfrac{1}{2}z)^{\nu+\mu}\, {}_2F_3(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu, 1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu; 1+\nu, 1+\mu, 1+\nu+\mu; -z^2).$$

From (48) we easily deduce the expansion

$$e^{\pm iz} J_\nu(z) = \pi^{-\frac{1}{2}} (2z)^\nu \sum_{n=0}^{\infty} \frac{\Gamma(\nu+n+\tfrac{1}{2})(\pm 2iz)^n}{n!\,\Gamma(2\nu+n+1)}.$$

7.2.8. Miscellaneous results

Differentiation formulas and recurrence relations follow. From (2) we find that

(50) $$\frac{d}{dz}[z^\nu J_\nu(z)] = z^\nu \sum_{m=0}^{\infty} (-1)^m (\tfrac{1}{2}z)^{2m+\nu-1}/[m!\,\Gamma(m+\nu)] = z^\nu J_{\nu-1}(z),$$

(51) $$\frac{d}{dz}[z^{-\nu} J_\nu(z)] = z^{-\nu} \sum_{m=1}^{\infty} (-1)^m (\tfrac{1}{2}z)^{2m+\nu-1}/[(m-1)!\,\Gamma(m+\nu+1)] = -z^{-\nu} J_{\nu+1}(z),$$

and hence by repeated differentiation

(52) $$\left(\frac{d}{z\,dz}\right)^m [z^\nu J_\nu(z)] = z^{\nu-m} J_{\nu-m}(z),$$

(53) $$\left(\frac{d}{z\,dz}\right)^m [z^{-\nu} J_\nu(z)] = (-1)^m z^{-\nu-m} J_{\nu+m}(z) \qquad m = 1, 2, 3, \ldots.$$

From (50) and (51) it is obvious that

(54) $$z J'_\nu(z) + \nu J_\nu(z) = z J_{\nu-1}(z),$$

(55) $z\,J_\nu'(z) - \nu\,J_\nu(z) = -z\,J_{\nu+1}(z)$

and hence

(56) $J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu z^{-1}\,J_\nu(z)$,

(57) $J_{\nu-1}(z) - J_{\nu+1}(z) = 2\,J_\nu'(z)$.

By virtue of (4), (5), (6) the same relations are valid for Bessel functions of the second and third kind. Relations (12), (13), and the previous results give similar formulas for the modified Bessel functions. For these see sec. 7.11.

From the recurrence relations the following inequality (Szász, 1950) may be derived

$$[J_\nu(x)]^2 - J_{\nu-1}(x)\,J_{\nu+1}(x) > (\nu+1)^{-1}\,[J_\nu(x)]^2 \qquad \nu > 0, \quad x \text{ real.}$$

WRONSKIANS

The Wronskian of two solutions w_1 and w_2 of (1) is a constant multiple of $\exp\,[-\int z^{-1}\,dz]$.

(58) $W\{w_1, w_2\} \equiv w_1\,w_2' - w_2\,w_1' = Cz^{-1}$.

The constant C can be computed from the first terms of the series expansions of the solutions involved. If we take $w_1 = J_\nu(z), w_2 = J_{-\nu}(z)$ we find from the series (2) that

$$\lim_{z\to 0} zW = -(2\nu)/[\Gamma(1-\nu)\,\Gamma(1+\nu)] = -2\pi^{-1}\sin(\nu\pi) = C,$$

and therefore we have

(59) $W[J_\nu, J_{-\nu}) = -2(\pi z)^{-1}\sin(\nu\pi)$.

If ν is an integer, this Wronskian vanishes, thus confirming the result of sec. 7.2.4 about the linear dependence of J_n and J_{-n}. For other Wronskians of Bessel functions or modified Bessel functions see sec. 7.11.

From (59) and (54) it follows that

(60) $J_{-\nu+1}(z)\,J_\nu(z) + J_{-\nu}(z)\,J_{\nu-1}(z) = 2(\pi z)^{-1}\sin(\nu\pi)$.

For other similar formulas see sec. 7.11.

ANALYTIC CONTINUATION

The Bessel function of the first kind of variable $ze^{im\pi}$ where m is any integer, may be expressed by (2) as

(61) $J_\nu(ze^{im\pi}) = e^{im\pi\nu}\,J_\nu(z) \qquad m = \pm 1, \pm 2, \pm 3, \ldots.$

For the corresponding relations for the other types of Bessel functions see sec. 7.11.

DIFFERENTIAL EQUATIONS

A large class of differential equations whose solutions may be expressed in terms of Bessel functions has been obtained by Lommel. One of Lommel's transformations is

$$z = \beta\zeta^{\gamma}, \quad w = \zeta^{-a} v$$

where ζ is the independent variable and v the new dependent variable. This tranformations carries (1) into

$$(62) \quad \zeta^2 \frac{d^2 v}{d\zeta^2} + (1 - 2a)\,\zeta \frac{dv}{d\zeta} + [(\beta\gamma\zeta^{\gamma})^2 + (a^2 - \nu^2\gamma^2)]\, v = 0.$$

If $w_1(z)$ and $w_2(z)$ are any two linearly independent solutions of Bessel's equation, the general solution of (62) is

$$(63) \quad v_1 = \zeta^{a} w_1(\beta\zeta^{\gamma}) \quad \text{and} \quad v_2 = \zeta^{a} w_2(\beta\zeta^{\gamma}).$$

For other differential equations whose solutions can be expressed in terms of Bessel functions see Kamke(1948, p. 440).

The general solution of the inhomogeneous Bessel equation

$$(64) \quad z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)\, w = f(z)$$

may be obtained by the method of variation of parameters in the form

$$(65) \quad w = Aw_1(z) + Bw_2(z) + u(z)$$

where $w_1(z)$ and $w_2(z)$ are two linearly independent solutions of the homogeneous equation (1), $u(z)$ is a particular solution of (64) defined by

$$(66) \quad Cu(z) = -w_1(z)\int_{z_0}^{z} t^{-1} w_2(t) f(t)\,dt + w_2(z)\int_{z_0}^{z} t^{-1} w_1(t) f(t)\,dt$$

and C is the constant in the Wronskian of w_1 and w_2 [cf. (58)].

The functions $J'_\nu(z)$ and $az\,J'_\nu(z) + b\,J_\nu(z)$ satisfy the following differential equations respectively,

$$(67) \quad z^2(z^2 - \nu^2)\frac{d^2 w}{dz^2} + z(z^2 - 3\nu^2)\frac{dw}{dz} + [(z^2 - \nu^2)^2 - (z^2 + \nu^2)]\, w = 0,$$

$$(68) \quad z^2[a^2(z^2 - \nu^2) + b^2]\frac{d^2 w}{dz^2} - z[a^2(z^2 + \nu^2) - b^2]\frac{dw}{dz}$$
$$+ [a^2(z^2 - \nu^2)^2 + 2abz^2 + b^2(z^2 - \nu^2)]\, w = 0.$$

7.3. Integral representations

7.3.1. Bessel coefficients

If Cauchy's theorem of residues is applied to 7.2(25) we obtain

(1) $2\pi i\, J_n(az) = a^n \int_C t^{-n-1} \exp\left[\tfrac{1}{2} z\,(t - a^2 t^{-1})\right] dt \qquad n = 0, 1, 2, \ldots.$

C is any simple closed contour in the t-plane around the origin. If in (1) we put $a = 1$ and choose C to be the unit circle around the origin, $t = e^{i\phi}$, we have

(2) $2\pi\, J_n(z) = \int_0^{2\pi} e^{i(z\sin\phi - n\phi)}\, d\phi = 2\int_0^{\pi} \cos(z\sin\phi - n\phi)\, d\phi$

$n = 0, 1, 2, \ldots.$

This is Bessel's representation.

7.3.2. Integral representations of the Poisson type

For general ν we have Poisson's integral representation [for a generalization of this formula see 7.8(11)]

(3) $\Gamma(\nu + \tfrac{1}{2})\, J_\nu(z) = 2\pi^{-1/2}\, (\tfrac{1}{2}z)^\nu \int_0^{\frac{1}{2}\pi} \cos(z\sin\phi)\, (\cos\phi)^{2\nu}\, d\phi$

$\operatorname{Re}\nu > -\tfrac{1}{2}.$

This result may be proved by expanding $\cos(z\sin\phi)$ into a series of powers of z and integrating term by term. In this process one encounters the integral

$$\int_0^{\frac{1}{2}\pi} (\sin\phi)^{2m}\, (\cos\phi)^{2\nu}\, d\phi$$

which is found to be equal to

$$\tfrac{1}{2}\,\Gamma(\nu + \tfrac{1}{2})\,\Gamma(m + \tfrac{1}{2})/\Gamma(m + \nu + 1)$$

by virtue of 1.5(19). Therefore, we have

$$\Gamma(\nu + \tfrac{1}{2})\, J_\nu(z) = \pi^{-1/2}\, (\tfrac{1}{2}z)^\nu \sum_{m=0}^{\infty} (-1)^m\, z^{2m}\, \frac{\Gamma(\nu + \tfrac{1}{2})\,\Gamma(m + \tfrac{1}{2})}{(2m)!\,\Gamma(\nu + m + 1)}.$$

Using the duplication formula from 1.2(15) of the gamma function for $(2m)! = \Gamma(2m + 1)$ and remembering also 7.2(2) the result (3) is established. Slight modifications of (3) are given in sec. 7.12.

Poisson's integral, in the form of 7.12(6), may be used to derive an inequality for $J_\nu(z)$. Let ν be real, $\nu > -\tfrac{1}{2}$ and $z = x + iy$ (x, y real); then we obtain

$$\Gamma(\nu + 1)\, |J_\nu(z)| \le \pi^{-1/2}\, (\tfrac{1}{2}|z|)^\nu \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{|y|}\, (\cos\phi)^{2\nu}\, d\phi$$

and by virtue of 1.5 (19)

(4) $|J_\nu(z)| \le |\tfrac{1}{2}z|^\nu\, e^{|y|}/\Gamma(\nu + 1)$

[see also 7.10(22)].

7.3.3. Representations by loop integrals

Bessel functions for unrestricted values of the order ν may be represented as loop integrals. Let a be a complex number with Re $a > 0$; then we have the representation

$$(5) \quad 2\pi i\, J_\nu(az) = z^\nu \int_{-\infty}^{(0+)} \exp[\tfrac{1}{2} a(t - z^2 t^{-1})]\, t^{-\nu-1}\, dt$$
$$= (z/2)^\nu \int_{-\infty}^{(0+)} \exp[a(t - \tfrac{1}{4} z^2 t^{-1})]\, t^{-\nu-1}\, dt$$
$$\operatorname{Re} a > 0, \qquad |\arg t| \leq \pi.$$

Here the symbol, $\int_{-\infty}^{(0+)}$ denotes, as usual, integration along a contour which starts at infinity on the negative real t-axis, encircles the origin counter-clockwise, and returns to its starting point. Clearly (5) is an extension of (1), for the integrand in (5) is one-valued, and the loop may be deformed into a closed contour around the origin, if ν is an integer. To prove (5), we use the expansion

$$\exp[-az^2/(4t)] = \sum_{m=0}^{\infty} (-1)^m\, (\tfrac{1}{4} a z^2)^m\, t^{-m}/m!$$

in (5) and integrate term by term. From 1.6 (6) we obtain

$$\int_{-\infty}^{(0+)} e^{at}\, t^{-m-\nu-1}\, dt = 2\pi i\, a^{m+\nu}/\Gamma(m + \nu + 1).$$

Therefore, we have

$$\int_{-\infty}^{(0+)} \exp[at - \tfrac{1}{4} a z^2 t^{-1}]\, t^{-\nu-1}\, dt$$
$$= 2\pi i (\tfrac{1}{2} z)^{-\nu} \sum_{m=0}^{\infty} (-1)^m\, (\tfrac{1}{2} a z)^{2m+\nu}/[m!\,\Gamma(m+\nu+1)],$$

and using 7.2(2) this establishes (5).

The corresponding loop integral for the other types of Bessel functions may be obtained using formulas 7.2(4) to 7.2(6) and formulas 7.2(12) and 7.2(13). For these see McLachlan and Meyers (1937).

When Re $\nu > -1$ and a is real and positive, the contour in (5) may be deformed into one parallel to the imaginary axis, leading to

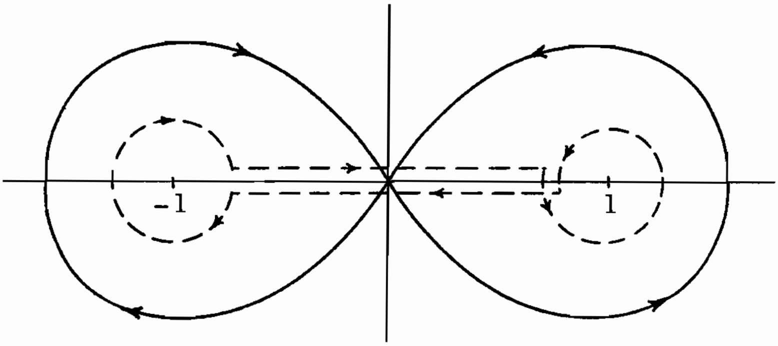
$$(6) \quad 2\pi i\, J_\nu(az) = z^\nu \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2} a (t - z^2 t^{-1})}\, t^{-\nu-1}\, dt \qquad c,\ a > 0, \quad \operatorname{Re} \nu > -1.$$

HANKEL'S REPRESENTATIONS

Generalizations of Poisson's integral (3) were given by Hankel. The first of these is

$$(7) \quad 2\pi i\, J_\nu(z) = \pi^{-\frac{1}{2}}\, \Gamma(\tfrac{1}{2} - \nu)\, (\tfrac{1}{2} z)^\nu \int^{(1+,-1-)} e^{izt} (t^2 - 1)^{\nu - \frac{1}{2}}\, dt,$$

$\nu + \frac{1}{2}$ not a negative integer. The path of integration is the figure eight indicated in the diagram below.



t - plane

The initial amplitude of $(t-1)$ and $(t+1)$ at the point of intersection with the positive real axis on the right-hand side of $t=1$ is zero. To prove (7) we replace the original contour by the dotted one. If we assume that $\operatorname{Re}(\nu+\frac{1}{2})>0$ and make the radii of the circles around ± 1 tend to zero, then we obtain

$$\int^{(1+,-1-)} e^{izt}(t^2-1)^{\nu-\frac{1}{2}}\,dt = 2i\cos(\nu\pi)\int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt$$

$$\operatorname{Re}\nu > -\tfrac{1}{2}.$$

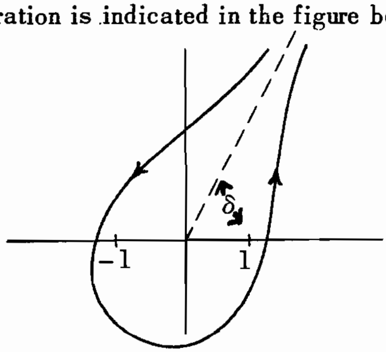
If the integral on the right-hand side is expressed by 7.12(7), we obtain (7). By the theory of analytic continuation the restriction $\operatorname{Re}\nu > -\frac{1}{2}$ may be omitted as long as $\nu+\frac{1}{2}$ is not a positive integer.

Another representation [for a related expression compare 7.8(13)] is

$$(8)\quad 2\pi i\, J_\nu(z) = \pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\nu)\, e^{i3\nu\pi}(\tfrac{1}{2}z)^{-\nu} \times \int_{\infty e^{i\delta}}^{(-1+,1+)} e^{izt}(t^2-1)^{-\nu-\frac{1}{2}}\,dt$$

$$\nu+\tfrac{1}{2}\neq 0,-1,-2,\ldots,\qquad \delta\le\arg t\le 2\pi+\delta,\qquad -\delta<\arg z<\pi-\delta.$$

The path of integration is indicated in the figure below,



t - plane

and the initial and final values of arg t are taken to be δ and $2\pi + \delta$. To prove (8) we take the contour to lie outside the unit circle. Then we have

$$\Gamma(\tfrac{1}{2}+\nu)(t^2-1)^{-\nu-\frac{1}{2}} = \sum_{m=0}^{\infty} \Gamma(\tfrac{1}{2}+\nu+m)\, t^{-2\nu-2m-1}/m!.$$

We insert this in (8) and integrate term by term. Then from 1.6(6) with $\zeta = ze^{-i\frac{1}{2}\pi}$ we obtain

$$\int_{\infty e^{i\delta}}^{(0+)} t^{-2\nu-2m-1}\, e^{izt} dt = 2\pi i z^{2\nu+2m}\, e^{-i3\pi(\nu+m)}/\Gamma(2\nu+2m+1)$$

$$-\delta < \arg z < \pi - \delta.$$

Thus we have

$$J_\nu(z) = \pi^{-\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m\, (\tfrac{1}{2}z)^{\nu+2m} \frac{2^{2m+2\nu}\,\Gamma(\tfrac{1}{2}+\nu+m)}{m!\,\Gamma(2m+2\nu+1)}.$$

Using the duplication formula from 1.2(15) for the gamma function, (8) is established.

7.3.4. Schläfli's, Gubler's, Sonine's and related integral representations

From the results of sec. 7.3.3 a number of representations in the form of definite integrals may be obtained.

SCHLÄFLI'S REPRESENTATIONS

In (5) we interchange a and z, put $a = 1$, and deform the loop into a path consisting of the real axis from $-\infty$ to -1 ($\arg t = -\pi$), the unit circle in the positive sense around the origin ($-\pi \le \arg t \le \pi$), and the real axis from -1 to $-\infty$ ($\arg t = \pi$). The result is Schläfli's representation

(9) $$\pi J_\nu(z) = \int_0^\pi \cos(z \sin\phi - \nu\phi)\, d\phi - \sin(\nu\pi) \int_0^\infty e^{-(z\sinh\beta+\nu\beta)} d\beta$$

$$\operatorname{Re} z > 0.$$

It still holds in case $\operatorname{Re} z = 0$ provided that $\operatorname{Re}\nu > 0$. Formula (9) reduces to (2) in case ν is an integer. Also 7.2(4) and (9) admit a similar expression for Neumann's function

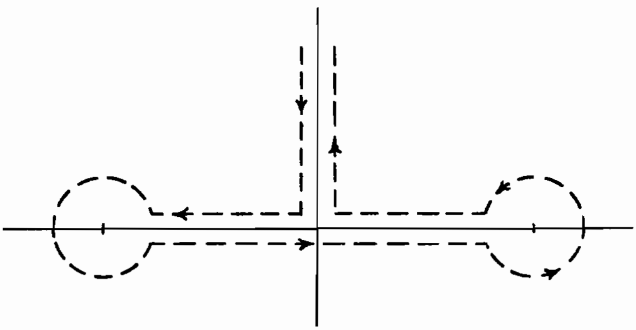
(10) $$\pi Y_\nu(z) = \int_0^\pi \sin(z \sin t - \nu t)\, dt - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\nu\pi)\, e^{-z\sinh t} dt$$

$$\operatorname{Re} z > 0.$$

[For the first integral on the right-hand side of (9) and (10) compare 7.5(32).] Generalizations of (9) and (10) are given in formulas 7.12(17) and 7.12(18).

GUBLER'S REPRESENTATIONS

From (8) another representation for $J_\nu(z)$ may be derived by specializing the contour.



We choose $\delta = \frac{1}{2}\pi$ and deform the contour into the dotted line. If Re $\nu < \frac{1}{2}$ and the radii of the circles around ± 1 tend to zero, we obtain the result

(11) $$\Gamma(\tfrac{1}{2}-\nu)\, J_\nu(z) = 2\pi^{-\frac{1}{2}}(\tfrac{1}{2}z)^{-\nu}\Big[\int_0^1 (1-t^2)^{-\nu-\frac{1}{2}}\cos(zt-\nu\pi)\,dt - \sin(\nu\pi)\int_0^\infty (1+t^2)^{-\nu-\frac{1}{2}} e^{-zt}dt\Big] \qquad \text{Re } z>0, \quad \text{Re } \nu < \tfrac{1}{2}.$$

This formula corresponds to Poisson's integral (3). If in (12) ν is replaced by $-\nu$ and this is combined with (3) and also 7.2(4), the corresponding expression for Neumann's function is

(12) $$\Gamma(\nu+\tfrac{1}{2})\, Y_\nu(z) = 2\pi^{-\frac{1}{2}}(\tfrac{1}{2}z)^{\nu}\Big[\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\sin(zt)\,dt - \int_0^\infty e^{-zt}(1+t^2)^{\nu-\frac{1}{2}}\,dt\Big] \qquad \text{Re } z>0, \quad \text{Re } \nu > -\tfrac{1}{2}.$$

By introducing Struve's function 7.5(78) in (12) we have

(13) $$[\mathbf{H}_\nu(z) - Y_\nu(z)]\,\Gamma(\nu+\tfrac{1}{2}) = 2\pi^{-\frac{1}{2}}(\tfrac{1}{2}z)^{\nu}\int_0^\infty e^{-zt}(1+t^2)^{\nu-\frac{1}{2}}\,dt \qquad \text{Re } z>0.$$

Now in (8) we take $\delta = 0$ and as a path of integration the dotted line. Replacing z by $ze^{i\frac{1}{2}\pi}$ and ν by $-\nu$ we obtain, as we suppose Re $\nu > -\frac{1}{2}$ in order that the radii of the indentations around $t = \pm 1$ may tend to zero,

(14) $$I_{-\nu}(z) = \pi^{-3/2}\,\Gamma(\tfrac{1}{2}-\nu)(\tfrac{1}{2}z)^{\nu}\Big[\sin(2\nu\pi)\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt + \cos(\nu\pi)\int_{-1}^1 e^{zt}(1-t^2)^{\nu-\frac{1}{2}}\,dt\Big] \qquad \text{Re } \nu > -\tfrac{1}{2}, \quad \text{Re } z>0.$$

Hence and by the aid of formulas 7.2(13), 7.2(12), and 7.2(14) we obtain the result

(15) $$\Gamma(\nu+\tfrac{1}{2})\, K_\nu(z) = \pi^{\frac{1}{2}}(\tfrac{1}{2}z)^{\nu}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt \qquad \text{Re } \nu > -\tfrac{1}{2}, \quad \text{Re } z>0.$$

Hence, with $t-1=v/z$, we have

(16) $\Gamma(\nu+\frac{1}{2})\,K_\nu(z)=(\frac{1}{2}\pi/z)^{\frac{1}{2}}\,e^{-z}\int_0^\infty e^{-v}\,v^{\nu-\frac{1}{2}}\,(1+\frac{1}{2}v/z)^{\nu-\frac{1}{2}}\,dv$

$|\arg z|<\pi, \quad \operatorname{Re}\nu>-\frac{1}{2},$

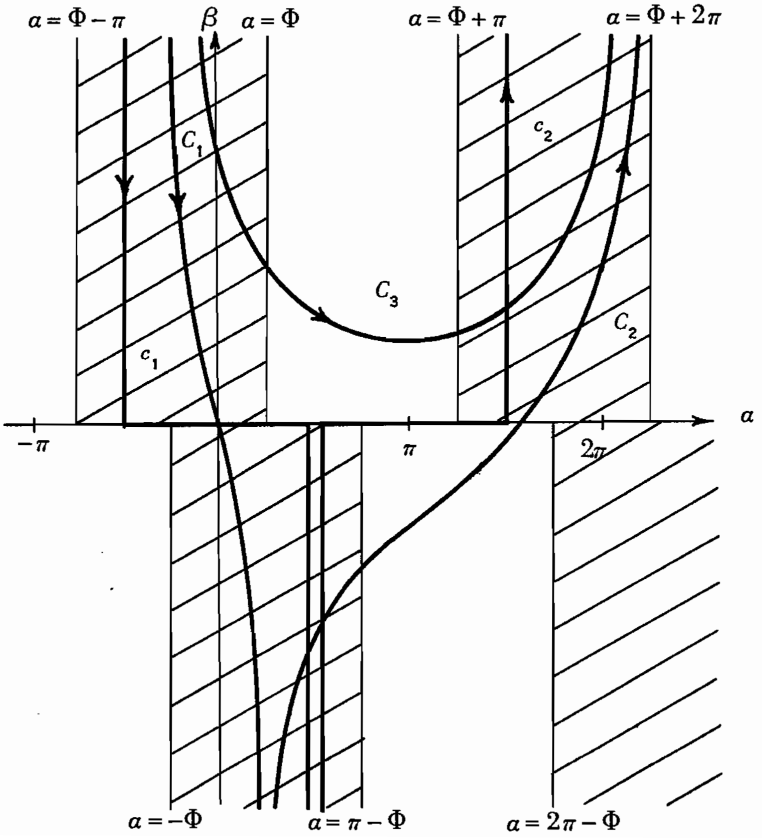
or more generally

(17) $\Gamma(\nu+\frac{1}{2})\,K_\nu(z)=(\frac{1}{2}\pi/z)^{\frac{1}{2}}\,e^{\pm z}\int_0^{\infty e^{i\delta}} e^{-t}\,t^{\nu-\frac{1}{2}}\,(1+\frac{1}{2}t/z)^{\nu-\frac{1}{2}}\,dt$

$\operatorname{Re}\nu>-\frac{1}{2}, \quad |\delta|<\frac{1}{2}\pi, \quad \delta-\pi<\arg z<\delta+\pi.$

7.3.5. Sommerfeld's integrals

If we evaluate $\int e^{iz\cos\tau}\,e^{i\nu(\tau-\frac{1}{2}\pi)}\,d\tau$ taken along the rectilinear contours c_1 (from $-\frac{1}{2}\pi+i\infty$ to $\frac{1}{2}\pi-i\infty$) and c_2 (from $\frac{1}{2}\pi-i\infty$ to $3/2\pi+i\infty$) (cf. figure),



we find from (9), (10), 7.2(5) and 7.2(6) that

(18) $\pi H_\nu^{(1)}(z) = \int_{c_1} e^{iz\cos\tau} e^{i\nu(\tau - \frac{1}{2}\pi)} d\tau,$

(19) $\pi H_\nu^{(2)}(z) = \int_{c_2} e^{iz\cos\tau} e^{i\nu(\tau - \frac{1}{2}\pi)} d\tau,$

both integrals being convergent for Re $z > 0$. The contour c_1 may be replaced by a contour C_1 from $-\eta + i\infty$ to $\eta - i\infty$, where η is a suitable number between 0 and π. With the notations

$$\Phi = \arg z, \quad \alpha = \operatorname{Re}\tau, \quad \beta = \operatorname{Im}\tau, \quad \tau = \alpha + i\beta,$$

it is easy to verify that $\operatorname{Re}(iz\cos\beta)$ is represented asymptotically by $-|z|\cosh\beta\sin(\Phi \mp \alpha)$ for large β. The upper or lower sign is to be taken according as $\beta \gtrless 0$. Therefore the integrand of (18) vanishes exponentially as $\tau \to \infty$ in the shaded part of the τ-plane. We may replace c_1 by C_1 as long as $-\eta < \Phi < \frac{1}{2}\pi$ or $-\frac{1}{2}\pi < \Phi < \pi - \eta$ according as $0 < \eta < \frac{1}{2}\pi$ or $\frac{1}{2}\pi < \eta < \pi$. Thus we have

(20) $\pi H_\nu^{(1)}(z) = \int_{C_1} e^{iz\cos\tau} e^{i\nu(\tau - \frac{1}{2}\pi)} d\tau$

and similarly

(21) $\pi H_\nu^{(2)}(z) = \int_{C_2} e^{iz\cos\tau} e^{i\nu(\tau - \frac{1}{2}\pi)} d\tau,$

C_2 being a contour from $\eta - i\infty$ to $2\pi - \eta + i\infty$. The integrals are convergent for

(22) $-\eta < \Phi = \arg z < \pi - \eta, \quad 0 \le \eta \le \pi,$

and by the theory of analytic continuation this is the range of validity for (20) and (21).

With these results it follows from 7.2(7) that

(23) $2\pi J_\nu(z) = \int_{C_3} e^{iz\cos\tau} e^{i\nu(\tau - \frac{1}{2}\pi)} d\tau,$

$$-\eta < \arg z < \pi - \eta, \quad 0 \le \eta \le \pi,$$

C_3 being a contour from $-\eta + i\infty$ to $2\pi - \eta + i\infty$.

Very often the contour integrals

(24) $\pi H_\nu^{(1)}(z) = -i \int_{-\infty}^{\infty + i\pi} e^{z\sinh\alpha - \nu\alpha} d\alpha,$

(25) $\pi H_\nu^{(2)}(z) = i \int_{-\infty}^{\infty - i\pi} e^{z\sinh\alpha - \nu\alpha} d\alpha,$

(26) $2\pi J_\nu(z) = -i \int_{\infty - i\pi}^{\infty + i\pi} e^{z\sinh\alpha - \nu\alpha} d\alpha,$

valid when $|\arg z| < \frac{1}{2}\pi$, are used. They simply are deduced from (20), (21), and (23), respectively, taking $\eta = \frac{1}{2}\pi$ and introducing the substitution $\tau = \frac{1}{2}\pi + i\alpha$.

SPECIAL CASES

We choose $\eta = 0$, take the contours C_1 and C_2 to be rectilinear, and obtain Heine's expressions

(27) $\pi H_\nu^{(1)}(z) = -ie^{-i\frac{1}{2}\nu\pi}\int_{-\infty}^{\infty} e^{iz\cosh t}\, e^{-\nu t}\,dt$ $\qquad 0 < \arg z < \pi,$

(28) $\pi H_\nu^{(2)}(z) = 2ie^{i\frac{1}{2}\nu\pi}[\int_0^{\infty} e^{iz\cosh t}\cosh(\nu t - i\nu\pi)\,dt$

$- i\int_0^{\pi} e^{-iz\cos t}\cos(\nu t)\,dt]$ $\qquad 0 < \arg z < \pi.$

If we take $\eta = \pi$ and C_1, C_2 to be rectilinear, we obtain

(29) $\pi H_\nu^{(1)}(z) = -2ie^{-i\frac{1}{2}\nu\pi}[\int_0^{\infty} e^{-iz\cosh t}\cosh(\nu t + i\nu\pi)\,dt$

$+ i\int_0^{\pi} e^{iz\cos t}\cos(\nu t)\,dt]$ $\qquad -\pi < \arg z < 0,$

(30) $\pi H_\nu^{(2)}(z) = ie^{i\frac{1}{2}\nu\pi}\int_{-\infty}^{\infty} e^{-iz\cosh t}\, e^{-\nu t}\,dt$ $\qquad -\pi < \arg z < 0.$

From (27) to (30) we obtain respectively using 7.2(7)

(31) $\pi J_\nu(z) = e^{i\frac{1}{2}\nu\pi}[\int_0^{\pi} e^{-iz\cos t}\cos(\nu t)\,dt - \sin(\nu\pi)\int_0^{\infty} e^{-\nu t + iz\cosh t}\,dt]$

$0 < \arg z < \pi,$

(32) $\pi J_\nu(z) = e^{-i\frac{1}{2}\nu\pi}[\int_0^{\pi} e^{iz\cos t}\cos(\nu t)\,dt - \sin(\nu\pi)\int_0^{\infty} e^{-\nu t - iz\cosh t}\,dt]$

$-\pi < \arg z < 0.$

In (27) let $e^t = v/a$; then we have

(33) $\pi H_\nu^{(1)}(az) = -ie^{-i\frac{1}{2}\nu\pi} a^\nu \int_0^{\infty} e^{i\frac{1}{2}z(v + a^2/v)}\, v^{-\nu-1}\,dv$

$\operatorname{Im} z > 0, \quad \operatorname{Im}(a^2 z) > 0.$

7.3.6. Barnes' integrals

A representation of the Bessel function of the first kind as a Mellin-Barnes integral (see 1.19) is

(34) $4\pi i J_\nu(x) = \int_{c-i\infty}^{c+i\infty} (\frac{1}{2}x)^{-s}\,\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)/\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}s)\,ds$

see errata!

$x > 0, \quad -\operatorname{Re}\nu < c < 1,$

and may be proved by evaluating the integral in terms of the residue of the integrand or applying Mellin's inversion formula to 7.7(19).

If the restriction $-\operatorname{Re}\nu < c < 1$ is removed, the integral still makes sense, but it need not represent a Bessel function. We put

(35) $4\pi i\, J_{\nu,m}(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} (\frac{1}{2}x)^{-s}\,\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)/\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}s)\,ds$

$x > 0, \quad \sigma < 1, \quad -2m - \operatorname{Re}\nu < \sigma < -(2m-1) - \operatorname{Re}\nu, \quad m = 1, 2, \ldots.$

the integral being taken along a line parallel to the imaginary axis. The evaluation of the integral in terms of the residues of the integrand gives

$$4\pi J_{\nu,m}(x) = 4\pi J_\nu(x) - 4\pi i \sum_{n=0}^{m-1} (-1)^n \frac{(\tfrac{1}{2}x)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}.$$

We define for arbitrary complex values of z and ν

$$(36)\quad J_{\nu,m}(z) = \sum_{n=m}^{\infty} (-1)^n (\tfrac{1}{2}z)^{\nu+2n}/[n!\,\Gamma(\nu+n+1)] \qquad m = 1, 2, 3, \ldots,$$

and call this the *cut* Bessel function of the first kind. From (33) we have

$$(37)\quad \frac{d}{dz}[z^\nu J_{\nu,m}(z)] = z^\nu J_{\nu-1,m}(z),$$

$$(38)\quad \frac{d}{dz}[z^{-\nu} J_{\nu,m}(z)] = -z^{-\nu} J_{\nu+1,m}(z).$$

7.3.7. Airy's integrals

Airy's formulas,

$$(39)\quad \int_0^\infty \cos(t^3 + 3tx)\,dt = (x/3)^{1/2} K_{1/3}(2x^{3/2}) \qquad x > 0,$$

$$(40)\quad \int_0^\infty \cos(t^3 - 3tx)\,dt = -\pi/3\, x^{1/2}\,[J_{1/3}(2x^{3/2}) + J_{-1/3}(2x^{3/2})] \qquad x > 0,$$

can be proved as follows. In (39) we substitute $t = 2x^{1/2}\sinh \frac{1}{3}v$. Since

$$4(\sinh v/3)^3 + 3\sinh(v/3) = \sinh v$$

we obtain

$$\int_0^\infty \cos(t^3 + 3tx)\,dt = 2x^{1/2}/3 \int_0^\infty \cos(2x^{3/2}\sinh v)\cosh(v/3)\,dv,$$

and using 7.12(25) this establishes (39).

To prove (40) we express the right-hand side of (39) by means of its power series [see 7.2(12) and 7.2(13)] and obtain

$$\int_0^\infty \cos(t^3 + 3tx)\,dt$$
$$= 1/3\,\pi\left[\sum_{m=0}^{\infty} \frac{x^{3m}}{m!\,\Gamma(-1/3+m+1)} - x\sum_{m=0}^{\infty}\frac{x^{3m}}{\Gamma(1/3+m+1)\,m!}\right].$$

Here we replace x by $-x$ and using 7.2(2) we obtain (40). For generalizations of the formulas (39) and (40) see Watson (1944, pp. 320-324).

7.4. Asymptotic expansions

The asymptotic behavior of Bessel functions is different according as the order ν, the variable z, or both of these quantities increase indefinitely. The power series expansions of 7.2(2) are asymptotic expansions when z is fixed and $\nu \to \infty$. It is comparatively easy to derive asymptotic expansions for the case that ν is fixed and $z \to \infty$; when both ν and z are large, the investigation becomes much more involved.

7.4.1. Large variable

We shall derive here the asymptotic expansion of the modified Bessel function of the third kind, $K_\nu(z)$. The corresponding expansions of other Bessel functions may be obtained by means of formulas 7.2(16), 7.2(17), and 7.2(8); the results are given in sec. 7.13.1.

We start with the integral representation 7.3(17),

$$\Gamma(\nu+\tfrac{1}{2})\, K_\nu(z) = (\tfrac{1}{2}\pi/z)^{\frac{1}{2}}\, e^{-z} \int_0^{\infty e^{i\delta}} e^{-t} t^{\nu-\frac{1}{2}} (1+\tfrac{1}{2}t/z)^{\nu-\frac{1}{2}}\, dt,$$

$$\operatorname{Re}\nu > -\tfrac{1}{2}, \qquad |\delta| < \tfrac{1}{2}\pi, \qquad \delta-\pi < \arg z < \delta+\pi,$$

substitute the binomial expansion with remainder term,

$$(1+\tfrac{1}{2}t/z)^{\nu-\frac{1}{2}} = \sum_{m=0}^{M-1} \frac{\Gamma(\nu+\frac{1}{2})}{m!\,\Gamma(\nu+\frac{1}{2}-m)}\, (\tfrac{1}{2}t/z)^m + r_M,$$

and use 1.1(6) obtaining

$$(1) \quad K_\nu(z) = (\tfrac{1}{2}\pi/z)^{\frac{1}{2}} e^{-z} \left[\sum_{m=0}^{M-1} \frac{\Gamma(\nu+\frac{1}{2}+m)}{m!\,\Gamma(\nu+\frac{1}{2}-m)}\, (2z)^{-m} + R_M\right]$$

$$-3\pi/2 < \arg z < 3\pi/2,$$

where the remainder is given by

$$(2) \quad (M-1)!\,\Gamma(\nu+\tfrac{1}{2}-M)\, R_M = (2z)^{-M} \int_0^\infty e^{-t} t^{\nu-\frac{1}{2}+M}\, dt \int_0^1 (1-v)^{M-1} (1+\tfrac{1}{2}vt/z)^{\nu-\frac{1}{2}-M}\, dv.$$

It is easy to see that for any fixed ν with $\operatorname{Re}\nu > -\frac{1}{2}$,

$$R_M = O(|z|^{-M}), \qquad z\to\infty, \qquad -3\pi/2+\epsilon \le \arg z \le 3\pi/2-\epsilon, \qquad \epsilon > 0.$$

By a more careful discussion of (2) it may be shown that the modulus of the remainder in (1) is less than the modulus of the first neglected term ($m = M$) if ν is real, $M > \nu - \frac{1}{2} > -1$, and $\operatorname{Re} z > 0$ (MacRobert 1947, p. 272; Watson, 1944, p. 207) and that the remainder is approximately equal to half of the first neglected term when ν and z are both real and $2z - M + \frac{1}{2}$ is small in comparison with z (compare Burnett, 1929). Airey (see 1937), modified (1) so as to obtain a much closer approximation suitable for numerical computation to high accuracy.

Using Hankel's symbol 1.20(3)

$$(3) \quad (\nu, m) = \frac{2^{-2m}}{m!} \{(4\nu^2-1^2) \cdots [4\nu^2-(2m-1)^2]\} = \frac{\Gamma(\frac{1}{2}+\nu+m)}{m!\,\Gamma(\frac{1}{2}+\nu-m)},$$

the asymptotic expansion may conveniently be written as

(4) $K_\nu(z) = (\frac{1}{2}\pi/z)^{\frac{1}{2}} e^{-z} [\sum_{m=0}^{M-1} (\nu, m)(2z)^{-m} + O(|z|^{-M})]$

$-3\pi/2 < \arg z < 3\pi/2.$

Since only ν^2 appears in the definition of (ν, m), the restriction Re $\nu > -\frac{1}{2}$ may be omitted.

7.4.2. Large order

The first reliable investigation of Bessel functions with large variable and order was carried out by Debye (1909) by means of the method of steepest descents. This method is based on the following consideration (Copson, 1935, p. 330; Watson, 1944, p. 235).

Suppose a function $F(z)$ is given in the form

(5) $F(z) = \int_C e^{-zf(\alpha)} g(\alpha)\, d\alpha$

where C is a contour in the complex α-plane joining two zeros of $e^{-zf(\alpha)}$. In many cases it is possible to choose C so that it passes through a zero α_0 of $f'(\alpha)$ and that the imaginary part of $f(\alpha)$ is constant along C. Thus we have $f'(\alpha_0) = 0$ and

(6) $\operatorname{Im}[f(\alpha)] = \text{constant} = \operatorname{Im}[f(\alpha_0)]$

along C so that $\operatorname{Re}[f(z)]$ changes as rapidly as possible when α traverses C. For large z, the modulus of the integrand has a sharp maximum at α_0 and only that part of C which is in the immediate neighborhood of α_0 will give a significant contribution to the contour integral (5).

For the sake of simplicity we assume that both order and variable are positive and put

(7) $z = x > 0, \quad \nu = p > 0.$

Moreover, we shall assume that the quantity v_0 determined by

(8) $\sinh v_0 = p/x, \quad \cosh v_0 = (1 + p^2/x^2)^{\frac{1}{2}}, \quad v_0 > 0$

is fixed as $p, x \to \infty$. We shall discuss $K_p(x)$ only; the corresponding expansions of other Bessel functions are listed in sec. 7.13.2.

An integral representation for $K_p(x)$ of the form (5) is immediately obtained from 7.2(15) and Sommerfeld's expression 7.3(20) in the form

(9) $K_p(x) = \frac{1}{2}i \int_C e^{-x\cos\alpha} e^{ip\alpha}\, d\alpha = \frac{1}{2}i \int_C e^{-xf(\alpha)}\, d\alpha$

where

(10) $f(\alpha) = \cos\alpha - ip\alpha/x.$

According to the results in sec. 7.3.5, the contour C starts at $-\eta+i\infty$, ends at $\eta-i\infty$, where $0\leq\eta\leq\pi$, and lies entirely within the strip $-\eta\leq \mathrm{Re}\ a\leq\eta$ of the complex a-plane. The condition $f'(a)=0$ leads to

(11) $\sin a = -ip/x = -i \sinh v_0$,

and this equation has an infinite number of solutions

(12) $a_m = -iv_0 + 2\pi m \qquad m = 0, \pm 1, \pm 2, \ldots.$

From these only a_0 lies within the strip $-\eta < \mathrm{Re}\ a < \eta$. Hence we have

(13) $a_0 = -i \log\{x^{-1}[p + (p^2+x^2)^{1/2}]\} = -iv_0$

and from (10)

(14) $f(a_0) = \cosh v_0 - v_0 \sinh v_0$.

The condition (6) shows that the path of steepest descent is the imaginary axis, and from (9) with $a = iv$ we obtain

(15) $K_p(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh v + pv}\,dv = \frac{1}{2}\int_{-\infty}^{\infty} e^{-xg(v)}\,dv,$

where

$$g(v) = \cosh v - v \sinh v_0.$$

The substitution

(16) $\tau = g(v) - g(v_0) = \cosh v - \cosh v_0 - (v-v_0)\sinh v_0$

maps the v-plane on the τ-plane. The mapping is conformal except at the points $v_m = v_0 + 2\pi i m$ where $d\tau/dv$ has a simple zero. Thus

(17) $\Phi(\tau) = dv/d\tau = [g'(v)]^{-1}$

may be represented in a neighborhood of $\tau = 0$ in the form

(18) $\Phi(\tau) = \sum_{n=1}^{\infty} b_n \tau^{\frac{1}{2}n-1}$,

and this expansion is convergent up to the next singular point τ, which corresponds to $v = v_0 \pm 2\pi i$.

As v increases from $-\infty$ to v_0, the variable τ decreases from ∞ to 0; and as v continues to increase from v_0 to ∞, the variable τ increases from 0 to ∞. We shall determine the coefficients b_n in (18) so that we may take $\arg\tau = 2\pi$ on the former, and $\arg\tau = 0$ on the latter part of the path of integration. Then we have

(19) $K_p(x) = \frac{1}{2} e^{-xf(v_0)} \int_0^{\infty} e^{-\tau x}[\Phi(\tau) - \Phi(\tau e^{i2\pi})]\,d\tau.$

Here we use (18) and apply Watson's lemma (Copson, 1935, p. 218) to obtain the desired asymptotic expansion

(20) $K_p(x) = e^{-xf(v_0)} \left[\sum_{n=0}^{M-1} b_{2n+1}\, x^{-n-\frac{1}{2}}\, \Gamma(n+\tfrac{1}{2}) + O(x^{-M-\frac{1}{2}})\right].$

The coefficients in (18) are obtained by Cauchy's theorem

(21) $4\pi i b_n = \int \tau^{-\frac{1}{2}n}\, \Phi(\tau)\, d\tau = \int [g(v) - g(v_0)]^{-\frac{1}{2}n}\, dv,$

the last integral being taken around a small closed contour encircling $v = v_0$ once in the positive direction.

Since $[g(v) - g(v_0)]^{-n-\frac{1}{2}}$ has a pole of order $2n+1$ at $v = v_0$ we may represent $(v - v_0)^{2n+1} [g(v) - g(v_0)]^{-n-\frac{1}{2}}$ as a Taylor series. We then have

$$(v - v_0)^{2n+1} [g(v) - g(v_0)]^{-n-\frac{1}{2}} = \sum_{l=0}^{\infty} A_l^{(n)} (v - v_0)^l$$

with

(22) $A_l^{(n)} = \frac{1}{l\,!} \left\{ \frac{d^l}{dv^l} (v - v_0)^{2n+1} [g(v) - g(v_0)]^{-n-\frac{1}{2}} \right\}_{v=v_0}.$

On the other hand, Cauchy's theorem gives

(23) $2\pi i\, A_l^{(n)} = \int (v - v_0)^{2n-l} [g(v) - g(v_0)]^{-n-\frac{1}{2}}\, dv,$

taken around a closed contour encircling $v = v_0$. A comparison between (21) and (23) gives for the coefficients in (20),

(24) $b_{2n+1} = \tfrac{1}{2} A_{2n}^{(n)} = \frac{1}{2(2n)\,!} \left\{ \frac{d^{2n}}{dv^{2n}} (v - v_0)^{2n+1} [g(v) - g(v_0)]^{-n-\frac{1}{2}} \right\}_{v=v_0}$

We thus obtain the asymptotic expansion

(25) $K_p(x) = 2^{-\frac{1}{2}} (p^2 + x^2)^{-\frac{1}{4}} \exp[-(p^2 + x^2)^{\frac{1}{2}} + p \sinh^{-1}(p/x)]$

$$\times \left[\sum_{m=0}^{M-1} 2^m a_m\, \Gamma(m + \tfrac{1}{2}) (p^2 + x^2)^{-\frac{1}{2}m} + O(x^{-M})\right] \qquad p,\ x > 0,$$

where

$$a_m = 2^{\frac{1}{2}-m} (1 + p^2/x^2)^{\frac{1}{4}+\frac{1}{2}m}\, b_{2m+1}\,.$$

The first few coefficients in (25) are

(26) $a_0 = 1, \quad a_1 = -\frac{1}{8} + \frac{5}{24} (1 + x^2/p^2)^{-1},$

$$a_2 = \frac{3}{128} - \frac{77}{576}(1+x^2/p^2)^{-1} + \frac{385}{3456}(1+x^2/p^2)^{-2}.$$

A similar expansion derived by the method of the stationary phase was given by J. Bijl (1937, p. 23). He gives the following result valid for $p \geq x^{1/2} \geq 1$.

$$(27)\quad |K_p(x) - 2^{-1/2}(p^2+x^2)^{-1/4}\exp[-(p^2+x^2)^{1/2}+p\sinh^{-1}(p/x)] \times \sum_{m=0}^{M-1} 2^m d_{2m}\Gamma(m+\tfrac{1}{2})(p^2+x^2)^{-\frac{1}{2}m}/(2m)!|$$
$$\leq Cw^{-2M}(p^2+x^2)^{-1/4}\exp[-(p^2+x^2)^{1/2}+p\sinh^{-1}(p/x)],$$

where $w = px^{-1/2}$ or $(p^2+x^2)^{1/4}p^{-1/3}$ according as $p \leq x^{3/4}$ or $> x^{3/4}$.

For the coefficients in (27) there exists the recurrence relation

$$(28)\quad d_m = -\sum \left[\binom{m-1}{l} pd_l + \binom{m-1}{l-1}(p^2+x^2)^{1/2} d_{l-1}\right]$$

with $d_0 = 1$, $d_1 = d_2 = 0$. Here $\binom{m-1}{-1}$ is interpreted as zero and the sum is formed over all l for which $m - l$ is odd and $0 \leq l \leq m-3$. From (28) it follows that

$$(29)\quad d_0 = 1,\quad d_2 = 0,\quad d_4 = -(p^2+x^2)^{1/2},\quad d_6 = 10p^2 - (p^2+x^2)^{1/2},$$
$$d_8 = 56p^2 + 35(p^2+x^2) - (p^2+x^2)^{1/2} \quad\text{and}$$
$$d_{10} = -2100p^2(p^2+x^2) + 246p^2 + 210(p^2+x^2) - (p^2+x^2)^{1/2}.$$

The corresponding expansions for $J_p(x)$ and $H_p^{(1)}(x)$ are obtained in a similar manner from Sommerfeld's expressions shown in 7.3(20) and 7.3(23) by the method of steepest descents (compare Debye, 1909; Watson, 1944, p. 235; Weyrich, 1937, p. 49). (For a discussion of the paths of steepest descents for various cases see Emde, 1937, 1939, and Emde and Rühle, 1934.) Different cases are to be distinguished according as p is larger, less or in the neighborhood of x. They are listed in formulas 7.13(11) to 7.13(16). Formulas for the upper bound of the remainder of the expansions 7.13(11) and 7.13(14) respectively, and recurrence relations for the coefficients have been given by Meijer (1933, p. 108), and Van Veen (1927, p. 27), respectively.

Recently (compare Schöbe, 1948) two different asymptotic expansions for the second Hankel function have been derived from the contour integral of 7.3(25). The terms of Schöbe's series are not elementary functions as in Debye's series shown in 7.13(11) and 7.13(13) but involve the second Hankel function of the orders 1/3 and −2/3. The first term is just Nicholson's formula 7.13 (27) and Watson's formula 7.3 (34) respectively.

See errata!

7.4.3. Transitional regions

The asymptotic expansions 7.13 (11), 7.13 (13) and 7.13 (15) for $H_p^{(1)}(x)$, valid in case $x > p$, $x < p$ and x nearly equal to p respectively, do not cover all possibilities since the restriction $x - p = O(x^{1/3})$ has to be imposed in the last case. In the transitional region, that is when p/x is nearly equal to 1 while $|x - p|$ is large, other formulas have to be used. These have been given by Nicholson (Watson, 1944, p. 248); Watson (1944, p. 249); Schöbe (1948); Tricomi (1949).

Nicholson's formulas for integer order n of the Bessel function of the first kind are

(30) $J_n(x) \sim \pi^{-1}\, 3^{-1/6}\, (\xi/x)^{1/3}\, K_{1/3}(\xi)$.

(31) $J_n(x) \sim 3^{-2/3} (\xi/x)^{1/3}\, [J_{1/3}(\xi) + J_{-1/3}(\xi)]$,

according as $x < n$ or $x > n$ and

(32) $$\xi = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/2} |x-n|^{-3/2}.$$

[For the $Y_n(x)$ see 7.13 (24) and 7.13 (26).] These formulas were derived by means of the principle of the stationary phase (Watson, 1944, p. 229). For this purpose we start with the integral representation 7.3 (2)

(33) $$\pi J_n(x) = \int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi.$$

The phase is stationary where $d/d\phi\,(n\phi - x\sin\phi) = 0$ or $\cos\phi = n/x$. Since n is supposed to be nearly equal to x, ϕ is small, and in the neighborhood of the stationary point we may replace $\sin\phi$ by $\phi - \phi^3/6$. Thus

$$\pi J_n(x) \sim \int_0^{\pi} \cos[x\phi^3/6 - (x-n)\phi]\, d\phi$$

$$\sim \int_0^{\infty} \cos[x\phi^3/6 - (x-n)\phi]\, d\phi.$$

This is Airy's integral 7.3 (39) and 7.3 (40) respectively, according as $x < n$ or $x > n$ and the desired results (30), (31) are established.

This method of deriving Nicholson's formula is a questionable one; moreover the range of validity and the order of magnitude of the error cannot be determined. [A rigorous theory of the method of the stationary phase has been given by van der Corput (1934, 1936). This method was applied by J. Bijl (1937) to derive asymptotic expansions for the Bessel functions.]

WATSON'S FORMULAS

A more precise form of Nicholson's formula was given by Watson (1944, p. 250)

$$(34)\quad e^{i\pi/6} H_p^{(2)}(x) = 3^{-1/2}\, w e^{-ip(w - w^3/3 - \tan^{-1} w)} \times H_{1/3}^{(2)}(pw^3/3) + O(p^{-1}).$$

Here the order p is not restricted to be an integer, and we have

$$(35)\quad w = (x^2/p^2 - 1)^{1/2},$$

where $\arg w = 0$ for $x > p$ and $\arg w = \frac{1}{2}\pi$ for $x < p$. The corresponding formulas for $J_p(x)$ and $Y_p(x)$ are listed in formulas 7.13(28) to 7.13(31). In case x is nearly equal to p, w can be replaced by $(\frac{1}{2}p)^{-1/2}\,(x-p)^{1/2}$ [$\arg(x-p)^{1/2} = 0$ or $\frac{1}{2}\pi$ for $x > p$ or $x < p$ respectively], and Nicholson's formulas (30), (31) are obtained.

From his asymptotic expansion, Schöbe (1948) derives the result (see end of sec. 7.3.2), see errata!

$$(36)\quad e^{i\pi/6} H_p^{(2)}(x) = 3^{-1/6}\left(\frac{\xi}{x}\right)^{1/3}\left(\frac{9}{10}+\frac{p}{10x}\right)^{-3/2} \times H_{1/3}^{(2)}\left[\xi\left(\frac{9}{10}+\frac{p}{10x}\right)^{-1/2}\right] + O(p^{-5/2}),$$

$$\xi = \frac{2}{3}\,(\tfrac{1}{2}x)^{-1/2}\,(x-p)^{3/2},$$

and $\arg(x-p)^{3/2}$ equal to 0 or $3\pi/2$ according as $x > p$ or $x < p$.

Another formula was given by Tricomi (1949). The results are

$$(37)\quad \pi J_p[p + (p/6)^{1/3} t] = (6/p)^{1/3} A_1(t) - 1/(10p)\,[3t^2 A_1'(t) + 2t\, A_1(t)] + O(p^{-5/3}),$$

$$(38)\quad \pi Y_p[p + (p/6)^{1/3} t] = (6/p)^{1/3} A_2(t) + 1/(10p)\,[3t^2 A_2'(t) + 2t\, A_2(t)] + O(p^{-5/3}).$$

Here, $A_1(t)$ and $A_2(t)$ denote the functions

$$(39)\quad A_1(t) = \pi/3\,(t/3)^{1/2}\,\{J_{-1/3}[2(t/3)^{3/2}] + J_{1/3}[2(t/3)^{3/2}]\},$$

$$(40)\quad A_2(t) = \pi/3\, t^{1/2}\,\{J_{-1/3}[2(t/3)^{3/2}] - J_{1/3}[2(t/3)^{3/2}]\}$$

[see Airy's integral 7.3(40)]

7.4.4. Uniform asymptotic expansions

DIFFERENTIAL EQUATION METHODS

The asymptotic formulas discussed so far have been obtained from integral representations for the Bessel functions mostly from Sommerfeld's formulas (see 7.3.5). Another approach uses the differential equation as its starting point.

For the following we restrict ourselves to positive real values of both order p and argument x and transform the Bessel equation of 7.2(1) by the substitution $x = pe^y$. The resulting equation is

$$(41) \quad w''(y) + p^2(e^{2y} - 1)\, w(y) = 0.$$

The asymptotic behavior of solutions of differential equations of the form

$$(42) \quad w''(y) + [p^2\, \Phi^2(y) - K(y)]\, w(y) = 0$$

in which p is a large parameter, has been investigated by several authors (Horn, 1899; Schlesinger, 1907; Birkhoff, 1908; Blumenthal, 1912; Jeffreys, 1925; Jordan, 1930). The basic principle is that approximately identical differential equations will have approximately identical solutions. In the work of earlier authors the comparison equation has a constant Φ, and therefore, all these methods fail in a region in which $\Phi(y)$ has a zero. In the case of the Bessel equation this failure occurs in the neighborhood of $y = 0$ or $x = p$.

Langer (1931, 1932, 1934) used a comparison equation in which $\Phi(y)$ is essentially a suitable power of y and was thus able to cope with zeros (of any order) of $\Phi^2(y)$. The solution of Langer's comparison equation may be expressed in terms of Bessel functions of order 1/3. The application of Langer's results to (28) leads to the following asymptotic formula which is valid uniformly in $0 < x < \infty$ (Langer, 1931, pp. 60-61).

$$(43) \quad e^{i\pi/6} H_p^{(2)}(x) = w^{-1/2}(w - \tan^{-1} w)^{1/2}$$
$$\times H_{1/3}^{(2)}(pw - p\tan^{-1} w) + O(p^{-4/3}) \qquad w = (x^2/p^2 - 1)^{1/2}.$$

For $x > p$, $\arg w$ and $\arg(w - \tan^{-1} w)$ are equal to zero; for $x < p$, $\arg w$ is equal to $\frac{1}{2}\pi$, and $\arg(w - \tan^{-1} w)$ is equal to $3\pi/2$. [The results for $J_p(x)$ and $Y_p(x)$ are listed in formulas 7.13(32) to 7.13(35).] For a comparison between numerical values of $J_p(x)$ and those obtained by Langer's formula (43) see Fock (1934), and for an extension of (43) to complex p and x, see Langer (1932).

In case of sufficiently small w (x nearly equal to p) $w - \tan^{-1} w$ may be replaced by $w^3/3$ and Watson's formula (34) is obtained.

The method of the "approximately identical" differential equations was also used by Cherry (1949, p. 121), to obtain uniform asymptotic expansions for the Bessel functions. The differential equation for

$$y^{1/2} J_p[a(1-y^2)^{1/2}]$$

is

$$(44)\quad \frac{d^2w}{du^2} + w\left[-p^2 + (y^{-2}-1)\left(\frac{5}{4}y^{-4} - \frac{1}{4}y^{-2} + a^2 - p^2\right)\right] = 0,$$

where

$$(45)\quad u = \tanh^{-1} y - y.$$

Near $y = 0$, the coefficient of w in (44) can be developed in the form

$$-p^2 + \frac{5}{36}u^{-2} + (a^2 - p^2 - 1/35)(3u)^{-2/3} + P(u^{2/3})$$

where P stands for a power series. Thus (44) is close to

$$(46)\quad \frac{d^2W}{du^2} + W\left(-p^2 + \frac{5}{36}u^{-2}\right) = 0.$$

But according to formulas 7.2(62) and 7.2(63) a solution of (46) is

$$(47)\quad W = (pu)^{1/2} K_{1/3}(pu),$$

and if (44) is written as

$$(48)\quad \frac{d^2w}{du^2} + w\left(-p^2 + \frac{5}{36}u^{-2}\right) = wf(u)$$

with

$$(49)\quad f(u) = \frac{5}{36}u^{-2} - (y^{-2}-1)\left(\frac{5}{4}y^{-4} - \frac{1}{4}y^{-2} + a^2 - p^2\right),$$

then, starting with the expression (47) in place of w on the right-handside of (48), we find the solution of (48) by an iterative procedure using the method of the variation of parameters. Further results may be found in Cherry (1949, 1950).

7.5. Related functions

There are certain polynomials and functions which are either similar, or in some ways analogous, to Bessel functions or which occur in investigations connected with Bessel functions. These polynomials and functions are thoroughly discussed in Watson's book (1944, Chapters 9 and 10). Here we shall give only a very brief account of the basic properties of some of these functions. For more detailed information the reader may refer to Watson's book.

7.5.1. Neumann's and related polynomials

Neumann's polynomials $O_n(z)$ are defined by the equation

$$(1)\quad (z-\xi)^{-1} = \sum_{n=0}^{\infty} \epsilon_n J_n(\xi)\, O_n(z)$$

$$\epsilon_0 = 1, \quad \epsilon_n = 2 \quad \text{if} \quad n \geq 1, \quad |\xi| < |z|,$$

and are of importance in the theory of the expansion of an arbitrary analytic function $f(z)$ as a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z).$$

In order to obtain an explicit expression for $O_n(z)$ we start with the identity

$$(2)\quad (z-\xi)^{-1} = z^{-1}\int_0^{\infty} e^{-x}\, e^{x\xi/z}\, dx \qquad \mathrm{Re}\, \xi/z < 1.$$

In 7.2(25) we put $\alpha = 1$, replace z by ξ, $t - t^{-1}$ by $2x/z$, and obtain

$$e^{x\xi/z} = \sum_{n=0}^{\infty} \{z^{-n}[x+(x^2+z^2)^{½}]^n + (-z)^n[x+(x^2+z^2)^{½}]^{-n}\} J_n(\xi).$$

This we substitute in (2), remark that term by term integration may be justified if $|\xi/z| < 1$ and compare the results with (1). Thus we obtain Neumann's integral representation

$$(3)\quad O_n(z) = ½\, z^{-n-1}\int_0^{\infty}\{[x+(x^2+z^2)^{½}]^n + [x-(x^2+z^2)^{½}]^n\}\, e^{-x}\, dx$$

$$= ½\int_0^{\infty e^{i\delta}} \{[t+(t^2+1)^{½}]^n + [t-(t^2+1)^{½}]^{-n}\}\, e^{-zt}\, dt,$$

where $n > 0$ and $|\delta + \arg z| < ½\pi$.

To exhibit the polynomial nature of $O_n(z)$, we substitute

$$[(t^2+1)^{½} \pm t]^n = {}_2F_1(-½n, ½n; ½; -t^2)$$

$$\pm\, nt\; {}_2F_1(½+½n, ½-½n; 3/2; -t^2)$$

in (3) and integrate term by term with the result that

$$(4)\quad O_{2n}(z) = ½n \sum_{m=0}^{n} \frac{(n+m-1)!}{(n-m)!}(½z)^{-2m-1},$$

$$(5)\quad O_{2n+1}(z) = ½(n+½)\sum_{m=0}^{n}\frac{(n+m)!}{(n-m)!}(½z)^{-2m-2},$$

or, after some algebra

(6) $O_n(z) = \tfrac{1}{4} \sum_{m=0}^{\leq \frac{1}{2}n} n(n-m-1)!\,(\tfrac{1}{2}z)^{2m-n-1}/m!$ $\qquad n \geq 1.$

In particular we have

(7) $O_0(z) = z^{-1}, \quad O_1(z) = z^{-2}, \quad O_2(z) = z^{-1} + 4z^{-3}.$

Evidently $O_n(z)$ is a polynomial in z^{-1} of degree $n+1$. From (6) we have the following inequality

(8) $|O_n(z)| \leq 2^{n-1}\, n!\, |z|^{-n-1} \exp(\tfrac{1}{4}|z|^2)$ $\qquad n > 1.$

Hence, and from 7.3 (4) it follows that the series

$$\sum_{n=0}^{\infty} a_n J_n(\xi)\, O_n(z)$$

is absolutely convergent whenever the series $\Sigma\, a_n(\xi/z)^n$ is absolutely convergent.

From the definition we have the relations

(9) $O_0'(z) = -O_1(z),$

(10) $2O_n'(z) = O_{n-1}(z) - O_{n+1}(z)$ $\qquad n \geq 1,$

(11) $(n-1)\,O_{n+1}(z) + (n+1)\,O_{n-1}(z) - 2t^{-1}(n^2-1)\,O_n(z)$
$= 2nt^{-1}(\sin \tfrac{1}{2}n\pi)^2,$

(12) $nz\,O_{n-1}(z) - (n^2-1)\,O_n(z) = (n-1)\,z\,O_n'(z) + n(\sin \tfrac{1}{2}n\pi)^2,$

(13) $nz\,O_{n+1}(z) - (n^2-1)\,O_n(z) = -(n+1)\,z\,O_n'(z) + n(\sin \tfrac{1}{2}n\pi)^2.$

From these relations it follows that $O_n(z)$ satisfies the differential equation

(14) $z^2 \dfrac{d^2v}{dz^2} + 3z\dfrac{dv}{dz} + (z^2 + 1 - n^2)\,v = z(\cos \tfrac{1}{2}n\pi)^2$
$+ n(\sin \tfrac{1}{2}n\pi)^2.$

If C denotes any simple closed contour around the origin, then from (6) and 7.2 (2) it follows that

(15) $\int_C O_m(z)\,O_n(z)\,dz = 0$ $\qquad m = n$ and $m \neq n,$

(16) $\int_C J_m(z)\,O_n(z)\,dz = 0$ $\qquad m \neq n,$

(17) $\int_C J_m(z)\,O_m(z)\,dz = \pi i$ $\qquad m \geq 1.$

For some purposes Schläfli's polynomial,

(18) $$S_0(z) = 0, \quad S_n(z) = \sum_{m=0}^{\leq \frac{1}{2} n} (n-m-1)!\,(\tfrac{1}{2}z)^{-n+2m}/m! \qquad n \geq 1,$$

may conveniently be used (Watson, 1944, Sections 9.3 - 9.34). It is connected with Neumann's polynomial by the relation

(19) $$n\, S_n(z) = 2z\, O_n(z) - 2(\cos \tfrac{1}{2} n\pi)^2.$$

The polynomials $\Omega_n(z)$ defined by the expansion

(20) $$(z^2 - \xi^2)^{-1} = \sum_{n=0}^{\infty} \epsilon_n [J_n(z)]^2\, \Omega_n(z) \qquad |\xi| < |z|,$$

have also been investigated by Neumann. (cf. Watson, 1944, Sections 9.4 and 9.41).

Both of Neumann's polynomials have been generalized by Gegenbauer (Watson, 1944, Sections 9.2, 9.5). The defining expansions are

(21) $$\xi^{\nu}/(z - \xi) = \sum_{n=0}^{\infty} A_{n,\nu}(z)\, J_{\nu+n}(\xi) \qquad |\xi| < |z|,$$

(22) $$\xi^{\nu+\mu}/(z-\xi) = \sum_{n=0}^{\infty} B_{n;\mu,\nu}(z)\, J_{\mu+\frac{1}{2}n}(\xi)\, J_{\nu+\frac{1}{2}n}(\xi).$$

7.5.2. Lommel's polynomials

Through repeated application of the recurrence relation, see 7.2(56), it follows that $J_{\nu+m}$ may be expressed in the form

(23) $$J_{\nu+m}(z) = J_{\nu}(z)\, R_{m,\nu}(z) - J_{\nu-1}(z)\, R_{m-1,\nu+1}(z),$$

where $R_{m,\nu}$ is a polynomial of degree m in z^{-1}; it is called Lommel's polynomial. Similarly we have

(24) $$(-1)^m J_{-\nu-m}(z) = J_{-\nu}(z)\, R_{m,\nu}(z) + J_{-\nu-1}(z)\, R_{m-1,\nu+1}(z).$$

From (23), (24), and 7.11(33) we find that

(25) $$R_{m,\nu}(z) = \tfrac{1}{2}\pi z (\sin \nu\pi)^{-1} [J_{\nu+m}(z)\, J_{-\nu+1}(z) + (-1)^m J_{-\nu-m}(z)\, J_{\nu-1}(z)].$$

Using the power series of 7.2(48) for the product of two Bessel functions we find from (25) after some reductions

(26) $$R_{m,\nu}(z) = \sum_{n=0}^{\leq \frac{1}{2} m} \frac{(-1)^n\,(m-n)!\,\Gamma(\nu+m-n)}{n!\,(m-2n)!\,\Gamma(\nu+n)}\,(\tfrac{1}{2}z)^{-m+2n}$$

$$=\frac{\Gamma(\nu+m)}{\Gamma(\nu)}(\tfrac{1}{2}z)^{-m}\ {}_2F_3(\tfrac{1}{2}-\tfrac{1}{2}m,-\tfrac{1}{2}m;\nu;-m,1-\nu-m;-z^2).$$

Hence we can find that

(27) $R_{m,\nu}(z)=(-1)^m R_{m,-\nu-m+1}(z).$

Since the Bessel functions of the second kind satisfy the same recurrence relations 7.2(56), we obtain a relation analogous to (25)

(28) $Y_{\nu+m}(z)=Y_\nu(z)R_{m,\nu}(z)-Y_{\nu-1}(z)R_{m-1,\nu+1}(z).$

Hence, from (25) and 7.11(36) we have

(29) $R_{m,\nu}(z)=-\tfrac{1}{2}\pi z\,[Y_{\nu+m}(z)J_{\nu-1}(z)-J_{\nu+m}(z)Y_{\nu-1}(z)].$

Let n be an integer, $m=2n$ and $\nu=\tfrac{1}{2}n$ in (25). Using (26) and 7.11(5) we obtain

(30) $$[J_{n+\frac{1}{2}}(z)]^2+[J_{-n-\frac{1}{2}}(z)]^2=[J_{n+\frac{1}{2}}(z)]^2+[Y_{n+\frac{1}{2}}(z)]^2$$

$$=2(\pi z)^{-1}\sum_{m=0}^{n}\frac{(2z)^{2m-2n}(2n-2m)!\,(2n-m)!}{m!\,(n-m)!\,(n-m)!}.$$

The recurrence and differentiation formulas satisfied by $R_{m,\nu}$ may be obtained from (25). For these formulas and also for the proof of Hurwitz's limit

(31) $\lim_{m\to\infty}[(\tfrac{1}{2}z)^{m+\nu}R_{m,\nu+1}(z)/\Gamma(\nu+m+1)]=J_\nu(z)$

see Watson (1944, sections 9.63, 9.65). For other results see Mcdonald (1926).

7.5.3. Anger-Weber functions

Anger's function $\mathbf{J}_\nu(z)$ and Weber's function $\mathbf{E}_\nu(z)$ are defined by integrals of the Bessel type

(32) $\mathbf{J}_\nu(z)\pm i\,\mathbf{E}_\nu(z)=\pi^{-1}\int_0^\pi e^{\pm i(\nu\phi-z\sin\phi)}d\phi.$

Hence, from 7.3(9) and 7.3(10), respectively, follow the expressions

(33) $$\mathbf{J}_\nu(z)=J_\nu(z)+\pi^{-1}\sin(\nu\pi)\int_0^\infty e^{-z\sinh t-\nu t}dt$$

$$=J_\nu(z)+\pi^{-1}\sin(\nu\pi)\int_0^\infty e^{-zv}[v+(1+v^2)^{\frac{1}{2}}]^{-\nu}(1+v^2)^{-\frac{1}{2}}dv$$
$\mathrm{Re}\,z>0,$

(34) $$\mathbf{E}_\nu(z)=-Y_\nu(z)-\pi^{-1}\int_0^\infty(e^{\nu t}+e^{-\nu t}\cos\nu\pi)\,e^{-z\sinh t}dt$$

$$=-Y_\nu(z)-\pi^{-1}\int_0^\infty e^{-zv}\{[v+(1+v^2)^{\frac{1}{2}}]^\nu$$

$$+\cos\nu\pi[v+(1+v^2)^{\frac{1}{2}}]^{-\nu}\}(1+v^2)^{-\frac{1}{2}}dv$$ $\mathrm{Re}\,z>0.$

From (33) is it evident that

(35) $\mathbf{J}_n(z) = J_n(z)$ $\qquad n = 0, \pm 1, \pm 2, \ldots .$

The expansion of the integrand of (32) in powers of z and term by term integration by the aid of 1.5(29) lead to

$$(36)\quad \mathbf{J}_\nu(z) = \cos(\tfrac{1}{2}\nu\pi) \sum_{n=0}^{\infty} \frac{(-1)^n (\tfrac{1}{2}z)^{2n}}{\Gamma(n+1+\tfrac{1}{2}\nu)\,\Gamma(n+1-\tfrac{1}{2}\nu)}$$

$$+ \sin(\tfrac{1}{2}\nu\pi) \sum_{n=0}^{\infty} \frac{(-1)^n (\tfrac{1}{2}z)^{2n+1}}{\Gamma(n+3/2+\tfrac{1}{2}\nu)\,\Gamma(n+3/2-\tfrac{1}{2}\nu)},$$

$$(37)\quad \mathbf{E}_\nu(z) = \sin(\tfrac{1}{2}\nu\pi) \sum_{n=0}^{\infty} \frac{(-1)^n (\tfrac{1}{2}z)^{2n}}{\Gamma(n+1+\tfrac{1}{2}\nu)\,\Gamma(n+1-\tfrac{1}{2}\nu)}$$

$$- \cos(\tfrac{1}{2}\nu\pi) \sum_{n=0}^{\infty} \frac{(-1)^n (\tfrac{1}{2}z)^{2n+1}}{\Gamma(n+3/2+\tfrac{1}{2}\nu)\,\Gamma(n+3/2-\tfrac{1}{2}\nu)}.$$

CONNECTIONS BETWEEN THE ANGER AND THE WEBER FUNCTIONS AND RECURRENCE RELATIONS

From (33) and (34) we have

(38) $\sin(\nu\pi)\,\mathbf{J}_\nu(z) = \cos(\nu\pi)\,\mathbf{E}_\nu(z) - \mathbf{E}_{-\nu}(z),$

(39) $\sin(\nu\pi)\,\mathbf{E}_\nu(z) = \mathbf{J}_{-\nu}(z) - \cos(\nu\pi)\,\mathbf{J}_\nu(z).$

If we differentiate (32), we obtain

$$2\,[\mathbf{J}_\nu'(z) + i\,\mathbf{E}_\nu'(z)] = \pi^{-1} \int_0^{\pi} \{e^{\,i[(\nu-1)\phi - z\sin\phi]} - e^{\,i[(\nu+1)\phi - z\sin\phi]}\}\,d\phi$$

and hence using (32) again

(40) $2\,\mathbf{J}_\nu'(z) = \mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z),$

(41) $2\,\mathbf{E}_\nu'(z) = \mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z).$

In a similar manner, from (32), we derive

(42) $\mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{J}_\nu(z) - 2(\pi z)^{-1}\sin(\nu\pi),$

(43) $\mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = 2\nu z^{-1}\,\mathbf{E}_\nu(z) - 2(\pi z)^{-1}(1 - \cos\nu\pi).$

From (33) and 7.2(1) we find that

$$\mathbf{J}_\nu''(z) + z^{-1}\,\mathbf{J}_\nu'(z) + (1 - \nu^2 z^{-2})\,\mathbf{J}_\nu(z)$$

$$= \pi^{-1} z^{-2} \sin(\nu\pi) \int_0^{\infty} \frac{d}{dt}\,[(-z\cosh t + \nu)\,e^{-z\sinh t - \nu t}]\,dt,$$

and thus it is evident that

(44) $\mathbf{J}_\nu''(z) + z^{-1}\,\mathbf{J}_\nu'(z) + (1 - \nu^2 z^{-2})\,\mathbf{J}_\nu(z) = \pi^{-1} z^{-2}(z - \nu)\sin(\nu\pi)$.

From (44) and (39) we find that

(45) $\mathbf{E}_\nu''(z) + z^{-1}\,\mathbf{E}_\nu'(z) + (1 - \nu^2 z^{-2})\,\mathbf{E}_\nu(z)$
$= -\pi^{-1} z^{-2}[z + \nu + (z - \nu)\cos(\nu\pi)]$

ASYMPTOTIC EXPANSIONS

The asymptotic expansion of $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$ for large z and fixed ν may easily be obtained by Watson's lemma. We substitute

(46) $[v + (1 + v^2)^{1/2}]^\nu (1 + v^2)^{-1/2} = {}_2F_1(\tfrac{1}{2} + \tfrac{1}{2}\nu, \tfrac{1}{2} - \tfrac{1}{2}\nu; \tfrac{1}{2}; -v^2)$
$+ \nu v\, {}_2F_1(1 + \tfrac{1}{2}\nu, 1 - \tfrac{1}{2}\nu; 3/2; -v^2)$

in (33) and (34) respectively, use 2.1(2), 1.1(5), and obtain

(47) $\mathbf{J}_\nu(z) = J_\nu(z) + (\pi z)^{-1}\sin(\nu\pi)\,[\sum_{n=0}^{M-1}(-1)^n 2^{2n}(\tfrac{1}{2} + \tfrac{1}{2}\nu)_n(\tfrac{1}{2} - \tfrac{1}{2}\nu)_n z^{-2n}$
$+ O(|z|^{-2M}) + \nu\sum_{n=0}^{M-1}(-1)^n 2^{2n}(1 + \tfrac{1}{2}\nu)_n(1 - \tfrac{1}{2}\nu)_n z^{-2n-1}$
$+ \nu\, O(|z|^{-2M-1})]$,

(48) $\mathbf{E}_\nu(z) = -Y_\nu(z) - (\pi z)^{-1}(1 + \cos\nu\pi)$
$\times[\sum_{n=0}^{M-1}(-1)^n 2^{2n}(\tfrac{1}{2} + \tfrac{1}{2}\nu)_n(\tfrac{1}{2} - \tfrac{1}{2}\nu)_n z^{-2n} + O(|z|^{-2M})]$
$-\nu(\pi z^{-1})(1 - \cos\nu\pi)[\sum_{n=0}^{M-1}(-1)^n 2^{2n}(1+\tfrac{1}{2}\nu)_n(1-\tfrac{1}{2}\nu)_n z^{-2n-1}$
$+ O(|z|^{-2M-1})]$.

For the asymptotic expansion of $J_\nu(z)$ and $Y_\nu(z)$ in (47) and (48) respectively see 7.13(3) and 7.13(4).

The case of large $|\nu|$ and $|z|$ is discussed in Watson (1944, p. 316).

7.5.4. Struve's functions

Struve's function is defined by a representation similar to Poisson's integral 7.3(3)

(49) $\Gamma(\nu + \tfrac{1}{2})\,\mathbf{H}_\nu(z) = 2\pi^{-1/2}(\tfrac{1}{2}z)^\nu \int_0^1 (1 - t^2)^{\nu - 1/2}\sin(zt)\,dt$
$= 2\pi^{-1/2}(\tfrac{1}{2}z)^\nu \int_0^{\frac{1}{2}\pi}\sin(z\cos\phi)(\sin\phi)^{2\nu}\,d\phi \qquad \operatorname{Re}\nu > -\tfrac{1}{2}$.

From this expression it may be shown (Watson, 1944, p. 337) that $\mathbf{H}_\nu(x)$ is positive when x is positive and $\nu \geq \frac{1}{2}$.

If (49) is transformed into a loop integral, the restriction on ν may be removed and we have

$$(50)\quad \mathbf{H}_\nu(z) = -i\pi^{-3/2}\,\Gamma(\tfrac{1}{2}-\nu)\,(\tfrac{1}{2}z)^\nu \int_0^{(1+)} (t^2-1)^{\nu-\frac{1}{2}} \sin(zt)\,dt \qquad \nu \neq 1/2, 3/2, 5/2, \ldots .$$

A further representation follows from 7.2(12)

$$(51)\quad \Gamma(\nu+\tfrac{1}{2})[\mathbf{H}_\nu(\xi z) - Y_\nu(\xi z)] = \pi^{-\frac{1}{2}}(\tfrac{1}{2}\xi)^{\nu-1} z^\nu \times \int_0^{\infty e^{i\beta}} e^{-zt}(1+t^2\xi^{-2})^{\nu-\frac{1}{2}}\,dt$$
$$\beta - \tfrac{1}{2}\pi < \arg\xi < \beta + \tfrac{1}{2}\pi; \quad -\tfrac{1}{2}\pi - \beta < \arg z < \tfrac{1}{2}\pi - \beta.$$

(For other integral representations cf. Meijer, 1935 a, p. 628, 744; 1939; 1940, p. 198, 366; Nielsen, 1904, p. 234).

The modified Struve function is

$$(52)\quad \mathbf{L}_\nu(z) = -ie^{-i\frac{1}{2}\nu\pi}\,\mathbf{H}_\nu(ze^{i\frac{1}{2}\pi})$$

Hence we have from (49)

$$(53)\quad \mathbf{L}_\nu(z)\,\Gamma(\nu+\tfrac{1}{2}) = 2\pi^{-\frac{1}{2}}(\tfrac{1}{2}z)^\nu \int_0^{\frac{1}{2}\pi} \sinh(z\cos\phi)\,(\sin\phi)^{2\nu}\,d\phi \qquad \mathrm{Re}\,\nu > -\tfrac{1}{2}.$$

From (51) we have

$$(54)\quad \mathbf{L}_\nu(x) = I_{-\nu}(x) - \frac{2\pi^{-\frac{1}{2}}(\tfrac{1}{2}x)^\nu}{\Gamma(\nu+\tfrac{1}{2})} \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} \sin(xt)\,dt \qquad x > 0, \quad \mathrm{Re}\,\nu < \tfrac{1}{2}.$$

A representation of $\mathbf{H}_\nu(z)$ as a series of ascending powers of z is obtained from (49) by expanding $\sin(z\cos\phi)$ in powers of z

$$(55)\quad \mathbf{H}_\nu(z) = \sum_{m=0}^\infty (-1)^m (\tfrac{1}{2}z)^{\nu+2m+1}/[\Gamma(m+3/2)\,\Gamma(\nu+m+3/2)]$$
$$= 2\pi^{-\frac{1}{2}}(\tfrac{1}{2}z)^{\nu+1}\,{}_1F_2(1;\,3/2+\nu,\,3/2;\,-\tfrac{1}{4}z^2)/\Gamma(\nu+3/2).$$

Hence it is evident that $(\tfrac{1}{2}z)^{-\nu}\mathbf{H}_\nu(z)$ is an entire function of ν and z. Furthermore we have

$$(56)\quad \mathbf{H}_\nu(ze^{im\pi}) = e^{i\pi(\nu+1)m}\,\mathbf{H}_\nu(z) \qquad m = 1, 2, 3, \ldots .$$

From (52) we obtain

$$(57)\quad \mathbf{L}_\nu(z) = \sum_{m=0}^\infty (\tfrac{1}{2}z)^{\nu+2m+1}/[\Gamma(m+3/2)\,\Gamma(\nu+m+3/2)]$$

$$= 2\pi^{-1/2} (\tfrac{1}{2}z)^{\nu+1} \, {}_1F_2(1; 3/2+\nu, 3/2; \tfrac{1}{4}z^2)/\Gamma(\nu+3/2).$$

From (55) we easily obtain the differentiation formulas

(58) $$\frac{d}{dz}[z^{\nu} \mathbf{H}_\nu(z)] = z^{\nu} \mathbf{H}_{\nu-1}(z),$$

(59) $$\frac{d}{dz}[z^{-\nu} \mathbf{H}_\nu(z)] = 2^{-\nu} \pi^{-1/2}/\Gamma(\nu+3/2) - z^{-\nu} \mathbf{H}_{\nu+1}(z).$$

Carrying out the differentiation on the left-hand sides of (58) and (59) and comparing the results we find that

(60) $$\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{H}_\nu(z) + \pi^{-1/2} (\tfrac{1}{2}z)^{\nu}/\Gamma(\nu+3/2),$$

(61) $$\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}_\nu'(z) - \pi^{-1/2} (\tfrac{1}{2}z)^{\nu}/\Gamma(\nu+3/2).$$

From (58) and (59) it follows that the Struve function satisfies the differential equation

(62) $$z^2 \mathbf{H}_\nu''(z) + z\mathbf{H}_\nu'(z) + (z^2-\nu^2)\mathbf{H}_\nu(z) = \pi^{-1/2} (\tfrac{1}{2}z)^{\nu-1}/\Gamma(\nu+\tfrac{1}{2}).$$

ASYMPTOTIC REPRESENTATIONS

In (51) we put $z = 1$, expand $(1 + t^2 \xi^{-2})^{\nu-1/2}$ into a series of ascending powers of t, integrate term by term and obtain for large ξ and fixed ν

(63) $$\mathbf{H}_\nu(\xi) = Y_\nu(\xi) + \pi^{-1} \sum_{m=0}^{M-1} [\Gamma(m+\tfrac{1}{2}) (\tfrac{1}{2}\xi)^{-2m+\nu-1}/\Gamma(\nu+\tfrac{1}{2}-m)] + O(|\xi|^{\nu-2M-1}) \qquad |\arg \xi| < \pi.$$

For the asymptotic expansion of $Y_\nu(\xi)$ see 7.13(4). Furthermore it may be proved that if ν is real and $\xi > 0$, the remainder after M terms is of the same sign as, and numerically less than, the first neglected term, provided $M + \tfrac{1}{2} - \nu \geq 0$.

For the case of large $|\nu|$ and $|\xi|$ see Watson (1944, p. 333).

If $\nu = n + \tfrac{1}{2}$ $(n = 0, 1, 2, \ldots,)$, then $(1 + t^2 \xi^{-2})^{\nu-1/2}$ in (51) is a polynomial, and we have

(64) $$\mathbf{H}_{n+1/2}(\xi) = Y_{n+1/2}(\xi) + \pi^{-1} \sum_{m=0}^{n} (\tfrac{1}{2}\xi)^{-2m+n-1/2} \Gamma(m+\tfrac{1}{2})/\Gamma(n+1-m).$$

$Y_{n+1/2}(\xi)$ is given by 7.11(2). Furthermore from (51) and (54) we obtain

(65) $$\mathbf{H}_{-(n+1/2)}(z) = (-1)^n J_{n+1/2}(z); \qquad \mathbf{L}_{-(n+1/2)}(z) = I_{n+1/2}(z) \qquad n = 0, 1, 2, \ldots.$$

For $n = 0$ we obtain from (64)

$$\mathbf{H}_{1/2}(z) = (\tfrac{1}{2}\pi z)^{-1/2}(1 - \cos z).$$

When n is a positive integer we may deduce from (37) and (55) (Watson, 1944, p. 337)

(66) $$\mathbf{H}_n(z) = \pi^{-1} \sum_{m=0}^{<\frac{1}{2}n} \Gamma(m+\tfrac{1}{2})(\tfrac{1}{2}z)^{n-2m-1}/\Gamma(n+\tfrac{1}{2}-m) - \mathbf{E}_n(z),$$

(67) $$\mathbf{H}_{-n}(z) = (-1)^{n+1}\pi^{-1} \sum_{m=0}^{<\frac{1}{2}n} \Gamma(n-m-\tfrac{1}{2})(\tfrac{1}{2}z)^{-n+2m+1}/\Gamma(m+3/2) - \mathbf{E}_{-n}(z).$$

For further results concerning Struve functions see Baudoux (1946).

7.5.5. Lommel's functions

We consider the inhomogeneous Bessel differential equation

(68) $$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2-\nu^2)w = z^{\mu+1},$$

μ, ν being unrestricted constants. A solution of (68) is

(69) $$s_{\mu,\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2-\nu^2][(\mu+3)^2-\nu^2]\cdots[(\mu+2m+1)^2-\nu^2]}$$
$$= z^{\mu-1}\sum_{m=0}^{\infty} \frac{(-1)^m(\tfrac{1}{2}z)^{2m+2}\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2})\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+m+3/2)\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m+3/2)}$$
$$= \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} \times {}_1F_2(1; \tfrac{1}{2}\mu-\tfrac{1}{2}\nu+3/2; \tfrac{1}{2}\mu+\tfrac{1}{2}\nu+3/2; -\tfrac{1}{4}z^2).$$

The solution (69) becomes nugatory when one of the numbers $\mu \pm \nu$ is an odd integer.

If the differential equation (68) is integrated by the method of variation of parameters and if that solution is determined which is approximately $[(\mu-\nu+1)(\mu+\nu+1)]^{-1} z^{\mu+1}$ for small z, one finds

(70) $$s_{\mu,\nu}(z) = \tfrac{1}{2}\pi(\sin\nu\pi)^{-1}\left[J_\nu(z)\int_0^z z^\mu J_{-\nu}(z)\,dz - J_{-\nu}(z)\int_0^z z^\mu J_\nu(z)\,dz\right]$$
$$= \tfrac{1}{2}\pi\left[Y_\nu(z)\int_0^z z^\mu J_\nu(z)\,dz - J_\nu(z)\int_0^z z^\mu Y_\nu(z)\,dz\right].$$

The two expressions in (70) for $s_{\mu,\nu}$ are identical when ν is not an integer. When ν is an integer, the former expression is not defined, but the latter is still valid.

Another particular integral of (68) is

(71) $$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + [2^{\mu-1}\Gamma(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2})\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2})/\sin(\nu\pi)]$$

$$\times \{\cos[\tfrac{1}{2}(\mu-\nu)\pi]\, J_{-\nu}(z) - \cos[\tfrac{1}{2}(\mu+\nu)\pi]\, J_{\nu}(z)\}$$

$$= s_{\mu,\nu}(z) + 2^{\mu-1}\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2})\,\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2})$$

$$\times \{\sin[\tfrac{1}{2}(\mu-\nu)\pi]\, J_{\nu}(z) - \cos[\tfrac{1}{2}(\mu-\nu)\pi]\, Y_{\nu}(z)\}.$$

When either of the numbers $\mu \pm \nu$ is an odd positive integer, $S_{\mu,\nu}$ may be represented by the following terminating series in descending powers of z (cf. Watson, 1944, p. 347):

(72) $$S_{\mu,\nu}(z) = z^{\mu-1}\{1 - [(\mu-1)^2 - \nu^2]\, z^{-2} + [(\mu-1)^2 - \nu^2]\,[(\mu-3)^2 - \nu^2]\, z^{-4} - \cdots\}.$$

In case $\mu + \nu$ or $\mu - \nu$ is an odd negative integer, $s_{\mu,\nu}$ in undefined, but $S_{\mu,\nu}(z)$ approaches a limit (Watson, 1944, p. 348).

RECURRENCE RELATIONS

From the definitions we have

(73) $$s_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu+1)^2 - \nu^2]\, s_{\mu,\nu}(z),$$

(74) $$s'_{\mu,\nu}(z) + (\nu/z)\, s_{\mu,\nu}(z) = (\mu+\nu-1)\, s_{\mu-1,\nu-1}(z),$$

(75) $$s'_{\mu,\nu}(z) - (\nu/z)\, s_{\mu,\nu}(z) = (\mu-\nu-1)\, s_{\mu-1,\nu+1}(z),$$

(76) $$(2\nu/z)\, s_{\mu,\nu}(z) = (\mu+\nu-1)\, s_{\mu-1,\nu-1}(z) - (\mu-\nu-1)\, s_{\mu-1,\nu+1}(z),$$

(77) $$2 s'_{\mu,\nu}(z) = (\mu+\nu-1)\, s_{\mu-1,\nu-1}(z) + (\mu-\nu-1)\, s_{\mu-1,\nu+1}(z).$$

From (71) it follows that the same relations are valid if in (73) to (77) $s_{\mu,\nu}(z)$ is replaced by $S_{\mu,\nu}(z)$.

SPECIAL CASES OF LOMMEL'S FUNCTIONS

Several of the functions associated with Bessel functions can be expressed in terms of Lommel's functions.

(78) $$O_{2n}(z) = z^{-1} S_{1,2n}(z), \quad O_{2n+1}(z) = (2n+1)\, z^{-1} S_{0,2n+1}(z),$$

(79) $$S_{2n}(z) = 4n\, S_{-1,2n}(z), \quad S_{2n+1}(z) = 2\, S_{0,2n+1}(z),$$

(80) $$A_{2n,\nu}(z) = 2^{\nu} z^{\nu-1}\Gamma(\nu+n)\,(\nu+2n)\, S_{1-\nu,\nu+2n}(z)/n\,!,$$

(81) $$A_{2n+1,\nu}(z) = 2^{\nu+1} z^{\nu-1}\Gamma(\nu+n+1)\,(\nu+2n+1)\, S_{-\nu,\nu+2n+1}(z)/n\,!,$$

(82) $$\mathbf{J}_{\nu}(z) = \sin(\nu\pi)\, s_{0,\nu}(z)/\pi - \nu\sin(\nu\pi)\, s_{1,\nu}(z),$$ see errata!

(83) $\mathbf{E}_\nu(z) = -(1+\cos\nu\pi)\, s_{0,\nu}(z)/\pi - \nu(1-\cos\nu\pi)\, s_{-1,\nu}(z)/\pi,$

(84) $\mathbf{H}_\nu(z) = 2^{1-\nu}\pi^{-1/2}\, s_{\nu,\nu}(z)/\Gamma(\nu+\tfrac{1}{2})$
$\qquad = Y_\nu(z) + 2^{1-\nu}\pi^{-1/2}\, S_{\nu,\nu}(z)/\Gamma(\nu+\tfrac{1}{2}).$

where the notations introduced in sec. 7.5 have been used.

Young's function (1912) is

(85) $C_\nu(t) = \sum_{m=0}^{\infty} (-1)^m z^{\nu+2m}/\Gamma(\nu+2m+1) = z^{1/2}\, s_{\nu-3/2,\,1/2}(z)/\Gamma(\nu-1).$

ASYMPTOTIC EXPANSION

The series (72) diverges in general, but it can be shown (Watson, 1944, p. 351) to be an asymptotic expansion of $S_{\mu,\nu}(z)$ when $|z|$ is large and $|\arg z| < \pi$.

INTEGRAL REPRESENTATIONS

The integral representation

(86) $s_{\mu,\nu}(z) = 2^{\mu}(\tfrac{1}{2}z)^{\frac{1}{2}(1+\nu+\mu)}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu)$
$\qquad \times \int_0^{\frac{1}{2}\pi} J_{\frac{1}{2}(1+\mu-\nu)}(z\sin\theta)\,(\sin\theta)^{\frac{1}{2}(1+\nu-\mu)}(\cos\theta)^{\nu+\mu}\,d\theta$

$\operatorname{Re}(\nu+\mu+1) > 0$

may be verified by expansion in ascending powers of z. For further integral representations see formulas 7.12(48) to 7.12(52) and Szymanski (1935); Meijer (1935a, 1938, 1939a, 1940, pp. 198,366).

Lommel has also investigated functions of two variables defined as

(87) $U_\nu(w,z) = \sum_{m=0}^{\infty} (-1)^m (w/z)^{\nu+2m} J_{\nu+2m}(z),$

(88) $V_\nu(w,z) = \cos(\tfrac{1}{2}w + \tfrac{1}{2}z^2/w + \tfrac{1}{2}\nu\pi) + U_{-\nu+2}(w,z).$

For the theory of these see Watson (1944, sections 16.5 to 16.59); see also Shastri (1938).

7.5.6. Some other notations and related functions

In Nielsen's *Handbuch der Theorie der Zylinderfunktionen*, some notations (for a list of those see Nielsen's book, p. 406) different from those introduced in sec. 7.5 are used. These notations are

$Z^\nu(z) = \mathbf{H}_\nu(z), \quad \Psi^\nu(z) = \mathbf{J}_\nu(z), \quad \Omega^\nu(z) = -\mathbf{E}_\nu(z),$

$\Pi^{\nu,\rho}(z) = 2^{2-\rho}\cos[\tfrac{1}{2}\pi(\nu-\rho)]\, s_{\rho-1,\nu}(z)/[\Gamma(\tfrac{1}{2}\rho-\tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\rho+\tfrac{1}{2}\nu)].$

Furthermore the following functions are investigated there:

$$\Pi^{\nu}(z) = \tfrac{1}{2}[\mathbf{J}_{\nu}(z) + \mathbf{J}_{-\nu}(z)], \qquad \mathrm{X}^{\nu}(z) = \tfrac{1}{2}[\mathbf{J}_{\nu}(z) - \mathbf{J}_{-\nu}(z)],$$

$$\pi \Phi^{\nu}(z) = i^{\nu} \int_0^{\pi} e^{iz\cos\phi} \cos(\nu\phi)\, d\phi,$$

$$\pi \Lambda^{\nu}(z) = i^{1-\nu} \int_0^{\pi} e^{iz\cos\phi} \sin(\nu\phi)\, d\phi.$$

The last two functions are generalizations of Hansen's integral see 7.12(2) for the Bessel coefficients. [See also formulas 7.12(40) to 7.12(45).]

7.6. Addition theorems

There are two types of expansions of Bessel functions which are known as addition theorems. Roughly speaking, Gegenbauer's type is connected with the theory of spherical wave functions (in $2\nu + 2$ dimensions), while Graf's type is more nearly related to the theory of cylindrical waves. This description is not quite accurate, and the two types coincide when $\nu = 0$. As a matter of fact these two types are developed as two different generalizations of Neumann's addition theorem for J_0.

7.6.1. Gegenbauer's addition theorem

Gegenbauer's addition theorem will be established for the modified Bessel function of the third kind, $K_{\nu}(z)$. We put

(1) $$w = (z^2 + Z^2 - 2zZ\cos\phi)^{1/2} = [(Z - ze^{-i\phi})(Z - ze^{i\phi})]^{1/2}$$

and assume at first that z, Z, ϕ, are real and $0 < z < Z$. With $z = 1$ and $a = w$ in 7.12(23) we have

(2) $$w^{-\nu} K_{\nu}(w) = \tfrac{1}{2} \int_0^{\infty} \exp[-t - (z^2 + Z^2 - 2z\,Z\cos\phi)/t]\, t^{-\nu-1}\, dt.$$

If $\nu \neq 0$, we use Sonine's expansion 7.10(5)

$$\exp[t^{-1} zZ\cos\phi]$$

$$= [2t/(zZ)]^{\nu}\, \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n)\, C_n^{\nu}(\cos\phi)\, I_{\nu+n}(zZ/t),$$

substitute in (2), and integrate term by term using here 7.7(37) in the process. Thus we obtain the addition theorem (for the C_n^{ν} see sec. 3.15),

(3) $$w^{-\nu} K_{\nu}(w) = (\tfrac{1}{2} zZ)^{-\nu}\, \Gamma(\nu) \times \sum_{n=0}^{\infty} (\nu + n)\, C_n^{\nu}(\cos\phi)\, I_{\nu+n}(z)\, K_{\nu+n}(Z)$$

$$\nu \neq 0, -1, -2, \dots, \qquad z < Z.$$

If we make ν tend to zero, we obtain, using 3.15(14),

$$(4)\quad K_0(w) = I_0(z)\,K_0(z) + 2\sum_{n=1}^{\infty} I_n(z)\,K_n(Z)\cos n\phi \qquad z < Z.$$

It follows from 7.2(12) and 7.2(13) that the series (3) converges like $\Sigma\, C_n^\nu(\cos\phi)\,(z/Z)^n$ and therefore from 3.15(1) that (3) and (4) hold, provided that $|ze^{\pm i\phi}| < |Z|$.

The addition theorems for the other Bessel functions follow from (3) by means of 7.2(16), 7.2(17), 7.2(7), and 7.2(8). Also see 7.15(28) to 7.15(32).

7.6.2. Graf's addition theorem

Graf's addition formula

$$(5)\quad J_\nu(w)\left(\frac{Z - ze^{-i\phi}}{Z - ze^{i\phi}}\right)^{\frac{1}{2}\nu} = \sum_{n=-\infty}^{\infty} J_{\nu+n}(Z)\,J_n(z)\,e^{in\phi},$$

where we have

$$|ze^{\pm i\phi}| < |Z|, \qquad w = (z^2 + Z^2 - 2zZ\cos\phi)^{\frac{1}{2}} = [(Z - ze^{-i\phi})(Z - ze^{i\phi})]^{\frac{1}{2}}$$

may be proved as follows. From 7.3(5) we obtain

$$(2\pi i)\,J_{\nu+n}(Z)\,J_n(z)\,e^{in\phi} = \int_{-\infty\exp(-i\beta)}^{(0+)} e^{\frac{1}{2}Z(t-t^{-1})}\,t^{-\nu-1}(e^{i\phi}/t)^n J_n(z)\,dt.$$

From 7.2(25) we have

$$(2\pi i)\sum_{n=-\infty}^{\infty} J_{\nu+n}(Z)\,J_n(z)\,e^{in\phi}$$
$$= \int_{-\infty\exp(-i\beta)}^{(0+)} \exp[\tfrac{1}{2}Z(t-t^{-1}) - \tfrac{1}{2}z(te^{-i\phi} - t^{-1}e^{i\phi})]\,t^{-\nu-1}\,dt.$$

Now we put $(Z - ze^{-i\phi})\,t = wv$, $(Z - ze^{i\phi})/t = w/v$ and take that value of the square root (1) which makes $w \to +Z$ when $z \to 0$. We then may take the contour to start from and end at $-\infty\exp(-i\alpha)$ where $\alpha = \arg w$. Thus we have

$$(2\pi i)\sum_{n=-\infty}^{\infty} J_{\nu+n}(Z)\,J_n(z)\,e^{in\phi} = w^{-\nu}(Z - ze^{-i\phi})^\nu$$
$$\times\int_{-\infty\exp(-i\alpha)}^{(0+)} \exp[\tfrac{1}{2}w(v - v^{-1})]\,v^{-\nu-1}\,dv.$$

Using 7.3(5) again we obtain (5).

Formula (5) may be written in a slightly different manner, introducing an angle ψ by means of the equations

$$Z - z\cos\phi = w\cos\psi, \qquad z\sin\phi = w\sin\psi,$$

so that in case of real ϕ and positive z, Z and w, ψ is the angle opposite to z in the triangle with the sides z, Z, w. We then have

$$(6)\quad e^{i\nu\psi} J_\nu(w) = \sum_{n=-\infty}^{\infty} J_{\nu+n}(Z)\, J_n(z)\, e^{in\phi}$$

$$|ze^{\pm i\phi}| < |Z| \quad \text{in case} \quad \nu \neq 0, \pm 1, \pm 2, \ldots .$$

For the other Bessel functions see formulas 7.15(33) to 7.15(36).

A duplication formula for the Bessel function of the first kind and for the modified Hankel function in case the order is half an odd integer has been derived by Cooke (1930). The results are

$$(7)\quad J_{m+\frac{1}{2}}(2z) = (-1)^m \pi^{\frac{1}{2}}\, m!\, z^{m+\frac{1}{2}}$$

$$\times \sum_{n=0}^{m} (-1)^n (2m-2n+1) J_{m-n+\frac{1}{2}}(z)\, J_{-(m-n+\frac{1}{2})}(z)/[n!(2m-n+1)!].$$

$$(8)\quad K_{m+\frac{1}{2}}(2z) = \pi^{-\frac{1}{2}}\, m!\, z^{m+\frac{1}{2}} \sum_{n=0}^{m} \frac{(-1)^n (2m-2n+1)[K_{m-n+\frac{1}{2}}(z)]^2}{n!\,(2m-n+1)!}$$

For other similar formulas see Cooke (1930).

7.7. Integral formulas

7.7.1. Indefinite integrals

From formulas 7.2(52) and 7.2(53), respectively, we have

$$(1)\quad \int z^{\nu+1} J_\nu(z)\, dz = z^{\nu+1} J_{\nu+1}(z),$$

$$(2)\quad \int z^{-\nu+1} J_\nu(z)\, dz = -z^{-\nu+1} J_{\nu-1}(z).$$

From sec. 7.2(57) we obtain

$$(3)\quad \int J_\nu(z)\, dz = 2 \sum_{n=0}^{m-1} J_{\nu+2n+1}(z) + \int J_{\nu+2m}(z)\, dz \qquad m = 1, 2, 3, \ldots .$$

Equations 7.2(4) to 7.2(6) show that (1) to (3) are valid for $Y_\nu(z)$ and $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$. For similar formulas see 7.14(1) to 7.14(13).

7.7.2. Finite integrals

Many definite integrals involving Bessel functions are of the convolution type

$$F * G(t) = \int_0^t F(v)\, G(t-v)\, dv$$

and may be evaluated by means of the convolution formula of the Laplace transform (Doetsch, 1937, p. 161; Widder, 1941, p. 84). According to this method if

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \equiv L\{F\}$$

and

$$g(s) = L\{G\}, \quad \text{then} \quad f(s)\, g(s) = L\{F * G\}.$$

The formula is valid for example if $L\{F\}$ and $L\{G\}$ are absolutely convergent.

As an example we shall prove Sonine's second integral in this manner. With

$$F_\mu(a, t) = a^{-\mu} t^{\frac{1}{2}\mu} J_\mu(a t^{\frac{1}{2}})$$

we have from (24) for Re $\mu > -1$

$$f_\mu(a, s) = L\{F_\mu(a, t)\} = 2^{-\mu} s^{-\mu-1} \exp(-\tfrac{1}{4} a^2 s^{-1})$$

Now we have

$$f_\mu(a, s)\, f_\nu(\beta, s) = 2 f_{\mu+\nu+1}[(a^2 + \beta^2)^{\frac{1}{2}}, s]$$

and this leads to Sonine's integral

$$\int_0^t \tau^{\frac{1}{2}\mu} J_\mu(a\tau^{\frac{1}{2}})\, (t-\tau)^{\frac{1}{2}\nu} J_\nu[\beta(t-\tau)^{\frac{1}{2}}]\, d\tau$$

$$= 2\ a^\mu \beta^\nu (a^2 + \beta^2)^{-\frac{1}{2}(\nu+\mu+1)} J_{\nu+\mu+1}\,[t(a^2 + \beta^2)^{\frac{1}{2}}]$$

$$\text{Re}\,\nu > -1, \quad \text{Re}\,\mu > -1.$$

Putting $t = 1$ and substituting $\tau = (\sin\theta)^2$ we obtain

(4) $$\int_0^{\frac{1}{2}\pi} J_\mu(a \sin\theta)\, J_\nu(\beta \cos\theta)\, (\sin\theta)^{\mu+1} (\cos\theta)^{\nu+1}\, d\theta$$

$$= a^\mu \beta^\nu (a^2 + \beta^2)^{-\frac{1}{2}(\nu+\mu+1)} J_{\nu+\mu+1}[(a^2 + \beta^2)^{\frac{1}{2}}]$$

$$\text{Re}\,\nu > -1, \quad \text{Re}\,\mu > -1.$$

A limiting case of (4) may be mentioned separately. If we divide both sides of (4) by β^ν and let $\beta \to 0$, we obtain Sonine's first integral

(5) $$\int_0^{\frac{1}{2}\pi} J_\mu(a \sin\theta)\, (\sin\theta)^{\mu+1} (\cos\theta)^{2\rho+1}\, d\theta$$

$$= 2^\rho \Gamma(\rho + 1)\, a^{-\rho-1} J_{\rho+\mu+1}(a) \qquad \text{Re}\,\rho > -1, \quad \text{Re}\,\mu > -1.$$

Other formulas of the convolution type are

(6) $$\int_0^t \tau^\mu J_\mu(\tau)\, (t-\tau)^\nu J_\nu(t-\tau)\, d\tau$$

$$= (2\pi)^{-\frac{1}{2}} \Gamma(\nu + \tfrac{1}{2}) \Gamma(\mu + \tfrac{1}{2}) t^{\nu+\mu+\frac{1}{2}} J_{\nu+\mu+\frac{1}{2}}(t)/\Gamma(\nu + \mu + 1)$$

$$\text{Re}\,\mu > -\tfrac{1}{2}, \quad \text{Re}\,\nu > -\tfrac{1}{2},$$

(see Hardy, 1921, p. 169) and

(7) $\int_0^t \tau^{-1/2} J_{2\nu}(a\tau^{1/2})(t-\tau)^{-1/2} \cos[\beta(t-\tau)^{1/2}]\, d\tau$

$$= \pi J_\nu\{\tfrac{1}{2}t^{1/2}[(a^2+\beta^2)^{1/2}+\beta]\} J_\nu\{\tfrac{1}{2}t^{1/2}[(a^2+\beta^2)^{1/2}-\beta]\}$$

Re $\nu > -\frac{1}{2}$,

which may be written as

(8) $\int_0^{\frac{1}{2}\pi} J_{2\nu}[2(z\zeta)^{1/2}\sin\theta]\cos[(z-\zeta)\cos\theta]\,d\theta = \tfrac{1}{2}\pi J_\nu(z) J_\nu(\zeta)$

Re $\nu > -\frac{1}{2}$.

Formulas (6) and (7) result from the convolution theorem in connection with (17) and (23), (25) respectively.

An integral formula involving Struve's function, corresponding to Sonine's first integral (5), is

(9) $\int_0^{\frac{1}{2}\pi} \mathbf{H}_\mu(z\sin\theta)(\sin\theta)^{\mu+1}(\cos\theta)^{2\rho+1}\,d\theta$

$$= \Gamma(\rho+1)\, 2^\rho z^{-\rho-1}\, \mathbf{H}_{\rho+\mu+1}(z) \qquad \text{Re}\,\rho > -1, \quad \text{Re}\,\mu > -3/2,$$

and may be established as follows. We expand the Struve function under the integral sign according to 7.5(55) and integrate term by term using 1.5(19).

In many cases the representations 7.2(47) to 7.2(49) of a product of two Bessel functions as a power series may be used for the evaluation of integrals involving Bessel functions. For instance we have from 7.2(2) and 1.5(19)

$$\int_0^{\frac{1}{2}\pi} J_\nu(2z\sin\theta)(\sin\theta)^\nu(\cos\theta)^{2\nu}\,d\theta$$
$$= \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-1)^m z^{\nu+2m}\,\Gamma(\nu+m+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}{m!\,\Gamma(\nu+m+1)\,\Gamma(2\nu+m+1)}$$

and by 7.2(49) this leads to the result

(10) $\int_0^{\frac{1}{2}\pi} J_\nu(2z\sin\theta)(\sin\theta)^\nu(\cos\theta)^{2\nu}\,d\theta$

$$= \tfrac{1}{2} z^{-\nu}\pi^{1/2}\,\Gamma(\nu+\tfrac{1}{2})\,[J_\nu(z)]^2 \qquad \text{Re}\,\nu > -\tfrac{1}{2}.$$

Similarly we prove Neumann's formula

(11) $\int_0^{\frac{1}{2}\pi} J_{\nu+\mu}(2z\cos\theta)\cos[(\mu-\nu)\theta]\,d\theta = \tfrac{1}{2}\pi J_\nu(z) J_\mu(z)$

Re $(\nu+\mu) > -1$

in the proof of which we use formulas 7.2(2), 1.5(19) and 7.2(49).

A generalization of Neumann's formula

(12) $\pi(2\alpha z)^{-\mu}(2\beta z)^{-\nu} J_\mu(\alpha z) J_\nu(\beta z)$

$$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{i\theta(\mu-\nu)} (\cos\theta)^{\nu+\mu} (\lambda z)^{-\nu-\mu} J_{\nu+\mu}(\lambda z)\, d\theta$$

$$\operatorname{Re}(\nu+\mu) > -1, \quad \lambda = [2\cos\theta(\alpha^2 e^{i\theta} + \beta^2 e^{-i\theta})]^{\frac{1}{2}}$$

may be proved as follows. We expand the Bessel function under the integral sign according to 7.2(2) and obtain

$$\pi(\alpha z)^{-\mu}(\beta z)^{-\nu} J_\mu(\alpha z) J_\nu(\beta z)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\, 2^{-m}\, z^{2m}}{m!\,\Gamma(m+\nu+\mu+1)}$$

$$\times \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{i\theta(\mu-\nu)} (\cos\theta)^{m+\nu+\mu} (\alpha^2 e^{i\theta} + \beta^2 e^{-i\theta})^m\, d\theta.$$

But the integral is expressible as a hypergeometric function ${}_2F_1$ [also compare 2.4(11)] and by 7.2(47) the truth of (12) is obvious. [For a related representation see 7.14(60).]

Another class of integral formulas may be derived from the addition theorem in sections 7.6 and 7.15. From 7.15(31) we have

(13) $$\pi[J_n(z)]^2 = \int_0^\pi J_0(2z\sin\phi)\cos(2n\phi)\, d\phi \qquad n = 0, 1, 2, \ldots,$$

or, more generally, if Z_ν denotes any Bessel function of the first, second, or third kind, we obtain from formulas 7.15(28), 7.15(29), and 3.15(17)

(14) $$\int_0^\pi w^{-\nu} Z_\nu(w)\, C_m^\nu(\cos\phi)\,(\sin\phi)^{2\nu}\, d\phi$$

$$= 2\pi\,\Gamma(m+2\nu)\,(2zy)^{-\nu} Z_{\nu+m}(y)\, J_{\nu+m}(z)/[m!\,\Gamma(\nu)]$$

$$w = (z^2 + y^2 - 2zy\cos\phi)^{\frac{1}{2}}, \quad \operatorname{Re}\nu > -\tfrac{1}{2}, \quad m = 0, 1, 2, \ldots.$$

For other formulas of a similar type see formulas 7.14(14) to 7.14(23) and Watson (1944, p. 373); Copson (1932); Rutgers (1941); B. N. Bose (1948); MacRobert (1947, p. 383).

7.7.3. Infinite integrals with exponential functions

The formula

(15) $$2^{\nu+\mu}\alpha^{-\mu}\beta^{-\nu}\gamma^{\lambda+\mu+\nu}\Gamma(\nu+1)\int_0^\infty J_\mu(\alpha t)\, J_\nu(\beta t)\, e^{-\gamma t}\, t^{\lambda-1}\, dt$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\,\Gamma(\mu+m+1)}$$

$$\times\ {}_2F_1(-m, -\mu-m; \nu+1; \beta^2\alpha^{-2})\,(-\tfrac{1}{4}\alpha^2\gamma^{-2})^m$$

$$\operatorname{Re}(\lambda+\mu+\nu) > 0, \quad \operatorname{Re}(\gamma \pm i\alpha \pm i\beta) > 0$$

may be proved by replacing the Bessel function product by its power series expansion 7.2(47), then integrating term by term, and also using 1.1(5). In some special cases the right-hand side of (15) reduces to simpler expressions. If, for example, we put $\lambda + \nu = \rho$ and let β tend to zero, we obtain Hankel's integral

(16) $$(2\gamma/a)^{\mu}\gamma^{\rho}\Gamma(\mu+1)\int_0^\infty e^{-\gamma t}J_\mu(at)\,t^{\rho-1}\,dt$$
$$=\Gamma(\mu+\rho)\,{}_2F_1(\tfrac12\rho+\tfrac12\mu,\ \tfrac12\rho+\tfrac12\mu+\tfrac12;\ \mu+1;\ -a^2\gamma^{-2})$$
$$=\Gamma(\mu+\rho)\,(1+a^2\gamma^{-2})^{-\frac12\mu-\frac12\rho}$$
$$\times\,{}_2F_1[\tfrac12\rho+\tfrac12\mu,\ \tfrac12+\tfrac12\mu-\tfrac12\rho;\mu+1;\ a^2/(a^2+\gamma^2)]$$
$$\mathrm{Re}(\rho+\mu)>0,\quad \mathrm{Re}(\gamma\pm ia)>0.$$

The second expression (16) is derived from the first one using the transformation formula of 2.10(6) for the hypergeometric function.

From the second formula in (16) we see that if $\rho = \mu + 1$

(17) $$\int_0^\infty e^{-\gamma t}J_\mu(at)\,t^{\mu}\,dt=\pi^{\frac12}\,(2a)^{\mu}\,\Gamma(\mu+\tfrac12)\,(\gamma^2+a^2)^{-\frac12-\mu}$$

see errata!

$$\mathrm{Re}(2\mu+1)>0,\quad \mathrm{Re}(\gamma\pm ia)>0.$$

If in (16) $\rho = 1$, we obtain from 2.8(4)

(18) $$\int_0^\infty e^{-\gamma t}J_\mu(at)\,dt=a^{-\mu}(\gamma^2+a^2)^{-\frac12}[(\gamma^2+a^2)^{\frac12}-\gamma]^{\mu}$$
$$\mathrm{Re}\,\mu>-1,\quad \mathrm{Re}(\gamma\pm ia)>0.$$

Furthermore from the second expression in (16) with $\gamma = 0$, using 2.1(14) we have

(19) $$\int_0^\infty J_\mu(at)\,t^{\rho-1}\,dt=2^{\rho-1}\,a^{-\rho}\,\Gamma(\tfrac12\mu+\tfrac12\rho)/\Gamma(1+\tfrac12\mu-\tfrac12\rho)$$
$$-\mathrm{Re}\,\mu<\mathrm{Re}\,\rho<3/2,\quad a>0.$$

In the same manner a number of similar integral formulas containing the square of the integration variable in the exponential function may be established. For example the relation

(20) $$2^{\nu+\mu+1}\,a^{-\mu}\beta^{-\nu}\gamma^{\nu+\mu+\lambda}\Gamma(\nu+1)\int_0^\infty J_\mu(at)\,J_\nu(\beta t)\,e^{-\gamma^2t^2}\,t^{\lambda-1}\,dt$$
$$=\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac12\nu+\frac12\mu+\frac12\lambda)}{m!\,\Gamma(m+\mu+1)}$$
$$\times\,{}_2F_1(-m,\ -\mu-m;\ \nu+1;\ \beta^2a^{-2})(-\tfrac14a^2\gamma^{-2})^{m}$$
$$\mathrm{Re}(\mu+\nu+\lambda)>0,\quad \mathrm{Re}\,\gamma^2>0$$

may be derived using the expression of 7.2(47) and integrating term by term. We shall now investigate some special cases in which (20) reduces to simpler expressions.

Let $\beta = a$; then we obtain using 2.1(14)

(21) $\int_0^\infty J_\mu(at)\, J_\nu(at)\, e^{-\gamma^2 t^2}\, t^{\lambda-1}\, dt$

$$= 2^{-\nu-\mu-1}\, \gamma^{-\nu-\lambda-\mu}\, a^{\nu+\mu} \frac{\Gamma(\tfrac12\lambda + \tfrac12\mu + \tfrac12\nu)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}$$

$$\times {}_3F_3(\tfrac12\nu+\tfrac12\mu+\tfrac12, \tfrac12\nu+\tfrac12\mu+1, \tfrac12\nu+\tfrac12\mu+\tfrac12\lambda; \mu+1, \nu+1, \mu+\nu+1; -a^2\gamma^{-2})$$

$\mathrm{Re}(\nu + \lambda + \mu) > 0, \quad \mathrm{Re}\,\gamma^2 > 0.$

Let β tend to zero in (20). Then the expression on the right-hand side of (20) reduces to a confluent hypergeometric function, and we obtain with $\nu + \lambda = \rho$

(22) $\Gamma(\mu + 1) \int_0^\infty J_\mu(at)\, e^{-\gamma^2 t^2}\, t^{\rho-1}\, dt$

$$= \tfrac12\gamma^{-\rho}\,\Gamma(\tfrac12\mu + \tfrac12\rho)\,(\tfrac12 a/\gamma)^\mu\; {}_1F_1(\tfrac12\mu + \tfrac12\rho;\; \mu + 1;\; -\tfrac14 a^2\gamma^{-2})$$

$$= \tfrac12\gamma^{-\rho}\,\Gamma(\tfrac12\mu + \tfrac12\rho)\,(\tfrac12 a/\gamma)^\mu \exp(-\tfrac14 a^2\gamma^{-2})$$

$$\times {}_1F_1(\tfrac12\mu - \tfrac12\rho + 1;\; \mu + 1;\; \tfrac14 a^2\gamma^{-2})$$

$\mathrm{Re}\,\gamma^2 > 0, \quad \mathrm{Re}(\mu + \rho) > 0.$

Furthermore we have

(23) $\int_0^\infty J_\mu(at)\, e^{-\gamma^2 t^2}\, dt = \tfrac12\pi^{1/2}\,\gamma^{-1} \exp(-2^{-3} a^2\gamma^{-2})\, I_{\frac12\mu}(2^{-3} a^2\gamma^{-2})$

$\mathrm{Re}\,\gamma^2 > 0, \quad \mathrm{Re}\,\mu > -1,$

(24) $\int_0^\infty J_\mu(at)\, e^{-\gamma^2 t^2}\, t^{\mu+1}\, dt = a^\mu (2\gamma^2)^{-\mu-1} \exp(-\tfrac14 a^2\gamma^{-2})$

$\mathrm{Re}\,\mu > -1, \quad \mathrm{Re}\,\gamma^2 > 0,$

(25) $\int_0^\infty J_\nu(at)\, J_\nu(\beta t)\, e^{-\gamma^2 t^2}\, t\, dt$

$$= \tfrac12\gamma^{-2} \exp[-\tfrac14\gamma^{-2}(a^2 + \beta^2)]\, I_\nu(\tfrac12 a\beta\gamma^{-2})$$

$\mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\gamma^2 > 0.$

Formulas (23) and (24) originate from (22), and (25) from (20).

A formula similar to (16)

(26) $\Gamma(\tfrac12 + \mu)\,\pi^{-1/2}\,(2\beta)^{-\nu}(a + \beta)^{\nu+\mu} \int_0^\infty e^{-at} K_\nu(\beta t)\, t^{\mu-1}\, dt$

$$= \Gamma(\mu + \nu)\,\Gamma(\mu - \nu)\; {}_2F_1[\nu + \mu, \nu + \tfrac12;\; \mu + \tfrac12;\; (a - \beta)/(a + \beta)]$$

$$= (2a)^{-2\nu-2\mu}(a + \beta)^{2\nu+2\mu}\,\Gamma(\mu + \nu)\,\Gamma(\mu - \nu)$$

$$\times {}_2F_1(\nu + \mu, \mu;\; 2\nu + 2\mu;\; 1 - \beta^2/a^2)$$

$\mathrm{Re}(\mu \pm \nu) > 0, \quad \mathrm{Re}(a + \beta) > 0$

may be proved by inserting here 7.3(15) for $K_\nu(\beta t)$, interchanging the order of integration, and then using 2.12(5). From (26) and 2.8(47) for $a = 0$ we have

$$(27)\quad \int_0^\infty K_\nu(\beta t)\, t^{\mu-1}\, dt = 2^{\mu-2}\, \beta^{-\mu}\, \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu)\, \Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu) \qquad \operatorname{Re}(\mu \pm \nu) > 0, \quad \operatorname{Re}\beta > 0.$$

Furthermore from (23) and 7.2(13) we obtain

$$(28)\quad \int_0^\infty K_\mu(at)\, e^{-\gamma^2 t^2}\, dt = \tfrac{1}{4}\pi^{\frac{1}{2}}\, \gamma^{-1} \sec(\tfrac{1}{2}\mu\pi) \exp(2^{-3} a^2/\gamma^2) \times K_{\frac{1}{2}\mu}(2^{-3} a^2/\gamma^2) \qquad -1 < \operatorname{Re}\mu < 1.$$

Furthermore one may consult Shabde (1935); Mohan (1942, p. 171); Sinha (1942).

7.7.4. The discontinuous integral of Weber and Schafheitlin

We shall now investigate the integral $\int_0^\infty J_\mu(at)\, J_\nu(bt)\, t^{-\rho}\, dt$ in which a, b, are positive real. It turns out that even when the integral converges for all positive a and b, its analytic expression is different, according as a is smaller, equal to, or larger than b. The results are

$$(29)\quad 2^\rho\, b^{\mu-\rho+1}\, \Gamma(\mu+1)\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\rho-\tfrac{1}{2}\mu) \times \int_0^\infty J_\mu(at)\, J_\nu(bt)\, t^{-\rho}\, dt = a^\mu\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\rho) \times {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\rho,\ \tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu-\tfrac{1}{2}\rho;\ \mu+1;\ a^2/b^2)$$
$$\operatorname{Re}(\nu + \mu - \lambda + 1) > 0, \quad \operatorname{Re}\rho > -1, \quad 0 < a < b,$$

with a corresponding expression for $0 < b < a$ [interchange a and b in (29)] and

$$(30)\quad \int_0^\infty J_\mu(at)\, J_\nu(at)\, t^{-\rho}\, dt = \frac{(\tfrac{1}{2}a)^{\rho-1}\, \Gamma(\rho)\, \Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+\tfrac{1}{2}-\tfrac{1}{2}\rho)}{2\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}\rho)\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+\tfrac{1}{2}\rho)\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2}\rho)}$$
$$\operatorname{Re}(\nu + \mu + 1) > \operatorname{Re}\rho > 0, \quad a > 0.$$

The proof of these results follows. We use (12) with $\alpha = a$, $\beta = b$, $z = t$ in the integrand of (29), interchange the order of integration, evaluate the integral with respect to t by (19), and obtain

$$\int_0^\infty J_\mu(at)\, J_\nu(bt)\, t^{-\rho} dt = \pi^{-1}\, a^\mu\, b^\nu\, 2^{\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\rho-\frac{1}{2}}\, \frac{\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2}\mu - \tfrac{1}{2}\rho + \tfrac{1}{2})}{\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2}\rho + \tfrac{1}{2})} \times \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{i\theta(\mu-\nu)} (\cos\theta)^{\frac{1}{2}(\nu+\mu+\rho-1)} \times (a^2 e^{i\theta} + b^2 e^{-i\theta})^{\frac{1}{2}(\rho-\nu-\mu-1)}\, d\theta.$$

But the integrand on the right-hand side is expressible as a hypergeometric function ${}_2F_1$, compare 2.4(11), and we immediately obtain the expressions (29) and (30) according as $b > a$ or $b = a$. In some special cases the hypergeometric function reduces to a simpler function. For instance the formulas 7.14(28) to 7.14(31) are derived from (29) and (30) by putting $\rho = \nu = \frac{1}{2}$.

An integral related to the Weber-Schafheitlin integral but with one Bessel function replaced by a modified Bessel function of the third kind can likewise be expressed in terms of hypergeometric functions, but it has no discontinuity at $a = b$. We obtain

$$(31)\quad 2^{\rho+1} a^{\nu-\rho+1}\Gamma(\nu+1)\int_0^\infty K_\mu(at)\, J_\nu(\beta t)\, t^{-\rho}\, dt$$

$$= \beta^\nu \Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\rho+\tfrac{1}{2}\mu+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\rho-\tfrac{1}{2}\mu+\tfrac{1}{2})$$

$$\times\, {}_2F_1(\tfrac{1}{2}\nu-\tfrac{1}{2}\rho+\tfrac{1}{2}\mu+\tfrac{1}{2}, \tfrac{1}{2}\nu-\tfrac{1}{2}\rho-\tfrac{1}{2}\mu+\tfrac{1}{2}; \nu+1; -\beta^2/a^2)$$

$$\mathrm{Re}(a \pm i\beta) > 0, \qquad \mathrm{Re}(\nu - \rho + 1 \pm \mu) > 0,$$

by expanding $J_\nu(\beta t)$ in a power series of 7.2(2) and integrating term by term using (27). Further integrals of a similar type are given in formulas 7.14(35) to 7.14(39). Here formulas 7.14(35) and 7.14(36) are consequences of (31). The other formulas were given by Dixon and Ferrar(1930).

7.7.5. Sonine and Gegenbauer's integrals and generalizations

Discontinuous integrals of a more general type than (29) to (30) have been investigated by Sonine and Gegenbauer. The integral

$$(32)\quad \int_0^\infty J_\mu(bt)\, J_\nu[a(t^2+z^2)^{\frac{1}{2}}]\,(t^2+z^2)^{-\frac{1}{2}\nu}\, t^{\mu+1}\, dt$$

$$= 0 \qquad a < b, \quad \mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1,$$

$$= b^\mu a^{-\nu} z^{1+\mu-\nu}(a^2-b^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}\, J_{\nu-\mu-1}[z(a^2-b^2)^{\frac{1}{2}}]$$

$$a > b, \quad \mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1,$$

may be established by replacing the second Bessel function under the integral sign by using 7.3(6), interchanging the order of integration, and using (24) and again 7.3(6).

Generalizations of (32) have been given by Bailey (1935 a), and by Gupta (1943). For instance according to Bailey, we have

$$(33)\quad \int_0^\infty J_\mu(bt)\, t^{\mu+1} \prod_{n=1}^{m} J_{\nu_n}[a_n(t^2+z_n^2)^{\frac{1}{2}}]\,(t^2+z_n^2)^{-\frac{1}{2}\nu_n}\, dt = 0$$

$$b > a_1 + a_2 + \cdots + a_m, \quad \mathrm{Re}(\nu_1 + \cdots + \nu_m + \tfrac{1}{2}m - \tfrac{1}{2}) > \mathrm{Re}\,\mu > -1,$$

$$(34)\quad \int_0^\infty J_\mu(bt)\, t^{\mu-1} \prod_{n=1}^{m} J_{\nu_n}[a_n(t^2+z_n^2)^{\frac{1}{2}}]\,(t^2+z_n^2)^{-\frac{1}{2}\nu_n}\, dt$$

$$= 2^{\mu-1} b^{-\mu}\Gamma(\mu) \prod_{n=1}^{m} z_n^{-\nu_n} J_{\nu_n}(z_n a_n)$$

$$b > a_1 + a_2 + \cdots + a_m, \quad \mathrm{Re}(\nu_1 + \nu_2 + \cdots + \nu_m + \tfrac{1}{2}m + 3/2) > \mathrm{Re}\,\mu > 0.$$

Another generalization of (32) is due to Sonine. To obtain this let us consider for a positive integer m and Re $a > 0$ the integral

$$\int_C z^{\rho-1} J_\mu[b(z^2+\zeta^2)^{1/2}](z^2+\zeta^2)^{-1/2\mu}(z^2-a^2)^{-m-1} H_\nu^{(1)}(az)\,dz,$$

where C is a contour consisting of the upper semicircle $|z| = R$ and its diameter, with an indentation at $z = 0$. When R approaches ∞, and the indentation shrinks to a point, the contribution of the circular arcs in C vanishes if $a \geq b$, $\mathrm{Re}(\pm\nu) < \mathrm{Re}\,\rho < (2m+4) + \mathrm{Re}\,\mu$. Expanding the integrand in ascending powers of $(z^2 - a^2)$ we find that the residue at the pole $z^2 = a^2$ is

$$\frac{2^{-m-1}}{m!}\left(\frac{d}{a\,da}\right)^m \{a^{\rho-2} J_\mu[b(a^2+\zeta^2)^{1/2}](a^2+\zeta^2)^{-1/2\mu} H_\nu^{(1)}(aa)\}.$$

From Cauchy's residue theorem and 7.2(16) we find that

(35) $$\int_0^\infty t^{\rho-1} J_\mu[b(t^2+\zeta^2)^{1/2}](t^2+\zeta^2)^{-1/2\mu}(t^2-a^2)^{-m-1}$$
$$\times [H_\nu^{(1)}(at) \pm e^{i\pi(\rho-\nu)} H_\nu^{(2)}(at)]\,dt$$
$$= \frac{\pi i}{m!} 2^{-m}\left(\frac{d}{a\,da}\right)^m \{a^{\rho-2} J_\mu[b(a^2+\zeta^2)^{1/2}](a^2+\zeta^2)^{-1/2\mu} H_\nu^{(1)}(aa)\}$$

$a \geq b$, $\quad \mathrm{Re}(\pm\nu) < \mathrm{Re}\,\rho < 2m + 4 + \mathrm{Re}\,\mu$, $\quad \mathrm{Re}(ia) < 0$, $\quad m = 0, 1, 2, \ldots$.

Similar formulas and special cases are listed in 7.14(46) to 7.14(59).

7.7.6. Macdonald's and Nicholson's formulas

Representations of a product of Bessel functions as an infinite integral have been given by Macdonald and Nicholson. Macdonald's formula

(36) $$\int_0^\infty \exp[-\tfrac{1}{2}t - \tfrac{1}{2}t^{-1}(z^2+Z^2)]\, K_\nu(zZ/t)\, t^{-1}\,dt = 2K_\nu(z)\,K_\nu(Z)$$

$$|\arg z| < \pi, \quad |\arg Z| < \pi, \quad |\arg(z+Z)| < \tfrac{1}{4}\pi$$

is an immediate consequence of

(37) $$\int_0^\infty \exp[-\tfrac{1}{2}t - \tfrac{1}{2}t^{-1}(x^2+X^2)]\, I_\nu(xX/t)\, t^{-1}\,dt = \begin{cases} 2I_\nu(x)\,K_\nu(X) \\ 2K_\nu(x)\,I_\nu(X) \end{cases}$$

according as $X > x$ or $X < x$. We prove (37) for positive real x and X and obtain (36) for positive z, Z, by 7.2(13); the extension to complex z, Z, follows from the theory of analytic continuation. Putting $a = x$, $\beta = X$, $\gamma^2 = \tfrac{1}{2}t$, in (25) we have

(38) $$I_\nu(xX/t) = t\,\exp[(x^2+X^2)/(2t)] \int_0^\infty J_\nu(xv)\,J_\nu(Xv)\,e^{-1/2tv^2}\,v\,dv.$$

Inserting this into (37) we obtain

$$\int_0^\infty \exp[-\tfrac{1}{2}t - \tfrac{1}{2}t^{-1}(x^2+X^2)]\, I_\nu(xX/t)\, t^{-1}\, dt$$
$$= \int_0^\infty J_\nu(xv)\, J_\nu(Xv)\, v\, dv \int_0^\infty e^{-\frac{1}{2}t(1+v^2)}\, dt$$
$$= 2\int_0^\infty (1+v^2)^{-1} J_\nu(xv)\, J_\nu(Xv)\, v\, dv$$

and using 7.14(57) this proves (37).

Nicholson's formulas

$$(39)\quad K_\mu(z)\,K_\nu(z) = 2\int_0^\infty K_{\nu+\mu}(2z\cosh t)\cosh[(\mu-\nu)t]\,dt$$
$$= 2\int_0^\infty K_{\nu-\mu}(2z\cosh t)\cosh[(\mu+\nu)t]\,dt \qquad \text{Re}\, z>0$$

may be proved as follows. From 7.12(21) we have

$$K_\nu(z)\,K_\mu(z) = \tfrac{1}{4}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-z(\cosh t+\cosh v)}\cosh(\mu t)\cosh(\nu v)\,dt\,dv.$$

Now we make the transformation $t+v=2\zeta$, $t-v=2\eta$, and after some reductions we obtain

$$K_\nu(z)\,K_\mu(z) = \tfrac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-2\zeta\cosh\zeta\cosh\eta}\cosh[(\mu+\nu)\zeta]$$
$$\times\cosh[(\mu-\nu)\eta]\,d\zeta\,d\eta.$$

With 7.12(21) this proves (39).

Another formula due to Nicholson is (Watson, 1944, p. 444)

$$(40)\quad [J_\nu(z)]^2 + [Y_\nu(z)]^2 = 8\pi^{-2}\int_0^\infty K_0(2z\sinh t)\cosh(2\nu t)\,dt$$
$$\text{Re}\, z>0.$$

For similar formulas, especially integrals for the product of two Bessel functions, see Watson (1944, p. 445); Chaundy (1931); Dixon and Ferrar (1930, 1933); Meijer (1935, p. 241, 1935 b, 1936, 1936 a, 1940, p. 366). For the sum or difference of a product of two Bessel functions, see Buchholz (1939, 1947).

7.7.7. Integrals with respect to the order

A formula due to Ramanujan (Watson, 1944, p. 449) valid for real y and a, $b>0$, $\text{Re}(\nu+\mu)>1$,

$$(41)\quad \int_{-\infty}^\infty a^{-\mu-x} J_{\mu+x}(a)\, b^{-\nu+x} J_{\nu-x}(b)\, e^{ixy}\,dx$$
$$= (2\cos\tfrac{1}{2}y)^{\frac{1}{2}(\nu+\mu)}(a^2e^{-i\frac{1}{2}y}+b^2e^{i\frac{1}{2}y})^{-\frac{1}{2}(\nu+\mu)}e^{i\frac{1}{2}y(\nu-\mu)}$$
$$\times J_{\nu+\mu}\{[2\cos\tfrac{1}{2}y\,(a^2e^{-i\frac{1}{2}y}+b^2e^{i\frac{1}{2}y})]^{\frac{1}{2}}\} \qquad |y|<\pi,$$
$$= 0 \qquad |y|>\pi$$

may be proved by applying Fourier's inversion formula to 7.7(12).

The cylindrical and the spherical wave function may be expressed, respectively, as

$$(42)\quad K_0[(a^2+b^2-2ab\cos\phi)^{1/2}] = (2/\pi)\int_0^\infty K_{ix}(a)\,K_{ix}(b)\cosh[(\pi-\phi)\,x]\,dx,$$

$$(43)\quad (a^2+b^2-2ab\cos\phi)^{-1/2}\,e^{-ik(a^2+b^2-2ab\cos\phi)^{1/2}} = -\tfrac{1}{2}\pi(ab)^{-1/2} \times\int_0^\infty x e^{\pi x} H^{(2)}_{ix}(ka)\,H^{(2)}_{ix}(kb)\tanh(\pi x)\,P_{-1/2+ix}(-\cos\phi)\,dx \qquad \operatorname{Im} k \le 0.$$

Equation (42) may be obtained from Macdonald's formula 7.7(36) and (43) from the residue theorem in connection with 7.15(41); (42) is a special case of a formula given by Crum (1940),

$$(44)\quad \int_{-\infty}^{\infty} K_{i(\xi+\eta)}(a)\,K_{i(\zeta+\eta)}(b)\,e^{(\pi-C)\eta}\,d\eta = K_{i(\xi-\zeta)}(c)\,e^{-\xi B-\zeta A},$$

where A, B, C, are the angles of the triangle whose sides are of lengths a, b, c.

Another generalization of (42) and (43) is

$$(45)\quad w^{-\nu}K_\nu(w) = \tfrac{1}{2}\Gamma(\nu)\,(\tfrac{1}{2}ab)^{-\nu} \times\int_{-\infty}^{\infty}\operatorname{sech}(\pi x)(\nu-\tfrac{1}{2}+ix)\,K_{\nu-1/2+ix}(a)\,I_{\nu-1/2+ix}(b) \times C^{\nu}_{-1/2+ix}(-\cos\phi)\,dx \qquad w=(a^2+b^2-2ab\cos\phi)^{1/2}$$

(for the definition of $C^{\nu}_{-1/2+ix}$ see sec. 3.15). For the proof of (45) use 7.6(3) and the residue theorem.

Other formulas are

$$(46)\quad \int_0^\infty K_{ix}(a)\cos(xy)\,dx = \tfrac{1}{2}\pi e^{-a\cosh y}$$

$$(47)\quad \int_0^\infty K_{ix}(a)\cosh(\tfrac{1}{2}\pi x)\cos(xy)\,dx = \tfrac{1}{2}\pi\cos(a\sinh y),$$

$$(48)\quad \int_0^\infty K_{ix}(a)\sinh(\tfrac{1}{2}\pi x)\sin(xy)\,dx = \tfrac{1}{2}\pi\sin(a\sinh y).$$

They may be derived from formulas 7.12(21), 7.12(25), and 7.12(26) respectively. For other results compare Ramanujan (1920, 1927, pp. 200, 224, 229); Fox (1929); MacRobert (1931, 1937); Crum (1940).

7.8. Relations between Bessel and Legendre functions

The Bessel and the modified Bessel functions may be expressed as a limiting case of the Legendre functions. In the expressions 3.2(14) and 3.4(6) for the Legendre functions we replace z by $\cosh(z/\nu)$ and x by

$\cos(x/\nu)$, respectively, to obtain

$$P_\nu^{-\mu}(\cosh z/\nu)\,\Gamma(\mu+1) = [\tanh(\tfrac{1}{2}z/\nu)]^\mu\, {}_2F_1\{-\nu,\ 1+\nu;\ 1+\mu;\ -[\sinh(\tfrac{1}{2}z/\nu)]^2\},$$

$$\mathrm{P}_\nu^{-\mu}(\cos x/\nu)\,\Gamma(\mu+1) = [\tan(\tfrac{1}{2}x/\nu)]^\mu\, {}_2F_1\{-\nu,\ 1+\nu;\ 1+\mu;\ [\sin(\tfrac{1}{2}x/\nu)]^2\}.$$

We now let ν approach ∞ and use 7.2(12) and 7.2(3) to obtain

(1) $$\lim_{\nu\to\infty} \nu^\mu P_\nu^{-\mu}(\cosh z/\nu) = \frac{(\tfrac{1}{2}z)^\mu}{\Gamma(\mu+1)}\, {}_0F_1(\mu+1; \tfrac{1}{4}z^2) = I_\mu(z),$$

(2) $$\lim_{\nu\to\infty} \nu^\mu \mathrm{P}_\nu^{-\mu}(\cos x/\nu) = \frac{(\tfrac{1}{2}x)^\mu}{\Gamma(\mu+1)}\, {}_0F_1(\mu+1; -\tfrac{1}{4}x^2) = J_\mu(x).$$

A similar relation (see Poole, 1934) may be derived from 3.2(41). It is

(3) $$\lim_{\mu\to\infty} \{Q_\nu^\mu[\mu/(iz)]\, e^{-i\mu\pi}/\Gamma(\mu)\} = \frac{ie^{i\frac{1}{2}\nu\pi}\,\pi^{\frac{1}{2}}(\tfrac{1}{2}z)^{\nu+1}}{\Gamma(\nu+3/2)}\, {}_0F_1(\nu+3/2; -z^2/4) = ie^{i\frac{1}{2}\nu\pi}(\tfrac{1}{2}\pi z)^{\frac{1}{2}}\, J_{\nu+\frac{1}{2}}(z).$$

Relations analogous to (1) and (2) may be obtained for Legendre functions of the second kind either from (1) and 3.3(4) or from (2) and 3.4(13). These relations are

(4) $$\begin{cases} \lim\limits_{\nu\to\infty} \nu^{-\mu} e^{-i\mu\pi} Q_\nu^\mu(\cosh z/\nu) = K_\mu(z) \\ \\ \lim\limits_{\nu\to\infty} \nu^\mu Q_\nu^{-\mu}(\cos x/\nu) = -\tfrac{1}{2}\pi\, Y_\mu(x). \end{cases}$$

We now turn to some integral relationships between Bessel and Legendre functions. Comparing the hypergeometric series on the right-hand side of 7.7(26) with 3.2(16) we obtain

(5) $$\Gamma(-\nu-\mu)\,\Gamma(\nu-\mu+1)\, P_\nu^\mu(z) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}}(z^2-1)^{-\frac{1}{2}\mu}\int_0^\infty e^{-tz} K_{\nu+\frac{1}{2}}(t)\, t^{-\mu-\frac{1}{2}}\, dt$$

$$\mathrm{Re}\, z > -1, \quad \mathrm{Re}(\nu-\mu+1) > 0, \quad \mathrm{Re}(\nu+\mu) < 0$$

and similarly from 7.7(16) and 3.2(41)

(6) $$Q_\nu^\mu(z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}(z^2-1)^{\frac{1}{2}\mu} e^{i\mu\pi}\int_0^\infty e^{-tz} I_{\nu+\frac{1}{2}}(t)\, t^{\mu-\frac{1}{2}}\, dt$$

$$\mathrm{Re}(\nu+\mu) > -1, \quad \mathrm{Re}\, z > 1.$$

Applying Whipple's formulas 3.3(13) and 3.3(14) to (5) and (6), respectively, we obtain four further integral representations

(7) $\Gamma(\nu-\mu+1)\,Q_\nu^\mu(z) = e^{i\mu\pi}(z^2-1)^{-\frac{1}{2}\nu-\frac{1}{2}}$

$$\times\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}}K_\mu(t)\,t^\nu\,dt \qquad \operatorname{Re}(\nu\pm\mu)>-1,$$

(8) $\Gamma(-\nu-\mu)\,P_\nu^\mu(z) = (z^2-1)^{\frac{1}{2}\nu}\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}}I_{-\mu}(t)\,t^\nu\,dt$

$$\operatorname{Re}(\nu+\mu)<0,$$

(9) $\Gamma(\nu+\mu+1)\,P_\nu^{-\mu}(z) = (z^2-1)^{-\frac{1}{2}\nu-\frac{1}{2}}\int_0^\infty e^{-tz(z^2-1)^{-\frac{1}{2}}}I_\mu(t)\,t^\nu\,dt$

$$\operatorname{Re}(\nu+\mu)>-1,$$

(10) $\Gamma(\nu+\mu+1)\,\mathrm{P}_\nu^{-\mu}(\cos\theta) = \int_0^\infty e^{-t\cos\theta}J_\mu(t\sin\theta)\,t^\nu\,dt$

$$\operatorname{Re}(\nu+\mu)>-1, \quad 0\le\theta<\tfrac{1}{2}\pi.$$

Equation (9) follows from(8) by means of 3.3(1) and (10) follows from (9) by means of 3.4(1).

A simple example of a representation of Bessel's functions of the first kind by means of an integral involving Legendre functions is Gegenbauer's generalization of Poisson's integral,

(11) $2^\nu\pi^{\frac{1}{2}}\,\Gamma(\nu+\tfrac{1}{2})\,\Gamma(n+2\nu)\,i^n z^{-\nu}J_{\nu+n}(z)/[n!\,\Gamma(2\nu)]$

$$=\int_0^\pi e^{iz\cos\phi}\,C_n^\nu(\cos\phi)(\sin\phi)^{2\nu}\,d\phi$$

$$\operatorname{Re}\nu>-\tfrac{1}{2}, \quad n=0,1,2,\dots,$$

This can be derived from Sonine's formula 7.10(5). We replace y by $\cos\phi$, multiply both sides by $C_n^\nu(\cos\phi)\,(\sin\phi)^{2\nu}$, integrate term by term with respect to ϕ, and use 3.15(17).

A similar formula

(12) $(2\pi/z)^{\frac{1}{2}}\,i^n(\sin\phi)^{\nu-\frac{1}{2}}\,C_n^\nu(\cos\phi)\,J_{\nu+n}(z)$

$$=\int_0^\pi e^{iz\cos\theta\cos\phi}J_{\nu-\frac{1}{2}}(z\sin\theta\sin\phi)\,C_n^\nu(\cos\theta)(\sin\theta)^{\nu+\frac{1}{2}}\,d\theta$$

$$\operatorname{Re}\nu>-\tfrac{1}{2}, \quad |\arg z|<\pi, \quad n=0,1,2,\dots.$$

may be derived from the addition theorem 7.15(17). For further formulas of these types see Meijer (1934, 1938); MacRobert (1936, 1940); Bailey (1935 a).

Finally we mention Whittaker's loop integral which is related to Hankel's integral 7.3(8). It is

(13) $\pi^{3/2}J_\nu(z) = (\tfrac{1}{2}z)^{1/2}e^{-\frac{1}{2}i\pi(\nu+\frac{1}{2})}\int_{\infty e^{i\delta}}^{(-1+,1+)} e^{izt}Q_{\nu-\frac{1}{2}}(t)\,dt$

$$-\tfrac{1}{2}\pi+\delta<\arg z<\tfrac{1}{2}\pi+\delta, \quad |\delta|<\tfrac{1}{2}\pi.$$

To prove this formula we assume that the contour lies entirely outside the circle $|t| = 1$; then we expand $Q_{\nu-\frac{1}{2}}(t)$ in descending powers of t according to 3.2(5) and proceed as in sec. 7.3. From (13) we obtain a corresponding expression for the second Hankel function

$$(14)\quad \pi^{3/2} H_\nu^{(2)}(z)\cos(\nu\pi) = (\tfrac{1}{2}z)^{\frac{1}{2}} e^{\frac{1}{2}(\nu+\frac{1}{2})i\pi} \int_{\infty e^{-i\delta}}^{(-1+,1+)} e^{izt} P_{\nu-\frac{1}{2}}(t)\,dt$$

$$-\tfrac{1}{2}\pi + \delta < \arg z < \tfrac{1}{2}\pi + \delta, \qquad |\delta| < \tfrac{1}{2}\pi,$$

where we have used 7.2(6) and 3.3(8).

An expansion of Legendre's function $\mathrm{P}_\nu(\cos\theta)$ in a series of Bessel functions,

$$(15)\quad \mathrm{P}_\nu(\cos\theta) = (\theta/\sin\theta)^{\frac{1}{2}} \sum_{m=0}^{\infty} a_m(\theta)(\nu+\tfrac{1}{2})^{-m} J_m[(\nu+\tfrac{1}{2})\theta]$$

has been given by Szegö (1933). The $a_m(\theta)$ are elementary functions, regular in $0 \leq \operatorname{Re}\theta < \pi$. In particular, $a_0 = 1$, $a_1 = 2^{-3}(\operatorname{ctn}\theta - \theta^{-1})$, etc. (15) is uniformly convergent in $0 \leq \theta \leq \theta_0 - \epsilon$ where $\epsilon > 0$ and

$$\theta_0 = 2(2^{\frac{1}{2}} - 1)\pi = (0.828\cdots)\pi.$$

This formula may be derived as follows. In 7.10(15) we put $s = 1$, $z = \theta$, $r^2 = 1 - t^2/\theta^2$, and $\nu = -\frac{1}{2}$ to obtain

$$2\cos t = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \sum_{m=0}^{\infty} 2^{1-m}(\theta^2 - t^2)^m \theta^{-m+\frac{1}{2}} J_{m-\frac{1}{2}}(\theta)/m!\,,$$

and hence

$$2(\cos t - \cos\theta) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \sum_{m=1}^{\infty} (\theta^2 - t^2)^m 2^{1-m} \theta^{\frac{1}{2}-m} J_{m-\frac{1}{2}}(\theta)/m!\,.$$

If we use this expansion in Mehler's integral 3.7(27), integrate term by term, and use 7.3(3) we obtain (15).

In the paper by Szegö already referred to, similar expansions are given for $P_\nu(\cosh\zeta)$, $\mathrm{Q}_\nu(\cos\theta)$ and $Q_\nu(\cosh\zeta)$ on pages 450, 449, and 448, respectively.

7.9. Zeros of Bessel functions

A detailed discussion of this subject is contained in Chapter XV of Watson's book. Some further results have been obtained since the original publication of Watson's book in 1922 and are not included in the 1944 edition. Here we shall discuss briefly the more important results.

GENERAL RESULTS

From general theorems on differential equations (Ince, 1944, Chap. X) follow the statements:

a) Any zero of any solution of 7.2(1) or 7.2(11) is a simple zero, the only possible exception being the origin.

b) The real zeros of two real linearly independent solutions of 7.2(1) separate one another. Here a real solution is defined by $a\,J_\nu(x) + b\,Y_\nu(x)$ with real a, b, ν, and positive real x.

BESSEL FUNCTIONS OF THE FIRST KIND

For the special case of the function $J_\nu(z)$ the following theorems may be proved.

The zeros of $J_\nu(z)$ and $J_\nu'(z)$ for real ν are symmetrical with respect to the axes of coordinates.

For real ν, $J_\nu(z)$ has an infinite number of real zeros (Watson, 1944, p. 478; Wilson, 1939).

If $\gamma_{\nu,1}$, $\gamma_{\nu,2}\cdots$ are the positive zeros of $J_\nu(x)$ arranged in ascending order of magnitude, then

$$0 < \gamma_{\nu,1} < \gamma_{\nu+1,1} < \gamma_{\nu,2} < \gamma_{\nu+1,2} < \gamma_{\nu,3} < \cdots \qquad \nu > -1,$$

(Watson, 1944, p. 479).

When $\nu > -1$ and A, B, C, D, are real numbers such that $AD - BC \neq 0$, then the positive zeros of $A\,J_\nu(x) + Bx\,J_\nu'(x)$ and $C\,J_\nu(x) + Dx\,J_\nu'(x)$ separate one another and no function of this type can have a repeated zero other than $x = 0$ (Watson, 1944, p. 480).

When A and B are real and $\nu > -1$, then the function

$$A\,J_\nu(x) + Bz\,J_\nu'(z)$$

has only real zeros except that it has two purely imaginary zeros when $A/B + \nu < 0$ (Watson, 1944, p. 482). For an asymptotic formula for these positive zeros see Moore (1920).

For $\nu > 1$ the function $J_{-\nu}(z)$ has an infinity of real zeros and also $2[\nu]$ conjugate complex zeros, among them two pure imaginary zeros when $[\nu]$ is an odd integer (Hurwitz's theorem). (For different proofs see Watson, 1944, p. 483; Obreschkoff, 1929; Pólya, 1929; Falkenberg, 1932; Hille Szegö, 1943).

A generalization of Hurwitz's theorem due to Hilb (1922) states that the principal branch of the function

$$A\,J_\nu(z) + B\,J_{-\nu}(z), \quad (A, B, \text{real}, \quad B \neq 0, \quad \nu > 0)$$

has $[\nu]$ complex zeros with a positive real part in case $[\nu]$ is even; when $[\nu]$ is odd there exist $[\nu] - 1$ or $[\nu] + 1$ complex zeros with a positive real part according as $(A/B) \gtrless 0$.

The number of zeros of $z^{-\nu} J_\nu(z)$ between the imaginary axis and the line on which

$$\operatorname{Re} z = m\pi + (\tfrac{1}{2}\operatorname{Re}\nu + \tfrac{1}{4})\pi$$

is equal to m for sufficiently large m, and all the zeros of $J_\nu(z)$ lie inside a strip $|\operatorname{Im} z| < A$ where A is bounded when ν is bounded.

Let γ_ν, γ_ν' and γ_ν'' be the smallest positive zeros of $J_\nu(x)$, $J_\nu'(x)$ and $J_\nu''(x)$ respectively; then we have (Watson, 1944, p. 485)

$$[\nu(\nu+2)]^{1/2} < \gamma_\nu < [2(\nu+1)(\nu+3)]^{1/2},$$

$$[\nu(\nu+2)]^{1/2} < \gamma_\nu' < [2\nu(\nu+1)]^{1/2},$$

when $\nu > 0$, and

$$[\nu(\nu-1)]^{1/2} < \gamma_\nu'' < (\nu^2-1)^{1/2}$$

when $\nu > 1$. For better bounds and for results on the following zeros see Mayr (1935).

The formula

$$\gamma_\nu = \nu + 1,855,757\nu^{1/3} + 103,315\,\nu^{-1/3} + O(\nu^{-1})$$

and similar formulas for other zeros of the Bessel functions of the first and second kind have been given by Tricomi (1948). For further information about the zeros of $J_\nu(x)$ and $J_\nu'(x)$ see Bickley (1943); Bickley and Miller, (1945); Gatteschi (1950); Olver (1950).

It has been proved by Siegel (1929) that $J_\nu(z)$ is not an algebraic number when ν is rational and z is an algebraic number other than zero. This theorem proves Bourget's conjecture that $J_\nu(z)$ and $J_{\nu+m}(z)$ ($m = 1, 2, 3, \ldots$) have no common zeros other than zero (Watson, 1944, p. 484).

Investigations about the zeros ν_n of $J_\nu(z)$ regarded as a function of ν, with fixed z have been carried out by Coulomb (1936). They show that for positive real values of z, the ν_n are real and simple and asymptotically near to negative integers (cf. also Gray and Mathews, 1922, p. 88).

The graph of $J_\nu(x)$ for fixed $\nu > -1$ and variable $x \geq 0$ resembles the graph of a damped oscillation. The successive areas of "half-waves" above and below the axis, form a decreasing sequence (Cooke, 1937).

The factorization theorem for entire functions (Copson, 1935, p. 158) leads to the representation of $z^{-\nu} J_\nu(z)$ as an infinite product (Watson, 1944, p. 497). We consider those zeros of $z^{-\nu} J_\nu(z)$ for a fixed $\nu \neq -1, -2, -3, \ldots$, which lie in the half-plane $\operatorname{Re} z > 0$ (those are symmetrical to the real axis) and arrange them according to non-decreasing real parts (in case there exist zeros on the imaginary axis only those with a positive imaginary part are considered). This sequence is denoted by $\gamma_{\nu,n}$ ($n = 1, 2, 3, \ldots$). Then we have

(1) $\Gamma(\nu+1)(\tfrac{1}{2}z)^{-\nu} J_\nu(z) = \prod_{n=1}^{\infty} (1 - z^2 \gamma_{\nu,n}^{-2})$.

A similar expansion is (Buchholz, 1947),

(2) $2\Gamma(\nu)(\tfrac{1}{2}z)^{1-\nu} J_\nu'(z) = \prod_{n=1}^{\infty} [1 - z^2 (\gamma_{\nu,n}')^{-2}]$.

Here the $\gamma_{\nu,n}'$ is a sequence formed of the zeros of $z^{1-\nu} J_\nu'(z)$ in the same manner as the sequence $\gamma_{\nu,n}$ was formed of the zeros of $z^{-\nu} J_\nu(z)$.

Forming the logarithmic derivative of (1) and using 7.2(51) we obtain

(3) $J_{\nu+1}(z)/J_\nu(z) = -2z \sum_{n=1}^{\infty} (z^2 - \gamma_{\nu,n}^2)^{-1}$.

Hence the following power series valid for $|z| < \gamma_\nu$ may be derived

(4) $\tfrac{1}{2} J_{\nu+1}(z)/J_\nu(z) = \sum_{n=1}^{\infty} S_{2n,\nu}\, z^{2n-1}$

where

(5) $S_{2l,\nu} = \sum_{n=1}^{\infty} \gamma_{\nu,n}^{-2l}$

and in particular (Nielsen, 1904, p. 360)

(6) $S_{2,\nu} = 2^{-2}/(\nu+1), \qquad S_{4,\nu} = 2^{-4}/[(\nu+1)^2(\nu+2)]$.

For further similar expansions and relations see sec. 7.15; Forsyth (1921); Buchholz (1947).

BESSEL FUNCTIONS OF THE SECOND KIND

The oldest result on zeros of Bessel functions of the second kind is a theorem by Schafheitlin (Watson, 1944, p. 482) according to which the principal branch of $Y_0(z)$ has no zeros with a positive real part other than real zeros. This result has been extended by Hilb (1922). When $[\nu]$ is even, than $Y_\nu(z)$ has $[\nu]$ complex zeros in $|\arg z| \leq \tfrac{1}{2}\pi$. When $[\nu]$ is odd, then $Y_\nu(z)$ has $[\nu] - 1$ or $[\nu] + 1$ complex zeros in the same range, according as $\cos(\nu\pi) \lesseqgtr 0$. Thus $Y_{2n}(z)$ and $Y_{2n+1}(z)$ $(n = 0, 1, 2, \ldots,)$ have $2n$ complex zeros in $|\arg z| \leq \tfrac{1}{2}\pi$.

$Y_n(z)$ (n an integer) has complex zeros in the left-half-plane on all branches and in the right-half-plane on all branches but the principal branch. Furthermore $Y_\nu(z)$ has positive real zeros only if ν is rational but not an integer. In the latter case $Y_\nu(z)$ has positive real zeros on the principal branch and other real zeros only if ν is rational but not an integer. In the latter case $Y_\nu(z)$ has real zeros only on the branch for which $2m\nu$ in 7.11(41) is an integer, (Hillmann, 1949).

For the zeros of linear combinations of $J_\nu(z)$ and $Y_\nu(z)$ see Watson, (1944, Chap. XV); Hilb (1922); Hillmann (1949). For a theorem similar to Bourget's hypothesis see Banerjee (1936).

For a combination of products of the Bessel functions of the first and second kind we have the theorem (Gray-Mathews, 1922, p. 82): If ν is real and a and b are positive, then

$$J_\nu(ax)\, Y_\nu(bx) - J_\nu(bx)\, Y_\nu(ax)$$

is a single-valued even function of x, whose zeros are all real and simple (see also Jahnke-Emde, 1945, p. 204; for similar combinations Carslaw and Jaeger, 1940; Kline, 1948).

BESSEL FUNCTIONS OF THE THIRD KIND

Investigations about the zeros on the principal branch of the first and second Hankel functions for real non-negative ν have been carried out by Falkenberg and Hilb (1916), and Falkenberg (1932). The results are: $H_\nu^{(1)}(z)$, $\nu \geq 0$, is free of zeros in $0 \leq \arg z \leq \pi$. The zeros, for $\nu \geq 0$, of $H_\nu^{(1)}$ in $-\pi < \arg z < 0$ and those of $H_\nu^{(2)}$ in $0 < \arg z < \pi$ lie symmetrically with respect to the imaginary axis.

There are no pure imaginary zeros except when $\nu = (2k - 1) + \frac{1}{2}$ $(k = 1, 2, 3, \ldots,)$ in which case there is one such zero.

The total number of the zeros of $H_\nu^{(1),(2)}(z)$ on the principal branch is equal to

$$\begin{array}{lll} 0 & \text{if} & 0 \leq \nu < 3/2, \\ 2k-1 & \text{if} & \nu = (2k-1) + \frac{1}{2}, \\ 2k & \text{if} & (2k-1) + \frac{1}{2} < \nu < 2k + \frac{1}{2} \qquad k = 1, 2, 3, \ldots. \end{array}$$

A theorem analogous to Bourget's hypothesis states that $H_\nu^{(1),(2)}(x)$ and $H_{\nu+m}^{(1),(2)}(x)$ have no common zeros when ν is real ≥ -1 and $m = 1, 2, 3, \ldots,$ (Banerjee, 1935).

MODIFIED BESSEL FUNCTIONS OF THE THIRD KIND

For $\nu \geq 0$, $K_\nu(z)$ has no zeros for which $|\arg z| \leq \frac{1}{2}\pi$. The number of zeros in $|\arg z| < \pi$ is the even integer nearest to $\nu - \frac{1}{2}$ unless $\nu - \frac{1}{2}$ is an integer, in which case the number is $\nu - \frac{1}{2}$ (Watson, 1944, p. 511).

When $\nu + 1$ is positive real, and m a positive integer, $K_\nu(z)$ and $K_{\nu+m}(z)$ have no common zero.

If $f(z)$ and $g(z)$ are given analytic functions without common zeros such that $g(z)/f(z)$ is meromorphic, and $\mathrm{Re}\,[g(z)/f(z)] \geq 0$ for $\mathrm{Re}\, z \geq 0$, then the function

$$F(z) = f(z)\, K_\nu'(z) - g(z)\, K_\nu(z)$$

has no zeros in the right-half of the complex plane (Erdélyi and Kermack, 1945).

The zeros of $K_\nu(z)$ and $I_\nu(az)\,K_\nu(bz) - K_\nu(az)\,I_\nu(bz)$ regarded as a function of ν are all purely imaginary, and these functions have an infinite number of zeros (Gray-Mathews, 1922, p. 88); compare also Pólya, (1926) and Bruijn (1950). The function $G(z)$ corresponding to equation (iii) in Polya's paper is $2K_{iz}(\lambda)$.

7.10. Series and integral representations of arbitrary functions

7.10.1. Neumann series

A Neumann series is a series of the type

$$(1)\quad \sum_{n=0}^{\infty} a_n\, J_{\nu+n}(z).$$

By the expansion 7.2(2) it is evident that its circle of convergence is identical with that of the power series $\Sigma\, a_n\,(\tfrac{1}{2}z)^{\nu+n}/\Gamma(\nu+n+1)$.

The Neumann series expansion of a function $f(z)$ which is given by a power series can easily be obtained. For this purpose we first give the Neumann series of a power of z

$$(2)\quad (\tfrac{1}{2}z)^{\nu} = \sum_{n=0}^{\infty} (\nu+2n)\,\Gamma(\nu+n)\,J_{\nu+2n}(z)/n!\,,$$

ν not a negative integer, which may be verified by inserting for $J_{\nu+n}(z)$ its power series, see 7.2(2), and rearranging the right-hand side in powers of z. All the coefficients except that of z^{ν} vanish.

Now let

$$f(z) = \sum_{l=0}^{\infty} b_l\, z^l.$$

If each power of z is replaced by its Neumann series (2), we obtain

$$f(z) = z^{-\nu} \sum_{l=0}^{\infty} b_l\, 2^{l+\nu} \sum_{m=0}^{\infty} (\nu+l+2m)\,\Gamma(\nu+l+m)\,J_{\nu+l+m}(z)/m!\,,$$

and hence

$$f(z) = z^{-\nu} \sum_{n=0}^{\infty} a_n\, J_{\nu+n}(z),$$

where

$$(3)\quad a_n = 2^{\nu+n}(\nu+n) \sum_{s=0}^{\le \frac{1}{2}n} 2^{-2s}\,\Gamma(\nu+n-s)\, b_{n-2s}/s!\,.$$

Conversely, the b_l may be expressed in terms of the a_n (Nielsen, 1904, p. 271) as

$$(4) \quad b_l \Gamma(\nu + l + 1) = 2^{-l-\nu} \sum_{m=0}^{\leq \frac{1}{2} l} (-1)^m \binom{\nu + l}{m} a_{l-2m} .$$

Some cases in which a simpler expression may be found for the sum in (3) are of special interest. For example we take

$$f(z) = e^{iz\gamma} = \sum_{l=0}^{\infty} (i\gamma)^l z^l / l ! .$$

Then we have from (3) after some algebra

$$a_n = i^n \gamma^n 2^{\nu+n} \Gamma(\nu + n + 1) \; {}_2F_1(-\tfrac{1}{2}n, \tfrac{1}{2} - \tfrac{1}{2}n; 1 - n - \nu; \gamma^{-2}) / n !,$$

or, introducing Gegenbauer's polynomial 3.15(8) we obtain Sonine's formula

$$(5) \quad z^\nu e^{i\gamma z} = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} i^n (\nu + n) C_n^\nu(\gamma) J_{\nu+n}(z) \quad \nu \neq 0, -1, -2, \ldots .$$

The expansion of a Bessel function as a Neumann series

$$(6) \quad (\tfrac{1}{2} az)^{\mu-\nu} J_\nu(az) \Gamma(\nu + 1)$$

$$= \sum_{n=0}^{\infty} {}_2F_1(-n, \mu + n; \nu + 1; a^2) \Gamma(\mu + n)(\mu + 2n) J_{\mu+2n}(z)/n !$$

may easily be established in a similar manner. We expand the left-hand side of (6) in a power series of z and use (3). In the same manner we obtain the Neumann series of Lommel's function 7.5(69),

$$(7) \quad s_{\mu,\nu}(z) = 2^{\mu+1} \sum_{n=0}^{\infty} \frac{(\mu + 1 + 2n) \Gamma(\mu + 1 + n)}{n ! [(2n + 1 + \mu)^2 - \nu^2]} J_{\mu+1+2n}(z).$$

Hence, using 7.5(82) to 7.5(84) similar expressions for Anger's, Weber's, and Struve's functions may be obtained. For further results compare sec. 7.15; Nielsen, 1904, Ch. XX; Watson, 1944, Ch. XVI; Baudoux, 1945, 1946.

The theory of the expansion of a function $f(x)$ of a real variable x in a Neumann series is based on the integral formulas [cf. 7.14(32)]

$$\int_0^\infty t^{-1} J_{\nu+2n+1}(t) J_{\nu+2m+1}(t)\, dt = \begin{cases} 0 & m \neq n, \\ (4n + 2\nu + 2)^{-1} & m = n, \quad \nu > -1. \end{cases}$$

Hence, we derive formally the expansion

$$(8) \quad f(x) = \sum_{n=0}^{\infty} (2\nu + 2 + 4n) J_{\nu+2n+1}(x) \int_0^\infty t^{-1} f(t) J_{\nu+2n+1}(t)\, dt \qquad \nu > -1.$$

The theory of this expansion has been given by Wilkins (1948, 1950). The special case $\nu = 0$ has been formerly investigated by Webb, Kapteyn, Bateman (Watson, 1944, p. 533); Korn (1931) and Titchmarsh (1948, p. 352). (For the term by term integration of a Neumann series see Hardy, 1926.)

A series of the type

$$(9)\quad \sum_{n=0}^{\infty} a_n J_{\mu+\frac{1}{2}n}(z)\, J_{\nu+\frac{1}{2}n}(z)$$

is called a Neumann series of the second kind. If the product of the two Bessel functions is replaced by its power series of 7.2(48) we obtain the relation

$$(10)\quad z^{-\nu-\mu} \sum_{n=0}^{\infty} a_n J_{\mu+\frac{1}{2}n}(z)\, J_{\nu+\frac{1}{2}n}(z) = \sum_{l=0}^{\infty} b_l z^l$$

where

$$(11)\quad \Gamma(\nu+1+\tfrac{1}{2}n)\,\Gamma(\mu+1+\tfrac{1}{2}n) b_l = 2^{-l-\nu-\mu} \sum_{m=1}^{\leq \frac{1}{2}l} (-1)^m \binom{l+\nu+\mu}{m} a_{l-2m}$$

and hence (Nielsen, 1904, p. 292)

$$(12)\quad a_n = 2^{\nu+\mu+n}(\nu+\mu+n) \times \sum_{s=0}^{\leq \frac{1}{2}n} 2^{-2s} b_{n-2s} \frac{\Gamma(\nu+\mu+n-s)\Gamma(\nu+1-s+\frac{1}{2}n)\Gamma(\mu+1-s+\frac{1}{2}n)}{s\,!\,\Gamma(\nu+\mu+n-2s+1)}$$

provided neither μ, nor ν, nor $\mu+\nu$ is a negative integer. Formula (12) gives the expansion of a power series in a Neumann series, and it may be shown that the Neumann series thus obtained converges uniformly within the interior of the circle of convergence of the power series.

A simple example is the expansion of a power of z. We easily obtain from (12)

$$(13)\quad \frac{(\frac{1}{2}z)^{\mu+\nu}}{\Gamma(\nu+1)\Gamma(\mu+1)} = \sum_{n=0}^{\infty} \frac{\nu+\mu+2n}{\nu+\mu+n} \binom{\nu+\mu+n}{n} J_{\nu+n}(z)\, J_{\mu+n}(z).$$

(For further results see Nielsen, 1904, Chap. XXI; Watson, 1944, p. 525; and Banerjee, 1939.) For series involving the product of an arbitrary number of Bessel functions see Stevenson (1928).

A modified form of Neumann's series is the series

$$(14)\quad \sum_{n=0}^{\infty} a_n z^n J_{\nu+n}(z).$$

From the loop integral, see 7.3(5), we immediately obtain the following equation

(15) $(s^2 - r^2)^{-\frac{1}{2}\nu} J_\nu[z(s^2 - r^2)^{\frac{1}{2}}] = \sum_{n=0}^{\infty} (\frac{1}{2} z r^2)^n s^{-\nu-n} J_{\nu+n}(zs)/n\,!$.

With $s = 1$ and $r^2 = 1 - \lambda^2$ we obtain the multiplication theorem of the Bessel function

(16) $J_\nu(\lambda z) = \lambda^\nu \sum_{n=0}^{\infty} [\frac{1}{2} z(1-\lambda^2)]^n J_{\nu+n}(z)/n\,!$.

Hence, making λ approach 0 we deduce that

(17) $(\frac{1}{2} z)^\nu = \Gamma(\nu+1) \sum_{n=0}^{\infty} (\frac{1}{2} z)^n J_{\nu+n}(z)/n\,!$

a formula analogous to (2).

Equation (17) is useful for the conversion of a power series into a series of the type mentioned above. We obtain

(18) $\sum_{l=0}^{\infty} b_l z^{2l} = z^{-\nu} \sum_{n=0}^{\infty} a_n z^n J_{\nu+n}(z)$

where

(19) $a_n = \sum_{s=0}^{n} \frac{\Gamma(\nu+s+1)}{(n-s)\,!} 2^{2s-n+\nu} b_s$,

and hence

(20) $\Gamma(\nu+n+1) b_n = \sum_{s=0}^{n} (-1)^s 2^{-\nu-n-s} a_{n-s}/s\,!$

(Nielsen, 1904, Ch. XXI).

7.10.2. Kapteyn series

Series of the form

(21) $\sum_{n=0}^{\infty} a_n J_{\nu+n}[(\nu+n)z]$

are known as Kapteyn series. From the inequality (Watson, 1944, p. 270)

(22) $|J_\alpha(\alpha z)| \leq \left(1 + \left|\frac{\sin\alpha\pi}{\alpha\pi}\right|\right) \left|z^\alpha e^{\alpha(1-z^2)^{\frac{1}{2}}} [1 + (1-z^2)^{\frac{1}{2}}]^{-\alpha}\right|$

it is evident that (21) converges throughout a domain in which

(23) $\sum_{n=0}^{\infty} a_n [w(z)]^n$

is absolutely convergent where

(24) $w(z) = ze^{(1-z^2)^{1/2}}/[1 + (1-z^2)^{1/2}]$.

The expansion of a power of z in a Kapteyn series

(25) $$(\tfrac{1}{2}z)^l = (\tfrac{1}{2}z)^{-\nu}(\nu+l)^2 \times \sum_{n=0}^{\infty} \Gamma(\nu+l+n)(\nu+l+2n)^{-\nu-l-1} J_{\nu+l+2n}[(\nu+l+2n)z]/n!$$

ν not a negative integer, may be verified by replacing each Bessel function on the right-hand side by its power series 7.2(2). The series (25) converges throughout the region

(26) $|w(z)| < 1$.

With (25) we may transform a power series into a Kapteyn series. If each power of z in

(27) $$f(z) = \sum_{l=0}^{\infty} b_l z^l$$

is replaced by its Kapteyn series (25), we find after some algebra

(28) $$f(z) = z^{-\nu} \sum_{n=0}^{\infty} a_n J_{\nu+n}[(\nu+n)z],$$

ν not a negative integer, where

(29) $$a_n = \tfrac{1}{2} \sum_{s=0}^{\leq \frac{1}{2}n} (\nu+n-2s)^2 \Gamma(\nu+n-s)(\tfrac{1}{2}\nu+\tfrac{1}{2}n)^{2s-n-\nu-1}.$$

The series in (29) is absolutely convergent when

$$|w(z)| < 1 \quad \text{and} \quad |w(z)| < |w(\rho)|$$

where ρ is the radius of convergence of (27).

A Kapteyn series of the second kind is a series of the type

(30) $$\sum_{n=0}^{\infty} a_n J_{\frac{1}{2}\nu+\frac{1}{2}n}[(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+n)z]\, J_{\frac{1}{2}\rho+\frac{1}{2}n}[(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+n)z].$$

It may be shown (Nielsen, 1904, p. 307) that

(31) $$(\tfrac{1}{2}z)^{\nu+\rho} = (\nu+\rho)\Gamma(1+\nu)\Gamma(1+\rho) \times \sum_{n=0}^{\infty} \binom{\nu+\rho+n-1}{n} (\nu+\rho+2n)^{-\nu-\rho-1} \times J_{\nu+n}[(\nu+\rho+2n)z]\, J_{\rho+n}[(\nu+\rho+2n)z],$$

where $\nu, \rho, \nu+\rho$, are not negative integers.

Now, let

(32) $$f(z) = \sum_{l=0}^{\infty} b_l z^l,$$

then (Nielsen, 1904, p. 308) we have

$$(33)\quad f(z) = z^{-\frac{1}{2}(\nu+\rho)} \sum_{n=0}^{\infty} a_n J_{\frac{1}{2}(\nu+n)}[(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+n)z]\, J_{\frac{1}{2}(\rho+n)}[(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+\tfrac{1}{2}n)z],$$

where

$$(34)\quad (\tfrac{1}{2}\nu + \tfrac{1}{2}\rho + \tfrac{1}{2}n)^{\frac{1}{2}\nu+\frac{1}{2}\rho+n+1}\, 2^{-\frac{1}{2}(\nu+\rho+n)} a_n$$
$$= \sum_{s=0}^{\leq \frac{1}{2}n} \frac{(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+n-2s)\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}n-s+1)\,\Gamma(\tfrac{1}{2}\rho+\tfrac{1}{2}n-s+1)}{(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+n)^{-s}}$$
$$\times \binom{\tfrac{1}{2}\nu+\tfrac{1}{2}\rho+n-s-1}{s} b_{n-2s}.$$

For further results and examples see Nielsen (1904, Chaps. XXII, XXIII); Watson (1944, Chap. XVII); Bailey, (1932); Budden (1926).

7.10.3. Schlömilch series

Series of the form

$$(35)\quad f(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m J_0(mx)$$

have been investigated by Schlömilch. There is an expansion theorem for an arbitrary function of the real variable x over the interval $(0, \pi)$ (Gray and Mathews, 1922, p. 40; Watson, 1944, p. 619).

For a function $f(x)$ which possesses, in the interval $0 \leq x \leq \pi$, a continuous derivative of bounded variation there is an expansion (35) with

$$(36)\quad a_0 = 2f(0) + 2\pi^{-1}\int_0^{\pi} v \int_0^{\frac{1}{2}\pi} f'(v\sin\phi)\,d\phi\,dv,$$

$$(37)\quad a_m = 2\pi^{-1}\int_0^{\pi} v\cos(mv)\int_0^{\frac{1}{2}\pi} f'(v\sin\phi)\,d\phi\,dv.$$

A generalized Schlömilch series is

$$(38)\quad \Sigma\,[a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)]\,(\tfrac{1}{2}mx)^{-\nu}.$$

The theory of such expansions may be found in Watson (1944, Chap. XIX) and Nielsen (1904, p. 134). In a paper by Cooke (1928), the results stated in Watson's book are partly simplified and extended. The theory is based on the formulas

$$(39)\quad \int_0^{\frac{1}{2}\pi} J_\nu(z\sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{-2\nu}\,d\theta$$
$$= 2^{-\nu}\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}-\nu)\,z^{\nu-1}\sin z \qquad -1 < \operatorname{Re}\nu < \tfrac{1}{2},$$

$$(40)\quad \int_0^{\frac{1}{2}\pi} \mathbf{H}_\nu(z\sin\theta)(\sin\theta)^{\nu+1}(\cos\theta)^{-2\nu}\,d\theta$$
$$= 2^{-\nu}\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}-\nu)\,z^{\nu-1}(1-\cos z) \qquad -3/2 < \operatorname{Re}\nu < 1/2,$$

which may easily be derived from equations 7.7(5) and 7.7(9) putting there $\mu = \nu$ and $\rho = -\nu - \frac{1}{2}$. Now, let us assume the validity of the expansions

(41) $$f(x) = \sum_{m=1}^{\infty} [a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)] (\tfrac{1}{2}mx)^{-\nu}$$
$$-\tfrac{1}{2} < \nu < \tfrac{1}{2}, \quad -\pi \le x \le \pi.$$

Here, we replace x by $x \sin\theta$, multiply both sides by

$$(\sin\theta)^{2\nu+1} (\cos\theta)^{-2\nu},$$

integrate with respect to θ from zero to $\frac{1}{2}\pi$ and use (39) and (40). Thus we obtain formally

$$\int_0^{\frac{1}{2}\pi} f(x\sin\theta)(\sin\theta)^{2\nu+1}(\cos\theta)^{-2\nu}\,d\theta$$

$$= \pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}-\nu) \sum_{m=1}^{\infty} [a_m \sin(mx) + b_m(1-\cos mx)]/(mx),$$

and hence for the coefficients of the expansion (41)

(42) $$\Gamma(\tfrac{1}{2}-\nu)\, a_m = m\pi^{-\frac{1}{2}}$$
$$\times \int_{-\pi}^{\pi} t\sin(mt) \int_0^{\frac{1}{2}\pi} f(t\sin\theta)(\sin\theta)^{2\nu+1}(\cos\theta)^{-2\nu}\,d\theta\, dt,$$

(43) $$\Gamma(\tfrac{1}{2}-\nu)\, b_m = -m\pi^{-\frac{1}{2}}$$
$$\times \int_{-\pi}^{\pi} t\cos(mt) \int_0^{\frac{1}{2}\pi} f(t\sin\theta)(\sin\theta)^{2\nu+1}(\cos\theta)^{-2\nu}\,d\theta\, dt.$$

The series (41) with the coefficients (42) and (43) is called the Schlömilch series of $f(x)$.

In the paper by Cooke, already referred to, it is proved that the class of functions for which (41) with the coefficients (42) and (43) is valid in any interval excluding 0, $\pm\pi$, is the class for which the theory of Fourier series applies. Furthermore theorems analogous to the Riemann-Lebesgue, Parseval, Riesz-Fischer theorems on Fourier series are established. In this connection see also Cooke (1927, 1929, 1930b, 1936); Wilton (1927); Jesmanowicz, (1938); Wilkins (1950a).

Let us now consider some simple examples of the Schlömilch expansion (41). We take $f(x) = (ax)^{-\nu}\,\mathbf{H}_\nu(ax)$ (a arbitrary). This is an odd function of x [compare 7.5(55)] and we have $a_m = 0$ in (41). On account of (40) we obtain from (43)

$$\pi b_m = -(-1)^m\, 2^{-\nu+1}\, m\sin(a\pi)/(m^2 - a^2)$$

and thus

(44) $$\pi(ax)^{-\nu}\,\mathbf{H}_\nu(ax) = -2^{-\nu+1}\sin(a\pi) \sum_{m=1}^{\infty} (-1)^m \frac{m}{m^2-a^2} (\tfrac{1}{2}mx)^{-\nu}\mathbf{H}_\nu(mx)$$
$$-\pi < x < \pi, \quad \mathrm{Re}\,\nu > -3/2$$

Dividing both sides of (44) by $\sin(a\pi)$ and making a approach zero we obtain [see 7.5(55)]

$$(45)\quad \pi^{-\frac{1}{2}}/\Gamma(\nu+3/2)+\sum_{m=1}^{\infty}(-1)^m\,(\tfrac{1}{2}mx)^{-\nu-1}\,\mathbf{H}_\nu(mx)=0$$

$$0<x<\pi,\qquad \operatorname{Re}\nu>-3/2.$$

Now let $f(x)=(ax)^{-\nu}J_\nu(ax)$. Here $f(x)$ is an even function of x, and therefore, $b_m=0$. From (42) and (39) we obtain

$$\pi a_m=-(-1)^m\,2^{-\nu+1}\sin(a\pi)\,m^2/[a(m^2-a^2)],$$

and therefore,

$$(46)\quad \pi(ax)^{-\nu}J_\nu(ax)=-2^{-\nu+1}a^{-1}\sin(a\pi)$$
$$\times\sum_{m=1}^{\infty}(-1)^m\,m^2(m^2-a^2)^{-1}(\tfrac{1}{2}mx)^{-\nu}J_\nu(mx)$$

$$0<x<\pi,\qquad \operatorname{Re}\nu>-\tfrac{1}{2}.$$

Making a approach zero in (46) we obtain

$$(47)\quad \tfrac{1}{2}/\Gamma(\nu+1)+\sum_{m=1}^{\infty}(-1)^m\,(\tfrac{1}{2}mx)^{-\nu}J_\nu(mx)=0$$

$$-\tfrac{1}{2}<\nu\le\tfrac{1}{2}\quad\text{and}\quad 0<x<\pi\quad\text{or}\quad \nu>\tfrac{1}{2}\quad\text{and}\quad 0<x\le\pi.$$

From (45) and (47) one sees that there are generalized Schlömilch series with non-vanishing coefficients which converge and whose sum vanishes almost everywhere. Such series are known as null series (Nielsen, 1904, Chap. XXV; Fox, 1926; Cooke, 1930). The existence of null series indicates that the Schlömilch expansion of a function, if it exists at all, is not unique.

For other results and examples concerning Schlömilch and related series, see Pennell (1932); Bennet (1932); Doetsch (1935); Erdélyi (1937); Kober (1935); Watson (1931); Infield, et. al. (1947); Magnus and Oberhettinger (1948, pp. 58-62). Expansions where the Bessel and Struve functions in (38) are replaced by their squares have been given by Thielmann (1934).

7.10.4. Fourier-Bessel and Dini series

Let $\nu>-1$ and let $x=\gamma_m$ and $x=\gamma_n$ be two positive zeros of $J_\nu(x)$ (in this case all the zeros of $J_\nu(x)$ are real; see sec. 7.9). Using 7.2(56) we then find from 7.14(9) and 7.14(10), respectively, that

$$(48)\quad \int_0^1 t\,J_\nu(\gamma_m t)\,J_\nu(\gamma_n t)\,dt=\begin{cases}0 & n\neq m,\\ \tfrac{1}{2}[J_{\nu+1}(\gamma_m)]^2 & n=m.\end{cases}$$

Similarly if λ_m and λ_n denote two positive zeros (see sec. 7.9) of the function $z\,J_\nu'(z) + a\,J_\nu(z)$, where $\nu \geq -\frac{1}{2}$ and a is any given constant, we infer from formulas 7.14(9), 7.14(10), 7.2(54) and 7.2(55) that

$$(49)\quad \int_0^1 t\,J_\nu(\lambda_m t)\,J_\nu(\lambda_n t)\,dt = 0 \qquad n \neq m,$$

$$= \tfrac{1}{2}\lambda_m^{-2}\{\lambda_m^2\,[J_\nu'(\lambda_m)]^2 + (\lambda_m^2 - \nu^2)\,[J_\nu(\lambda_m)]^2\} \qquad n = m$$

The integral formula (48) expresses an orthogonal property of Bessel functions and suggest the expansion of an arbitrary function $f(x)$ of a real variable x in the form

$$(50)\quad f(x) = \sum_{m=1}^{\infty} a_m J_\nu(\gamma_m x)$$

with

$$(51)\quad \tfrac{1}{2}[J_{\nu+1}(\gamma_m)]^2\,a_m = \int_0^1 t\,f(t)\,J_\nu(\gamma_m t)\,dt,$$

where $\gamma_1, \gamma_2, \gamma_3, \ldots$ are the positive zeros of the function $J_\nu(x)$ arranged in ascending order of magnitude. This expansion is called the Fourier-Bessel expansion of $f(x)$.

Similarly from (49) we have

$$(52)\quad f(x) = \sum_{m=1}^{\infty} b_m\,J_\nu(\lambda_m x)$$

with

$$(53)\quad \{\lambda_m^2\,[J_\nu'(\lambda_m)]^2 + (\lambda_m^2 - \nu^2)\,[J_\nu(\lambda_m)]^2\}\,b_m$$

$$= 2\lambda_m^2 \int_0^1 t\,J_\nu(\lambda_m t)\,f(t)\,dt,$$

where $\nu \geq -\frac{1}{2}$ and $\lambda_1, \lambda_2, \ldots$ are the positive zeros of the function $z\,J_\nu'(z) + a\,J_\nu(z)$ arranged in ascending order of magnitude. This expansion is called the Dini expansion of $f(x)$.

The theory of the Fourier-Bessel and Dini expansion is given in Watson (1944, Chap. XVIII) and the following theorem may be stated: Let $t^{\frac{1}{2}}\,f(t)$ be absolutely integrable over (0, 1) and let $\nu > -\frac{1}{2}$; then if $0 < x < 1$, the expansions (50) and (52) behave in the same way as an ordinary Fourier series (see also Moore, 1911; Stone, 1927; MacRobert, 1931; Titchmarsh, 1946, p. 70). For the behavior near $x = 1$ and $x = 0$ see Watson (1944, pp. 594, 602, 615) and Young (1941); for the Gibbs phenomenon Cooke (1927), Wilton (1928), Moore (1930). For series similar to (50) and (52) but with the square of the Bessel function see Thielmann (1934).

Let for example $f(x) = x^{\nu}$, then we obtain from (50), (53) and 7.7(1)

(54) $x^{\nu} = \sum_{m=1}^{\infty} 2\, J_{\nu}(\gamma_m x)/[\gamma_m\, J_{\nu+1}(\gamma_m)]$ $\qquad 0 \le x < 1,$

(55) $$x^{\nu} = \sum_{m=1}^{\infty} \frac{2\lambda_m\, J_{\nu}(\lambda_m x)\, J_{\nu+1}(\lambda_m)}{(\lambda_m^2 - \nu^2)\,[J_{\nu}(\lambda_m)]^2 + \lambda_m^2\,[J'_{\nu}(\lambda_m)]^2}$$

$0 \le x \le 1, \quad a + \nu > 0.$

If $f(x) = J_{\nu}(xz)$, then we obtain from 7.14(9)

(56) $$\frac{J_{\nu}(xz)}{J_{\nu}(z)} = 2 \sum_{m=1}^{\infty} \frac{\gamma_m\, J_{\nu}(\gamma_m x)}{(\gamma_m^2 - z^2)\, J_{\nu+1}(\gamma_m)} \qquad 0 \le x < 1,$$

(57) $J_{\nu}(xz)$
$$= 2 \sum_{m=1}^{\infty} \frac{\lambda_m^2\, J_{\nu}(\lambda_m x)\,[\lambda_m\, J_{\nu+1}(\lambda_m)\, J_{\nu}(z) - z\, J_{\nu}(\lambda_m)\, J_{\nu+1}(z)]}{(\lambda_m^2 - z^2)\{\lambda_m^2\,[J'_{\nu}(\lambda_m)]^2 + (\lambda_m^2 - \nu^2)[J_{\nu}(\lambda_m)]^2\}}$$

$0 \le x \le 1.$

For further examples see sec. 7.15.

An expansion in series of Bessel functions which is suitable for a positive finite interval has been given by Titchmarsh(1923a, XIII-XVI) (see also MacRobert, 1931).

Let $f(x)$ be defined for $a < x < b$ $(a > 0)$. Then the expansion in question is

(58) $f(x) = \sum_{m=1}^{\infty} a_m\, [J_{\nu}(\gamma_m x)\, Y_{\nu}(\gamma_m b) - Y_{\nu}(\gamma_m x)\, J_{\nu}(\gamma_m b)],$

where $z = \gamma_m$ is the m-th positive root of

$$J_{\nu}(az)\, Y_{\nu}(bz) - Y_{\nu}(az)\, J_{\nu}(bz) = 0,$$

and

(59) $\{[J_{\nu}(\gamma_m a)]^2 - [J_{\nu}(\gamma_m b)]^2\}\, a_m$
$$= \tfrac{1}{2}\pi\, \gamma_m^2\, [J_{\nu}(\gamma_m a)]^2 \int_a^b [J_{\nu}(\gamma_m t)\, Y_{\nu}(\gamma_m b) - Y_{\nu}(\gamma_m t)\, J_{\nu}(\gamma_m b)]\, t f(t)\, dt.$$

GENERALIZED DIRICHLET SERIES

Series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n (\lambda_n s)^{\frac{1}{2}} K_{\nu}(\lambda_n s),$$

$$s = \sigma + i\tau, \quad \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty$$

have been investigated by Greenwood (1941). For $\nu = \frac{1}{2}$ they reduce to the Dirichlet series

$$f(s) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

For various theorems concerning these series see Greenwood (1941).

7.10.5. Integral representations of arbitrary functions

The theory of Schlömilch series (compare 7.10.3) gives a method of expressing an arbitrary function as a series of the functions J_ν and $\mathbf{H}_\nu$. Similar methods may be applied to corresponding expressions of an arbitrary function as an integral involving Bessel and related functions. We always suppose for the following that $f(t)$ is a real function of the real variable t and is of bounded variation in the neighborhood of $t = x$. If $f(t)$ is not continuous at $t = x$, in the following formulas $f(x)$ must be replaced by $\frac{1}{2}[f(x+0) + f(x-0)]$. The conditions on ν in some of the following expansion formulas have been relaxed by Cherry (1949a).

The simplest type of such a representation is Hankel's integral formula

$$(60)\quad f(x) = \int_0^\infty J_\nu(tx)\, t\, dt \int_0^\infty f(v)\, J_\nu(vt)\, v\, dv,$$

valid if $\nu \geq -\frac{1}{2}$ and

$$\int_0^\infty t^{\frac{1}{2}} |f(t)|\, dt$$

is convergent, or $\nu > -1$ and

$$\int_0^\infty t^{\frac{1}{2}} |f(t)|\, dt \quad \text{and} \quad \int_0^1 t^{\nu+1} |f(t)|\, dt$$

are convergent. The theory of the expression (60) has been thoroughly discussed by Watson (1944, Chap. XIV); Titchmarsh (1948, p. 240) and Tricomi. In case $\nu = \pm\frac{1}{2}$, (60) reduces to Fourier's sine and cosine integral respectively.

A generalization of Hankel's integral is due to Hardy (1925), who gave the formula

$$(61)\quad f(x) = \int_0^\infty G_\nu(xt)\, t\, dt \int_0^\infty F_\nu(vt)\, v f(v)\, dv,$$

where

$$(62)\quad F_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{\nu+2a+2m}}{\Gamma(a+m+1)\,\Gamma(a+m+\nu+1)}$$

$$= \frac{2^{2-\nu-2a} s_{\nu+2a-1,\nu}(z)}{\Gamma(a)\,\Gamma(\nu+a)}.$$

(63) $G_\nu(z) = \cos(a\pi)\, J_\nu(z) + \sin(a\pi)\, Y_\nu(z)$,

valid under the following conditions (Cooke, 1925):

(i) $a > -1, \quad a + \nu > -1, \quad \nu + 2a < 3/2, \quad |\nu| \le 3/2$,
(ii) $t^\sigma f(t)$ integrable over $(0, \delta)$, $\sigma = \min(1 + \nu + 2a, \tfrac{1}{2})$, $\delta > 0$,
(iii) $t^{1/2} f(t)$ integrable over (δ, ∞).

The theory of the expansion formula (61) has been given by Cooke (1925).

SPECIAL CASES OF HARDY'S FORMULA

If $a = 0$, we obtain $F_\nu(z) = J_\nu(z)$, $G_\nu(z) = J_\nu(z)$. This case reduces to Hankel's formula (60). If $a = \tfrac{1}{2}$, we obtain $F_\nu(z) = \mathbf{H}_\nu(z)$, $G_\nu(z) = Y_\nu(z)$. This leads to

(64) $f(x) = \int_0^\infty Y_\nu(xt)\, t\, dt \int_0^\infty \mathbf{H}_\nu(vt)\, v f(v)\, dv.$

If $a = -\tfrac{1}{2}$, we obtain

(65) $f(x)$

$$= -\int_0^\infty Y_\nu(xt)\, t\, dt \int_0^\infty \left[\frac{(vt)^{\nu-1}\, \pi^{-1/2}}{2^{\nu-1}\, \Gamma(\nu + \tfrac{1}{2})} - \mathbf{H}_\nu(vt)\right] v f(v)\, dv.$$

If $\nu = \tfrac{1}{2}$, we obtain

$$F_\nu(z) = (\tfrac{1}{2}\pi z)^{-1/2}\, C_{2a+1}(z), \qquad G_\nu(z) = (\tfrac{1}{2}\pi z)^{-1/2} \sin(z - a\pi)$$

where $C_{2a+1}(z)$ is Young's function 7.5 (85).

Weber and Orr's formula

$$(66)\quad f(x) = \int_a^\infty \frac{J_\nu(tx)\, Y_\nu(at) - J_\nu(at)\, Y_\nu(tx)}{[J_\nu(at)]^2 + [Y_\nu(at)]^2}\, t\, dt$$

see errata!

$$\times \int_0^\infty [J_\nu(vt)\, Y_\nu(at) - Y_\nu(vt)\, J_\nu(at)]\, v f(v)\, dv,$$

valid for ν real and $\int_0^\infty t^{1/2} |f(t)|\, dt$ convergent, reduces to Fourier's sine integral in case $\nu = \pm\tfrac{1}{2}$ (Titchmarsh, 1923; Watson, 1944, p. 468).

Another formula due to Titchmarsh (1925) is

(67) $f(x) = \pi \int_0^\infty \Gamma_\nu(xt)\, t\, dt \int_0^\infty (d/dt)\, [t\, \Lambda_\nu(vt)]\, v f(v)\, dv$

where

(68) $\Gamma_\nu(z) = \sin(a\pi)\{[J_\nu(z)]^2 - [Y_\nu(z)]^2\} - 2\cos(a\pi)\, J_\nu(z)\, Y_\nu(z)$,

$$(69)\quad \Lambda_\nu(z) = \sum_{m=0}^\infty \frac{(-1)^m\, \Gamma(\nu + m + a + \tfrac{1}{2})\, \pi^{-1/2}\, z^{2\nu+2a+2m}}{\Gamma(a + m + 1)\, \Gamma(\nu + a + m + 1)\, \Gamma(2\nu + a + m + 1)},$$

valid under the following conditions (Cooke, 1925)

(i) $a > -1, \quad a + 2\nu > -1, \quad 1 > a + \nu \geq -\tfrac{1}{2}, \quad |\nu| \leq 1,$
(ii) $t^{\sigma} f(t)$ integrable over $(0, \delta)$, $\quad \sigma = \min(1 + 2\nu + 2a, 1)$,
(iii) $t f(t)$ integrable over (δ, ∞), $\quad \delta > 0$.

The theory of the expansion formula (67) has been given by Cooke (1925).

Special cases of (68) and (69) are

$$a = 0, \quad \Gamma_\nu(z) = -2 J_\nu(z)\, Y_\nu(z), \quad \Lambda_\nu(z) = [J_\nu(z)]^2,$$

$$a = -\nu, \qquad \Lambda_\nu(z) = J_\nu(z)\, J_{-\nu}(z),$$

$$a = -2\nu, \qquad \Lambda_\nu(z) = [J_{-\nu}(z)]^2.$$

A generalization of Laplace's integral involving Bessel functions has been given by Meijer (1940, pp. 599, 702):

$$(70)\quad f(x) = (\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} I_\nu(xt)\,(xt)^{1/2}\, dt \int_0^\infty K_\nu(tv)\,(tv)^{1/2} f(v)\, dv.$$

As $K_\nu(z) = K_{-\nu}(z)$ 7.2 (14) we also have

$$(71)\quad f(x) = (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} [I_\nu(xt) + I_{-\nu}(xt)]\,(xt)^{1/2}\, dt$$
$$\times \int_0^\infty K_\nu(vt)\,(vt)^{1/2} f(v)\, dv$$

(cf. also Boas, 1942). In case $\nu = \pm\tfrac{1}{2}$, (71) reduces to Laplace's formula.

Other integral representations of arbitrary functions are

$$(72)\quad f(x) = -\tfrac{1}{2} \int_{-i\infty}^{i\infty} t\, J_t(x)\, dt \int_0^\infty H_t^{(2)}(v)\, v^{-1} f(v)\, dv$$

(Kontorovich and Lebedev, 1938),

$$(73)\quad f(x) = \pi^{-2} \int_{-\infty}^{\infty} e^{\frac{1}{2}\pi(x+t)} K_{i(x+t)}(a)\, dt$$
$$\times \int_{-\infty}^{\infty} e^{\frac{1}{2}\pi(t+v)} K_{i(t+v)}(a)\, f(v)\, dv \qquad a > 0,$$

(Crum, 1940),

$$(74)\quad f(x) = \tfrac{1}{2} \int_0^\infty \frac{J_{it}(e^x) + J_{-it}(e^x)}{\sinh(\pi t)}\, t\, dt \int_{-\infty}^{\infty} [J_{it}(e^v) + J_{-it}(e^v)]\, f(v)\, dv$$

(Titchmarsh, 1946, p. 83),

$$(75)\quad x f(x) = 2\pi^{-2} \int_0^\infty K_{it}(x)\, t \sinh(\pi t)\, dt \int_0^\infty K_{it}(v)\, f(v)\, dv$$

(Lebedev, 1946),

$$(76)\quad f(x) = (\pi i)^{-1} \int_{\sigma-i\infty}^{\sigma+i\infty} t K_t(x)\, dt \int_0^\infty v^{-1} f(v)\, I_t(v)\, dv$$

(Lebedev, 1947). For further examples see Hardy (1927), and Hardy and Titchmarsh (1933).

DUAL INTEGRAL EQUATIONS INVOLVING BESSEL FUNCTIONS

In some problems of potential - and electromagnetic or acoustic radiation theory the unknown function satisfies one integral equation over part of the range $(0, \infty)$ and a different equation over the rest of the range (Nicholson, 1924; King, 1935, 1936; Sommerfeld, 1943). The pair of equations (Titchmarsh, 1948, p. 337; Busbridge, 1938)

$$\int_0^\infty y^\alpha f(y)\, J_\nu(xy)\, dy = g(x) \qquad 0 < x < 1,$$

(77)

$$\int_0^\infty f(y)\, J_\nu(xy)\, dy = 0 \qquad x > 1$$

has the solution

(78) $\Gamma(\tfrac{1}{2}\alpha)\, f(x)$

$$= (2x)^{1-\frac{1}{2}\alpha} \int_0^1 t^{1+\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha}(xt)\, dt \int_0^1 g(vt)\, v^{\nu+1} (1-v^2)^{\frac{1}{2}\alpha-1}\, dv,$$

supposed $\alpha > 0$.

The special case $\alpha = 1$, $\nu = 1$, $g(x) = 1$ has the solution

$$\tfrac{1}{2}\pi f(x) = x^{-2} \sin x - x^{-1} \cos x.$$

The pair (Tranter, 1951)

$$\int_0^\infty y\, \Phi(y)\, J_\nu(xy)\, dy = f(x) \qquad 0 < x < 1,$$

(79)

$$\int_0^\infty \Phi(y)\, J_\nu(xy)\, dy = F(x) \qquad x > 1$$

has the solution

(80) $\Phi(y) = H(y) + (\tfrac{1}{2}\pi y)^{\frac{1}{2}} \int_0^1 t^{\nu+\frac{1}{2}} L(t)\, J_{\nu+\frac{1}{2}}(ty)\, dt,$

where

(81) $H(y) = F(1)\, J_{\nu+1}(y) + y \int_1^\infty x\, F(x)\, J_\nu(xy)\, dx,$

(82) $L(t) = (2/\pi)\, t^{-2\nu} \int_0^t x^{\nu+1} [f(x) - \int_0^\infty y H(y)\, J_\nu(xy)\, dy]\, (t^2 - x^2)^{-\frac{1}{2}}\, dx.$

The solution of

$$\int_0^\infty \Phi(y)\, J_\nu(xy)\, dy = G(x) \qquad 0 < x < 1,$$

(83)

$$\int_0^\infty y\, \Phi(y)\, J_\nu(xy)\, dy = g(x) \qquad x > 1$$

is

(84) $\Phi(y) = K(y) + (\tfrac{1}{2}\pi y)^{\frac{1}{2}} \int_0^1 t^{\nu+\frac{1}{2}} \xi(t)\, J_{\nu-\frac{1}{2}}(ty)\, dt,$

where

(85) $K(y) = \int_1^\infty x\, g(x)\, J_\nu(xy)\, dx,$

(86) $\tfrac{1}{2}\pi t^{2\nu} \xi(t) = M(0) + t \int_0^t (t^2 - x^2)^{-\frac{1}{2}} M'(x)\, dx$

(87) $M(x) = x^\nu G(x) - x^\nu \int_0^\infty K(y)\, J_\nu(xy)\, dy.$

SECOND PART: FORMULAS

7.11. Elementary relations and miscellaneous formulas

SPHERICAL BESSEL FUNCTIONS

In (1) to (13) $n = 0, 1, 2, \ldots$,

(1) $$J_{n+\frac{1}{2}}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} \Big[\sin(z - \tfrac{1}{2}n\pi) \sum_{m=0}^{\leq \frac{1}{2}n} (-1)^m (n+\tfrac{1}{2}, 2m)(2z)^{-2m} + \cos(z - \tfrac{1}{2}n\pi) \sum_{m=0}^{\leq \frac{1}{2}n-\frac{1}{2}} (-1)^m (n+\tfrac{1}{2}, 2m+1)(2z)^{-2m-1}\Big],$$

(2) $$Y_{n+\frac{1}{2}}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} \Big[\sin(z - \tfrac{1}{2}n\pi) \sum_{m=0}^{\leq \frac{1}{2}n-\frac{1}{2}} (-1)^m (n+\tfrac{1}{2}, 2m+1)(2z)^{-2m-1} - \cos(z - \tfrac{1}{2}n\pi) \sum_{m=0}^{\leq \frac{1}{2}n} (-1)^m (n+\tfrac{1}{2}, 2m)(2z)^{-2m}\Big],$$

(3) $$H^{(1)}_{n+\frac{1}{2}}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} i^{-n-1} e^{iz} \sum_{m=0}^{n} i^m (n+\tfrac{1}{2}, m)(2z)^{-m},$$

(4) $$H^{(2)}_{n+\frac{1}{2}}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} i^{n+1} e^{-iz} \sum_{m=0}^{n} (-i)^m (n+\tfrac{1}{2}, m)(2z)^{-m},$$

(5) $$J_{-n-\frac{1}{2}}(z) = (-1)^{n+1} Y_{n+\frac{1}{2}}(z); \qquad Y_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z),$$

(6) $$H^{(1)}_{-n-\frac{1}{2}}(z) = i(-1)^n H^{(1)}_{n+\frac{1}{2}}(z); \qquad H^{(2)}_{-n-\frac{1}{2}}(z) = -i(-1)^n H^{(2)}_{n+\frac{1}{2}}(z).$$

(7) $$J_{n+\frac{1}{2}}(z) = (-1)^n (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} z^{n+1} \left(\frac{d}{zdz}\right)^n \frac{\sin z}{z},$$

(8) $$Y_{n+\frac{1}{2}}(z) = -(-1)^n (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} z^{n+1} \left(\frac{d}{zdz}\right)^n \frac{\cos z}{z},$$

(9) $$H^{(1)}_{n+\frac{1}{2}}(z) = -i(-1)^n (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} z^{n+1} \left(\frac{d}{zdz}\right)^n \frac{e^{iz}}{z},$$

(10) $$H^{(2)}_{n+\frac{1}{2}}(z) = i(-1)^n (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} z^{n+1} \left(\frac{d}{zdz}\right)^n \frac{e^{-iz}}{z},$$

(11) $$\psi_n(z) = (\tfrac{1}{2}\pi/z)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z) = (-1)^n z^n \left(\frac{d}{zdz}\right)^n \frac{\sin z}{z},$$

(12) $$\zeta_n^{(1)}(z) = (\tfrac{1}{2}\pi/z)^{\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(z) = -i(-1)^n z^n \left(\frac{d}{zdz}\right)^n \frac{e^{iz}}{z},$$

(13) $\zeta_n^{(2)}(z) = (\frac{1}{2}\pi/z)^{\frac{1}{2}} H_{n+\frac{1}{2}}^{(2)}(z) = i\,(-1)^n z^n \left(\frac{d}{zdz}\right)^n \frac{e^{-iz}}{z}.$

(14) $J_{\frac{1}{2}}(z) = Y_{-\frac{1}{2}}(z) = (\frac{1}{2}\pi z)^{-\frac{1}{2}} \sin z,$

(15) $Y_{\frac{1}{2}}(z) = -J_{-\frac{1}{2}}(z) = -(\frac{1}{2}\pi z)^{-\frac{1}{2}} \cos z,$

(16) $I_{\frac{1}{2}}(z) = (\frac{1}{2}\pi z)^{-\frac{1}{2}} \sinh z,$

(17) $H_{\frac{1}{2}}^{(1)}(z) = -i\,H_{-\frac{1}{2}}^{(1)}(z) = -i(\frac{1}{2}\pi z)^{-\frac{1}{2}} e^{iz},$

(18) $H_{\frac{1}{2}}^{(2)}(z) = i\,H_{-\frac{1}{2}}^{(2)}(z) = i(\frac{1}{2}\pi z)^{-\frac{1}{2}} e^{-iz}.$

RECURRENCE RELATIONS AND DIFFERENTIATION FORMULAS FOR MODIFIED BESSEL FUNCTIONS

(19) $\left(\frac{d}{zdz}\right)^m [z^\nu I_\nu(z)] = z^{\nu-m} I_{\nu-m}(z),$

(20) $\left(\frac{d}{zdz}\right)^m [z^{-\nu} I_\nu(z)] = z^{-\nu-m} I_{\nu+m}(z),$

(21) $\left(\frac{d}{zdz}\right)^m [z^\nu K_\nu(z)] = (-1)^m z^{\nu-m} K_{\nu-m}(z),$

(22) $\left(\frac{d}{zdz}\right)^m [z^{-\nu} K_\nu(z)] = (-1)^m z^{-\nu-m} K_{\nu+m}(z).$

(23) $I_{\nu-1}(z) - I_{\nu+1}(z) = 2\nu z^{-1} I_\nu(z),$

(24) $I_{\nu-1}(z) + I_{\nu+1}(z) = 2\,I'_\nu(z),$

(25) $K_{\nu-1}(z) - K_{\nu+1}(z) = -2\nu z^{-1} K_\nu(z),$

(26) $K_{\nu-1}(z) + K_{\nu+1}(z) = -2K'_\nu(z).$

WRONSKIANS AND RELATED FORMULAS

$$W(w_1, w_2) = w_1 w_2' - w_1' w_2$$

(27) $W(J_\nu, J_{-\nu}) = -2(\pi z)^{-1} \sin(\nu\pi),$

(28) $W(J_\nu, Y_\nu) = 2(\pi z)^{-1},$

(29) $W(J_\nu, H_\nu^{(1),(2)}) = \pm 2i(\pi z)^{-1}$,

(30) $W(H_\nu^{(1)}, H_\nu^{(2)}) = -4i(\pi z)^{-1}$,

(31) $W(I_\nu, I_{-\nu}) = -2(\pi z)^{-1}\sin(\nu\pi)$,

(32) $W(I_\nu, K_\nu) = -z^{-1}$.

(33) $J_\nu(z)J_{-\nu+1}(z) + J_{-\nu}(z)J_{\nu-1}(z) = 2(\pi z)^{-1}\sin(\nu\pi)$,

(34) $H_\nu^{(1)}(z)H_{\nu-1}^{(2)}(z) - H_{\nu-1}^{(1)}(z)H_\nu^{(2)}(z) = -4i(\pi z)^{-1}$,

(35) $J_\nu(z)Y_{\nu-1}(z) - Y_\nu(z)J_{\nu-1}(z) = 2(\pi z)^{-1}$,

(36) $J_{\nu-1}(z)H_\nu^{(1)}(z) - J_\nu(z)H_{\nu-1}^{(1)}(z) = 2(\pi i z)^{-1}$,

(37) $J_\nu(z)H_{\nu-1}^{(2)}(z) - J_{\nu-1}(z)H_\nu^{(2)}(z) = 2(\pi i z)^{-1}$,

(38) $I_\nu(z)I_{-\nu+1}(z) - I_{-\nu}(z)I_{\nu-1}(z) = -2(\pi z)^{-1}\sin(\nu\pi)$,

(39) $K_{\nu+1}(z)I_\nu(z) + K_\nu(z)I_{\nu+1}(z) = z^{-1}$.

FUNCTIONS OF VARIABLE $ze^{i\pi m}$, (m integer)

(40) $J_\nu(ze^{im\pi}) = e^{im\pi\nu}J_\nu(z)$,

(41) $$Y_\nu(ze^{im\pi}) = e^{-im\pi\nu}Y_\nu(z) + 2i\,\frac{\sin(m\pi\nu)}{\sin(\nu\pi)}\cos(\pi\nu)\,J_\nu(z),$$

(42) $$H_\nu^{(1)}(ze^{im\pi}) = -\frac{\sin[(m-1)\pi\nu]}{\sin(\pi\nu)}H_\nu^{(1)}(z) - e^{-i\pi\nu}\frac{\sin(m\pi\nu)}{\sin(\pi\nu)}H_\nu^{(2)}(z),$$

(43) $$H_\nu^{(2)}(ze^{im\pi}) = \frac{\sin[(m+1)\pi\nu]}{\sin(\pi\nu)}H_\nu^{(2)}(z) + e^{i\pi\nu}\frac{\sin(m\pi\nu)}{\sin(\pi\nu)}H_\nu^{(1)}(z),$$

(44) $I_\nu(ze^{im\pi}) = e^{im\pi\nu}I_\nu(z)$,

(45) $$K_\nu(ze^{im\pi}) = e^{-im\pi\nu}K_\nu(z) - i\pi\frac{\sin(m\pi\nu)}{\sin(\pi\nu)}I_\nu(z).$$

In case ν is an integer equal to n, then

$$\lim_{\nu\to n}\frac{\sin(l\pi\nu)}{\sin(\pi\nu)} = l(-1)^{n(l+1)},$$

where l is equal to $m-1$, m or $m+1$ respectively.

7.12. Integral representations

BESSEL COEFFICIENTS

(1) $\pi J_n(z) = \int_0^\pi \cos(z \sin\phi - n\phi)\, d\phi,$

(2) $\pi J_n(z) = i^{-n} \int_0^\pi e^{iz\cos\phi} \cos(n\phi)\, d\phi,$

(3) $\pi J_{2n}(z) = 2 \int_0^{\frac{1}{2}\pi} \cos(z \sin\phi) \cos(2n\phi)\, d\phi,$

(4) $\pi J_{2n+1}(z) = 2 \int_0^{\frac{1}{2}\pi} \sin(z \sin\phi) \sin[(2n+1)\phi]\, d\phi.$

In (1) to (4) n equals 0, 1, 2, ...

POISSON'S INTEGRAL

(5) $\Gamma(\nu + \frac{1}{2}) J_\nu(z) = 2\pi^{-\frac{1}{2}} (\frac{1}{2}z)^\nu \int_0^{\frac{1}{2}\pi} \cos(z \sin\phi) (\cos\phi)^{2\nu}\, d\phi,$

(6) $\qquad = \pi^{-\frac{1}{2}} (\frac{1}{2}z)^\nu \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{iz\sin\phi} (\cos\phi)^{2\nu}\, d\phi,$

(7) $\qquad = \pi^{-\frac{1}{2}} (\frac{1}{2}z)^\nu \int_{-1}^{1} e^{izt} (1 - t^2)^{\nu-\frac{1}{2}}\, dt,$

(8) $\qquad = 2\pi^{-\frac{1}{2}} (\frac{1}{2}z)^\nu \int_0^1 (1 - t^2)^{\nu-\frac{1}{2}} \cos(zt)\, dt,$

(9) $\qquad = \pi^{-\frac{1}{2}} (\frac{1}{2}z)^\nu \int_0^\pi e^{iz\cos\phi} (\sin\phi)^{2\nu}\, d\phi,$

(10) $\Gamma(\nu + \frac{1}{2}) I_\nu(z) = \pi^{-\frac{1}{2}} (\frac{1}{2}z)^\nu \int_{-1}^{1} e^{-zt} (1 - t^2)^{\nu-\frac{1}{2}}\, dt,$

in (5) to (10) $\operatorname{Re} \nu > -\frac{1}{2}$.

HEINE'S FORMULA

(11) $\pi Y_\nu(z) = e^{i\frac{1}{2}\nu\pi} \{ i \int_0^\pi e^{-iz\cos t} \cos(\nu t)\, dt$

$\qquad - \int_0^\infty e^{iz\cosh t} [\cosh(\nu t - i\nu\pi) + e^{-i\nu\pi} \cosh(\nu t)]\, dt \}$

$0 < \arg z < \pi.$

MEHLER - SONINE FORMULAS

(12) $\Gamma(\frac{1}{2} - \nu) J_\nu(x) = 2\pi^{-\frac{1}{2}} (\frac{1}{2}x)^{-\nu} \int_1^\infty (t^2 - 1)^{-\nu-\frac{1}{2}} \sin(xt)\, dt,$

(13) $\Gamma(\frac{1}{2} - \nu) Y_\nu(x) = -2\pi^{-\frac{1}{2}} (\frac{1}{2}x)^{-\nu} \int_1^\infty (t^2 - 1)^{-\nu-\frac{1}{2}} \cos(xt)\, dt,$

in both formulas $x > 0$, $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$.

(14) $\pi J_\nu(x) = 2\int_0^\infty \sin(x\cosh t - \tfrac{1}{2}\nu\pi)\cosh(\nu t)\,dt,$

(15) $\pi Y_\nu(x) = -2\int_0^\infty \cos(x\cosh t - \tfrac{1}{2}\nu\pi)\cosh(\nu t)\,dt,$

in formulas (14) and (15) $x > 0$, $-1 < \operatorname{Re}\nu < 1$.

(16) $\pi J_\nu(x) = \int_0^\infty e^{-\nu t}\sin(x\cosh t - \tfrac{1}{2}\nu\pi)\,dt + \int_0^{\frac{1}{2}\pi}\cos(x\sin t - \nu t)\,dt$

$x > 0, \quad \operatorname{Re}\nu \geq 0.$

Generalizations of Schläfli's integrals (Lambe, 1931)

(17) $\pi\left(\dfrac{x+y}{x-y}\right)^{\frac{1}{2}\nu} J_\nu[(x^2-y^2)^{\frac{1}{2}}] = \int_0^\pi e^{y\cos t}\cos(x\sin t - \nu t)\,dt$

$- \sin(\nu\pi)\int_0^\infty e^{-\nu t} e^{-y\cosh t - x\sinh t}\,dt \qquad \operatorname{Re}(x+y) > 0.$

(18) $\pi\left(\dfrac{x+y}{x-y}\right)^{\frac{1}{2}\nu} Y_\nu[(x^2-y^2)^{\frac{1}{2}}] = \int_0^\pi e^{y\cos t}\sin(x\sin t - \nu t)\,dt$

$- \int_0^\infty (e^{\nu t + y\cosh t} + e^{-\nu t - y\cosh t}\cos\nu\pi)\, e^{-x\sinh t}\,dt$

$\operatorname{Re}x > \operatorname{Re}y > 0.$

MODIFIED HANKEL FUNCTIONS

(19) $\Gamma(\tfrac{1}{2}-\nu)K_\nu(z) = \pi^{\frac{1}{2}}(\tfrac{1}{2}z)^{-\nu}\int_1^\infty e^{-zt}(t^2-1)^{-\nu-\frac{1}{2}}\,dt$

$\operatorname{Re}z > 0, \quad \operatorname{Re}\nu < \tfrac{1}{2},$

(20) $\Gamma(\tfrac{1}{2}+\nu)K_\nu(z) = \pi^{\frac{1}{2}}(\tfrac{1}{2}z)^{\nu}\int_0^\infty e^{-z\cosh t}(\sinh t)^{2\nu}\,dt$

$\operatorname{Re}z > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2},$

(21) $K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)\,dt \qquad \operatorname{Re}z > 0,$

(22) $\Gamma(\nu+\tfrac{1}{2})K_\nu(z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} z^\nu e^{-z}\int_0^\infty e^{-zt}t^{\nu-\frac{1}{2}}(1+\tfrac{1}{2}t)^{\nu-\frac{1}{2}}\,dt$

$\operatorname{Re}z > 0, \quad \operatorname{Re}\nu > -\tfrac{1}{2},$

(23) $K_\nu(az) = \tfrac{1}{2}a^\nu\int_0^\infty e^{-\frac{1}{2}z(t+a^2t^{-1})}t^{-\nu-1}\,dt$

$\operatorname{Re}z > 0, \quad \operatorname{Re}(a^2z) > 0,$

(24) $K_\nu(az) = \tfrac{1}{2}e^{i\frac{1}{2}\nu\pi}a^\nu\int_0^\infty e^{i\frac{1}{2}z(t-a^2t^{-1})}t^{-\nu-1}\,dt$

$\operatorname{Im}z > 0, \quad \operatorname{Im}(a^2z) > 0,$

(25) $K_\nu(x)\cos(\tfrac{1}{2}\nu\pi) = \int_0^\infty \cos(x\sinh t)\cosh(\nu t)\,dt,$

(26) $K_\nu(x)\sin(\tfrac{1}{2}\nu\pi) = \int_0^\infty \sin(x\sinh t)\sinh(\nu t)\,dt,$

in formulas (25) and (26) $x > 0$, $-1 < \operatorname{Re}\nu < 1$.

(27) $K_\nu(z) = \pi^{-1/2} (2z)^\nu \Gamma(\nu + 1/2) \int_0^\infty (t^2 + z^2)^{-\nu-1/2} \cos t \, dt$

$\operatorname{Re} \nu > -1/2, \quad |\arg z| < 1/2 \pi.$

HANKEL FUNCTIONS

(28) $i \Gamma(1/2 - \nu) H_\nu^{(1)}(z) = 2\pi^{-1/2} (1/2 z)^{-\nu} \int_1^\infty e^{izt} (t^2 - 1)^{-\nu-1/2} dt$

$\operatorname{Im} z > 0, \quad \operatorname{Re} \nu < 1/2,$

(29) $-i \Gamma(1/2 - \nu) H_\nu^{(2)}(z) = 2\pi^{-1/2} (1/2 z)^{-\nu} \int_1^\infty e^{-izt} (t^2 - 1)^{-\nu-1/2} dt$

$\operatorname{Im} z < 0, \quad \operatorname{Re} \nu < 1/2,$

(30) $i \Gamma(1/2 - \nu) H_\nu^{(1)}(z) = (1/2 z)^{-\nu} 2\pi^{-1/2} \int_0^\infty t^{-2\nu} (1 + t^2)^{-1/2} e^{iz(1+t^2)^{1/2}} dt$

$\operatorname{Im} z > 0, \quad \operatorname{Re} \nu < 1/2,$

(31) $-i \Gamma(1/2 - \nu) H_\nu^{(2)}(z) = 2\pi^{-1/2} (1/2 z)^{-\nu} \int_0^\infty t^{-2\nu} (1 + t^2)^{-1/2} e^{-iz(1+t^2)^{1/2}} dt$

$\operatorname{Im} z < 0, \quad \operatorname{Re} \nu < 1/2,$

(32) $\Gamma(\nu + 1/2) H_\nu^{(1)}(z) = (1/2 \pi z)^{-1/2} e^{i(z - 1/2 \nu\pi - 1/4 \pi)}$

$\times \int_0^{\infty e^{i\delta}} e^{-t} t^{\nu-1/2} (1 + 1/2 itz^{-1})^{\nu-1/2} dt$

$\operatorname{Re} \nu > -1/2, \quad |\delta| < 1/2 \pi, \quad \delta - 1/2 \pi < \arg z < \delta + \frac{3}{2}\pi,$

(33) $\Gamma(\nu + 1/2) H_\nu^{(2)}(z) = (1/2 \pi z)^{-1/2} e^{-i(z - 1/2 \nu\pi - 1/4 \pi)}$

$\times \int_0^{\infty e^{i\delta}} e^{-t} t^{\nu-1/2} (1 - 1/2 itz^{-1})^{\nu-1/2} dt$

$\operatorname{Re} \nu > -1/2, \quad |\delta| < 1/2 \pi, \quad -3\pi/2 + \delta < \arg z < 1/2 \pi + \delta,$

BARNES' REPRESENTATION

(34) $2\pi^2 H_\nu^{(1)}(z) = -e^{-i 1/2 \nu\pi} \int_{-C-i\infty}^{-C+i\infty} \Gamma(-\nu - s) \Gamma(-s) (-1/2 iz)^{\nu+2s} ds$

$|\arg(-iz)| < 1/2 \pi,$

(35) $2\pi^2 H_\nu^{(2)}(z) = e^{i 1/2 \nu\pi} \int_{-C-i\infty}^{-C+i\infty} \Gamma(-\nu - s) \Gamma(-s) (1/2 iz)^{\nu+2s} ds$

$|\arg(iz)| < 1/2 \pi,$

C is any positive number exceeding $\operatorname{Re} \nu$.

(36) $2\pi i J_\nu(x) = \int_{-i\infty}^{i\infty} \Gamma(-s) [\Gamma(\nu + s + 1)]^{-1} (1/2 x)^{\nu+2s} ds$

$x > 0, \quad \operatorname{Re} \nu > 0,$

(37) $\pi^{5/2} H_\nu^{(1)}(z) = -e^{i(z-\nu\pi)} \cos(\nu\pi) (2z)^\nu$

$\times \int_{-i\infty}^{i\infty} \Gamma(-s) \Gamma(-2\nu - s) \Gamma(\nu + s + 1/2) (-2iz)^s ds$

$|\arg(-iz)| < 3\pi/2, \quad 2\nu$ not an odd integer,

(38) $$\pi^{5/2} H_\nu^{(2)}(z) = e^{-i(z-\nu\pi)} \cos(\nu\pi)\,(2z)^\nu \times \int_{-i\infty}^{i\infty} \Gamma(-s)\,\Gamma(-2\nu-s)\,\Gamma(\nu+s+\tfrac{1}{2})\,(2iz)^s\,ds$$

$|\arg(iz)| < 3\pi/2$, 2ν not an odd integer,

(39) $$2\pi^2 i K_\nu(z) = (\tfrac{1}{2}\pi/z)^{1/2} e^{-z} \cos(\nu\pi) \times \int_{-i\infty}^{i\infty} \Gamma(s)\,\Gamma(\tfrac{1}{2}-s-\nu)\,\Gamma(\tfrac{1}{2}-s+\nu)\,(2z)^s\,ds$$

$|\arg z| < 3\pi/2$, 2ν not an odd integer.

INTEGRALS EXPRESSED IN TERMS OF RELATED FUNCTIONS

(40) $$\int_0^{\frac{1}{2}\pi} \cos(z\cos\phi)\cos\nu\phi\,d\phi = \pi[4\cos(\tfrac{1}{2}\nu\pi)]^{-1}\,[\mathbf{J}_\nu(z)+\mathbf{J}_{-\nu}(z)]$$
$$= -\nu\sin(\tfrac{1}{2}\nu\pi)\,s_{-1,\nu}(z) = \pi[4\sin(\tfrac{1}{2}\nu\pi)]^{-1}\,[\mathbf{E}_\nu(z)-\mathbf{E}_{-\nu}(z)],$$

(41) $$\int_0^{\frac{1}{2}\pi} \sin(z\cos\phi)\cos\nu\phi\,d\phi = \pi[4\sin(\tfrac{1}{2}\nu\pi)]^{-1}\,[\mathbf{J}_\nu(z)-\mathbf{J}_{-\nu}(z)]$$
$$= \cos(\tfrac{1}{2}\nu\pi)\,s_{0,\nu}(z) = -\pi[4\cos(\tfrac{1}{2}\nu\pi)]^{-1}\,[\mathbf{E}_\nu(z)+\mathbf{E}_{-\nu}(z)]$$

(42) $$\int_0^{\pi} \cos(z\sin\phi)\cos\nu\phi\,d\phi = -\nu\sin(\nu\pi)\,s_{-1,\nu}(z),$$

(43) $$\int_0^{\pi} \cos(z\sin\phi)\sin\nu\phi\,d\phi = -\nu(1-\cos\nu\pi)\,s_{-1,\nu}(z)$$

(44) $$\int_0^{\pi} \sin(z\sin\phi)\sin\nu\phi\,d\phi = \sin(\nu\pi)\,s_{0,\nu}(z),$$

(45) $$\int_0^{\pi} \sin(z\sin\phi)\cos\nu\phi\,d\phi = (1+\cos\nu\pi)\,s_{0,\nu}(z),$$

(46) $$\int_0^{\infty} e^{nt-z\sinh t}\,dt = \tfrac{1}{2}[S_n(z)-\pi\mathbf{E}_n(z)-\pi Y_n(z)]$$

$n = 0, 1, 2, \ldots,$ $\operatorname{Re} z > 0,$

(47) $$\int_0^{\infty} e^{-nt-z\sinh t}\,dt = \tfrac{1}{2}(-1)^{n+1}[S_n(z)+\pi\mathbf{E}_n(z)+\pi Y_n(z)]$$

$n = 0, 1, 2, \ldots,$ $\operatorname{Re} z > 0,$

(48) $$S_{\mu,\nu}(z) = z^\mu \int_0^\infty e^{-tz}\,{}_2F_1(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\ \tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu;\ \tfrac{1}{2};\ -t^2)\,dt,$$

$\operatorname{Re} z > 0,$

(49) $$S_{\mu,\nu}(z) = z^{\mu+1} \int_0^\infty te^{-tz}\,{}_2F_1(\tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\ \tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu;\ 3/2;\ -t^2)\,dt,$$

$\operatorname{Re} z > 0,$

(50) $$S_{0,\nu}(z) = \int_0^\infty e^{-z\sinh t}\cosh(\nu t)\,dt,$$

(51) $\nu S_{0,\nu}(z) = z \int_0^\infty e^{-z\sinh t} \sinh(\nu t) \cosh t \, dt,$

(52) $S_{1,\nu}(z) = z \int_0^\infty e^{-z\sinh t} \cosh(\nu t) \cosh t \, dt,$

in (50) to (52) Re $z > 0$.

7.13. Asymptotic expansions

7.13.1. Large variable

(1) $H_\nu^{(1)}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} [\sum_{m=0}^{M-1} (\nu, m) (-2iz)^{-m} + O(|z|^{-M})]$

$$-\pi < \arg z < 2\pi,$$

(2) $H_\nu^{(2)}(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} e^{-i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} [\sum_{m=0}^{M-1} (\nu, m) (2iz)^{-m} + O(|z|^{-M})]$

$$-2\pi < \arg z < \pi,$$

For an appraisal of the remainder after the M-th term for complex ν and $-\tfrac{1}{2}\pi < \arg z < 3\pi/2$ and for $-3\pi/2 < \arg z < \tfrac{1}{2}\pi$ see Watson (1944, p.219). These results have been extended to the range $-\pi < \arg z < 2\pi$ and $-2\pi < \arg z < \pi$ by Meijer (1932, pp. 656, 852, 948, 1079). For the asymptotic behavior of a function expressed as an infinite Hankel function series see Meixner (1949).

(3) $J_\nu(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} \{\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$

$$\times [\sum_{m=0}^{M-1} (-1)^m (\nu, 2m) (2z)^{-2m} + O(|z|^{-2M})]$$

$$-\sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) [\sum_{m=0}^{M-1} (-1)^m (\nu, 2m+1) (2z)^{-2m-1} + O(|z|^{-2M-1})]\}$$

$$-\pi < \arg z < \pi,$$

(4) $Y_\nu(z) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} \{\sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi)$

$$\times [\sum_{m=0}^{M-1} (-1)^m (\nu, 2m) (2z)^{-2m} + O(|z|^{-2M})]$$

$$+ \cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) [\sum_{m=0}^{M-1} (-1)^m (\nu, 2m+1) (2z)^{-2m-1} + O(|z|^{-2M-1})]\}$$

$$-\pi < \arg z < \pi.$$

For formulas for the remainder after the M-th term see Watson (1944, pp. 206, 209) and in case of complex ν Meijer, (1932, ref. above). For further formulas see Burnett (1929).

(5) $$I_\nu(z) = (2\pi z)^{-1/2} \{e^z [\sum_{m=0}^{M-1} (-1)^m (\nu, m)\, (2z)^{-m} + O(|z|^{-M})]$$

$$+ ie^{-z+i\nu\pi} [\sum_{m=0}^{M-1} (\nu, m)\, (2z)^{-m} + O(|z|^{-M})]\}$$

$$-\tfrac{1}{2}\pi < \arg z < 3\pi/2,$$

(6) $$I_\nu(z) = (2z)^{-1/2}\, \pi^{-3/2} \cos(\pi\nu) \{\sum_{m=0}^{M-1} [e^z - i(-1)^m e^{-i\pi\nu - z}]$$

$$\times \Gamma(m + \tfrac{1}{2} - \nu)\Gamma(m + \tfrac{1}{2} + \nu)(2z)^{-m}/m! + e^z O(|z|^{-M})\}$$

$$-3\pi/2 < \arg z < \tfrac{1}{2}\pi,$$

(7) $$K_\nu(z) = (\tfrac{1}{2}\pi/z)^{1/2}\, e^{-z} [\sum_{m=0}^{M-1} (\nu, m)\, (2z)^{-m} + O(|z|^{-M})]$$

$$-3\pi/2 < \arg z < 3\pi/2,$$

Throughout these formulas

$$(\nu, m) = 2^{-2m} \{(4\nu^2 - 1)(4\nu^2 - 3^2) \cdots [4\nu^2 - (2m-1)^2]\}/m!$$
$$= \Gamma(\tfrac{1}{2} + \nu + m)/[m!\, \Gamma(\tfrac{1}{2} + \nu - m)].$$

7.13.2. Large order

(8) $$2\pi I_p(x) = 2^{1/2} (p^2 + x^2)^{-1/4} \exp[(p^2 + x^2)^{1/2} - p \sinh^{-1}(p/x)]$$

$$\times [\sum_{m=0}^{M-1} (-2)^m a_m \Gamma(m + \tfrac{1}{2})\, (p^2 + x^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$$p,\ x > 0,$$

(9) $$a_0 = 1, \quad a_1 = -\frac{1}{8} + \frac{5}{24}(1 + x^2/p^2)^{-1},$$

$$a_2 = \frac{3}{128} - \frac{77}{576}(1 + x^2/p^2)^{-1} + \frac{385}{3456}(1 + x^2/p^2)^{-2}, \ldots .$$

(For other expansions of $I_p(x)$ see Lehmer, 1944; Montroll, 1946.)

(10) $$K_p(x) = 2^{-1/2} (p^2 + x^2)^{-1/4} \exp[-(p^2 + x^2)^{1/2} + p \sinh^{-1}(p/x)]$$

$$\times [\sum_{m=0}^{M-1} 2^m a_m \Gamma(m + \tfrac{1}{2})\, (p^2 + x^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$$p,\ x > 0, \quad a_m \text{ as in (9)},$$

(11) $$\pi H_p^{(1)}(x) = 2^{1/2} (x^2 - p^2)^{-1/4} \exp[i(x^2 - p^2)^{1/2} + ip \sin^{-1}(p/x)]$$

$$\times e^{-i\frac{1}{2}\pi(p+\frac{1}{2})} [\sum_{m=0}^{M-1} 2^m b_m \Gamma(m + \tfrac{1}{2})(-i)^m (x^2 - p^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$$x > p > 0,$$

(12) $b_0 = 1, \quad b_1 = \frac{1}{8} - \frac{5}{24}(1 - x^2/p^2)^{-1},$

$$b_2 = \frac{3}{128} - \frac{77}{576}(1 - x^2/p^2)^{-1} + \frac{385}{3456}(1 - x^2/p^2)^{-2}, \ldots,$$

(13) $\pi H_p^{(1)}(x) = -i\, 2^{1/2}(p^2 - x^2)^{-1/4} \exp[-(p^2 - x^2)^{1/2} + p \cosh^{-1}(p/x)]$

$$\times [\sum_{m=0}^{M-1} (-1)^m 2^m b_m \Gamma(m + \tfrac{1}{2})(p^2 - x^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$p > x > 0, \quad b_m$ as in (12),

(14) $2\pi J_p(x) = 2^{1/2}(p^2 - x^2)^{-1/4} \exp[(p^2 - x^2)^{1/2} - p \sinh^{-1}(p/x)]$

$$\times [\sum_{m=0}^{M-1} 2^m b_m \Gamma(m + \tfrac{1}{2})(p^2 - x^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$p > x > 0, \quad b_m$ as in (12),

(15) $\pi H_p^{(1)}(x) \sim -2/3 \sum_{m=0}^{\infty} e^{2(m+1)\pi i/3} B_m(\epsilon x) \sin[(m+1)\pi/3]$

$$\times \Gamma(m + 1/3)(x/6)^{-(m+1)/3}$$

$p \approx x, \quad p, x > 0, \quad \epsilon = 1 - p/x, \quad \epsilon = o(x^{-2/3}),$

(16) $B_0(\epsilon x) = 1, \quad B_1(\epsilon x) = \epsilon x, \quad B_2(\epsilon x) = \tfrac{1}{2}(\epsilon x)^2 - \frac{1}{20},$

$$B_3(\epsilon x) = \frac{1}{6}(\epsilon x)^3 - \frac{1}{15}\epsilon x, \quad B_4(\epsilon x) = \frac{1}{24}(\epsilon x)^4 - \frac{1}{24}(\epsilon x)^2 + \frac{1}{280},$$

$$B_5(\epsilon x) = \frac{1}{120}(\epsilon x)^5 - \frac{1}{60}(\epsilon x)^3 + \frac{43}{8400}\epsilon x.$$

(For B_6, B_7, B_9, see Airey, 1916, p. 520.)

PURE IMAGINARY ORDER

(17) $2\pi J_{ip}(x) = 2^{1/2}(p^2 + x^2)^{-1/4} \exp[i(p^2 + x^2)^{1/2} - ip \sinh^{-1}(p/x) - \tfrac{1}{4} i\pi]$

$$\times e^{\frac{1}{2}p\pi} [\sum_{m=0}^{M-1} (2i)^m a_m \Gamma(m + \tfrac{1}{2})(p^2 + x^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$p, x > 0, \quad a_m$ as in (9),

(18) $K_{ip}(x) = 2^{-1/2}(x^2 - p^2)^{-1/4} \exp[-(x^2 - p^2)^{1/2} - p \sin^{-1}(p/x)]$

$$\times [\sum_{m=0}^{M-1} (-1)^m 2^m b_m \Gamma(m + \tfrac{1}{2})(x^2 - p^2)^{-\frac{1}{2}m} + O(x^{-M})]$$

$x > p > 0, \quad b_m$ as in (12)

(19) $K_{ip}(x) = 2^{1/2} (p^2 - x^2)^{-1/4} e^{-1/2 p\pi}$

$$\times \{ \sum_{m=0}^{M-1} 2^m b_m \Gamma(m + 1/2) (p^2 - x^2)^{-1/2 m}$$

$$\times \sin[1/2 \pi m + p \cosh^{-1}(p/x) - (p^2 - x^2)^{1/2} + 1/4 \pi] + O(x^{-M})\}$$

$p > x > 0, \quad b_m$ as in (12),

(20) $K_{ip}(x) \sim 1/3\pi \, e^{-1/2 p\pi} \sum_{m=0}^{\infty} (-1)^m C_m(\epsilon x) \sin[(m+1)\pi/3]$

$$\times \Gamma(1/2 m + 1/3) (x/6)^{-(m+1)/3}$$

$p \approx x, \quad p, x > 0, \quad \epsilon = 1 - p/x, \quad \epsilon = o(x^{-2/3}),$

(21) $C_0(\epsilon x) = 1, \quad C_1(\epsilon x) = \epsilon x, \quad C_2(\epsilon x) = 1/2 (\epsilon x)^2 + \frac{1}{20},$

$$C_3(\epsilon x) = \frac{1}{6} (\epsilon x)^3 + \frac{1}{15} \epsilon x, \quad C_4(\epsilon x) = \frac{1}{24} (\epsilon x)^4 + \frac{1}{24} (\epsilon x)^2 + \frac{1}{280},$$

$$C_5(\epsilon x) = \frac{1}{120} (\epsilon x)^5 + \frac{1}{60} (\epsilon x)^3 + \frac{43}{4800} \epsilon x,$$

(22) $\pi H^{(1)}_{ip}(x) = 2^{1/2} (p^2 + x^2)^{-1/4} \exp[i(p^2 + x^2)^{1/2} - ip \sinh^{-1}(p/x)]$

$$\times e^{1/2 p\pi - i 1/4 \pi} [\sum_{m=0}^{M-1} (-i)^m 2^m b_m \Gamma(m + 1/2) (p^2 + x^2)^{-1/2 m} + O(x^{-M})]$$

$p, x > 0, \quad b_m$ as in (12).

7.13.3. Transitional regions

NICHOLSON'S FORMULAS

($x \sim n$, n integer > 0)

(23) $J_n(x) \sim 3^{-2/3} (\xi/x)^{1/3} [J_{1/3}(\xi) + J_{-1/3}(\xi)],$

(24) $Y_n(x) \sim 3^{-1/6} (\xi/x)^{1/3} [J_{1/3}(\xi) - J_{-1/3}(\xi)],$

$x > n, \quad \xi = \frac{2}{3} (1/2 x)^{-1/2} (n - x)^{3/2},$

(25) $J_n(x) \sim \pi^{-1} 3^{-1/6} (\xi/x)^{1/3} K_{1/3}(\xi),$

(26) $Y_n(x) \sim -3^{-1/6} (\xi/x)^{1/3} [I_{1/3}(\xi) + I_{-1/3}(\xi)],$

$n > x, \quad \xi = \frac{2}{3} (1/2 x)^{-1/2} (x - n)^{3/2},$

(27) $e^{i\pi/6} H^{(2)}_n(x) \sim 3^{-1/6} (\xi/x)^{1/3} H^{(2)}_{1/3}(\xi)$

$\xi = \frac{2}{3} (1/2 x)^{-1/2} (x - n)^{3/2}, \quad$ for $x > n,$

$\arg(x - n) = 0; \quad$ for $x < n, \quad \arg(x - n) = \pi.$

WATSON'S FORMULAS

(28) $J_p(x) = 3^{-\frac{1}{2}} w\,[J_{1/3}(pw^3/3)\cos\delta - Y_{1/3}(pw^3/3)\sin\delta] + O(p^{-1})$,

(29) $Y_p(x) = 3^{-\frac{1}{2}} w\,[J_{1/3}(pw^3/3)\sin\delta + Y_{1/3}(pw^3/3)\cos\delta] + O(p^{-1})$,

$x > p, \quad \delta = pw - pw^3/3 - p\tan^{-1} w + \pi/6, \quad w = (x^2/p^2 - 1)^{\frac{1}{2}}$,

(30) $J_p(x) = 3^{-\frac{1}{2}}\pi^{-1} w e^{p\alpha} K_{1/3}(pw^3/3) + O(p^{-1})$,

(31) $Y_p(x) = -3^{-\frac{1}{2}} w e^{p\alpha}[I_{1/3}(pw^3/3) + I_{-1/3}(pw^3/3)] + O(p^{-1})$

$x < p, \quad \alpha = p(w + w^3/3 - \tanh^{-1} w), \quad w = (1 - x^2/p^2)^{\frac{1}{2}}$,

7.13.4. Uniform asymptotic expressions

LANGER'S FORMULAS

(32) $J_p(x) = w^{-\frac{1}{2}}(w - \tan^{-1} w)^{\frac{1}{2}}\,[J_{1/3}(z)\cos(\pi/6) - Y_{1/3}(z)\sin(\pi/6)]$
$+ O(p^{-4/3})$,

(33) $Y_p(x) = w^{-\frac{1}{2}}(w - \tan^{-1} w)^{\frac{1}{2}}\,[J_{1/3}(z)\sin(\pi/6) + Y_{1/3}(z)\cos(\pi/6)]$
$+ O(p^{-4/3})$,

$x > p, \quad w = (x^2/p^2 - 1)^{\frac{1}{2}}, \quad z = p(w - \tan^{-1} w)$,

(34) $J_p(x) = \pi^{-1} w^{-\frac{1}{2}}(\tanh^{-1} w - w)^{\frac{1}{2}} K_{1/3}(z) + O(p^{-4/3})$,

(35) $Y_p(x) = -w^{-\frac{1}{2}}(\tanh^{-1} w - w)^{\frac{1}{2}}\{I_{1/3}(z) + I_{-1/3}(z)\} + O(p^{-4/3})$

$x < p, \quad w = (1 - x^2/p^2)^{\frac{1}{2}}, \quad z = p(\tanh^{-1} w - w)$.

7.14. Integral formulas

7.14.1. Finite integrals

(1) $\int z^{\nu+1} I_\nu(z)\,dz = z^{\nu+1} I_{\nu+1}(z)$,

(2) $\int z^{-\nu+1} I_\nu(z)\,dz = z^{-\nu+1} I_{\nu-1}(z)$,

(3) $\int z^{\nu+1} K_\nu(z)\,dz = -z^{\nu+1} K_{\nu+1}(z)$,

(4) $\int z^{-\nu+1} K_\nu(z)\,dz = -z^{-\nu+1} K_{\nu-1}(z)$,

(5) $\int z^\nu J_\nu(z)\,dz = 2^{\nu-1}\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})\,z\,[J_\nu(z)\,\mathbf{H}_{\nu-1}(z) - \mathbf{H}_\nu(z)\,J_{\nu-1}(z)]$,

(6) $\int z^\nu K_\nu(z)\,dz = 2^{\nu-1}\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})\,z\,[K_\nu(z)\,\mathbf{L}_{\nu-1}(z) + \mathbf{L}_\nu(z)\,K_{\nu-1}(z)]$,

(7) $\int z^{\mu} J_\nu(z)\, dz = (\mu + \nu - 1)\, z\, J_\nu(z)\, S_{\mu-1,\nu-1}(z) - z\, J_{\nu-1}(z)\, S_{\mu,\nu}(z)$,

(5) and (7) are also valid, when the Bessel function of the first kind is replaced by the Bessel function of the second or third kind.

Let $w_\nu(z)$ and $W_\mu(z)$ be any Bessel function of the first, second, or third kind and the order ν and μ respectively; then

(8) $$\int [(\beta^2 - a^2)\, z + (\nu^2 - \mu^2)/z]\, w_\nu(az)\, W_\mu(\beta z)\, dz$$
$$= z\, [a\, W_\mu(\beta z)\, w_\nu'(az) - \beta\, w_\nu(az)\, W_\mu'(\beta z)]$$
$$= az\, W_\mu(\beta z)\, w_{\nu-1}(az) - \beta z\, W_{\mu-1}(\beta z)\, w_\nu(az)$$
$$+ (\mu - \nu)\, W_\mu(\beta z)\, w_\nu(az),$$

(9) $$\int z w_\nu(az)\, W_\nu(\beta z)\, dz = z\, (\beta^2 - a^2)^{-1}$$
$$\times [\beta W_{\nu+1}(\beta z)\, w_\nu(az) - a\, W_\nu(\beta z)\, w_{\nu+1}(az)],$$

(10) $$\int z w_\nu(az)\, W_\nu(az)\, dz$$
$$= \tfrac{1}{4} z^2\, [2 w_\nu(az) W_\nu(az) - w_{\nu+1}(az) W_{\nu-1}(az) - w_{\nu-1}(az) W_{\nu+1}(az)],$$

(11) $$\int z^{-1}\, w_\nu(az)\, W_\nu(az)\, dz = (2\nu)^{-1}\, w_\nu(az)\, W_\nu(az)$$
$$+ (2\nu)^{-1}\, az \left[w_{\nu+1}(az) \frac{\partial W_\nu(az)}{\partial \nu} - w_\nu(az) \frac{\partial W_{\nu+1}(az)}{\partial \nu} \right].$$

Let $v_\nu(z)$ and $V_\mu(z)$ be any modified Bessel function of the first or second kind and the order ν and μ respectively, then,

(12) $$\int [(\beta^2 - a^2)\, z + (\mu^2 - \nu^2)/z]\, v_\nu(az)\, V_\mu(\beta z)\, dz$$
$$= z\, [-a\, V_\mu(\beta z)\, v_\nu'(az) + \beta\, v_\nu(az)\, V_\mu'(\beta z)],$$

(13) $$\int z\, [v_\nu(az)]^2\, dz$$
$$= -\tfrac{1}{2} z^2\, \{[v_\nu'(az)]^2 - [v_\nu(az)]^2\, (1 + a^{-2} z^{-2} \nu^2)\}.$$

For other indefinite integrals see Watson (1944, pp. 163-138); Thielmann (1929); McLachlan (1934, p. 115); McLachlan and Meyers (1936, p. 437); Straubel (1941, 1942); Picht (1949); Horton (1950); Luke (1950).

(14) $$\int_0^{\frac{1}{2}\pi} J_\mu[z\, (\sin\theta)^2]\, J_\nu[z\, (\cos\theta)^2]\, (\sin\theta \cos\theta)^{-1}\, d\theta$$
$$= \tfrac{1}{2} (\nu^{-1} + \mu^{-1})\, J_{\nu+\mu}(z)/z \qquad \operatorname{Re}\nu > 0, \quad \operatorname{Re}\mu > 0,$$

(15) $$\int_0^{\frac{1}{2}\pi} J_\mu[z\, (\sin\theta)^2]\, J_\nu[z\, (\cos\theta)^2]\, \operatorname{ctn}\theta\, d\theta = \tfrac{1}{2} J_{\nu+\mu}(z)/\mu$$
$$\operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > 0,$$

(16) $\int_0^{\frac{1}{2}\pi} J_\mu[z(\sin\theta)^2]\, J_\nu[z(\cos\theta)^2]\sin\theta\cos\theta\, d\theta$

$$= z^{-1}\sum_{m=0}^{\infty}(-1)^m J_{\nu+\mu+2m+1}(z) \qquad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > -1.$$

(17) $\int_0^{\frac{1}{2}\pi} J_\mu[z(\sin\theta)^2]\, J_\nu[z(\cos\theta)^2]\,(\sin\theta)^{2\lambda-1}(\cos\theta)^{2\delta-1}\, d\theta,$

(Bailey, 1930, p. 419, 1930c, p. 203; Rutgers, 1931.)

(18) $\int_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)\, J_\nu(z\sin\theta)\,(\sin\theta)^{2\delta+1}(\cos\theta)^{2\mu+1}\, d\theta,$

(19) $\int_0^{\frac{1}{2}\pi} J_\lambda(z\sin\theta)\, J_\nu(z\cos\theta)\,(\sin\theta)^{2\delta+1}(\cos\theta)^{2\mu+1}\, d\theta,$

(Bailey, 1938, p. 145.)

(20) $\int_0^{\frac{1}{2}\pi} [J_\nu(z\sin\theta)]^2(\sin\theta)^{2\delta+1}(\cos\theta)^{2\mu+1}\, d\theta$

(Bailey, 1938, p. 141.)

(21) $\int_0^{\frac{1}{2}\pi} [J_\nu(z\sin\theta)]^2\sin\theta\, d\theta = \sum_{m=0}^{\infty} J_{2\nu+2m+1}(z) \qquad \operatorname{Re}\nu > -1,$

See errata!

(22) $\int_0^z t^\lambda \sin(z-t)\, J_\nu(t)\, dt,$

(23) $\int_0^z t^\lambda \cos(z-t)\, J_\nu(t)\, dt,$

(Bailey, 1930c, p. 204, 205.)

(24) $\sin\pi(\nu+\mu)\int_0^{\frac{1}{2}\pi} K_{\mu+\nu}(2z\cos\theta)\cos[(\mu-\nu)\theta]\, d\theta$

$$= \tfrac{1}{2}\pi\,[I_{-\nu}(z)\, I_{-\mu}(z) - I_\nu(z)\, I_\mu(z)]. \qquad |\operatorname{Re}(\mu+\nu)| < 1.$$

7.14.2. Infinite integrals

INTEGRALS WITH EXPONENTIAL FUNCTIONS

(25) $\int_0^\infty Y_{2\nu}(at)\, e^{-\gamma^2 t^2}\, dt$

$$= -\tfrac{1}{2}\pi^{\frac{1}{2}}\gamma^{-1}\exp\left(-\frac{1}{8}a^2/\gamma^2\right)$$
$$\times\left[I_\nu\left(\frac{1}{8}a^2/\gamma^2\right)\tan\nu\pi + \frac{1}{\pi}K_\nu\left(\frac{1}{8}a^2/\gamma^2\right)\sec\nu\pi\right],$$

$$|\operatorname{Re}\nu| < \tfrac{1}{2}.$$

(26) $\int_0^\infty e^{-t}t^{-1}H_\nu^{(1)}(2x^2/t)\, dt = 2K_\nu(2x)\, H_\nu^{(1)}(2x),$

(Hardy, 1927.)

(27) $\int_0^\infty I_\nu(at)\, e^{-\gamma^2 t^2}\, dt = \frac{1}{2}\pi^{\frac{1}{2}}\, \gamma^{-1} \exp\left(\frac{1}{8}\, a^2/\gamma^2\right) I_{\frac{1}{2}\nu}\left(\frac{1}{8}\, a^2/\gamma^2\right)$

$\operatorname{Re}\nu > -1, \quad \operatorname{Re}\gamma^2 > 0.$

[see also 7.14(60) to 7.14(79)].

SPECIAL CASES OF THE WEBER-SCHAFHEITLIN INTEGRAL

(28) $\int_0^\infty t^{-1} J_\mu(at) \sin(bt)\, dt = \mu^{-1} \sin[\mu \sin^{-1}(b/a)] \qquad b < a,$

$= a^\mu \mu^{-1} \sin(\frac{1}{2}\pi\mu)\, [b + (b^2 - a^2)^{\frac{1}{2}}]^{-\mu} \qquad b > a,$

$\operatorname{Re}\mu > -1.$

(29) $\int_0^\infty t^{-1} J_\mu(at) \cos(bt)\, dt = \mu^{-1} \cos[\mu \sin^{-1}(b/a)] \qquad b < a,$

$= \mu^{-1} a^\mu \cos(\frac{1}{2}\mu\pi)\, [b + (b^2 - a^2)^{\frac{1}{2}}]^{-\mu} \qquad b > a,$

$\operatorname{Re}\mu > 0.$

(30) $\int_0^\infty J_\mu(at) \cos(bt)\, dt = (a^2 - b^2)^{-\frac{1}{2}} \cos[\mu \sin^{-1}(b/a)] \qquad b < a.$

$= -a^\mu \sin(\frac{1}{2}\mu\pi)\, (b^2 - a^2)^{-\frac{1}{2}} [b + (b^2 - a^2)^{\frac{1}{2}}]^{-\mu} \qquad b > a,$

$\operatorname{Re}\mu > -1.$

(31) $\int_0^\infty J_\mu(at) \sin(bt)\, dt = (a^2 - b^2)^{-\frac{1}{2}} \sin[\mu \sin^{-1}(b/a)] \qquad b < a,$

$= a^\mu \cos(\frac{1}{2}\mu\pi)\, (b^2 - a^2)^{-\frac{1}{2}}\, [b + (b^2 - a^2)^{\frac{1}{2}}]^{-\mu} \qquad b > a,$

$\operatorname{Re}\mu > -2.$

For the corresponding integrals for the Neumann function, see Nielsen, (1904, p. 195).

(32) $\frac{1}{2}\pi(\nu^2 - \mu^2) \int_0^\infty J_\mu(at)\, J_\nu(at)\, t^{-1}\, dt = \sin\frac{1}{2}(\nu - \mu)\, \pi,$

$\operatorname{Re}(\nu + \mu) > 0.$

(33) $\int_0^\infty J_\mu(at)\, J_\nu(at)\, t^{-(\nu+\mu)}\, dt = \dfrac{\pi^{\frac{1}{2}}\, (\frac{1}{2}a)^{\nu+\mu}\, \Gamma(\mu + \nu)}{a\, \Gamma(\frac{1}{2} + \nu + \mu)\, \Gamma(\mu + \frac{1}{2})\, \Gamma(\nu + \frac{1}{2})},$

$\operatorname{Re}(\nu + \mu) > 0.$

(34) $\Gamma(\nu - \mu) \int_0^\infty J_\mu(at)\, J_\nu(bt)\, t^{\mu-\nu+1}\, dt$

$= 2^{\mu-\nu+1} a^\mu b^{-\nu} (b^2 - a^2)^{\nu-\mu-1} \qquad b > a,$

$= 0 \qquad b < a,$

$\operatorname{Re}\nu > \operatorname{Re}\mu > -1,$

INTEGRALS RELATED WITH THE WEBER-SCHAFHEITLIN INTEGRAL

(35) $2^{\rho+1}\,\Gamma(\nu+1)\,a^{\nu+1-\rho}\int_0^\infty K_\mu(at)\,I_\nu(bt)\,t^{-\rho}\,dt$

$$= b^\nu\,\Gamma(\tfrac12-\tfrac12\rho+\tfrac12\mu+\tfrac12\nu)\,\Gamma(\tfrac12-\tfrac12\rho-\tfrac12\mu+\tfrac12\nu)$$

$$\times\,{}_2F_1(\tfrac12-\tfrac12\rho+\tfrac12\mu+\tfrac12\nu,\ \tfrac12-\tfrac12\rho-\tfrac12\mu+\tfrac12\nu;\ \nu+1;\ b^2/a^2),$$

$\operatorname{Re}(\nu-\rho+1\pm\mu)>0, \quad a>b,$

(36) $2^{\rho+2}\,\Gamma(1-\rho)\int_0^\infty K_\mu(at)\,K_\nu(\beta t)\,t^{-\rho}\,dt$

$$= \alpha^{\rho-\nu-1}\,\beta^\nu\,{}_2F_1(\tfrac12+\tfrac12\nu+\tfrac12\mu-\tfrac12\rho,\ \tfrac12+\tfrac12\nu-\tfrac12\mu-\tfrac12\rho;\ 1-\rho;\ 1-\beta^2/\alpha^2)$$

$$\times\,\Gamma(\tfrac12+\tfrac12\nu+\tfrac12\mu-\tfrac12\rho)\,\Gamma(\tfrac12+\tfrac12\nu-\tfrac12\mu-\tfrac12\rho)$$

$$\times\,\Gamma(\tfrac12-\tfrac12\nu+\tfrac12\mu-\tfrac12\rho)\,\Gamma(\tfrac12-\tfrac12\nu-\tfrac12\mu-\tfrac12\rho)$$

$\operatorname{Re}(\alpha+\beta)>0, \quad \operatorname{Re}(\rho\pm\mu\pm\nu+1)>0,$

(37) $\tfrac12\pi\int_0^\infty Y_\mu(at)\,J_\nu(bt)\,t^{-\rho}\,dt = \sin\tfrac12\pi(\nu-\mu-\rho)$

$$\times\int_0^\infty K_\mu(at)\,I_\nu(bt)\,t^{-\rho}\,dt \qquad a>b, \quad \operatorname{Re}(\nu-\rho+1\pm\mu)>0,$$

(38) $\int_0^\infty Y_\nu(bt)\,J_\mu(at)\,t^{-\rho}\,dt$

$$= -\int_0^\infty \{Y_\mu(at)\,J_\nu(bt) + (4/\pi^2)\cos[\tfrac12\pi(\rho+\nu+\mu)]K_\mu(at)K_\nu(bt)\}\,t^{-\rho}\,dt$$

$a>b, \quad \operatorname{Re}(\rho+\nu-\mu)>-1, \quad \operatorname{Re}\rho>-1,$

(39) $\int_0^\infty J_\nu(\beta t)\,K_\mu(\alpha t)\,t^{\nu+\mu+1}\,dt$

$$= (2\beta)^\nu\,(2\alpha)^\mu\,\Gamma(\nu+\mu+1)\,(\alpha^2+\beta^2)^{-\nu-\mu-1}$$

$\operatorname{Re}(\nu+1)>|\operatorname{Re}\mu|, \quad \operatorname{Re}\alpha>|\operatorname{Im}\beta|.$

For further combinations see Dixon and Ferrar (1930).

INTEGRALS INVOLVING PRODUCTS OF THREE AND MORE BESSEL FUNCTIONS

(40) $\int_0^\infty t^{\rho-1}\,J_\mu(at)\,J_\nu(bt)\,J_\lambda(ct)\,dt,$

Watson,(1934).

(41) $\int_0^\infty t^{\rho-1}\,J_\mu(at)\,J_\nu(bt)\begin{Bmatrix} J_\lambda(ct) \\ K_\lambda(ct)\end{Bmatrix}dt,$

(42) $\int_0^\infty t^{\rho-1}\, I_\mu(at)\, K_\lambda(ct) \left\{ \begin{array}{c} I_\nu(bt) \\ K_\nu(bt) \end{array} \right\} dt,$

(43) $\int_0^\infty t^{\rho-1}\, K_\mu(at)\, K_\nu(bt)\, K_\rho(ct)\, dt,$

(Bailey, 1935 a, 1936.)

(44) $\int_0^\infty [J_\nu(ax)]^2\, [J_\nu(bx)]^2\, x^{1-2\nu}\, dx$

$$= \frac{a^{2\nu-1}\,\Gamma(\nu)}{2\pi b\,\Gamma(\nu+\frac{1}{2})\,\Gamma(2\nu+\frac{1}{2})}\; {}_2F_1(\nu,\, \tfrac{1}{2}-\nu;\, 2\nu+\tfrac{1}{2};\, a^2/b^2)$$

$0 < \mathrm{Re}\,\nu,$

(45) $\int_0^\infty J_\nu(ax)\, Y_\nu(ax)\, J_\nu(bx)\, Y_\nu(bx)\, x^{2\nu+1}\, dx$

$$= \frac{a^{2\nu}\, b^{-2-4\nu}\,\Gamma(3\nu+1)}{2\pi\,\Gamma(1/2-\nu)\,\Gamma(2\nu+3/2)}\; {}_2F_1(\nu+1/2,\, 3\nu+1;\, 2\nu+3/2;\, a^2/b^2)$$

$-1/3 < \mathrm{Re}\,\nu < 1/2.$

(For other formulas see Nicholson, 1920, 1927; Titchmarsh, 1927; Mitra, 1933; Mayr, 1933; Sinha, 1943.)

INTEGRALS OF THE SONINE GEGENBAUER TYPE

(46) $\int_0^\infty J_\mu(bt)\, K_\nu[a(t^2+z^2)^{\frac{1}{2}}]\,(t^2+z^2)^{-\frac{1}{2}\nu}\, t^{\mu+1}\, dt$

$$= b^\mu\, a^{-\nu}\, z^{\mu-\nu+1}\,(a^2+b^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}\, K_{\nu-\mu-1}[z(a^2+b^2)^{\frac{1}{2}}]$$

$\mathrm{Re}\,\mu > -1, \quad \mathrm{Re}\,z > 0.$

(47) $\int_0^\infty J_\mu(bt)\, K_\nu[a(t^2-y^2)^{\frac{1}{2}}]\,(t^2-y^2)^{-\frac{1}{2}\nu}\, t^{\mu+1}\, dt$

$$= \tfrac{1}{2}\pi e^{-i\pi(\nu-\mu-\frac{1}{2})}\, b^\mu\, a^{-\nu}\, y^{1+\mu-\nu}\,(a^2+b^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}$$
$$\times H^{(2)}_{\nu-\mu-1}[y(a^2+b^2)^{\frac{1}{2}}]$$

$\mathrm{Re}\,\nu < 1, \quad \mathrm{Re}\,\mu > -1, \quad \arg(t^2-y^2)^{\frac{1}{2}} = 0 \quad \text{if} \quad t > y,$
$\arg(t^2-y^2)^\sigma = \pi\sigma \quad \text{if} \quad t < y,$ where $\sigma = \frac{1}{2}$ and $-\frac{1}{2}\nu$, respectively.

(48) $\int_0^\infty J_\mu(bt)\, H^{(2)}_\nu[a(t^2+x^2)^{\frac{1}{2}}]\,(t^2+x^2)^{-\frac{1}{2}\nu}\, t^{\mu+1}\, dt$

$$= a^{-\nu}\, b^\mu\, x^{1+\mu-\nu}(a^2-b^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}\, H^{(2)}_{\nu-\mu-1}[x(a^2-b^2)^{\frac{1}{2}}]$$

$a > b,$

$$= 2i\pi^{-1}\, b^\mu\, a^{-\nu}\, x^{1+\mu-\nu}(b^2-a^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}\, K_{\nu-\mu-1}[x(b^2-a^2)^{\frac{1}{2}}]$$

$a < b,$

$\mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1, \quad x > 0.$

(49) $\int_0^\infty H_\nu^{(2)}[a(t^2+x^2)^{1/2}]\,(t^2+x^2)^{-1/2\nu}\,t^{2\mu+1}\,dt$

$= 2^\mu a^{-\mu-1} x^{1+\mu-\nu}\,\Gamma(\mu+1)\,H_{\nu-\mu-1}^{(2)}(ax)$

$\mathrm{Re}(\tfrac{1}{2}\nu-\tfrac{1}{4}) > \mathrm{Re}\,\mu > -1.$

(50) $\int_0^\infty K_\nu[a(t^2+z^2)^{1/2}]\,(t^2+z^2)^{-1/2\nu}\,t^{2\mu+1}\,dt$

$= 2^\mu a^{-\mu-1} z^{1+\mu-\nu}\,\Gamma(\mu+1)\,K_{\nu-\mu-1}(az)$

$a>0, \quad \mathrm{Re}\,\mu > -1.$

(51) $\int_0^\infty J_\mu(bt)\,(t^2+z^2)^{-\nu}\,t^{\mu+1}\,dt = (\tfrac{1}{2}b)^{\nu-1} z^{1+\mu-\nu} K_{\nu-\mu-1}(bz)/\Gamma(\nu)$

$\mathrm{Re}(2\nu-\tfrac{1}{2}) > \mathrm{Re}\,\mu > -1, \quad \mathrm{Re}\,z>0.$

(52) $\int_0^\infty J_0(bt)\,e^{-a(t^2-y^2)^{1/2}}\,(t^2-y^2)^{-1/2}\,t\,dt = e^{-iy(a^2+b^2)^{1/2}}\,(a^2+b^2)^{-1/2}$

$\arg(t^2-y^2)^{1/2} = \tfrac{1}{2}\pi \quad \text{if } t<y,$

(53) $\pi e^{-iy(a^2+b^2)^{1/2}}\,(a^2+b^2)^{-1/2} = 2\int_0^\infty \cos(bt)\,K_0[a(t^2-b^2)^{1/2}]\,dt$

$= -\pi i \int_0^\infty \cos(bt)\,H_0^{(2)}[a(b^2-t^2)^{1/2}]\,dt,$ see errata!

(For similar formulas see Watson, 1944, p. 417-418; Mayr, 1932; Gupta, 1943 b.)

(54) $e^{i\frac{1}{2}\pi(\rho-\nu)}\int_0^\infty t^{\rho-1}\,J_\mu[b(t^2+y^2)^{1/2}]\,(t^2+y^2)^{-1/2\mu}\,(t^2-a^2)^{-m-1}$

$\times\{\cos[\tfrac{1}{2}\pi(\rho-\nu)]\,J_\nu(at) + \sin[\tfrac{1}{2}\pi(\rho-\nu)]\,Y_\nu(at)\}\,dt$

$= \frac{\pi i}{m!}\,2^{-m-1}\left(\frac{d}{a\,da}\right)^m \{a^{\rho-2}\,J_\mu[b(a^2+y^2)^{1/2}]\,(a^2+y^2)^{-1/2\mu}H_\nu^{(1)}(a\,a)\}$

$a \geq b, \quad \mathrm{Re}(\pm\nu) < \mathrm{Re}\,\rho < 2m+4+\mathrm{Re}\,\mu, \quad \mathrm{Re}(i\,a)<0, \quad m=0,1,2,\ldots,$

(55) $\int_0^\infty t^{\rho-1}\,J_\mu[b(t^2+y^2)^{1/2}]\,(t^2+y^2)^{-1/2\mu}\,(t^2+\beta^2)^{-m-1}$

$\times\{\cos[\tfrac{1}{2}\pi(\rho-\nu)]\,J_\nu(at) + \sin[\tfrac{1}{2}\pi(\rho-\nu)]\,Y_\nu(at)\}\,dt$

$= (-1)^{m+1}\frac{2^{-m}}{m!}\left(\frac{d}{\beta\,d\beta}\right)^m \{\beta^{\rho-2} J_\mu[b(y^2-\beta^2)^{1/2}](y^2-\beta^2)^{-1/2\mu}K_\nu(a\beta)\}$

$a \geq b, \quad \mathrm{Re}(\pm\nu) < \mathrm{Re}\,\rho < 2m+4+\mathrm{Re}\,\mu, \quad \mathrm{Re}\,\beta>0, \quad m=0,1,2,\ldots,$

(56) $\int_0^\infty t^{\nu+1}\,J_\mu[b(t^2+y^2)^{1/2}]\,(t^2+y^2)^{-1/2\mu}\,(t^2+\beta^2)^{-1}\,J_\nu(at)\,dt$

$= \beta^\nu J_\mu[b(y^2-\beta^2)^{1/2}]\,(y^2-\beta^2)^{-1/2\mu}K_\nu(a\beta)$

$a \geq b, \quad \mathrm{Re}\,\beta>0, \quad -1<\mathrm{Re}\,\nu<2+\mathrm{Re}\,\mu,$

(57) $\int_0^\infty t^{\nu-\mu+1} J_\mu(bt) J_\nu(at) (t^2+\beta^2)^{-1}\, dt = \beta^{\nu-\mu} I_\mu(b\beta) K_\nu(a\beta)$

$a \geq b, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu-\mu) < 2, \quad \operatorname{Re}\beta > 0.$

(58) $\int_0^\infty t^{\nu+1} J_\nu(at) (t^2+\beta^2)^{-1}\, dt = \beta^\nu K_\nu(a\beta)$

$a > 0, \quad \operatorname{Re}\beta > 0, \quad -1 < \operatorname{Re}\nu < 3/2.$

(59) $\int_0^\infty t^{\nu+1} J_\nu(at) (t^2+\beta^2)^{-\mu-1}\, dt = a^\mu \beta^{\nu-\mu} 2^{-\mu} K_{\nu-\mu}(a\beta)/\Gamma(\mu+1)$

$-1 < \operatorname{Re}\nu < 2\operatorname{Re}\mu + 3/2.$

For similar integrals see Watson (1944 p. 434-435.).

PRODUCTS OF BESSEL FUNCTIONS

(60) $K_\mu(Z) K_\nu(z) = \int_{-\infty}^\infty e^{-(\mu-\nu)t} \left(\dfrac{Ze^t + ze^{-t}}{Ze^{-t} + ze^t}\right)^{\frac{1}{2}(\nu+\mu)}$

$\times K_{\nu+\mu}[(Z^2 + z^2 + 2Zz\cosh 2t)^{\frac{1}{2}}]\, dt \qquad \operatorname{Re}(Z) > 0, \quad \operatorname{Re} z > 0.$

(61) $2\pi J_\mu(X) J_\nu(x) = \int_{-\pi}^\pi e^{i\nu\theta} \left(\dfrac{X - xe^{-i\theta}}{X - xe^{i\theta}}\right)^{\frac{1}{2}(\nu+\mu)} [\cos\nu\pi\, J_{\mu+\nu}(w)$

$- \sin\nu\pi\, Y_{\nu+\mu}(w)]\,d\theta - 2\sin\nu\pi \int_0^\infty e^{-\nu t} \left(\dfrac{X + xe^t}{X + xe^{-t}}\right)^{\frac{1}{2}(\nu+\mu)}$

$\times [\cos\nu\pi\, J_{\mu+\nu}(\Phi) - \sin\nu\pi\, Y_{\mu+\nu}(\Phi)]\, dt$

$X > x > 0, \quad \operatorname{Re}(\mu-\nu) < \frac{1}{2}, \qquad w = (X^2 + x^2 - 2xX\cos\theta)^{\frac{1}{2}},$
$\Phi = (X^2 + x^2 + 2Xx\cosh t)^{\frac{1}{2}},$

(Dixon and Ferrar, 1933, p. 193, 194).

(62) $J_\mu(z) J_\nu(z) + Y_\mu(z) Y_\nu(z)$

$= 4\pi^{-2} \int_0^\infty K_{\mu+\nu}(2z\sinh t)\, [e^{(\mu-\nu)t}\cos\nu\pi + e^{-(\mu-\nu)t}\cos\mu\pi]\, dt$

$\operatorname{Re} z > 0, \quad |\operatorname{Re}(\nu+\mu)| < 1.$

(63) $J_\mu(z) J_\nu(z) + Y_\mu(z) Y_\nu(z)$

$= 4\pi^{-2} \int_0^\infty K_{\nu-\mu}(2z\sinh t)\, [e^{(\mu+\nu)t} + e^{-(\mu+\nu)t}\cos(\mu-\nu)\,t]\, dt$

$\operatorname{Re} z > 0, \quad |\operatorname{Re}(\nu-\mu)| < 1.$

(64) $J_\mu(x) J_\nu(x) - Y_\mu(x) Y_\nu(x) = 4\pi^{-1} \int_0^\infty Y_{\mu+\nu}(2x\cosh t) \cosh[(\mu-\nu)t]\, dt$

$x > 0.$

(65) $J_\mu(x)\,Y_\nu(x)+J_\nu(x)\,Y_\mu(x)=-4\pi^{-1}\int_0^\infty J_{\mu+\nu}(2x\cosh t)\cosh[(\mu-\nu)t]\,dt$

$x>0.$

(66) $J_\mu(z)\,Y_\nu(z)-J_\nu(z)\,Y_\mu(z)$

$=4\pi^{-2}\int_0^\infty K_{\nu+\mu}(2z\sinh t)\,[e^{(\nu-\mu)t}\sin(\mu\pi)-e^{(\mu-\nu)t}\sin(\nu\pi)]\,dt$

$\operatorname{Re} z>0,\quad |\operatorname{Re}(\nu+\mu)|<1.$

(67) $J_\mu(z)\,Y_\nu(z)-J_\nu(z)\,Y_\mu(z)$

$=4\pi^{-2}\sin[(\mu-\nu)\pi]\int_0^\infty K_{\nu-\mu}(2z\sinh t)\,e^{-(\nu+\mu)t}\,dt$

$\operatorname{Re} z>0,\quad |\operatorname{Re}(\nu-\mu)|<1,$

(68) $K_\nu(x)\,I_\mu(x)=\int_0^\infty J_{\nu+\mu}(2x\sinh t)\,e^{(\nu-\mu)t}\,dt$

$\operatorname{Re}(\nu-\mu)<3/2,\quad \operatorname{Re}(\nu+\mu)>-1,\quad x>0.$

(69) $[K_\nu(x)]^2\sin(\nu\pi)=\pi\int_0^\infty J_0(2x\sinh t)\sinh(2\nu t)\,dt$

$|\operatorname{Re}\nu|<3/4,\quad x>0.$

(70) $[K_\nu(x)]^2\cos(\nu\pi)=-\pi\int_0^\infty Y_0(2x\sinh t)\cosh(2\nu t)\,dt$

$|\operatorname{Re}\nu|<3/4,\quad x>0.$

(71) $I_\nu(x)\,K_\mu(x)+I_\mu(x)\,K_\nu(x)=2\int_0^\infty J_{\nu+\mu}(2x\sinh t)\cosh[(\mu-\nu)t]\,dt$

$\operatorname{Re}(\nu+\mu)>-1,\quad |\operatorname{Re}(\mu-\nu)|<3/2,\quad x>0.$

(72) $I_\nu(x)\,K_\mu(x)-I_\mu(x)\,K_\nu(x)=2\int_0^\infty J_{\nu+\mu}(2x\sinh t)\sinh[(\mu-\nu)t]\,dt$

$\operatorname{Re}(\nu+\mu)>-1,\quad |\operatorname{Re}(\mu-\nu)|<3/2,\quad x>0.$

(73) $I_\mu(x)\,K_\nu(x)-\cos[(\nu-\mu)\pi]\,I_\nu(x)\,K_\mu(x)$

$=\sin[\pi(\mu-\nu)]\int_0^\infty Y_{\nu-\mu}(2x\sinh t)\,e^{-(\nu+\mu)t}\,dt$

$x>0,\quad |\operatorname{Re}(\nu-\mu)|<1,\quad \operatorname{Re}(\nu+\mu)>-1/2,$

(74) $J_\nu(z)\,\dfrac{\partial Y_\nu(z)}{\partial\nu}-Y_\nu(z)\,\dfrac{\partial J_\nu(z)}{\partial\nu}=-4\pi^{-1}\int_0^\infty K_0(2z\sinh t)\,e^{-2\nu t}\,dt$

$\operatorname{Re} z>0.$

(75) $I_\nu(x)\,\dfrac{\partial K_\nu(x)}{\partial\nu}-K_\nu(x)\,\dfrac{\partial I_\nu(x)}{\partial\nu}=\pi\int_0^\infty Y_0(2x\sinh t)\sinh(2\nu t)\,dt$

$+\cos(\nu\pi)\,[K_\nu(x)]^2\qquad x>0,\quad |\operatorname{Re}\nu|<3/4.$

For most of these formulas see Dixon and Ferrar (1930) and Meijer (1936, p. 519).

(76) $$H_\nu^{(2)}(x)\,H_\mu^{(2)}(y) = (\tfrac{1}{2}\pi)^{-1}\, i \int_{-\infty}^{\infty} e^{-(\nu-\mu)t}\left(\frac{xe^{-t}+ye^{t}}{xe^{t}+ye^{-t}}\right)^{\frac{1}{2}(\nu+\mu)} \times H_{\nu+\mu}^{(2)}[(x^2+y^2+2xy\cosh 2t)^{\frac{1}{2}}]\,dt \qquad |\mathrm{Re}\,(\nu-\mu)| < 3/2,$$

(77) $$2\pi K_\mu(x)\,I_\nu(y) = \int_{-\pi}^{\pi} e^{-i\nu\phi}\left(\frac{x-ye^{i\phi}}{x-ye^{-i\phi}}\right)^{\frac{1}{2}(\nu+\mu)} \times K_{\nu+\mu}[(x^2+y^2-2xy\cos\phi)^{\frac{1}{2}}]\,d\phi - 2\sin(\nu\pi) \times \int_0^\infty e^{\nu t}\left(\frac{x+ye^{-t}}{x+ye^{t}}\right)^{\frac{1}{2}(\nu+\mu)} K_{\nu+\mu}[(x^2+y^2+2xy\cosh t)^{\frac{1}{2}}]\,dt \qquad x > y,$$

Dixon and Ferrar (1933).

(78) $$I_\nu(z)\,K_\nu(\xi) = \int_0^\infty J_{2\nu}[2(z\xi)^{\frac{1}{2}}\sinh t]\, e^{-(\xi-z)\cosh t}\,dt \qquad \mathrm{Re}\,\nu > -\tfrac{1}{2}, \quad \mathrm{Re}\,(\xi - z) > 0,$$

(79) $$K_\nu(z)\,K_\nu(\xi) = 2\cos(\nu\pi)\int_0^\infty K_{2\nu}[2(z\xi)^{\frac{1}{2}}\sinh t]\, e^{-(\xi+z)\cosh t}\,dt \qquad -\tfrac{1}{2} < \mathrm{Re}\,\nu < \tfrac{1}{2}, \quad \mathrm{Re}\,(z^{\frac{1}{2}} + \xi^{\frac{1}{2}})^2 \geq 0.$$

[see also 7.14(25) to 7.14(27)].

INTEGRALS INVOLVING STRUVE'S FUNCTIONS

(80) $$\int_0^\infty t^{\mu-\nu-1}\,\mathbf{H}_\nu(t)\,dt = \frac{\Gamma(\tfrac{1}{2}\mu)\,2^{\mu-\nu-1}\tan(\tfrac{1}{2}\mu\pi)}{\Gamma(\nu-\tfrac{1}{2}\mu+1)} \qquad -1 < \mathrm{Re}\,\mu < 1, \quad \mathrm{Re}\,\nu > \mathrm{Re}\,\mu - 3/2,$$

(81) $$\int_0^\infty \mathbf{H}_\nu(t)\,\mathbf{H}_\mu(t)\,t^{-\mu-\nu}\,dt = \frac{\pi^{\frac{1}{2}}\,\Gamma(\mu+\nu)\,2^{-\mu-\nu}}{\Gamma(\mu+\nu+\tfrac{1}{2})\,\Gamma(\mu+\tfrac{1}{2})\,\Gamma(\nu+\tfrac{1}{2})} \qquad \mathrm{Re}\,(\mu+\nu) > 0,$$

(82) $$\int_0^\infty \mathbf{H}_\nu(2zt)(t^2-1)^{-\nu-\frac{1}{2}}\,t^{-\nu}\,dt = \tfrac{1}{2}\pi^{\frac{1}{2}}\,\Gamma(\tfrac{1}{2}-\nu)\,z^\nu\,[J_\nu(z)]^2 \qquad z > 0, \quad |\mathrm{Re}\,\nu| < \tfrac{1}{2}.$$

For further integrals involving Struve's functions see Mohan (1942); Horton (1950).

7.15. Series of Bessel functions

SERIES OF THE NEUMANN TYPE

(1) $$z^\nu e^{yz} = 2^\nu\,\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\,C_n^\nu(y)\,I_{\nu+n}(z),$$

(2) $$(\tfrac{1}{2}z)^{\mu-\nu} J_\nu(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)\,\Gamma(\nu+1-\mu)\,(\mu+2n)}{n!\,\Gamma(\nu+1-\mu-n)\,\Gamma(\nu+n+1)} J_{\mu+2n}(z).$$

(If $\nu-\mu$ is a non-negative integer this expression reduces to a finite sum.)

(3) $$J_\nu(z\sin\theta) = (\tfrac{1}{2}\pi z)^{-\frac{1}{2}} (\sin\theta)^\nu \sum_{n=0}^{\infty} \frac{(\nu+\frac{1}{2}+2n)\,\Gamma(n+\frac{1}{2})}{\Gamma(n+\nu+1)} \Gamma(\nu+\tfrac{1}{2}) \times C_{2n}^{\nu+\frac{1}{2}}(\cos\theta)\, J_{\nu+\frac{1}{2}+2n}(z),$$

(4) $$\mathbf{H}_\nu(z)\,\Gamma(\nu+\tfrac{1}{2}) = 4\pi^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\nu+1+2n)\,\Gamma(\nu+1+n)}{n!\,(2n+2\nu+1)\,(2n+1)} J_{\nu+1+2n}(z),$$

(5) $$\mathbf{J}_\nu(z)\,\pi = \sin\nu\pi \left[\nu^{-1} J_0(z) + \sum_{n=1}^{\infty} \left(\frac{1}{\nu+n} + \frac{(-1)^n}{\nu-n}\right) J_n(z)\right],$$

(6) $$(\tfrac{1}{2}z)^{\gamma-\nu-\mu} J_\mu(\alpha z)\, J_\nu(\beta z) = [\alpha^\mu \beta^\nu/\Gamma(\nu+1)] \sum_{m=0}^{\infty} (\gamma+2m)\, J_{\gamma+2m}(z) \times \left[\sum_{n=0}^{\infty} \frac{(-1)^n\,\Gamma(\gamma+m+n)\,\alpha^{2n}}{n!\,(m-n)!\,[\Gamma(n+\mu+1)]^2}\, {}_2F_1(-n, -n-\mu; \nu+1; \beta^2/\alpha^2)\right],$$

(7) $$(\tfrac{1}{2}z)^{\gamma-\mu-\nu} J_\mu(\alpha z)\, J_\nu(\beta z) = \alpha^\mu \beta^\nu/[\Gamma(\mu+1)\,\Gamma(\nu+1)] \times \sum_{m=0}^{\infty} \frac{(\gamma+2m)\,\Gamma(\gamma+m)}{m!} F_4(-m, \gamma+m; \mu+1, \nu+1; \alpha^2, \beta^2)\, J_{\gamma+2m}(z),$$

(Bailey, 1935.)

(8) $$\tfrac{1}{2}z\, J_\mu(z\cos\phi\cos\Phi)\, J_\nu(z\sin\phi\sin\Phi)$$
$$= (\cos\phi\cos\Phi)^\mu (\sin\phi\sin\Phi)^\nu \sum_{n=0}^{\infty} (-1)^n (\mu+\nu+2n+1)\, J_{\mu+\nu+2n+1}(z)$$
$$\times \frac{\Gamma(\mu+\nu+n+1)\,\Gamma(\nu+n+1)}{n!\,\Gamma(\mu+n+1)\,[\Gamma(\nu+1)]^2}\, {}_2F_1[-n, \mu+\nu+n+1; \nu+1; (\sin\phi)^2]$$
$$\times {}_2F_1(-n, \mu+\nu+n+1; \nu+1; (\sin\Phi)^2]$$

μ, ν not a negative integer.

(Watson, 1944, p. 370; Bailey, 1929).

(9) $$z^\nu = 2^\nu\,\Gamma(1+\tfrac{1}{2}\nu) \sum_{n=0}^{\infty} (\tfrac{1}{2}z)^{\frac{1}{2}\nu+n} J_{\frac{1}{2}\nu+n}(z)/n!$$

(10) $$\Gamma(\nu-\mu)\, J_\nu(z) = \Gamma(\mu+1) \sum_{n=0}^{\infty} \frac{\Gamma(\nu-\mu+n)}{\Gamma(\nu+n+1)\,n!} (\tfrac{1}{2}z)^{\nu-\mu+n} J_{\mu+n}(z),$$

$\nu \neq \mu$, μ not a negative integer.

(11) $J_\nu(z\cos\theta)\, J_\nu(z\sin\theta) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}z\sin 2\theta)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} J_{\nu+2n}(z)$

ν not a negative integer,

(12) $(z+h)^{\pm\frac{1}{2}\nu} J_\nu[(z+h)^{\frac{1}{2}}] = \sum_{m=0}^{\infty} \frac{(\pm\frac{1}{2}h)^m}{m!} z^{\pm\frac{1}{2}\nu-\frac{1}{2}m} J_{\nu\mp m}(z^{\frac{1}{2}}),$

$|h| < |z|,$

(13) $(z+h)^{\pm\frac{1}{2}\nu} Y_\nu[(z+h)^{\frac{1}{2}}] = \sum_{m=0}^{\infty} \frac{(\pm\frac{1}{2}h)^m}{m!} z^{\pm\frac{1}{2}\nu-\frac{1}{2}m} Y_{\nu\mp m}(z^{\frac{1}{2}}),$

$|h| < |z|,$

(14) $H_0^{(1)}[z(1-a)^{\frac{1}{2}}] = \sum_{m=-\infty}^{\infty} (\frac{1}{2}az)^{m-\nu} H_{m-\nu}^{(1)}(z)/\Gamma(m-\nu+1),$

(15) $H_1^{(1)}[z(1-a)^{\frac{1}{2}}] = (1-a)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} (\frac{1}{2}az)^{m-\nu} H_{m-\nu+1}^{(1)}(z)/\Gamma(m-\nu+1),$

(16) $(\frac{1}{2}\pi z)^{-\frac{1}{2}} \cos[(z^2-2zt)^{\frac{1}{2}}] = \sum_{m=-\infty}^{\infty} t^{m-\nu} J_{m-\nu-\frac{1}{2}}(z)/\Gamma(m-\nu+1),$

(17) $(\frac{1}{2}\pi z)^{-\frac{1}{2}} \sin[(z^2+2zt)^{\frac{1}{2}}] = \sum_{m=-\infty}^{\infty} t^{m-\nu} J_{-(m-\nu-\frac{1}{2})}(z)/\Gamma(m-\nu+1),$

(18) $(s^2-r^2)^{-\frac{1}{2}\nu} H_\nu^{(1)}[z(s^2-r^2)^{\frac{1}{2}}] = \sum_{m=0}^{\infty} (\frac{1}{2}zr^2)^m s^{-\nu-m} H_{\nu+m}^{(1)}(zs)/m!\,,$

(19) $(s^2-r^2)^{-\frac{1}{2}\nu} K_\nu[z(s^2-r^2)^{\frac{1}{2}}] = \sum_{m=0}^{\infty} (\frac{1}{2}zr^2)^m s^{-\nu-m} K_{\nu+m}(zs)/m!\,,$

(20) $(\nu\pi)^{-1}\sin(\nu\pi) = J_\nu(z)\,J_{-\nu}(z) + 2\sum_{n=1}^{\infty} J_{n+\nu}(z)\,J_{n-\nu}(z),$

(21) $J_\nu(2z\cos\theta) = [J_{\frac{1}{2}\nu}(z)]^2 + 2\sum_{n=1}^{\infty} J_{\frac{1}{2}\nu-n}(z)\,J_{\frac{1}{2}\nu+n}(z)\cos(2n\theta)$

$\mathrm{Re}\,\nu > 0, \quad -\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi,$

(22) $[J_\nu(z)]^2 = 2\sum_{n=1}^{\infty} (-1)^{n-1} J_{\nu+n}(z)\,J_{\nu-n}(z)$ $\qquad \mathrm{Re}\,z > 0,$

(23) $J_{2\nu}(2z) = \frac{1}{2}\pi z^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n [n!\,\Gamma(3/2-n)]^{-1} J_{\nu+n}(z)\,J_{\nu-\frac{1}{2}+n}(z),$

(24) $J_0[x(t-t^{-1})^{\frac{1}{2}}] = J_0(x)\,I_0(x) + \sum_{n=1}^{\infty} [(-t)^n + t^{-n}]\,J_n(x)\,I_n(x),$

(25) $J_0[x(t+t^{-1})] = [J_0(x)]^2 + \sum_{n=1}^{\infty} (-1)^n (t^{2n} + t^{-2n}) [J_n(x)]^2,$

(26) $\operatorname{ber}(2^{1/2}x) = J_0(x)\, I_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x)\, I_{2n}(x),$

(27) $\operatorname{bei}(2^{1/2}x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x)\, I_{2n+1}(x).$

(For further examples see Bailey, 1935, p. 235; Wise, 1935; Banerjee, 1939; Bateman and Rice, 1935; Fox, 1927; Rice, 1944; Rutgers, 1942; Nielsen, 1904, Ch. XIX to XXI.)

ADDITION THEOREMS AND RELATED SERIES

$w = (z^2 + Z^2 - 2zZ\cos\phi)^{1/2}$ and $C_n^{\nu}(z)$ is Gegenbauer's polynomial (see section 3.15).

(28) $w^{-\nu} H_{\nu}^{(1),(2)}(w) = (\tfrac{1}{2}zZ)^{-\nu}\Gamma(\nu) \sum_{n=0}^{\infty} (\nu+n) C_n^{\nu}(\cos\phi) J_{\nu+n}(z) H_{\nu+n}^{(1),(2)}(Z)$

$\nu \neq 0, -1, -2, \dots, \quad |ze^{\pm i\phi}| < |Z|,$

(29) $H_0^{(1),(2)}(w) = J_0(z)\, H_0^{(1),(2)}(Z) + 2 \sum_{n=1}^{\infty} J_n(z)\, H_n^{(1),(2)}(Z) \cos(n\phi)$

$|ze^{\pm i\phi}| < |Z|,$

(30) $w^{-\nu} J_{\nu}(w) = (\tfrac{1}{2}zZ)^{-\nu}\Gamma(\nu) \sum_{n=0}^{\infty} (\nu+n) C_n^{\nu}(\cos\phi) J_{\nu+n}(z)\, J_{\nu+n}(Z)$

$\nu \neq 0, -1, -2, \dots,$

(31) $J_0(w) = J_0(z)\, J_0(Z) + 2 \sum_{n=1}^{\infty} J_n(z)\, J_n(Z) \cos(n\phi),$

(32) $w^{-\nu} J_{-\nu}(w) = (\tfrac{1}{2}zZ)^{-\nu}\Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n (\nu+n) C_n^{\nu}(\cos\phi) J_{-\nu-n}(z) J_{\nu+n}(Z)$

$\nu \neq 0, -1, -2, \dots, \quad |ze^{\pm i\phi}| < |Z|,$

Let $e^{i\psi} = (Z - ze^{-i\phi})/w$ and $|ze^{\pm i\phi}| < |Z|$.

(33) $Y_{\nu}(w)\, e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} Y_{\nu+n}(Z)\, J_n(z)\, e^{in\phi},$

(34) $H_{\nu}^{(1),(2)}(w)\, e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} H_{\nu+n}^{(1),(2)}(Z)\, J_n(z)\, e^{in\phi},$

(35) $K_\nu(w)\, e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} K_{\nu+n}(Z)\, I_n(z)\, e^{in\phi}$,

(36) $I_\nu(w)\, e^{i\nu\psi} = \sum_{n=-\infty}^{\infty} (-1)^n I_{\nu+n}(Z)\, I_n(z)\, e^{in\phi}$,

(37) $(2z \sin \tfrac{1}{2}\phi)^{-\nu} J_\nu(2z \sin \tfrac{1}{2}\phi)$

$$= 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu + n)\, [z^{-\nu} J_{\nu+n}(z)]^2\, C_n^\nu(\cos\phi)$$

$\nu \neq 0, -1, -2, \ldots$,

(38) $J_0(2z \sin \tfrac{1}{2}\phi) = [J_0(z)]^2 + 2 \sum_{n=1}^{\infty} [J_n(z)]^2 \cos(n\phi)$,

or

(39) $t^\nu J_\nu[z(t + t^{-1})] = \sum_{n=-\infty}^{\infty} t^{2n} J_{\nu-n}(z)\, J_n(z)$,

$|t| < 1$ in case $\nu \neq 0, \pm 1, \pm 2, \ldots$,

(40) $t^\nu I_\nu[z(t^{-1} - t)] = \sum_{n=-\infty}^{\infty} (-1)^n t^{2n} J_{\nu-n}(z)\, J_n(z)$

$|t| < 1$ in case $\nu \neq 0, \pm 1, \pm 2, \ldots$,

(41) $(z^2 + Z^2 - 2zZ \cos\phi)^{-\frac{1}{2}} \exp[\pm i [z^2 + Z^2 - 2zZ \cos\phi)^{\frac{1}{2}}]$

$$= \pm i\pi (zZ)^{-\frac{1}{2}} \sum_{n=0}^{\infty} (n + \tfrac{1}{2})\, J_{n+\frac{1}{2}}(z)\, H_{n+\frac{1}{2}}^{(1),(2)}(Z)\, P_n(\cos\phi)$$

$|ze^{\pm i\phi}| < |Z|$,

(42) $(\tfrac{1}{2}z)^{2\nu}\, \Gamma(2\nu) = \Gamma(\nu)\, \Gamma(1+\nu) \sum_{n=0}^{\infty} (\nu + n)\, \Gamma(2\nu + n)\, [J_{\nu+n}(z)]^2 / n!$,

(43) $(\sin\alpha \sin\beta)^{\frac{1}{2}-\nu} J_{\nu-\frac{1}{2}}(z \sin\alpha \sin\beta)\, e^{iz\cos\alpha\cos\beta}$

$$= 2^{2\nu-\frac{1}{2}} (\pi z)^{-\frac{1}{2}} [\Gamma(\nu)]^2 \sum_{n=0}^{\infty} \frac{i^n n!(\nu+n)}{\Gamma(2\nu + n)} J_{\nu+n}(z)\, C_n^\nu(\cos\alpha)\, C_n^\nu(\cos\beta),$$

(44) $\cos(z \cos\phi) = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n (\nu + 2n)\, z^{-\nu} J_{\nu+2n}(z)\, C_{2n}^\nu(\cos\phi)$,

(45) $\sin(z \cos\phi) = 2^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (-1)^n (\nu + 2n + 1)\, z^{-\nu} J_{\nu+2n+1}(z)\, C_{2n+1}^\nu(\cos\phi)$.

SERIES OF THE KAPTEYN TYPE

(46) $\nu\pi\,\mathbf{J}_\nu(\nu z) = \sin\nu\pi[1 - 2\nu^2 \sum_{n=1}^{\infty} (-1)^n J_n(nz)/(n^2 - \nu^2)]$,

(47) $\nu\pi\,\mathbf{E}_\nu(\nu z) = 2(\sin\tfrac{1}{2}\nu\pi)^2 - 4\nu^2 \sum_{n=1}^{\infty} [\sin(\tfrac{1}{2}\nu\pi + \tfrac{1}{2}n\pi)]^2 J_n(nz)/(n^2 - \nu^2)$,

(48) $(1 - z^2)^{-1/2} = 1 + 2 \sum_{n=1}^{\infty} [J_n(nz)]^2$,

(49) $[(1 - z^2)^{-1/2} - 1] = \sum_{n=0}^{\infty} J_n[(n + \tfrac{1}{2})z]\, J_{n+1}[(n + \tfrac{1}{2})z]$,

(50) $z^{-1}\sin z = 1 - z \sum_{n=1}^{\infty} (4n^2 - 1)^{-1} [J_n(nz)]^2$,

(51) $(1 - z)^{-1} = 1 + 2 \sum_{n=1}^{\infty} J_n(nz)$,

SCHLÖMILCH AND RELATED SERIES

(52) $\Gamma(\nu + 1) \sum_{m=1}^{\infty} \cos(mt)(\tfrac{1}{2}mx)^{-\nu} J_\nu(mx) = -\tfrac{1}{2}$ $0 < x < t \le \pi$,

$= \tfrac{1}{2} + \pi^{1/2} x^{-1} (1 - t^2/x^2)^{\nu - 1/2}$ $0 < t < x < \pi$,

$\operatorname{Re}\nu > -\tfrac{1}{2}$, (Cooke, 1928),

(53) $\sum_{m=1}^{\infty} (\tfrac{1}{2}mx)^{-\mu} J_\mu(mx)\, (\tfrac{1}{2}my)^{-\nu} J_\nu(my) = -[2\Gamma(\mu + 1)\Gamma(\nu + 1)]^{-1}$

$+ \pi^{1/2} [y\,\Gamma(\mu + 1)\Gamma(\nu + \tfrac{1}{2})]^{-1}\, {}_2F_1(\tfrac{1}{2} - \nu, \tfrac{1}{2}; \mu + 1; x^2/y^2)$

$\pi > y > x > 0$, $\mu, \nu > -\tfrac{1}{2}$, (Cooke, 1928),

(54) $\Gamma(\nu + 3/2) \sum_{m=1}^{\infty} \cos(mt)\, (\tfrac{1}{2}mx)^{-\nu-1}\, \mathbf{H}_\nu(mx) = -(\nu + \tfrac{1}{2})\, \pi^{-1/2}$

$0 < x < t \le \pi$, $\operatorname{Re}\nu > -1$, (Cooke, 1930, p. 58),

$= -\pi^{-1/2} + \pi^{1/2} x^{-1} (1 - t^2/x^2)^{\nu + 1/2}\, {}_2F_1(\nu + \tfrac{1}{2}, \tfrac{1}{2}; \nu + 3/2; 1 - t^2/x^2)$

$0 < t < x < \pi$, $\operatorname{Re}\nu > -1$, (Cooke, 1930, p. 58),

(55) $x^\nu = -2\Gamma(\nu + 1) \sum_{m=1}^{\infty} (-1)^m (\tfrac{1}{2}\pi/a)^{-\nu} m^{-\nu} J_\nu(m\pi x/a)$

$0 < x < a$, $\nu \ge 0$,

(56) $\pi J_\nu(x) = 2^{3-\nu} \sum_{m=1}^{\infty} m^{1-\nu}(4m^2-1)^{-1} \mathbf{H}_\nu(2mx)$

$$0 \le x \le \pi, \quad \nu \ge -\tfrac{1}{2},$$

(57) $x^{\nu-1}\pi^{\frac{1}{2}} - \pi\Gamma(\nu+\tfrac{1}{2})(\tfrac{1}{2}a)^{1-\nu}\mathbf{H}_\nu(ax) + \pi i\,\Gamma(\nu+\tfrac{1}{2})(\tfrac{1}{2}a)^{1-\nu}J_\nu(ax)$

$$= 2\Gamma(\nu+\tfrac{1}{2}) \sum_{m=1}^{\infty} m(m^2-a^2)^{-1}[1-(-1)^m e^{ia\pi}](\tfrac{1}{2}m)^{1-\nu}J_\nu(mx)$$

$$0 < x < \pi, \quad \nu \ge \tfrac{1}{2}.$$

For (55) to (57) see Pennel (1932).

EXPANSIONS OF THE FOURIER-BESSEL TYPE

In the following formulas ν and z are arbitrary, but $\nu \ne -1, -2, -3, \ldots$. The zeros of $z^{-\nu}J_\nu(z)$ arranged in ascending magnitudes of $\mathrm{Re}(\gamma_{\nu,n}) > 0$, are $\pm\gamma_{\nu,n}$, $(n = 1, 2, 3, \ldots,)$. Then (Buchholz, 1947)

(58) $\dfrac{\pi J_\nu(xz)}{4J_\nu(z)}[J_\nu(z)Y_\nu(Xz) - J_\nu(Xz)Y_\nu(z)]$

$$= \sum_{n=1}^{\infty} J_\nu(x\gamma_{\nu,n}) J_\nu(X\gamma_{\nu,n}) [J_{\nu+1}(\gamma_{\nu,n})]^{-2}(z^2-\gamma_{\nu,n}^2)^{-1}$$

$$0 \le x \le X \le 1.$$

(59) $J_\nu(xz)/J_\nu(z) = 2\sum_{n=1}^{\infty} \gamma_{\nu,n} J_\nu(x\gamma_{\nu,n})[(\gamma_{\nu,n}^2 - z^2) J_{\nu+1}(\gamma_{\nu,n})]^{-1}$

$$= x^\nu + 2z^2 \sum_{n=1}^{\infty} J_\nu(\gamma_{\nu,n}x)[\gamma_{\nu,n}(\gamma_{\nu,n}^2 - z^2) J_{\nu+1}(\gamma_{\nu,n})]^{-1}$$

$$0 \le x < 1,$$

(60) $J_{\nu+1}(xz)/J_\nu(z) = 2z\sum_{n=1}^{\infty} J_{\nu+1}(\gamma_{\nu,n}x)[(\gamma_{\nu,n}^2 - z^2) J_{\nu+1}(\gamma_{\nu,n})]^{-1},$

(61) $\tfrac{1}{2}\log X = -\sum_{n=1}^{\infty} J_0(x\gamma_n) J_0(X\gamma_n)[\gamma_n J_1(\gamma_n)]^{-2}$

$$0 \le x \le X \le 1, \quad \gamma_n = \gamma_{0,n},$$

(62) $[J_0(z)]^{-1} = 1 - 2\sum_{n=1}^{\infty} [\gamma_{0,n}(z^2-\gamma_{0,n}^2)^{-1} + \gamma_{0,n}^{-1}][J_1(\gamma_{0,n})]^{-1},$

(63) $[J_0(z)]^{-2} = 1 + 4\sum_{n=1}^{\infty} [\gamma_{0,n}^2(z^2-\gamma_{0,n}^2)^{-2} + (z^2-\gamma_{0,n}^2)^{-1}][J_1(\gamma_{0,n})]^{-2}$

(64) $$[J_1(z)]^{-1} = 2z^{-1} + 2z \sum_{n=1}^{\infty} [(z^2 - \gamma_{1,n}^2)^{-1} [J_0(\gamma_{1,n})]^{-1}$$

(65) $$[J_1(z)]^{-2} = 4z^{-2} + 1$$
$$+ 4 \sum_{n=1}^{\infty} [\gamma_{1,n}^2 (z^2 - \gamma_{1,n}^2)^{-2} + (z^2 - \gamma_{1,n}^2)^{-1}] [J_0(\gamma_{1,n})]^{-2}.$$

For (62) to (65) see Forsyth (1921).

REFERENCES

Airey, J. R., 1916: *Philos. Mag.* 31, 520-528; 32, 7-14, 237-238.

Airey, J. R., 1935: *Philos. Mag.* 19, 230-235.

Airey, J. R., 1935a: *Philos. Mag.* 19, 236-243.

Airey, J. R., 1937: *Philos. Mag.* 24, 521-552.

Bailey, W. N., 1929: *Proc. Cambridge Philos. Soc.* 25, 48-49.

Bailey, W. N., 1929a: *Proc. Cambridge Philos. Soc.* 26, 82-87.

Bailey, W. N., 1930: *Proc. London Math. Soc.* (2), 30, 415-421.

Bailey, W. N., 1930a: *Proc. London Math. Soc.* (2), 30, 422-424.

Bailey, W. N., 1930b: *J. London Math. Soc.* 5, 258-265.

Bailey, W. N., 1930c: *Proc. London Math. Soc.* (2), 31, 200-208.

Bailey, W. N., 1932: *Proc. London Math. Soc.* 33, 154-159.

Bailey, W. N., 1935: *Quart. J. Math. Oxford Ser.* 6, 233-238.

Bailey, W. N., 1935a: *Proc. London Math. Soc.* (2), 40, 37-48.

Bailey, W. N., 1936: *J. London Math. Soc.* 11, 16-20.

Bailey, W. N., 1937: *Quart. J. Math. Oxford Ser.* 6, 241-248.

Bailey, W. N., 1938: *Quart. J. Math. Oxford Ser.* 9, 141-147.

Banerjee, D. P., 1935: *J. Indian Math. Soc.*, N.S., 1, 266-268.

Banerjee, D. P., 1936: *J. Indian Math. Soc.*, N. S., 2, 211-212.

Banerjee, D. P., 1939: *Quart. J. Math. Oxford Ser.* 10, 261-265.

Basu, K., 1923: *Bull. Calcutta Math. Soc.* 14, 25-30.

Bateman, Harry and S. O. Rice, 1935: *Proc. Nat. Acad. Sci. U.S.A.* 21, 173-179.

Baudoux, P., 1945: *Acad. Roy. Belgique Bull. Cl. Sci.* (31), 471-478.

Baudoux, P., 1945a: *Acad. Roy. Belgique Bull. Cl. Sci.* (31), 669-681.

Baudoux, P., 1946: *Acad. Roy. Belgique Bull. Cl. Sci.* (32), 127-131.

Bell, E. T., 1926: *Philos. Mag.* 1, 304-312.

Bennet, W. R., 1932: *Bull. Amer. Math. Soc.* 38, 843-848.

Bickley, W. G., 1943: *Philos. Mag.* 34, 37-49.

Bickley, W. G. and J. C. P. Miller, 1945: *Philos. Mag.* 36, 121-133; 200-210.

Bijl, Jan, 1937: *Dissertation Groningen.*

Birkhoff, G. D., 1908: *Trans. Amer. Math. Soc.* 9, 219-231.

Blumenthal, Otto, 1912: *Arch. der Math. und Phys.* 19, 136-152.

Boas, R. P., 1942: *Proc. Nat. Acad. Sci. U.S.A.* 28, 21-27.

Boas, R. P., 1942a: *Bull. Amer. Math. Soc.* 48, 286-294.

REFERENCES

Bose, B. N., 1944: *Bull. Calcutta Math. Soc.* 36, 126.

Bose, B. N., 1945: *Bull. Calcutta Math. Soc.* 37, 77.

Bose, B. N., 1948: *Bull. Calcutta Math. Soc.* 40, 8-14.

Bose, S. K., 1946: *Bull. Calcutta Math. Soc.* 38, 177-180.

Bose, S. K., 1946a: *Bull. Calcutta Math. Soc.* 38, 181-184.

Bradley, W. F., 1936: *Proc. London Math. Soc.* 31, 209-214.

Bruijn, N. G., 1948: *Philos. Mag.* 39, 134-140.

Bruijn, N. G., 1950: *Duke Math. J.* 17, 197-225.

Buchholz, Herbert, 1939: *Philos. Mag.* 27, 407-420.

Buchholz, Herbert, 1947: *Z. Angew. Math. Mech.* 25/27, 245-252.

Budden, R. F., 1926: *Proc. London Math. Soc.* 24, 471-478.

Burchnall, J. L., 1951: *Canadian J. Math.* 3, 62-68.

Burnett, B. H., 1929: *Proc. Cambridge Phil. Soc.* 26, 145-151.

Busbridge, I. W., 1938: *Proc. London Math. Soc.* (2) 44, 115-129.

Carslaw, H. S., and J. C. Jaeger, 1940: *Proc. London Math. Soc.* 46, 361-388.

Chaundy, T. W., 1931 : *Quart. J. Math. Oxford Ser.* 2, 144-154.

Cherry, T. M., 1949: *J. London Math. Soc.* 24, 121-130.

Cherry, T. M., 1949a: *Proc. London Math. Soc.* 51, 14-45.

Cherry, T. M., 1950: *Trans. Amer. Math. Soc.*, 86, 224-257.

Cooke, R. G., 1925: *Proc. London Math. Soc.* 24, 381-420.

Cooke, R. G., 1927: *Proc. London Math. Soc.* 27, 171-192.

Cooke, R. G., 1928: *Proc. London Math. Soc.* 28, 207-241.

Cooke, R. G., 1929: *J. London Math. Soc.* 4, 18-21.

Cooke, R. G., 1930: *J. London Math. Soc.* 5, 54-58.

Cooke, R. G., 1930a: *J. London Math. Soc.* 5, 58-61.

Cooke, R. G., 1930b: *Proc. London Math. Soc.* 30, 144-164.

Cooke, R. G., 1932: *J. London Math. Soc.* 7, 281-283.

Cooke, R. G., 1936: *Proc. London Math. Soc.* 41, 176-190.

Cooke, R. G., 1937: *J. London Math. Soc.* 12, 180-185.

Copson, E. T., 1932: *Proc. London Math. Soc.* (2) 33, 145-153.

Copson, E. T., 1933: *Quart. J. Math. Oxford Ser.* 4, 134-139.

Copson, E. T., 1935: *Functions of a complex variable*, Oxford.

REFERENCES

Copson, E. T., and W. L. Ferrar, 1937: *Proc. Edinburgh Math. Soc.* 5, 160-168.

Corput, Van der, J. G., 1934: *Compositio Math.* 1, 15-38.

Corput, Van der, J. G., 1936: *Compositio Math.* 3, 328-372.

Costello, J. C., 1936: *Philos. Mag.* 2, 308-318.

Coulomb, M. J., 1936: *Bull. Sci. Math.* 60, 297-302.

Crum, M. M., 1940: *Quart. J. Math. Oxford Ser.* 11, 49-52.

Dalzell, D. P., 1945: *J. London Math. Soc.* 20, 213-218.

Davis, H. T., 1924: *Amer. J. Math.* 46, 95-109.

Debye, Peter, 1909: *Math. Ann.* 67, 535-558.

Dixon, A.L., and W. L. Ferrar, 1930: *Quart. J. Math. Oxford Ser.* 1, 122-145.

Dixon, A.L., and W. L. Ferrar, 1930a: *Quart. J. Math. Oxford Ser.* 1, 236-238.

Dixon, A.L., and W. L. Ferrar, 1933: *Quart. J. Math. Oxford Ser.* 4, 193-208; 297-304.

Dixon, A.L., and W. L. Ferrar, 1935: *Quart. J. Math. Oxford Ser.* 6, 166-174.

Dixon, A. L., and W. L. Ferrar, 1937: *Quart. J. Math. Oxford Ser.* 8, 66-74.

Doetsch, Gustav, 1935: *Compositio Math.* 1, 85-87.

Doetsch, Gustav, 1937: *Theorie und Anwendung der Laplace Transformation*, J. Springer, Berlin.

Dougall, John, 1919: *Proc. Edinburgh Math. Soc.* 37, 33-47.

Emde, Fritz, and Rudolf Rühle, 1934: *Jber. Deutsch. Verein* 43.

Emde, Fritz, 1937: *Z. Angew. Math. Mech.* 17, 324-346.

Emde, Fritz, 1939: *Z. Angew. Math. Mech.* 19, 101-118.

Erdélyi, Arthur, 1937: *Compositio Math.* 4, 406-423.

Erdélyi, Arthur, 1939: *Proc. Edinburgh Math. Soc.* 6, 94-104.

Erdélyi, Arthur, and W.O. Kermack, 1945: *Proc. Cambridge Philos. Soc.* 41, 74-75.

Falkenberg, Hans, 1932: *Math. Z.* 35, 457-463.

Falkenberg, Hans and Ernst Hilb, 1916: *Goettinger Machrichten*, p. 190-196.

Ferrar, W. L., 1937: *Compositio Math.* 4, 394-405.

Fock, V., 1934 *C. R. (Doklady) Acad. Sci. URSS* N.S. 1, 99-102.

Forsyth, A. R., 1921: *Messenger of Math.* 50, 129-149.

Fox, Cyril, 1926: *Proc. London Math. Soc.* 24, 479-493.

Fox, Cyril, 1927: *Proc. London Math. Soc.* 26, 35-87; 201-210.

REFERENCES

Fox, Cyril, 1929: *Proc. Cambridge, Philos. Soc.* 25, 130-131.

Gatteschi, L., 1950: *Revista di Mathematicadella Universita di Parma,* 1, 347-362.

Gray, Andrew, and G. B. Mathew, 1922: *A treatise on Bessel functions*, Macmillan & Co., Ltd., London.

Greenwood, R. E., 1941: *Ann. Of Math.* 42, 778-805.

Gupta, H. C., 1943: *Proc. Nat. Acad. Sci. India Sect. A,* 13, 225-231.

Gupta, H. C., 1943 a: *Proc. Benares Math. Soc.* 5, 1-16.

Gupta, H. C., 1943b: *Bull. Calcutta Math. Soc.* 35, 7-11.

Hardy, G. H., 1921: *Messenger of Math.* 50, 165-171.

Hardy, G. H., 1925: *Proc. London Math. Soc.* 23, IX

Hardy, G. H., 1926: *Messenger of Math.* 55, 140-144.

Hardy, G. H., 1927: *Messenger of Math.* 56, 186-192.

Hardy, G. H., 1927a: *Messenger of Math.* 57, 113-120.

Hardy, G. H., and E. C. Titchmarsh, 1933: *Proc. London Math. Soc.* 35, 116-155.

Hilb, Ernst, 1922: *Math. Z.* 15, 274-279.

Hille, Einar and Gábor Szegö, 1943: *Bull. Amer. Math. Soc.* 49, 605-610.

Hillmann, Abraham, 1949: *Bull. Amer. Math. Soc.* 55, 198-200.

Horn, Jakob, 1899: *Math. Ann.* 52, 271-292.

Horton, C. W., 1950: *J. Math. Physics* 29, 31-37.

Ince, E. L., 1944: *Ordinary differential equations*, Longmans.

Infield, L., Smith, V. G. and W. Z. Chien, 1947: *J. of Math. and Phys.* 26, 22-28.

Jahnke, Eugene, and Fritz Emde, 1933: *Funktionentafeln mit Formeln und Kurven,* B. G. Teubner and 1945, New York.

Jeffreys, Harold, 1925: *Proc. London Math. Soc.* 23, 428-436.

Jesmanowicz, L., 1938: *C. R. Soc. Sci. Varsovie,* 31, 43-59.

Jordan, Henri, 1930: *J. Reine Angew. Math.* 162, 17-59.

Kamke, Erich, 1948: *Differentialgleichungen. Lösungsmethoden und Lösungen,* Chelsea, New York.

King, L. V., 1935: *Proc. Roy. Soc. A.* 153, 1-16.

King, L. V., 1936: *Philos. Mag.* (7) 21, 118-144.

Kishore, Raj, 1929: *Bull. Calcutta Math. Soc.* 21, 187-190.

Klein, Felix, 1933: *Vorlesungen über die hypergeometrische Funktion,* J. R. Springer, Berlin.

REFERENCES

Kline, Morris, 1948: *J. Math. Phys.* 27, 37-48.

Kline, Morris, 1950: *Proc. Amer. Math. Soc.* 1, 543-552.

Kober, Hermann, 1935: *Math. Z.* 39, 609-624.

Kober, Hermann, 1937: *Quart. J. Math. Oxford Ser.* 8, 186-199.

Kontorowich, M. J. and N. N. Lebedev, 1938: *Ž. eksper, Teor. Fizike, Moskwa, Leningrad* 8, 1192-1206.

Korn, Arthur, 1931: *S. B. Preuss. Akad. Wissensch. Phys. Math. Kl. H.* 22/23, 437-449.

Koshliakov, N. S., 1926: *Messenger of Math.* 55, 152-160.

Krall, H. L., and O. Frink, 1949: *Trans. Amer. Math. Soc.* 65, 100-115.

Lambe, C. G., 1931: *J. London Math. Soc.* 6, 257-259.

Langer, R. E., 1931: *Trans. Amer. Math. Soc.* 33, 23-64.

Langer, R. E., 1932: *Trans. Amer. Math. Soc.* 34, 447-480.

Langer, R. E., 1934: *Bull. Amer. Math. Soc.* 40, 545-582.

Lebedev, N. N., 1946: *C. R. (Doklady) Acad. Sci. URSS* (N.S.) 52, 655-658.

Lebedev, N. N., 1947: *Doklady Akad. Nauk SSSR* (N.S.) 58, 1007-1010.

Lehmer, D. H., 1944: *Math. tables and other aids to computation* 1, 133-134.

Lense, Josef, 1933: *Jber. Deutsch. Math. Verein* 43, 146-153.

Luke, Y. L., 1950: *J. Math. Physics* 29, 27-30.

McLachlan, N. W., 1934: *Bessel functions for engineers*, Oxford.

McLachlan, N. W., and A. L. Meyers, 1936: *Philos. Mag.* 21, 425-436, 437-448.

McLachlan, N. W. and A. L. Meyers, 1937: *Philos. Mag.* 23, 762-774.

McLachlan, N. W., 1938: *Philos. Mag.* 26, 394-408, 457-473.

MacRobert, T. M., 1930: *Proc. Edinburgh Math. Soc. Ser. II* 1, 28.

MacRobert, T. M., 1931: *Proc. Roy. Soc. Edinburgh* 51, 116-126.

MacRobert, T. M., 1936: *Philos. Mag.* 21, 697-703.

MacRobert, T. M., 1937: *Proc. Roy. Soc. Edinburgh* 57, 19-25.

MacRobert, T. M., 1940: *Quart. J. Math. Oxford Ser. II*, 95-99.

MacRobert, T. M., 1947: *Functions of a complex variable*, Macmillan & Co., Ltd., London.

Magnus, Wilhelm and Fritz Oberhettinger, 1948: *Formeln und Sätze für die speziellen Funktionen der Mathematischen Physik*, second edition, Springer.

REFERENCES

Mayr, K., 1932: *Akad. Wiss. Wien. S. B.* 141, 227-265.

Mayr, K., 1933: *Akad. Wiss. Wien. S. B.* 142, 1-17.

Mayr, K., 1935: *Akad. Wiss. Wien, S. B.* 144, 277-292.

Mcdonald, J. H., 1926: *Trans. Amer. Math. Soc.* 28, 384-390.

Meijer, C. S., 1932: *Nederl. Akad. Wetensch., Proc.* 35, 656-667, 852-866, 948-958, 1079-1096.

Meijer, C. S., 1933: *Math. Ann.* 108, 321.

Meijer, C. S., 1933 a: *Dissertation Groningen.*

Meijer, C. S., 1934: *Nederl. Akad. Wetensch., Proc.* 37, 805-812.

Meijer, C. S., 1935: *Quart. J. Math. Oxford Ser.* 6, 241-248, 528-535.

Meijer, C. S., 1935 a: *Nederl. Akad. Wetensch., Proc.* 38, 628-634, 744-749.

Meijer, C. S., 1935 b: *Proc. London Math. Soc.* 40, 1-22.

Meijer, C. S., 1936: *Nederl. Akad. Wetensch., Proc.* 39, 394-403, 519-527.

Meijer, C. S., 1936 a: *Math. Ann.* 112, 469-489.

Meijer, C. S., 1938: *Nederl. Akad. Wetensch., Proc.* 41, 151-154.

Meijer, C. S., 1939: *Compositio Math.* 6, 348-367.

Meijer, C. S., 1939a: *Nederl. Akad. Wetensch., Proc.* 42, 355-369, 872-879, 938-947.

Meijer, C. S., 1940: *Nederl. Akad. Wetensch., Proc.* 43, 198-210, 366-378, 599-608, 702-711.

Meixner, Josef, 1949: *Math. Nach.* 3, 9-13.

Mitra, Subodchandra, 1925: *Bull. Calcutta Math. Soc.* 15, 83-85.

Mitra, Subodchandra, 1933: *Bull. Calcutta Math. Soc.* 25, 81-98.

Mitra, Subodchandra, 1936: *Math. Z.* 41, 680-685.

Mohan, Brij, 1942: *Bull. Calcutta Math. Soc.* 34, 55-59, 171-175.

Mohan, Brij, 1942 a: *Quart. J. Math. Oxford Ser.* 13, 40-47.

Mohan, Brij, 1942 b: *Proc. Nat. Acad. Sci. India* 12, 231-235.

Montroll, E. W., 1946: *J. Math. Physics* 25, 37-49.

Moore, C. N., 1920: *Trans. Amer. Math. Soc.* 21, 107-156.

Moore, C. N., 1926: *Trans. Amer. Math. Soc.* 12, 181-206.

Moore, C. N., 1930: *Trans. Amer. Math. Soc.* 32, 408-416.

Mordell, L. J., 1930: *J. London Math. Soc.* 5, 203-208.

Müller, R., 1940: *Z. Angew. Math. Mech.* 20, 61-62.

Newson, C. V. and A. Frank, 1940: *Bull. Mat.* 13, 11-14.

Nicholson, J. W., 1920: *Quart. J. Math.* 48, 321-329.

REFERENCES

Nicholson, J. W., 1924: *Philos. Trans. Roy. Soc. A*, 224, 303-369.

Nicholson, J. W., 1927: *Quart. J. Math.* 50, 297-314.

Nielsen, Niels, 1904: *Die Zylinderfunktionen und ihre Anwendungen*, Leipzig, B.G. Teubner.

Obreschkoff, Nikolai, 1929: *Jber. Deutsch Math. Verein.*,38, 156-161.

Olver, F. W. J., 1950: *Proc. Cambridge Philos. Soc.* 46, 570-580.

Pennel, W. O., 1932: *Bull. Amer. Math. Soc.* 38, 115-122.

Picht, Johannes, 1949: *Z. Angew. Math. Mech.* 29, 155-157.

Pol, Balthasar van der, and K. F. Niessen, 1932: *Philos. Mag.* 13, 537-572.

Pólya, Georg, 1926: *J. London Math. Soc.* 1, 98-99.

Pólya, Georg, 1929: *Jber. Deutsch, Math. Verein*, 38, 161-168.

Poole, E. C., 1934: *Quart. J. Math. Oxford Ser.* 5, 186-194.

Rayleigh, J., W., 1945: *The theory of sound*, Dover, New York.

Ramanujan, Srinivasa, 1920: *Quart. J. Math.* 48, 294-310.

Ramanujan, Srinivasa, 1927: *Collected papers*, Cambridge.

Rice, S. O., 1935: *Quart. J. Math. Oxford Ser.* 6, 52-64.

Rice, S. O., 1944: *Philos. Mag.* 35, 686-693.

Rosen, Joseph, 1939: *Tohoku Math. J.* 45, 230-238.

Rutgers, J. G., 1931: *Nederl. Akad. Wetensch, Proc.* 34, 148-159; 239-256; 427-437.

Rutgers, J. G., 1941: *Nederl. Akad. Wetensch., Proc.* 44, 464-474; 636-647; 744-753; 840-851; 978-988; 1092-1098.

Rutgers, J. G., 1942: *Nederl. Akad. Wetensch., Proc.* 45, 929-936; 987-993.

Schlesinger, Ludwig, 1907: *Math. Ann.* 63, 277-300.

Schöbe, Waldemar, 1948: *Arch. Math.* 1, 230-232.

Shabde, N. G., 1935: *Bull. Calcutta Math. Soc.* 27, 165-170.

Shabde, N. G., 1938: *Bull. Calcutta Math. Soc.* 30, 29-30.

Shabde, N. G., 1939: *Proc. Benares Math. Soc.* 1, 55-59.

Shastri, N. A., 1938: *Philos. Mag.* 25, 930-950.

Siegel, C. L., 1929: *Abh. Preuss, Akad. Wiss. Nr.* 1.

Sinha, S., 1942-43: *Bull. Calcutta, Math. Soc.* 34, 35, 67-77; 37-42.

Sircar, H., 1945: *Bull. Calcutta Math. Soc.* 37, 1-4.

Sommerfeld, Arnold, 1943: *Ann. Phys.* 42, 389-420.

REFERENCES

Stevenson, Georg, 1928: *Amer. J. Math.* 50, 569-590.

Stone, M. H., 1927: *Ann. Math.* (2) 28; 271-290.

Straubel, Rudolph, 1941: *Ing. Arch.* 12, 325-336.

Straubel, Rudolph, 1942: *Ing. Arch.* 13, 14-20.

Svetlov, A., 1934: *C. R. (Doklady) Acad. Sci. URSS.* (2), 445-448.

Szász, Otto, 1950: *Proc. Amer. Math. Soc.* 1, 256-267.

Szegö, Gábor, 1933: *Proc. London Math. Soc.* 36, 427.

Szymanski, Piotr, 1935: *Proc. London Math. Soc.* 40, 71-82.

Temple, G., 1927: *Proc. London Math. Soc.* 26, 518-530.

Thielmann, H.P., 1929: *Proc. U.S.A. Acad.* 15, 731-733.

Thielmann, H. P., 1934: *Bull. Amer. Math. Soc.* 40, 695-698.

Titchmarsh, E. C., 1923: *Proc. London Math. Soc.* 22, 15-28.

Titchmarsh, E. C., 1923 a: *Proc. London Math. Soc.* 22, xiii-xvi.

Titchmarsh, E. C., 1925: *Proc. London Math. Soc.* 23, xii.

Titchmarsh, E. C., 1927: *J. London Math. Soc.* 2, 97-99.

Titchmarsh, E. C., 1946: *Eigenfunction expansions*, Oxford.

Titchmarsh, E. C, 1948: *Introduction to the theory of Fourier integrals*, Oxford.

Tranter, C. J., 1951: *Quart. J. Math. Oxford Ser.* 2, 60-66.

Tricomi, Francesco, 1935: *Rend. Lincei*, (6), 22, 564-576.

Tricomi, Francesco, 1949: *Atti Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat.* 83, 3-20.

Truesdell, C. A., 1947: *Proc. Nat. Acad. Sci. U. S. A.* 33, 82-93.

Truesdell, C. A., 1948: *A unified theory of special functions*, Princeton University Press, Princeton, N. J.

Varma, D. S., 1936: *Proc. London Math. Soc.* 42, 9-17.

Varma, D. S., 1936a: *Bull. Calcutta Math. Soc.* 28, 209-211.

Veen, S. C., 1927: *Math. Ann.* 97, 696-710.

Watson, G. N., 1928: *J. London Math. Soc.* 3, 22-27.

Watson, G. N., 1931: *Quart. J. Math. Oxford Ser.* 2, 298-309.

Watson, G. N., 1934: *J. London Math. Soc.* 9, 16-22.

Watson, G. N., 1938: *J. London Math. Soc.* 13, 41-44.

Watson, G. N., 1944: *A treatise on the theory of Bessel functions*, Cambridge.

REFERENCES

Weinstein, Alexander, 1948: *Trans. Amer. Math. Soc.* 63, 342-354.

Weyrich, Rudolf, 1937: *Die Zylinderfunktionen und ihre Anwendungen*, Leipzig, B. G. Teubner.

Whittaker, E. T., and G. N. Watson, 1946: *A course of modern analysis*, Cambridge.

Widder, D.V., 1941: *The Laplace transform*, University Press, Princeton, N.J.

Wilkins, J. E., 1948: *Bull. Amer. Math. Soc.* 54, 232-234.

Wilkins, J. E., 1948a: *Trans. Amer. Math. Soc.* 64, 359-385.

Wilkins, J. E., 1950: *Trans. Amer. Math. Soc.* 69, 55-65.

Wilkins, J. E., 1950a: *Amer. J. Math.* 75, 187-191.

Wilson, R., 1939: *Proc. Edinburgh Math. Soc.* 6, 17-18.

Wilton, J. R., 1925: *Proc. London Math. Soc.* 23, VIII.

Wilton, J. R., 1927: *Messenger of Math.* 56, 175-181.

Wilton, J. R., 1928: *Proc. London Math. Soc.* 27, 81-104.

Wilton, J. R., 1928a: *J. Math.* 159, 144-153.

Wise, W. H., 1935: *Bull. Amer. Math. Soc.* 41, 700-706.

Wright, E. M., 1934: *Proc. London Math. Soc.* 28, 257-270.

Wright, E. M., 1940: *Philos. Trans. Royal Soc.* (*A*) 238, 423-451.

Wright, E. M., 1940a: *Quart. J. Math. Oxford Ser.* 11, 36-48.

Young, L. C., 1941: *Proc. London Math. Soc.* 47, 290-308.

Young, W. H., 1912: *Quart. J. Math. Oxford Ser.* 43, 161-177.

Young, W. H., 1920: *Proc. London Math. Soc.* 18, 163-200.

CHAPTER VIII

FUNCTIONS OF THE PARABOLIC CYLINDER AND OF THE PARABOLOID OF REVOLUTION

8.1. Introduction

Let x_1, x_2, x_3, be Cartesian coordinates in the three-dimensional space. We define coordinates of the parabolic cylinder ξ, η, ζ, by

(1) $x_1 = \xi\eta, \quad x_2 = \tfrac{1}{2}\xi^2 - \tfrac{1}{2}\eta^2, \quad x_3 = \zeta$

and coordinates of the paraboloid of revolution ξ, η, ϕ, by

(2) $x_1 = \xi\eta \cos\phi, \quad x_2 = \xi\eta \sin\phi, \quad x_3 = \tfrac{1}{2}\xi^2 - \tfrac{1}{2}\eta^2.$

Let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

be Laplace's operator, and let f be any function of x_3 only. The partial differential equation

(3) $\Delta u + f(x_3)\, u = 0$

transformed to the coordinates of the parabolic cylinder is

(4) $$(\xi^2 + \eta^2)^{-1}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + \frac{\partial^2 u}{\partial \zeta^2} + f(\zeta)\, u = 0$$

and it has particular solutions of the form $U(\xi)\, V(\eta)\, W(\zeta)$ where U, V, W, satisfy the ordinary differential equations

(5) $$\frac{d^2 U}{d\xi^2} + (\sigma\xi^2 + \lambda)\, U = 0, \quad \frac{d^2 V}{d\eta^2} + (\sigma\eta^2 - \lambda)\, V = 0,$$

(6) $$\frac{d^2 W}{d\zeta^2} + [f(\zeta) - \sigma]\, W = 0,$$

with arbitrary constants σ, λ. Again, with a constant k^2, the partial differential equation

$$\Delta u + k^2 u = 0$$

transformed to the coordinates of the paraboloid of revolution is

$$(7)\quad (\xi^2+\eta^2)^{-2}\left[\xi^{-1}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right)+\eta^{-1}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right)\right]$$

$$+(\xi\eta)^{-2}\frac{\partial^2 u}{\partial\phi^2}+k^2 u=0,$$

and it has particular solutions of the form $U(\xi)\ V(\eta)\ W(\phi)$, where U satisfies the ordinary differential equation

$$(8)\quad \frac{d^2 U}{d\xi^2}+\xi^{-1}\frac{dU}{d\xi}+(k^2\xi^2-4\mu^2\xi^{-2}+\lambda)\,U=0.$$

V satisfies an equation similar to (8) except that the sign of λ is reversed, and W satisfies

$$(9)\quad \frac{d^2 W}{d\phi^2}+4\mu^2 W=0.$$

For solutions which are one-valued and continuous on the paraboloids $\xi=$ constant or $\eta=$ constant, 2μ must be an integer.

In the case of a more than three-dimensional space several generalizations of this approach to the investigation of (3) are possible. For some of them see P. Humbert (1920 a, b, c, d).

The solutions of (5) and (8) can be expressed in terms of confluent hypergeometric functions. Although (8) contains two essentially independent constants, and therefore is as general as the confluent hypergeometric equation 6.1(2) itself, the special cases where 2μ is an integer and where k, λ, are real are particularly important for certain boundary value problems. These cases, and the solutions of (5) will be discussed in this chapter.

PARABOLIC CYLINDER FUNCTIONS

8.2. Definitions and elementary properties

By a simple change of variable, 8.1(5) can be transformed into

$$(1)\quad \frac{d^2y}{dz^2}+(\nu+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y=0.$$

The solutions of (1) are called *parabolic cylinder functions* or *Weber-Hermite functions.* They can be expressed in terms of confluent hypergeometric functions. If we define

(2) $D_\nu(z) = 2^{\frac{1}{2}(\nu-1)} e^{-\frac{1}{4}z^2} z\, \Psi(1/2-\nu/2;\, 3/2;\, z^2/2)$

(3) $\qquad = 2^{\frac{1}{2}(\nu+\frac{1}{2})} z^{-\frac{1}{2}} W_{\frac{1}{2}(\nu+\frac{1}{2}),\,-\frac{1}{4}}(\tfrac{1}{2}z^2)$

(4) $$= 2^{\frac{1}{2}\nu} e^{-z^2/4} \left[\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)} \Phi(-\tfrac{1}{2}\nu;\, \tfrac{1}{2};\, \tfrac{1}{2}z^2) + \frac{z}{2^{\frac{1}{2}}} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2}\nu)} \Phi(1/2-\nu/2;\, 3/2;\, z^2/2) \right]$$

[see 6.1 (1), 6.9 (2) and sec. 6.5, for the notations] we find that

(5) $D_\nu(z), \quad D_\nu(-z), \quad D_{-\nu-1}(iz), \quad D_{-\nu-1}(-iz),$

satisfy (1). The values of $D_\nu(z)$ and of its derivative at $z = 0$ are seen from (4). Since a solution of (1) is completely determined by its value and the value of its derivative at $z = 0$, and since there are precisely two linearly independent solutions of (1), we find the following relations:

(6) $$D_\nu(z) = \frac{\Gamma(\nu+1)}{(2\pi)^{\frac{1}{2}}} [e^{\nu\pi i/2} D_{-\nu-1}(iz) + e^{-\nu\pi i/2} D_{-\nu-1}(-iz)]$$

(7) $$= e^{-\nu\pi i} D_\nu(-z) + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{-(\nu+1)\pi i/2} D_{-\nu-1}(iz)$$

(8) $$= e^{\nu\pi i} D_\nu(-z) + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} e^{(\nu+1)\pi i/2} D_{-\nu-1}(-iz)$$

and those which can be obtained from these by substituting $-z$ for z. These relations show how any three of the solutions in (5) are connected.

The parabolic cylinder functions are entire functions of z. If $\nu = n$ is a non-negative integer, we find from (4) that

(9) $$e^{z^2/4} D_n(z) = 2^{-\frac{1}{2}n} H_n(2^{-\frac{1}{2}} z)$$

is a polynomial; $H_n(x)$ is called the *Hermite polynomial* of degree n (see Chap. 10). If ν is not an integer, $D_\nu(z)$ and $D_\nu(-z)$ are linearly independent. For all values of ν, $D_\nu(z)$ and $D_{-\nu-1}(\pm iz)$ are linearly independent. The Wronskian determinants are

(10) $$D_\nu(z) \frac{d}{dz} D_\nu(-z) - D_\nu(-z) \frac{d}{dz} D_\nu(z) = (2\pi)^{\frac{1}{2}}/\Gamma(-\nu),$$

(11) $$D_\nu(z) \frac{d}{dz} D_{-\nu-1}(iz) - D_{-\nu-1}(iz) \frac{d}{dz} D_\nu(z) = -i \exp(-\tfrac{1}{2}\nu\pi i).$$

If ν and z are real, the values of $D_\nu(z)$ are also real. For the differential equation 8.1 (5) we can also give real and linearly independent solutions in terms of the D_ν if σ, λ, are real. If we assume $\sigma > 0$, we can transform 8.1 (5) into

$$(12)\quad \frac{d^2y}{dx^2} + (\tfrac{1}{4}x^2 - \rho)\,y = 0$$

where $\xi = (4\sigma)^{-\frac{1}{4}}x$, $\rho = -\lambda(4\sigma)^{-\frac{1}{2}}$, and we find that the real and imaginary parts of

$$(13)\quad D_{i\rho-\frac{1}{2}}\left(\pm\frac{1+i}{2^{\frac{1}{2}}}x\right)$$

satisfy (12). Other sets of solutions of (12) which are real on the real axis are

$$\frac{\Gamma(\tfrac{3}{4}-\tfrac{1}{2}\rho)}{2^{\frac{1}{2}i\rho+\frac{3}{4}}\,\pi^{\frac{1}{2}}}\,[D_{i\rho-\frac{1}{2}}(e^{i\pi/4}x) + D_{i\rho-\frac{1}{2}}(-e^{i\pi/4}x)] = y_0(x)$$

$$-\frac{\Gamma(\tfrac{1}{4}-\tfrac{1}{2}i\rho)}{2^{\frac{1}{2}i\rho+\frac{3}{4}}\,\pi^{\frac{1}{2}}(1+i)}\,[D_{i\rho-\frac{1}{2}}(e^{i\pi/4}x) + D_{i\rho-\frac{1}{2}}(-e^{i\pi/4}x)] = y_1(x)$$

$$\mathrm{Re}\{2^{\frac{1}{2}}e^{3\pi\rho/4}[(1+e^{-2\pi\rho})^{\frac{1}{2}}-1]^{\frac{1}{2}}e^{-i(\gamma'/2+\pi/8)}D_{i\rho-\frac{1}{2}}(xe^{i\pi/4})\} = y_2(x)$$

$$-\mathrm{Im}\{2^{\frac{1}{2}}e^{3\pi\rho/4}[(1+e^{-2\pi\rho})^{\frac{1}{2}}+1]^{\frac{1}{2}}e^{-i(\gamma'/2+\pi/8)}D_{i\rho-\frac{1}{2}}(xe^{i\pi/4})\} = y_3(x)$$

where $\gamma' = \arg\Gamma(\tfrac{1}{2}+i\rho)$. y_0 and y_1 behave simply at $x = 0$:

$$y_0(0) = 1, \quad y_1(0) = 0, \quad y_0'(0) = 0, \quad y_1'(0) = 1,$$

and y_2 and y_3 behave simply at $x = \infty$:

$$y_2 = (2/x)^{\frac{1}{2}}\,e^{\frac{1}{4}\pi\rho}[(1+e^{-2\pi\rho})^{\frac{1}{2}}+1]^{\frac{1}{2}}\sin[g(x)]\,[1+O(x^{-1})]$$

$$y_3 = (2/x)^{\frac{1}{2}}\,e^{\frac{1}{4}\pi\rho}[(1+e^{-2\pi\rho})^{\frac{1}{2}}-1]^{\frac{1}{2}}\cos[g(x)]\,[1+O(x^{-1})],$$

where

$$g(x) = \tfrac{1}{4}x^2 - \rho\log x + \tfrac{1}{4}\pi + \tfrac{1}{2}\gamma'.$$

We also have

$$y_3(-x) = y_2(x).$$

J. C. P. Miller has made y_2 and y_3 the basis for the computation of numerical tables; y_0 and y_1 have been discussed by Wells and Spence, (1945) and by Darwin (1949).

From (2) and 6.6 (6), 6.6 (7) we find

(14) $D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0,$

and from 6.6(10) we have

(15) $$\frac{d^m}{dz^m}[e^{\frac{1}{4}z^2} D_\nu(z)] = (-1)^m (-\nu)_m\, e^{\frac{1}{4}z^2} D_{\nu-m}(z),$$

(16) $$\frac{d^m}{dz^m}[e^{-\frac{1}{4}z^2} D_\nu(z)] = (-1)^m\, e^{-\frac{1}{4}z^2} D_{\nu+m}(z) \qquad m = 1, 2, 3, \ldots .$$

Therefore, we obtain from (15), (16) and from Taylor's theorem

(17) $$D_\nu(x+y) = e^{\frac{1}{2}(xy+\frac{1}{2}y^2)} \sum_{m=0}^{\infty} (-y)^m (m!)^{-1} D_{\nu+m}(x)$$

$$= e^{-\frac{1}{2}(xy+\frac{1}{2}y^2)} \sum_{m=0}^{\infty} \binom{\nu}{m} y^m D_{\nu-m}(x)$$

and for $\nu = 0$ this gives the generating function of the $D_n(z)$ [i.e., of the Hermite polynomials, see (9) and Chap. 10].

(18) $$e^{-\frac{1}{4}z^2+zt-\frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_n(z).$$

If ν is a negative integer, the $D_\nu(z)$ can be expressed in terms of the error function

(19) $$D_{-m-1}(z) = 2^{\frac{1}{2}} \frac{(-1)^m}{m!} e^{-\frac{1}{4}z^2} \frac{d^m}{dz^m} [e^{\frac{1}{2}z^2} \operatorname{Erfc}(2^{-\frac{1}{2}} z)],$$

and if $\nu = -\frac{1}{2}$ in terms of a modified Bessel function of the third kind,

(20) $D_{-\frac{1}{2}}(z) = (\frac{1}{2}z\pi)^{\frac{1}{2}} K_{\frac{1}{4}}(\frac{1}{4}z^2).$

8.3. Integral representations and integrals

INTEGRAL REPRESENTATIONS OF PARABOLIC CYLINDER FUNCTIONS

$D_\nu(z)$

(1) $$= \frac{2^{\frac{1}{2}\nu}}{\Gamma(-\frac{1}{2}\nu)} e^{-\frac{1}{4}z^2} \int_0^\infty e^{-\frac{1}{2}tz^2} t^{-1-\frac{1}{2}\nu} (1+t)^{\frac{1}{2}(\nu-1)} dt$$
$\operatorname{Re}\nu < 0, \quad |\arg z| \le \frac{1}{4}\pi,$

(2) $$= \frac{2^{\frac{1}{2}(\nu-1)}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu)} z e^{-\frac{1}{4}z^2} \int_0^\infty e^{-\frac{1}{2}tz^2} t^{-\frac{1}{2}(1+\nu)} (1+t)^{\frac{1}{2}\nu} dt$$
$\operatorname{Re}\nu < 1, \quad |\arg z| \le \frac{1}{4}\pi,$

(3) $$= \frac{e^{-\frac{1}{4}z^2}}{\Gamma(-\nu)} \int_0^\infty e^{-zt-\frac{1}{2}t^2} t^{-\nu-1} dt \qquad \operatorname{Re}\nu < 0,$$

(4) $D_\nu(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{\frac{1}{4}z^2} \int_0^\infty e^{-\frac{1}{2}t^2} t^\nu \cos(zt - \tfrac{1}{2}\nu\pi)\, dt$

$\text{Re}\,\nu > -1,$

(5) $D_{-2\nu}[(a/z)^{\frac{1}{2}}]$

$= e^{-\frac{1}{4}a/z} z^\nu 2^{\nu-1} [\Gamma(2\nu)]^{-1} \int_0^\infty e^{-zt} t^{\nu-1} \exp[-(2at)^{\frac{1}{2}}]\, dt$

$\text{Re}\,\nu > 0,$

(6) $D_\nu(z) = 2^{\frac{1}{2}\nu - \frac{1}{4}} z^{\frac{1}{2}} e^{-\frac{1}{4}z^2} (2\pi i)^{-1}$

$\times \int_{-i\infty}^{i\infty} \frac{\Gamma(s)\,\Gamma(\frac{1}{2}\nu + \frac{1}{4} - s)\,\Gamma(\frac{1}{2}\nu - \frac{1}{4} - s)}{\Gamma(\frac{1}{2}\nu + \frac{1}{4})\,\Gamma(\frac{1}{2}\nu - \frac{1}{4})} (\tfrac{1}{2}z^2)^s\, ds$

$|\arg z| < \tfrac{3}{4}\pi, \quad \nu \neq 1/2, -1/2, -3/2, \ldots,$

(7) $D_\nu(z)\; D_{-\nu-1}(z) = 2\int_0^\infty J_{\nu+\frac{1}{2}}(t^2) \cos(zt - \tfrac{1}{2}\nu\pi)\, e^{-zt}\, dt$

$\text{Re}\, z > 0, \quad \text{Re}\,\nu > -1,$

$D_\nu(ze^{\frac{1}{4}\pi i})\, D_\nu(ze^{-\frac{1}{4}\pi i})$

(8) $= \frac{\pi^{\frac{1}{2}}}{\Gamma(-\nu)} \int_0^\infty J_{-\nu-\frac{1}{2}}(\tfrac{1}{2}t^2)\, e^{-zt}\, dt \qquad \text{Re}\,\nu < 0, \quad \text{Re}\, z \geq 0,$

(9) $= \frac{2^{3/2}}{\pi^{\frac{1}{2}}\Gamma(-\nu)} \int_0^\infty K_{\nu+\frac{1}{2}}(t^2) \cos(zt - \tfrac{1}{2}\nu\pi)\, e^{-zt}\, dt$

$0 < -\text{Re}\,\nu < 1,$

(10) $= \frac{1}{\pi}\int_0^\infty (\cosh t)^\nu (\sinh t)^{\nu-1} \exp(-\tfrac{1}{2}z^2 \sinh t)\, dt$

$\text{Re}\,\nu > 0, \quad |\arg z| \leq \tfrac{1}{4}\pi,$

The integral representations (1), (2), (3), (4), (5), can be proved by verifying that the right-hand sides satisfy the differential equation 8.2(1) and assume the correct initial values at $z = 0$. Equations (1) and (2) can also be derived from 6.5(2), 6.5(6) and 8.2(1). In (6), the path of integration must be chosen in such a way that it separates the poles of $\Gamma(s)$ from those of the two other gamma-factors in the numerator of the integrand; the formula is a consequence of 6.11(9).

The integral representations (7), (8), (9), were proved by Meijer (1935b, 1937a) and (10) was proved by Bailey (1937). J_ν, K_ν denote Bessel functions of order ν; see 7.2(2), 7.2(13).

There exists a very large number of other integral representations for the D_ν or a product of two parabolic cylinder functions. For representations of D_ν see Meijer (1934, 1935 a, 1938 a). An integral representation of D_ν involving other confluent hypergeometric functions was given by Meijer (1941). For results related to (7), (8), (9), see also Meijer (1937b).

INTEGRALS INVOLVING PARABOLIC CYLINDER FUNCTIONS

(11) $$\int_0^\infty e^{-zt} t^{-1+\beta/2} D_{-\nu}[2(kt)^{1/2}]\,dt = \frac{2^{1-\beta-\nu/2}\,\pi^{1/2}\,\Gamma(\beta)}{\Gamma(\tfrac12\nu+\tfrac12\beta+\tfrac12)}(z+k)^{-\beta/2} F\left(\frac{\nu}{2},\frac{\beta}{2};\frac{\nu+\beta+1}{2};\frac{z-k}{z+k}\right)$$
$\operatorname{Re}\beta>0,\quad \operatorname{Re} z/k>0,$

(12) $$\int_0^\infty e^{\frac14 t^2} D_{-\nu}(t)\, t^{2c-1}\,\Phi(a,c;-\tfrac12 pt^2)\,dt = \frac{\pi^{1/2}}{2^{c+\frac12\nu}}\frac{\Gamma(2c)\,\Gamma(\tfrac12\nu-c+a)}{\Gamma(\tfrac12\nu)\,\Gamma(a+\tfrac12+\tfrac12\nu)} F(a,c+\tfrac12;a+\tfrac12+\tfrac12\nu;1-p)$$
$|1-p|<1,\quad \operatorname{Re}c>0,\quad \operatorname{Re}\nu>2\operatorname{Re}(c-a),$

(13) $$\int_0^\infty e^{\frac14 t^2} D_{-\nu}(t)\, t^{2c-2}\,\Phi(a,c;-\tfrac12 pt^2)\,dt = \frac{\pi^{1/2}}{2^{c+\frac12\nu-\frac12}}\frac{\Gamma(2c-1)\,\Gamma(\tfrac12\nu+\tfrac12-c+a)}{\Gamma(\tfrac12+\tfrac12\nu)\,\Gamma(a+\tfrac12\nu)} F(a,c-\tfrac12;a+\tfrac12\nu;1-p)$$
$|1-p|<1,\quad \operatorname{Re}c>\tfrac12,\quad \operatorname{Re}\nu>2\operatorname{Re}(c-a)-1,$

(14) $$\int_0^\infty t^\nu e^{-\frac14 t^2} D_{2\nu}(t)\, J_{\nu-1}(tz)\,dt = 2^\nu\,\Gamma(\nu+\tfrac12)\,\pi^{-1/2} z^{\nu-1} e^{-\frac14 z^2}\,\Phi(-\nu,\tfrac12;\tfrac12 z^2)$$
$\operatorname{Re}\nu>-\tfrac12,$

(15) $$(2\pi\mu)^{-1/2}\int_{-\infty}^\infty e^{-(x-y)^2/(2\mu)}\, e^{\frac14 y^2} D_\nu(y)\,dy = (1-\mu)^{\frac12\nu} e^{x^2/(4-4\mu)} D_\nu[x(1-\mu)^{-1/2}]$$
$0<\operatorname{Re}\mu<1,$

(16) $$\int_{-\infty}^\infty e^{ixy-(1+\lambda)y^2/4} D_\nu[y(1-\lambda)^{1/2}]\,dy = (2\pi)^{1/2}\lambda^{\frac12\nu} e^{-(1+\lambda)x^2/(4\lambda)} D_\nu[i(\lambda^{-1}-1)^{1/2}x]$$
$\operatorname{Re}\lambda>0,$

(17) $$\int_0^\infty (xy)^{1/2} J_\nu(xy)\, y^{\nu-\frac12} e^{\frac14 y^2} D_{-2\nu}(y)\,dy = x^{\nu-\frac12} e^{\frac14 x^2} D_{-2\nu}(x)$$
$\operatorname{Re}\nu>-\tfrac12,$

(18) $$\int_0^\infty D_\nu(y) e^{-\frac14 y^2} y^\nu (x^2+y^2)^{-1}\,dy = (\tfrac12\pi)^{1/2}\,\Gamma(\nu+1)\, x^{\nu-1} e^{\frac14 x^2} D_{-\nu-1}(x)$$
$\operatorname{Re}\nu>-1,$

(19) $\int_0^\infty e^{-3t^2/4}\, t^{\nu-1}\, D_\nu(t)\, dt = 2^{-\frac{1}{2}\nu}\,\Gamma(\nu)\,\cos(\tfrac{1}{4}\nu\pi)$ Re $\nu > 0$,

(20) $\int_0^\infty e^{-\frac{1}{4}t^2}\, t^{\mu-1}\, D_{-\nu}(t)\, dt$

$$= \frac{\pi^{\frac{1}{2}}\, 2^{-\frac{1}{2}\mu-\frac{1}{2}\nu}\,\Gamma(\mu)}{\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})} \qquad \text{Re}\, \mu > 0,$$

(21) $\int_0^\infty D_\mu(\pm t)\, D_\nu(t)\, dt$

$$= \frac{\pi\, 2^{\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}}{\mu-\nu}\left[\frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu)\,\Gamma(-\frac{1}{2}\nu)} \mp \frac{1}{\Gamma(-\frac{1}{2}\mu)\,\Gamma(\frac{1}{2}-\frac{1}{2}\nu)}\right]$$

Re $\mu >$ Re ν if lower signs are taken.

(22) $$\int_0^\infty [D_\nu(t)]^2\, dt = \pi^{\frac{1}{2}}\, 2^{-3/2}\, \frac{\psi(\frac{1}{2}-\frac{1}{2}\nu) - \psi(-\frac{1}{2}\nu)}{\Gamma(-\nu)}$$

(23) $\int_0^\infty [D_n(t)]^2\, dt = (2\pi)^{\frac{1}{2}}\, n\,!$ $\qquad n = 0, 1, 2, \ldots$.

In these formulas, F, Φ, J, ψ, denote the hypergeometric and the confluent hypergeometric series, the Bessel function of the first kind and the logarithmic derivative of the Γ-function.

Equation (11) follows from 6.11 (12) and the inversion formula of the Laplace transformation. According to 2.1 (26), 2.1 (2), the right-hand side in (11) reduces to an elementary function if $\beta = \nu + 1$ or if $\nu = -2n$, $n = 0, 1, 2, \ldots$. For the proof of (12), and (13) see Erdélyi (1936). More general formulas of this type involving an ${}_pF_q$ (sec. 4.1) instead of Φ have been given by Mitra (1946). A proof of (14) was given by Meijer, (1938). To prove (15) it suffices to express D_ν by (3) and to interchange the order of integrations; if μ tends to 1, the right-hand side in (15) tends to x^ν. Formula (16) is essentially the same as (15) and formulas (17), (18) are due to R. S. Varma (1936, 1937); Watson (1910) has proved (19), and formulas (20), (21) were given by Erdélyi (1938); for $\nu = \mu$, we obtain (22) and (23) from (21). We also see from (21), that the $D_n(t)$, $n = 0, 1, 2, \ldots$, form an orthogonal system in $(-\infty, \infty)$.

8.4. Asymptotic expansions

From 8.3 (6) it can be shown that (see Whittaker-Watson, 1927) for large values of $|z|$ and a fixed value of ν

(1) $$D_\nu(z) = z^\nu e^{-\frac{1}{4}z^2}\left[\sum_{n=0}^{N} \frac{(-\frac{1}{2}\nu)_n\,(\frac{1}{2}-\frac{1}{2}\nu)_n}{n\,!\,(-\frac{1}{2}z^2)^n} + O|z^2|^{-N-1}\right]$$

$-\frac{3}{4}\pi < \arg z < \frac{3}{4}\pi$,

$$(2)\quad D_\nu(z) = z^\nu e^{-\frac14 z^2}\Bigg[\sum_{n=0}^{N} \frac{(-\frac12\nu)_n(\frac12-\frac12\nu)_n}{n!\,(-\frac12 z^2)^n} + O|z^2|^{-N-1}$$

$$-\frac{(2\pi)^{\frac12}}{\Gamma(-\nu)} e^{\nu\pi i} z^{-\nu-1} e^{\frac14 z^2}\sum_{n=0}^{N}\frac{(\frac12\nu)_n(\frac12+\frac12\nu)_n}{n!\,(\frac12 z^2)^n} + O|z^2|^{-N-1}\Bigg] \qquad \pi/4 < \arg z < 5\pi/4,$$

$$(3)\quad D_\nu(z) = z^\nu e^{-\frac14 z^2}\Bigg[\sum_{n=0}^{N} \frac{(-\frac12\nu)_n(\frac12-\frac12\nu)_n}{n!\,(-\frac12 z^2)^n} + O|z^2|^{-N-1}$$

$$-\frac{(2\pi)^{\frac12}}{\Gamma(-\nu)} e^{-\nu\pi i} z^{-\nu-1} e^{\frac14 z^2}\sum_{n=0}^{N}\frac{(\frac12\nu)_n(\frac12+\frac12\nu)_n}{n!\,(\frac12 z^2)^n} + O|z^2|^{-N-1}\Bigg] \qquad -5\pi/4 < \arg z < -\pi/4,$$

where the notation

$$(4)\quad (a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1, 2, 3, \dots,$$

is used.

The behavior of $D_\nu(z)$ for $|\nu| \to \infty$ and for arbitrary values of z which satisfy $|z| < |\nu|^{\frac12}$ has been completely discussed by Schwid (1935). His results are based on Langer's method (1932). As a special case we have the following result which, in the form given here, has been stated by Cherry (1949):

If $|z|$ is bounded and $|\arg(-\nu)| \le \frac12\pi$, then, for $|\nu| \to \infty$

$$(5)\quad D_\nu(z) = 2^{-\frac12}\exp[\tfrac12\nu\log(-\nu) - \tfrac12\nu - (-\nu)^{\frac12} z][1 + O|\nu|^{-\frac12}].$$

8.5. Representation of functions in terms of the $D_\nu(x)$

8.5.1. Series

From 6.12(3) we have as a special case, for positive real values of x:

$$(1)\quad D_\nu(x) = \frac{2^{\frac12\nu}}{\Gamma(-\frac12\nu)}\sum_{n=0}^{\infty}\frac{(-1)^n D_{2n}(x)}{n!\,2^n(n-\frac12\nu)}$$

$$= \frac{2^{\frac12\nu-\frac12}}{\Gamma(\frac12-\frac12\nu)}\sum_{n=0}^{\infty}\frac{(-1)^n D_{2n+1}(x)}{n!\,2^n(n+\frac12-\frac12\nu)}.$$

This can be considered as an interpolation formula for the function $D_\nu(x)$ of ν, the points of interpolation being the non-negative even or odd integers. An expansion of $D_\nu(x)\,D_\mu(x)$ in terms of the $D_n(2^{\frac12}x)$, $(n = 0, 1, 2, \dots,)$ has been given by Dhar (1935). Shanker (1939) proved the *addition theorem*

(2) $$D_\nu(x\cos t + y\sin t) = \exp[\tfrac14(x\sin t - y\cos t)^2] \times \sum_{n=0}^{\infty}\binom{\nu}{n}(\tan t)^n D_{\nu-n}(x)\,D_n(y)$$ see errata!

which holds for real values of t, x, y and $0 \le t < \pi/4$, Re $\nu \ge 0$.

Erdélyi (1936) proved the expansion [see 8.4(4)]

(3) $$W_{\kappa,\mu}(\tfrac12 z^2) = 2^{-\kappa} z^{\frac12}\left[\sum_{l=0}^{p-1}\frac{(-1)^l(\tfrac12-2\mu)_l(\tfrac12+2\mu)_l}{(2z)^l}D_{2\kappa-\frac12-l}(z)+R_p\right]$$

in which R_p denotes a remainder term. If μ is half of an odd integer, the series terminates. In all other cases the series is divergent in general, but the remainder term can be estimated, in particular if $|\arg z| < \tfrac14\pi$ and p is large, showing the asymptotic nature of the expansion

From the expansion 6.12(6), an expansion of $D_\nu(z)$ in terms of the Bessel functions can be derived, where the Bessel functions become elementary functions because their order is half an odd integer. In particular, we have

$$D_\nu(z) = \frac{\pi^{\frac12}2^{\frac12\nu}}{\Gamma(\tfrac34-\kappa)}\{\cos\zeta - 2^{-6}\kappa^{-2}\zeta[(1-2\zeta^2/3)\sin\zeta - \zeta\cos\zeta] + \cdots\}$$
$$-\frac{\pi^{\frac12}2^{\frac12\nu}}{\kappa^{\frac12}\Gamma(\tfrac14-\kappa)}\{\sin\zeta - 2^{-6}\kappa^{-2}[(1-\zeta^2)\sin\zeta - \zeta(1-2\zeta^2/3)\cos\zeta] + \cdots\}$$

where

$$\kappa = \tfrac12\nu + \tfrac14 > 0, \qquad \zeta = (2\kappa)^{\frac12} z,$$

and the terms indicated by $\cdots$ are of the order of κ^{-3} provided that ζ is bounded.

The Sturm-Liouville problems connected with 8.2(12) lead to certain orthogonal sets of functions for a finite interval $(0, x_0)$. These are essentially parabolic cylinder functions whose order is of the type $i\rho - \tfrac12$ (ρ real) and for which the variable has an argument, $\tfrac14\pi$ or $-\tfrac34\pi$, (see sec. 8.2). For an application see Magnus (1941); for Sturm-Liouville problems in general see Chap. 10 in the book by Ince (1944).

8.5.2. Representation by integrals with respect to the parameter

Cherry's theorem (1949). *If $f(x)$ is of bounded variation in any finite interval of the real variable x and is absolutely integrable in $(-\infty, \infty)$, then*

(4) $$-4\pi i\, f(x) = \int_{-\frac12-i\infty}^{-\frac12+i\infty}\frac{e^{\frac12(\nu+\frac12)\pi i}}{\sin\nu\pi}\,d\nu\int_{-\infty}^{\infty}[D_\nu(hx)D_{-\nu-1}(\bar{h}t) + D_\nu(-hx)\,D_{-\nu-1}(-\bar{h}t)]\,f(t)\,dt$$

where

(5) $h = e^{\frac{1}{4} i\pi}, \quad \bar{h} = e^{-\frac{1}{4} i\pi}.$

The condition that f *be absolutely integrable, can be replaced by*

$$(6) \quad f(x) = e^{-\frac{1}{4} i x^2} \left(\frac{c_1}{x^a} + \frac{c_2}{x^{a+1}} \right) [1 + O(|x|^{-1}]$$

for $x \to \pm \infty$, *where* a *is real and* $> \frac{1}{2}$ *and where* c_1, c_2, *are constants* (which may be different for $x \to +\infty$ and $x \to -\infty$). Condition (6) is needed in some boundary value problems (see Magnus, 1940). Equation (4) is analogous to the inversion theorem for Fourier integrals. It can be simplified if $f(x)$ is an even or an odd function of x.

Cherry (1949) has applied (4) to the function $f(x) = D_\mu(hx)$ for $x > 0$, $f(x) \equiv 0$ for $x < 0$. In a formal sense [although (4) and (6) are not satisfied], Erdélyi's formula for the expression of a plane wave in coordinates of the parabolic cylinder is a special case of Cherry's theorem, viz.:

$$(7) \quad -2i(2\pi)^{\frac{1}{2}} \exp[-\tfrac{1}{4} i (\xi^2 - \eta^2) \cos\phi - \tfrac{1}{2} i \xi\eta \sin\phi]$$

$$= \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \frac{d\nu}{\sin \nu\pi} \left[\frac{(\tan \frac{1}{2}\phi)^\nu}{\cos \frac{1}{2}\phi} D_\nu(-h\xi) D_{-\nu-1}(h\eta) \right.$$

$$\left. + \frac{(\operatorname{ctn} \frac{1}{2}\phi)^\nu}{\sin \frac{1}{2}\phi} D_{-\nu-1}(h\xi) D_\nu(-h\eta) \right]$$

(cf. Erdélyi, 1941). Here h is given by (5), and (7) holds for all real values of ξ, η. For the diffraction problem of a plane wave incident on a half-plane, Cherry (1949), gives the formula

$$(8) \quad -2i\, D_0[h(\zeta \cos \tfrac{1}{2}\phi + \eta \sin \tfrac{1}{2}\phi)]\, D_{-1}[h(\eta \cos \tfrac{1}{2}\phi - \zeta \sin \tfrac{1}{2}\phi)]$$

$$= \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \frac{d\nu}{\sin \nu\pi} \frac{(\tan \frac{1}{2}\phi)^\nu}{\cos \frac{1}{2}\phi} D_\nu(-h\zeta) D_{-\nu-1}(h\eta).$$

for the secondary wave ("Sommerfeld's wave").

A special case of 6.15(15) is the expression of a "cylindrical wave" in terms of solutions of 8.2(1), viz.

$$(9) \quad 2^{\frac{1}{2}} \pi^2 H_0^{(2)}[\tfrac{1}{2} k (\xi^2 + \eta^2)]$$

$$= \int_{c-i\infty}^{c+i\infty} D_\nu[k^{\frac{1}{2}}(1+i)\xi]\, D_{-\nu-1}[k^{\frac{1}{2}}(1+i)\eta]\, \Gamma(-\tfrac{1}{2}\nu)\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu)\, d\nu$$

where $-1 < c < 0$, ξ, η, real, $\operatorname{Re} ik \geq 0$. Another expression for the left-hand side in (9) in terms of an integral taken over the parameter of parabolic cylinder functions can be obtained from Cherry's theorem; see also Magnus (1941).

Erdélyi (1941) also proved the following formulas which can be considered as linear and bilinear continuous generating functions of D_ν [see also 6.2(20)]:

$$(10)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} D_\nu(z)\, t^\nu\, \Gamma(-\nu)\, d\nu = e^{-\frac{1}{4}z^2 - zt - \frac{1}{2}t^2}$$

$$c<0, \qquad |\arg t| < \pi/4,$$

$$(11)\quad \frac{(\frac{1}{2}\pi)^{\frac{1}{2}}}{2\pi i}\int_{c-i\infty}^{c+i\infty}[D_\nu(x)\,D_{-\nu-1}(iy) + D_\nu(-x)\,D_{-\nu-1}(-iy)]\frac{t^{-\nu-1}\,d\nu}{\sin(-\nu\pi)}$$

$$= (1+t^2)^{-\frac{1}{2}}\exp\left[\frac{1}{4}\,\frac{1-t^2}{1+t^2}(x^2+y^2) + i\,\frac{txy}{1+t^2}\right]$$

$$-1<c<0, \qquad |\arg t| < \tfrac{1}{2}\pi,$$

8.6. Zeros and descriptive properties

For any fixed value of ν the formulas 8.4(1) to 8.4(3) give a description of $D_\nu(z)$ for large values of $|z|$; if ν and z are real, then $D_\nu(z)$ is also real in spite of the appearance of 8.4(2) and 8.4(3). If ν is real, $D_\nu(z)$ has $[\nu+1]$ real zeros, where $[\nu+1]$ denotes the greatest positive integer less than $\nu+1$ or zero if such a positive integer does not exist. This result can be derived from a discussion of the differential equation 8.2(1). If $\nu = n = 0, 1, 2, \ldots$, $D_n(z)$ has exactly n real zeros (and no other zeros). For other results about real zeros of the solutions of 8.2(1) which are real on the real axis see Auluck (1941); for asymptotic formulas for the real zeros of $D_\nu(x)$ if ν is real see Tricomi (1947).

FUNCTIONS OF THE PARABOLOID OF REVOLUTION

The results of the two following sections comprise only a small part of the formulas which arise from boundary value problems of $\Delta u + \kappa^2 u = 0$ for the paraboloid of revolution. The whole subject has been thoroughly investigated by Buchholz; the formulas in sections 8.7 and 8.8 indicate which type of results can be found in the papers to which a reference is made.

8.7. The solutions of a particular confluent hypergeometric equation

If in 8.1(8) k, μ, λ are arbitrary complex constants, we have a differential equation which is equivalent to the confluent hypergeometric equation. However, if k and λ are real and 2μ is an integer, 8.1(8) may be reduced to

$$(1)\quad \frac{d^2u}{d\xi^2} + \xi^{-1}\frac{du}{d\xi} + (4\xi^2 - p^2\xi^{-2} - 4\tau)\,u = 0$$

where

(2) $p = 0, 1, 2, \dots,$ and τ, ξ, real.

Equation (1) has the solutions

(3) $\xi^{-1} M_{\pm i\tau, \frac{1}{2}p}(\pm i \xi^2), \quad \xi^{-1} W_{\pm i\tau, \frac{1}{2}p}(\pm i \xi^2)$

(for the notations see sec. 6.9). They are connected by the relations

(4) $e^{-\frac{1}{4} i\pi(p+1)} M_{i\tau, \frac{1}{2}p}(i\xi) = e^{\frac{1}{4} i\pi(p+1)} M_{-i\tau, \frac{1}{2}p}(-i\xi)$

(5) $$= \frac{p!\exp[\pi\tau - \frac{1}{4} i \pi(p+1)]}{\Gamma(\frac{1}{2} + \frac{1}{2}p - i\tau)} W_{-i\tau, \frac{1}{2}p}(-i\xi) + \frac{p!\exp[\pi\tau + \frac{1}{4} i \pi(p+1)]}{\Gamma(\frac{1}{2} + \frac{1}{2}p + i\tau)} W_{i\tau, \frac{1}{2}p}(i\xi)$$

where ξ denotes a real, positive variable, and $\arg \pm i\xi = \pm \frac{1}{2}\pi$. For $\xi \to \infty$ and fixed values of τ, p we have:

(6) $W_{i\tau, \frac{1}{2}p}(i\xi) = \xi^{i\tau} e^{-\frac{1}{2}i\xi - \frac{1}{4}\pi\tau}[1 + O(\xi^{-1})]$

(7) $W_{-i\tau, \frac{1}{2}p}(-i\xi) = \xi^{-i\tau} e^{\frac{1}{2}i\xi - \frac{1}{4}\pi\tau}[1 + O(\xi^{-1})]$.

The corresponding expression of the M-functions in (3) can be derived from (4), (5), (6), and (7).

The function

(8) $e^{-\frac{1}{4} i\pi(p+1)} M_{i\tau, \frac{1}{2}p}(i\xi) = e^{\frac{1}{4} i\pi(p+1)} M_{-i\tau, \frac{1}{2}p}(-i\xi)$

is real for real, positive values of ξ (if τ, p are real).

If p and ξ are fixed and τ is large, 6.13(8) gives the following asymptotic representations:

(9) $$W_{\pm i\tau, \frac{1}{2}p}(\pm i\xi) = 2^{\frac{1}{2}} e^{\mp\frac{1}{4}i\pi} e^{\mp i\tau} \tau^{\pm i\tau}(\xi\tau)^{-\frac{1}{4}} e^{-\frac{1}{2}\pi\tau} \times \cosh[(\tau - 2(\tau\xi)^{\frac{1}{2}} \pm \frac{1}{4} i\pi][1 + O(\tau^{-\frac{1}{2}})],$$

(10) $$W_{\mp i\tau, \frac{1}{2}p}(\pm i\xi) = 2^{\frac{1}{2}} e^{\pm i\tau} \tau^{\mp i\tau}(\xi\tau)^{-\frac{1}{4}} e^{-\frac{1}{2}\pi\tau} \times \cos[\mp i\tau - 2(\tau\xi)^{\frac{1}{2}} - \frac{1}{4}\pi][1 + O(\tau^{-\frac{1}{2}})],$$

where τ, ξ, are real and positive.

Erdélyi (1937) investigated the case where $|\tau|$ and $|\xi|$ are both large but where τ/ξ is a fixed negative number. This result is

(11) $$M_{-i\tau,\mu}[i\tau(2\sinh\beta)^2] = \Gamma(2\mu+1)e^{\frac{1}{2}i\pi(\mu+\frac{1}{2})}\tau^{-\mu}\left(\frac{2\tanh\beta}{\pi}\right)^{\frac{1}{2}} \times \sin[\tau(\sinh 2\beta + 2\beta) - (\mu - \frac{1}{4})\pi][1 + O(\tau^{-\frac{1}{2}})]$$

where τ, β, $\mu + \frac{1}{2}$, are real and positive.

For the solution of certain boundary-value problems, the following functions are needed. Let ζ_0 be a fixed real positive number. Then there exists a sequence of real numbers τ_n, $n = 1, 2, 3, \ldots$, such that

$$\tau_1 < \tau_2 < \tau_3 \cdots \quad \text{and} \quad M_{i\tau_n, \frac{1}{2}p}(i\zeta_0) = 0.$$

The functions

$$(12) \quad \left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} M_{i\tau_n, \frac{1}{2}p}(i\zeta)$$

are orthogonal in $(0, \zeta_0)$. In order to compute the τ_n for a given ζ_0 and in order to find the normalizing factors for the functions (12), Buchholz (1943) gave the formula

$$(13) \quad (i\zeta)^{-\frac{1}{2}-\frac{1}{2}p} M_{i\tau, \frac{1}{2}p}(i\zeta)$$

$$= \pi^{\frac{1}{2}} \frac{\Gamma(1+\frac{1}{2}p)}{\Gamma(\frac{1}{2}+\frac{1}{2}p)} \sum_{l=0}^{\infty} \frac{\Gamma(l+\frac{1}{2}+\frac{1}{2}p)}{\Gamma(l+1+\frac{1}{2}p)} \frac{(\frac{1}{4}\zeta)^{l+\frac{1}{2}}}{l!}$$

$$\times \prod_{r=0}^{l} \left[1 + \frac{\tau^2}{(r+\frac{1}{2}+\frac{1}{2}p)^2}\right]$$

$$\times \left[J_{l-\frac{1}{2}}(\tfrac{1}{2}|\zeta|) + \frac{\tau \operatorname{sgn} \zeta}{l+\frac{1}{2}+p} J_{l+\frac{1}{2}}(\tfrac{1}{2}|\zeta|)\right]$$

where τ, ζ, are real, $\tau > 0$, $\zeta \neq 0$, and also similar formulas for the partial derivatives

$$\frac{\partial}{\partial\tau}, \quad \frac{\partial}{\partial\zeta}, \quad \frac{\partial^2}{\partial\tau\,\partial\zeta}$$

of the function (13).

8.8. Integrals and series involving functions of the paraboloid of revolution

As a consequence of 6.15(15) we have

$$(1) \quad \frac{e^{i(x+y)}}{x+y} = \frac{-i}{2(xy)^{\frac{1}{2}}} \int_{-i\infty}^{i\infty} W_{-s,0}(-2ix)\, W_{s,0}(-2iy) \frac{ds}{\cos \pi s}.$$

This is the representation of a spherical wave with the center in the focus of the paraboloid in terms of the functions of the paraboloid in revolution. Formula (1) was first proved by Meixner (1933) who also derived the formula

$$(2)\quad \frac{2\pi\, p!\, p!}{(2p+1)!}\Gamma(p+1+2ia)\,\Gamma(p+1-2ia)\,\frac{(xy)^{\frac{1}{2}(p+1)}}{(x+y)^{p+1}}M_{-2ia,\frac{1}{2}p}(x+y)$$

$$= \int_{-\infty}^{\infty}\Gamma[\tfrac{1}{2}p+\tfrac{1}{2}+i(a+\tau)]\,\Gamma[\tfrac{1}{2}p+\tfrac{1}{2}+i(a-\tau)]$$

$$\times\,\Gamma[\tfrac{1}{2}p+\tfrac{1}{2}-i(a+\tau)]\,\Gamma[\tfrac{1}{2}p+\tfrac{1}{2}-i(a-\tau)]$$

$$\times\, M_{ia+i\tau,\frac{1}{2}p}(x)\, M_{ia+\frac{1}{2}i\tau,\frac{1}{2}p}(y)\, d\tau$$

$$\operatorname{Re} x \geq 0, \quad \operatorname{Re} y \geq 0, \quad p = 0, 1, 2, \ldots .$$

The integral representations for more complicated types of waves with a singularity at the focus of the paraboloid of revolution were given by Buchholz (1947). One of his results is

$$(3)\quad (xy)^{\frac{1}{2}}\, i^{n}\left[\frac{\pi}{2(x+y)}\right]^{\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}(x+y)\, P^{p}_{n}\left(\frac{x-y}{x+y}\right)$$

$$= \frac{(-1)^{p-1}}{2\pi}\;\frac{(n+p)!}{p!\,p!\,(n-p)!}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\tfrac{1}{2}p+\tfrac{1}{2})\,\Gamma(-s+\tfrac{1}{2}p+\tfrac{1}{2})$$

$$\times\; {}_3F_2\; W_{-s,\frac{1}{2}p}(-2ix)\, W_{-s,\frac{1}{2}p}(-2iy)\, ds$$

where

$$x > y \geq 0, \qquad \sigma < \tfrac{1}{2} + \tfrac{1}{2}p,$$

and where ${}_3F_2$ stands for

$${}_3F_2 = {}_3F_2(-n+p, n+p+1, -s+\tfrac{1}{2}+\tfrac{1}{2}p; p+1, p+1;\ 1),$$

[see 4.1(1) for the definition of the generalized hypergeometric series].

If $n = p$, (3) becomes equivalent to 6.15(15). It should be noted, that $H^{(1)}_{n+\frac{1}{2}}$ is an elementary function, see 7.2(6).

The expression of a spherical wave, the center of which is at an arbitrary point, in terms of the functions of the paraboloid of revolution was also given by Buchholz (1947). If

$$R = \{[(x_1 - y_1) - (x_0 - y_0)]^2 + 4x_0 y_0 + 4x_1 y_1$$

$$-8(x_0 y_0 x_1 y_1)^{\frac{1}{2}}\cos(\phi_1 - \phi_2)\}^{\frac{1}{2}}$$

and if x_0, y_0, x_1, y_1, are real and positive and $x_0 > x_1$, $y_0 > y_1$, then, for a real $\sigma < \frac{1}{2}$,

$$(4)\quad (x_0 y_0 x_1 y_1)^{\frac{1}{2}} \frac{e^{iR}}{iR} = -2 \sum_{p=0}^{\infty} (2 - \delta_{0,p}) \frac{\cos p(\phi_0 - \phi_1)}{p!\,p!}$$

$$\times (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s + \tfrac{1}{2} + \tfrac{1}{2}p)\,\Gamma(-s + \tfrac{1}{2} + \tfrac{1}{2}p)$$

$$\times [M_{-s,\frac{1}{2}p}(-2ix_1)\,M_{s,\frac{1}{2}p}(-2iy_1)\,W_{-s,\frac{1}{2}p}(-2ix_0)W_{s,\frac{1}{2}p}(-2iy_0)]ds$$

where $\delta_{0,0} = 1$ and $\delta_{0,p} = 0$ if $p > 0$.

For the plane wave Buchholz (1947) gives a mixed series and integral representation:

$$(5)\quad \exp[i(x - y)\cos\theta + 2(xy)^{\frac{1}{2}} \sin\theta \cos\phi]$$

$$= \frac{1}{(xy)^{\frac{1}{2}} \sin\theta} \sum_{p=0}^{\infty} \frac{2 - \delta_{0,p}}{p!\,p!}\, i^p \cos(p\phi)$$

$$\times (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s + \tfrac{1}{2} + \tfrac{1}{2}p)\,\Gamma(-s + \tfrac{1}{2} + \tfrac{1}{2}p)(\tan \tfrac{1}{2}\theta)^{2s}$$

$$\times M_{s,\frac{1}{2}p}(-2ix)\,M_{s,\frac{1}{2}p}(-2iy)\,ds.$$

There correspond certain series expansions to the integral representations in this section. In the simplest case, the formula corresponding to (1) is

$$(6)\quad \frac{e^{i(x+y)}}{x + y} = \frac{1}{(xy)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n\, W_{-n-\frac{1}{2},0}(-2ix)\, W_{-n-\frac{1}{2},0}(-2iy).$$

For a large number of other series and integrals see Buchholz (1943, 1947, 1948, 1949).

REFERENCES

Appell, Paul, and M. J. Kampé de Fériet, 1926: *Fonctions hypérgeométriques et hypersphériques. Polynomes d'Hermite.* Gauthier-Villars.

Auluck, F. C., 1941: *Proc. Nat. Inst. Sci., India* 7, 133-140.

Bailey, W. N., 1937: *Quart. J. Math. Oxford Ser.* 8, 51-53.

Buchholz, Herbert, 1943: *Z. Angew. Math. Mech.* 23, 47-58, 101-118.

Buchholz, Herbert, 1947: *Z. Physik,* 124, 196-218.

Buchholz, Herbert, 1948: *Ann. Physik* (6) 2, 185-210.

Buchholz, Herbert, 1949: *Math. Z.,* 52, 355-383.

Cherry, T. M., 1949: *Proc. Edinburgh Math. Soc.* (2), 8, 50-65.

Darwin, C. G., 1949: *Quart. J. Mech. Appl. Math.* 2, 311-320.

Dhar, S. C., 1935: *J. Indian Math. Soc.* (*N.S.*) 1, 105-108.

Erdélyi, Arthur, 1936: *Math. Ann.* 113, 347-356.

Erdélyi, Arthur, 1937: *Akad. Wiss. Wien. S.-B. IIa* 146, 589-604.

Erdélyi, Arthur, 1938: *J. Indian Math. Soc.* (*N.S.*) 3, 169-181.

Erdélyi, Arthur, 1941: *Proc. Royal Soc. Edinburgh,* 61, 61-70.

Humbert, Pierre, 1920 a: *C. R. Acad. Sci. Paris,* 170, 564.

Humbert, Pierre, 1920 b: *C. R. Acad. Sci. Paris,* 170, 832.

Humbert, Pierre, 1920 c: *C. R. Acad. Sci. Paris,* 170, 1482.

Humbert, Pierre, 1920 d: *C. R. Acad. Sci. Paris,* 171, 428.

Ince, E. L., 1944: *Ordinary differential equations,* Longmans.

Langer, R. E., 1932: *Trans. Amer. Math. Soc.* 34, 447-480.

Magnus, Wilhelm, 1940: *Jber. Deutsch. Math. Verein,* 50, 140-161.

Magnus, Wilhelm, 1941: *Z. Physik,* 118, 343-356.

Meijer, C. S., 1934: *N. Archiv. V. Wiskunde* (2), 18, 35-57.

Meijer, C. S., 1935 a: *Proc. Kon. Akad. Wetensch. Amsterdam* 38, 528-535.

Meijer, C. S., 1935 b: *Quart. J. Math. Oxford Ser.* 6, 241-248.

Meijer, C. S., 1937 a: *Kon. Akad. Wetensch. Amsterdam,* 40, 259-262.

Meijer, C. S., 1937b: *Kon. Akad. Wetensch. Amsterdam* 40, 871-879.

Meijer, C. S., 1938: *Kon. Akad. Nederland. Wetensch.* 41, 744-755.

Meijer, C. S., 1938a: *Proc. Kon. Nederland Akad. Wetensch. Amsterdam* 41, 42-44.

Meijer, C. S., 1941: *Proc. Kon. Akad. Wetensch. Amsterdam* 44, 590-598.

Meixner, Joseph, 1933: *Math. Z.* 36, 677-707.

Mitra, S. C., 1927: *Proc. Benares Math. Soc.* 9, 21-23.

REFERENCES

Mitra, S. C., 1946: *Proc. Edinburgh Math. Soc.* 7, (2), 171-173.

Schwid, N., 1935: *Trans. Amer. Math. Soc.* 37, 339-362.

Shanker, Hari, 1939: *J. Indian Math. Soc.* (*N.S.*) 3, 226-230.

Shanker, Hari, 1939: *J. Indian Math. Soc.* (*N.S.*) 3, 228-230.

Taylor, W. C., 1939: *J. Math. Physics*, 18, 34-49.

Tricomi, Francesco, 1947: *Ann. Mat. Pura Appl.* (4), 26, 283-300.

Varma, R. S., 1927: *Proc. Benares Math. Soc.* 9, 31-42.

Varma, R. S., 1936: *Proc. London Math. Soc.* (2), 42, 9-17.

Varma, R. S., 1937: *J. Indian Math. Soc.* (*N.S.*) 2, 269-275.

Watson, G. N., 1910: *Proc. London Math. Soc.* 8, 393-421.

Wells, C. P., and R. D. Spence, 1945: *J. Math. Phys. Mass. Inst. Tech.* 24, 51-64.

Whittaker, E. T., and G. N. Watson, 1927: *A course of modern analysis*, Cambridge.

CHAPTER IX

THE INCOMPLETE GAMMA FUNCTIONS AND RELATED FUNCTIONS

9.1. Introduction

A considerable number of functions occurring in applied mathematical work can be expressed in terms of the *incomplete gamma functions*,

(1) $\gamma(a, x) = \int_0^x e^{-t} t^{a-1}\, dt$ Re $a > 0$,

(2) $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1}\, dt = \Gamma(a) - \gamma(a, x)$,

which in their turn are closely connected with the particular case $a = 1$ of the confluent hypergeometric functions $\Phi(a, c; x)$ and $\Psi(a, c; x)$. By 6.5(1), 6.5(2), and 6.5(6) we have

(3) $\gamma(a, x) = a^{-1} x^a e^{-x} \Phi(1, 1 + a; x) = a^{-1} x^a \Phi(a, 1 + a; -x)$,

(4) $\Gamma(a, x) = x^a e^{-x} \Psi(1, 1 + a; x) = e^{-x} \Psi(1 - a, 1 - a; x)$.

When $a = 1$, the confluent hypergeometric equation 6.1(2) has the elementary solution

$$e^x x^{1-c}$$

so that the special confluent hypergeometric functions to be discussed in this chapter satisfy simple differential equations of the *first order*.

In many ways it is advantageous to adopt the slightly modified function

(5) $$\gamma^*(a, x) = \frac{x^{-a}}{\Gamma(a)} \int_0^x e^{-t} t^{a-1}\, dt$$

$$= \frac{e^{-x}}{\Gamma(1+a)} \Phi(1, 1 + a; x) = \frac{1}{\Gamma(1 + a)} \Phi(a, 1 + a; -x)$$

as the basic function because this is a single-valued entire function of both a and x and is real for real values of a and x.

The following functions are expressible in terms of the incomplete gamma functions: the exponential and the logarithmic integral, sine and cosine integrals, error functions and Fresnel integrals and their generalizations. Definitions and notations of these functions vary considerably.

The notations to be used here will be explained in the sections dealing with these functions.

THE INCOMPLETE GAMMA FUNCTIONS

9.2. Definitions and elementary properties

The incomplete gamma functions were first investigated for real x by Legendre (1811, Vol. 1, pp. 339-343 and later works). The significance of the decomposition

(1) $\Gamma(a) = \gamma(a, x) + \Gamma(a, x)$

was recognized by Prym (1877) who seems to have been the first to investigate the functional behavior of these functions (which he denotes by P and Q).

There are several notations for these functions. At present the most frequent notation besides the one adopted here is the notation used in astrophysics and nuclear physics,

$$E_n(x) = \int_1^\infty e^{-xu} u^{-n} du = x^{n-1} \Gamma(1-n, x).$$

The alternative notation $K_n(x)$ is sometimes used. For the formulas in this notation see Placzek (1946), Le Caine (1948), and Busbridge (1950).

The older theory of the incomplete gamma functions is presented, and references to the literature are given in Nielsen (1906a, especially in Chap. XV, and 1906b). A more recent account is found in Böhmer (1939).

It is customary to define the incomplete gamma functions by the incomplete Eulerian integrals of the second kind 9.1(1) and 9.1(2). However, in order to avoid convergence difficulties in 9.1 (1) when Re $a \leq 0$ we shall adopt 9.1(3) and 9.1(4) as the *definitions of the incomplete gamma functions* with the remark that x^a and Ψ are defined uniquely by the conventions of Chap. VI. Apart from the notation, 9.1(2) was known to Legendre. While $\gamma^*(a, x)$ is an entire function of both a and x, the function $\gamma(a, x)$ itself fails to be defined for $a = 0, -1, -2, \ldots$. The function $\Gamma(a, x)$ is an entire function of a, but in general, except when a is an integer, it is a many-valued function of x with a branch-point at $x = 0$.

The recurrence relations

(2) $\gamma(a+1, x) = a\,\gamma(a, x) - x^a e^{-x}$,

(3) $\Gamma(a+1, x) = a\,\Gamma(a, x) + x^a e^{-x}$,

are simple consequences of the definitions and can be derived from the incomplete Eulerian integrals of the second kind by integration by parts.

They can be used as an alternative definition of the functions under consideration.

We have the convergent expansions in ascending powers of x,

$$(4)\quad \gamma(a, x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^{a+n}}{(a)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-)^n x^{a+n}}{n!\; a+n},$$

$$(5)\quad \Gamma(a, x) = \Gamma(a) - \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{x^{a+n}}{a+n},$$

valid for all x, and $a \neq 0, -1, -2, \ldots$, with

$$(a)_0 = 1, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \cdots (a+n-1) \qquad n = 1, 2, \ldots,$$

and the asymptotic expansions in descending powers of x,

$$(6)\quad \Gamma(a, x) = x^{a-1} e^{-x} \left[\sum_{m=0}^{M-1} \frac{(1-a)_m}{(-x)^m} + O(|x|^{-M}) \right]$$

$$|x| \to \infty, \quad -3\pi/2 < \arg x < 3\pi/2, \quad M = 1, 2, \ldots,$$

$$(7)\quad \gamma(a, x) = \Gamma(a) - x^{a-1} e^{-x} \left[\sum_{m=0}^{M-1} \frac{(1-a)_m}{(-x)^m} + O(|x|^{-M}) \right].$$

Either from the power series expansion or from the definitions one obtains the differentiation formulas

$$(8)\quad \frac{d\gamma(a, x)}{dx} = -\frac{d\Gamma(a, x)}{dx} = x^{a-1} e^{-x},$$

$$(9)\quad \frac{d^n}{dx^n} [x^{-a} \gamma(a, x)] = (-)^n x^{-a-n} \gamma(a+n, x),$$

$$(10)\quad \frac{d^n}{dx^n} [e^x \gamma(a, x)] = (-)^n (1-a)_n e^x \gamma(a-n, x),$$

$$(11)\quad \frac{d^n}{dx^n} [x^{-a} \Gamma(a, x)] = (-)^n x^{-a-n} \Gamma(a+n, x),$$

$$(12)\quad \frac{d^n}{dx^n} [e^x \Gamma(a, x)] = (-)^n (1-a)_n e^x \Gamma(a-n, x),$$

the last four for $n = 0, 1, 2, \ldots$.

The continued fraction expansion

$$(13)\quad \Gamma(a, x) = \cfrac{e^{-x} x^{a}}{x + \cfrac{1-a}{1 + \cfrac{1}{x + \cfrac{2-a}{1 + \cdots}}}}$$

is due to Legendre and can be derived from (3). Cther continued fractions have been obtained by Schlömich (1871), and Tannery (1882).

Whenever a is a positive integer, the confluent hypergeometric functions $\Phi(a, c; x)$ and $\Psi(a, c; x)$ may be expressed in terms of incomplete gamma functions by means of the formulas

$$(14)\quad \Phi(n+1, a+1; x) = \frac{a}{n!} \frac{\partial^n}{\partial x^n} [e^{x} x^{n-a} \gamma(a, x)] \qquad n = 0, 1, 2, \ldots,$$

$$(15)\quad \Psi(n+1, a+1; x) = \frac{1}{n!(1-a)_n} \frac{\partial^n}{\partial x^n} [e^{x} x^{n-a} \Gamma(a, x)] \qquad n = 0, 1, 2, \ldots.$$

The first formula is meaningless for negative integers a, but it retains a meaning if it is divided by $\Gamma(a+1)$ before a approaches a negative integer. The second formula looses its meaning when a is a positive integer.

9.2.1. The case of integer a

In this section

$$(16)\quad e_n(x) = \sum_{m=0}^{n} \frac{x^m}{m!} \qquad n = 0, 1, 2, \ldots,$$

is the truncated exponential series, and $E_n(x)$ is the integral defined in sec. 9.2. We have

$$(17)\quad \gamma(1+n, x) = n!\,[1 - e^{-x} e_n(x)],$$

$$(18)\quad \Gamma(1+n, x) = n!\, e^{-x} e_n(x),$$

$$(19)\quad \Gamma(1-n, x) = x^{1-n} E_n(x).$$

By repeated integrations by parts we also have

(20) $\Gamma(-n, x) = \dfrac{(-)^n}{n!}\left[E_1(x) - e^{-x} \sum_{m=0}^{n-1} (-)^m \dfrac{m!}{x^{m+1}}\right]$

$n = 1, 2, 3, \ldots .$

The function $\gamma(a, x)$ does not exist when $a = -n$, but we have from 9.1 (5)

(21) $\gamma^*(-n, x) = x^n$.

It may be pointed out that for positive integer a and integer c, the confluent hypergeometric functions $\Phi(a, c; x)$ and $\Psi(a, c; x)$ may be expressed in terms of the functions discussed here. For Φ, with $a = 1 + n$ and $c = 2, 3, \ldots$, this follows from (14). For other integers c, we have to divide (14) by $\Gamma(c + 1)$, and write (14) in terms of γ^* before letting c be an integer. For Ψ with $a = 1 + n$ and $c = 1, 0, -1, -2, \ldots$, we have (15) and (19). The case $c = 2, 3, \ldots$, can be reduced to the former one by applying 6.5 (6).

When a is close to an integer, we may obtain useful approximations to incomplete gamma functions by evaluating their derivatives with respect to a for an integer a. By manipulating the integral representation 9.1 (5) one can prove

(22) $\left.\dfrac{\partial \gamma^*(a, x)}{\partial a}\right|_{a=0} = -\log x - E_1(x),$

and other results follow by application of the recurrence relations.

9.3. Integral representations and integral formulas

The basic integral representations are the incomplete Eulerian integrals of the second kind, 9.1 (1) and 9.1 (2). The first of these fails to converge when Re $a \le 0$. It may be replaced by a loop integral

(1) $\gamma(a, x) = -(2i \sin \pi a)^{-1} x^a \int_1^{(0+)} e^{-xu} (-u)^{a-1}\, du$

where $-\pi \le \arg(-u) \le \pi$ on the loop of integration, x is arbitrary, $\neq 0$, and a is not an integer. With the unit circle $-u = \cos\theta + i \sin\theta$, $-\pi \le \theta \le \pi$, as the path of integration one obtains

(2) $\gamma(a, x) = x^a \operatorname{cosec} \pi a \int_0^{\pi} e^{x\cos\theta} \cos(a\theta + x \sin\theta)\, d\theta.$

A real integral for Re $a \le 0$, $x \le 0$ may be derived from 6.11 (13).

For $\Gamma(a, x)$ the basic integral representations are 9.1 (2) and

(3) $\Gamma(a, x) = \dfrac{e^{-x} x^a}{\Gamma(1-a)} \displaystyle\int_0^{\infty} \frac{e^{-t} t^{-a}}{x + t}\, dt.$

The latter integral is obtained when 6.5(2) is applied to the last Ψ function in 9.1(4). Legendre's continued fraction 9.2(13) is a consequence of (3).

Other integral representations are

(4) $$\gamma(a, x) = x^{\frac{1}{2}a} \int_0^\infty e^{-t} t^{\frac{1}{2}a-1} J_a[2(xt)^{\frac{1}{2}}]\, dt \qquad \text{Re } a > 0,$$

(5) $$\Gamma(a, x) = \frac{2x^{\frac{1}{2}a} e^{-x}}{\Gamma(1-a)} \int_0^\infty e^{-t} t^{-\frac{1}{2}a} K_a[2(xt)^{\frac{1}{2}}]\, dt \qquad \text{Re } a < 1,$$

(6) $$\Gamma(2-2a)\,\Gamma(a, -ix)\,\Gamma(a, ix) = 2\int_0^\infty e^{-xt} t^{-2a} \left[\frac{1}{t+2i}\, {}_2F_1\left(1, \frac{1}{2}; \frac{3}{2} - a; \frac{t}{t+2i}\right) + \frac{1}{t-2i}\, {}_2F_1\left(1, \frac{1}{2}; \frac{3}{2} - a; \frac{t}{t-2i}\right)\right] dt \quad \text{Re } a < 1, \quad \text{Re } x > 0.$$

The last of these is due to Tricomi (1950 a).

Some of the more important integral formulas are

(7) $$\int_0^\infty e^{-st} t^{\beta-1} \gamma(a, t)\, dt = \frac{\Gamma(a+\beta)}{a(1+s)^{a+\beta}}\, {}_2F_1\left(1, a+\beta; a+1; \frac{1}{1+s}\right)$$
$$\text{Re}(a+\beta) > 0, \quad \text{Re } s > 0,$$

(8) $$\int_0^\infty e^{-st} t^{\beta-1} \Gamma(a, t)\, dt = \frac{\Gamma(a+\beta)}{\beta(1+s)^{a+\beta}}\, {}_2F_1\left(1, a+\beta; \beta+1; \frac{s}{1+s}\right)$$
$$\text{Re } \beta > 0, \quad \text{Re}(a+\beta) > 0, \quad \text{Re } s > -\tfrac{1}{2},$$

(9) $$\int_0^\infty e^{-st} \gamma(a, t^2)\, dt = 2^{1-a}\Gamma(2a)\, s^{-1} e^{s^2/8} D_{-2a}(2^{-\frac{1}{2}} s)$$
$$\text{Re } a > -\tfrac{1}{2}, \quad a \neq 0, \quad \text{Re } s > 0,$$

(10) $$\Gamma(a)\, x^{a-\beta} \int_0^1 e^{-xt} t^{a-\beta-1} \gamma(\beta, x - xt)\, dt = \Gamma(\beta)\,\Gamma(a-\beta)\,\gamma(a, x) \qquad \text{Re } a > \text{Re } \beta > -1, \quad a\beta \neq 0.$$

The hypergeometric function reduces to an elementary function if $\beta = 1$ in (7) or $a = 1$ in (8); in (9), D is the parabolic cylinder function. It may be noted that (3) to (9) are Laplace integrals. For other integrals see Nielsen (1906b, c), Le Caine (1948), and Busbridge (1950).

9.4. Series

The power series and continued fraction expansions were mentioned in sec. 9.2. Using the expansion

$$\frac{1}{x+t} = \sum_{n=0}^{\infty} \frac{(-t)_n}{(x)_{n+1}} \qquad t \geq 0, \quad \operatorname{Re} x > 0$$

in 9.3 (3), we obtain the expansion in *inverse factorials*

$$(1) \quad \Gamma(a, x) = e^{-x} x^a \sum_{n=0}^{\infty} \frac{c_n}{(x)_{n+1}} \qquad \operatorname{Re} x > 0,$$

where

$$c_n = \frac{1}{\Gamma(1-a)} \int_0^{\infty} e^{-t} t^{-a} (-t)_n \, dt = (-)^n \frac{n!}{\Gamma(1-a)} \int_0^{\infty} e^{-t} t^{-a} \binom{t}{n} dt.$$

From 9.1 (1), we have

$$\gamma(a, x+y) - \gamma(a, x) = e^{-x} x^{a-1} \int_0^y e^{-u} \left(1 + \frac{u}{x}\right)^{a-1} du.$$

If $|y| < |x|$, we may expand $(1 + u/x)^{a-1}$ in the binomial series, integrate term by term, and use 9.1 (17). Thus we obtain *Nielsen's expansion*

$$(2) \quad \Gamma(a, x) - \Gamma(a, x+y) = \gamma(a, x+y) - \gamma(a, x)$$

$$= e^{-x} x^{a-1} \sum_{n=0}^{\infty} (1-a)_n (-x)^{-n} [1 - e^{-y} e_n(y)] \qquad |y| < |x|,$$

which is useful for numerical computation.

Incomplete gamma functions occur in a large number of series expansions, many of which may be obtained by specializing parameters in the expansions of Chap. VI and will not be given in full. It is noteworthy that with $h = 0$, $a = -1$, the coefficients in 6.12 (7) can be expressed in terms of the truncated exponential series; 6.12 (6) becomes

$$(3) \quad \gamma(a, x) = \Gamma(a) e^{-x} x^{\frac{1}{2}a} \sum_{n=0}^{\infty} e_n(-1) x^{\frac{1}{2}n} I_{n+a}(2x^{\frac{1}{2}}),$$

and is rapidly convergent for all $x \neq 0$ provided that a is not a negative integer. In the expansion 6.12 (11) the coefficients may be expressed in terms of Laguerre polynomials.

If x and y are positive and $x \geq y$, we have

$$(4) \quad \Gamma(a, x) \gamma(a, y) = e^{-x-y} (xy)^a \sum_{n=0}^{\infty} \frac{n!}{(n+1)(a)_{n+1}} L_n^{(a)}(x) L_n^{(a)}(y).$$

The limiting case as $y \to 0$ of this expansion is

(5) $$\Gamma(a, x) = e^{-x} x^{a} \sum_{n=0}^{\infty} \frac{L_n^{(a)}(x)}{n+1} \qquad x > 0$$

and it coincides with the particular case $a = 1$ of the expansion 6.12 (3) of the Ψ-function in a series of Laguerre polynomials.

For other expansions see Nielsen (1906 a, sections 82, and 83).

9.5. Asymptotic representations

For $a \to \infty$, $x = o(|a|)$, the first series 9.2 (4) is an asymptotic expansion; for $x \to \infty$ and $a = o(|x|)$, we have 9.2 (6). If x and a are of the same order of magnitude, an expansion may be obtained from 6.13 (17), but it is not at all easy to find the general form of that expansion or to discuss conditions under which it represents $\gamma(a, x)$ asymptotically as both a and x increase. Considerable complications arise when x and $a + 1$ are nearly equal, more precisely if $a \to \infty$ and $x = a + 1 + o(|a|)$.

Tricomi (1950 b) has made a through investigation of the problem. He introduces the parameter

(1) $$z = \frac{a^{1/2}}{x - a}$$

and distinguishes two cases according as z is small or large.

If $z \to 0$ and $|\arg z| < 3\pi/4$, he proves that $\Gamma(1 + a, x)$ is asymptotically represented by

(2) $$e^{-x} x^{1+a} \sum_{n=0}^{\infty} l_n(a)\, n!\, (x - a)^{-n-1}$$

where the coefficients

(3) $$n!\, l_n(a) = \left\{ \frac{d^n}{dt^n} [e^{-at}(1+t)^a] \right\}_{t=0} = L_n^{(a-n)}(a)$$

are certain polynomials of degree $[n/2]$ in a. These polynomials have been studied extensively (Tricomi 1951). In particular, we have

(4) $$\Gamma(1 + a, x) = \frac{e^{-x} x^{a+1}}{x - a} \left[1 - \frac{a}{(x-a)^2} + \frac{2a}{(x-a)^3} + O(|a|^2 |x - a|^{-4}) \right].$$

If $z \to \infty$ (when x and a are nearly equal), one has to distinguish two cases according as Re a is positive or negative. In the latter case Tricomi uses the function

(5) $$\gamma_1(a, x) = \Gamma(a)\, x^{a}\, \gamma^*(a, -x) \qquad x > 0.$$

He then finds when $a \to +\infty$ and y is bounded,

(6) $\gamma[1+a,\ a+(2a)^{1/2}y] = \Gamma(1+a)\,[1/2 + \pi^{-1/2}\operatorname{Erf}(y) + O(a^{-1/2})]$,

(7) $\Gamma(a)\,\gamma_1[1-a,\ a+2(2a)^{1/2}y]$
$= -\pi \operatorname{ctn}(a\pi) + 2\pi^{1/2}\operatorname{Erfi}(y) + O(a^{-1/2})$.

For $a = n$ we have in particular

(8) $e_n\,[n+(2n)^{1/2}y]$
$= \exp[n+(2n)^{1/2}y]\cdot[1/2 - \pi^{-1/2}\operatorname{Erf}(y) + O(n^{-1/2})]$.

See also Furch (1939) and a contribution by Blanch in Placzek (1946).

9.6. Zeros and descriptive properties

Information about zeros for real a and x may be derived from the results of sec. 6.16. It turns out that $\gamma(a, x)$ has

(i) no real zeros (apart from $x = 0$) if $a \geq 0$,
(ii) one negative zero x' and no positive zero if $1 - 2n < a < 2 - 2n$, where ($n = 1, 2, 3, \ldots,$)
(iii) one negative zero x' and one positive zero x'' if $-2n < a < 1 - 2n$, $n = 1, 2, \ldots$.

The general behavior of these zeros as functions of a can be seen from the altitude chart (p. 142) of γ^*.

Approximations to the zeros for large a have been obtained by Tricomi (1950b); he proves that

(1) $x' = -(1-a)\,[1 + 2^{1/2}(1-a)^{-1/2}\,y^*(a) + O(|a|^{-1})]$,

(2) $$x'' = -ra - \frac{r}{1+r}\log\frac{1+r(-a\pi/2)^{1/2}}{\sin a\pi} + O\,[|a|^{-1}(\log|a|)^2].$$

Here $y^*(a)$ is the unique positive root of the equation

(3) $\operatorname{Erf}(y) = (\pi/2)^{1/2}\operatorname{ctn}(ay)$,

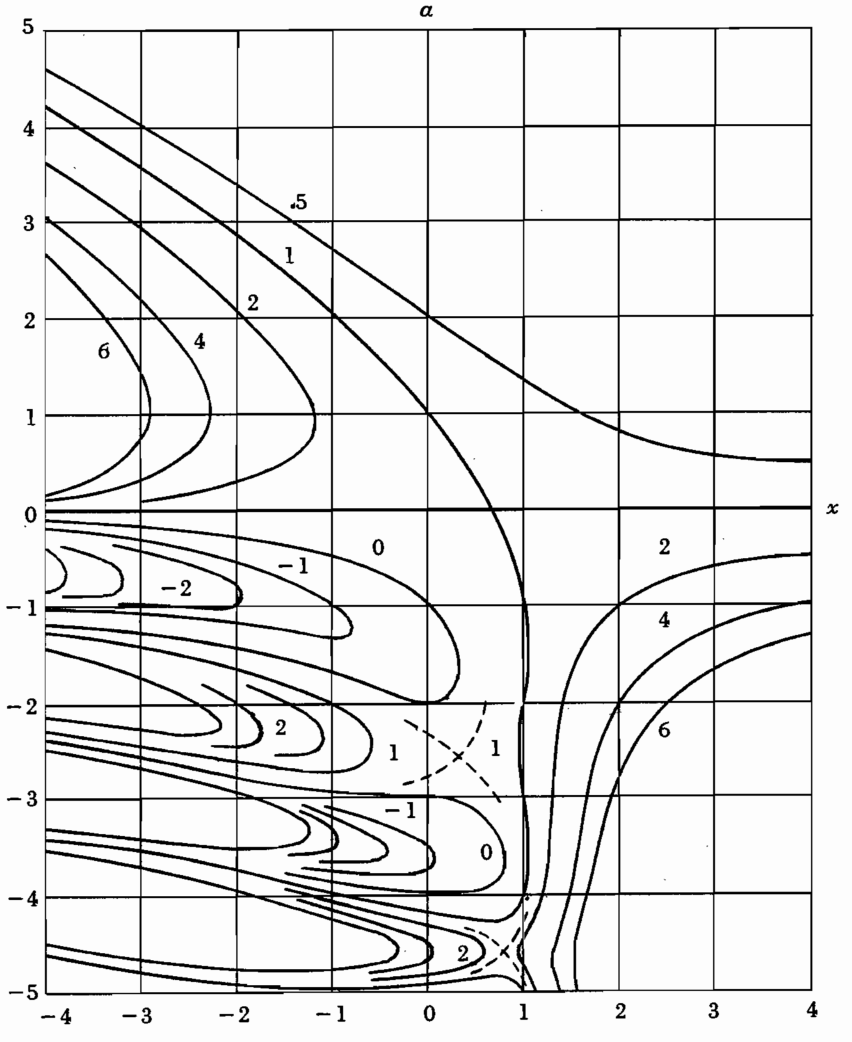
and $r = 0.278463\cdots$ is the unique positive root of the equation

(4) $1 + x + \log x = 0$.

If $a > 0$ is fixed, clearly $\gamma(a, x)$ is a monotonic increasing function of x for $x > 0$, and increases from zero to $\Gamma(a)$ as x increases from zero to ∞. It can be shown that for a fixed $x > 0$, the function $\Gamma(a, x)/\Gamma(a)$ is a monotonic decreasing function of a for $a > 0$. In the other quadrants of the real a, x, plane the incomplete gamma functions were investigated by Tricomi (1951), who puts

(5) $\Gamma(a, x) = -a^{-1}e^{-x}x^{a}G(a, x)$, $\qquad a \leq 0, \quad x \geq 0$,

(6) $\gamma_1(a, x) = a^{-1}e^{x}x^{a}g_1(a, x)$ $\qquad a \geq 0, \quad x \leq 0$,



Altitude chart of $\gamma^*(a, x)$

(7) $\gamma^*(-a, -x) = \Gamma(a+1)\, e^x\, k(a, x)$ $\qquad a \geq 0, \quad x \geq 0,$

and proves

$$\frac{\partial G}{\partial x} < 0, \quad \frac{\partial G}{\partial a} < 0, \quad \frac{\partial g_1}{\partial x} < 0, \quad \frac{\partial g_1}{\partial a} > 0, \qquad |k| \leq 1$$

throughout their domains of definition, $|k| \leq \frac{1}{2}$ for $a \geq 1$, and furthermore that k as a function of x has only one maximum or minimum if $0 < a < 1$, while it has two maxima or minima if $a > 1$.

The altitude chart (p. 142) is taken from Tricomi's paper. It shows the curves $\gamma^*(a, x) =$ constant.

SPECIAL INCOMPLETE GAMMA FUNCTIONS

9.7. The exponential and logarithmic integral

The principal functions to be considered are

(1) $E_1(x) = -\mathrm{Ei}(-x) = \int_x^\infty e^{-t} t^{-1}\, dt = \Gamma(0, x) = e^{-x}\, \Psi(1, 1, x),$

(2) $E^*(x) = -\,⨍_{-x}^\infty e^{-t} t^{-1}\, dt$ $\qquad x > 0,$

(3) $\mathrm{li}(x) = \displaystyle\int_0^x \frac{dt}{\log t} = \mathrm{Ei}(\log x) = -E_1(-\log x).$

In (2), the integral is a Cauchy principal value, i.e.,

$$\lim \left(\int_{-x}^{-\epsilon} + \int_\epsilon^\infty\right) \text{ as } \epsilon \to 0, \qquad \epsilon > 0;$$

this function is denoted by $\overline{\mathrm{Ei}}(x)$ in Jahnke-Emde (p. 2). We have the following relations between the functions defined in (1) and (2)

(4) $-E_1(xe^{\pm i\pi}) = E^*(x) \pm i\pi$ $\qquad x > 0.$

The following formulas, and some others, can be obtained by making $a \to 0$ in the results of the first part of this Chapter:

(5) $\mathrm{Ei}(-x) = \gamma + \log x + \displaystyle\sum_{n=1}^\infty \frac{(-x)^n}{n!\, n}$

$$= \gamma + \log x - e^{-x} \sum_{n=1}^\infty (1 + \tfrac{1}{2} + \cdots + 1/n) \frac{x^n}{n!},$$

(6) $E^*(x) = \gamma + \log x + \displaystyle\sum_{n=1}^\infty \frac{x^n}{n!\, n}$

where γ is Euler's constant of sec. 1.7.2,

(7) $$E_1(x) = x^{-1} e^{-x} \left[\sum_{m=0}^{M-1} \frac{m!}{(-x)^m} + O(|x|^{-M}) \right]$$

$|x| \to \infty, \quad -3\pi/2 < \arg x < 3\pi/2, \quad M = 1, 2, \ldots,$

(8) $$E^*(x) = x^{-1} e^{x} \left[\sum_{m=0}^{M-1} \frac{m!}{x^m} + O(|x|^{-M}) \right]$$

$x \to \infty, \quad x > 0, \quad M = 1, 2, \ldots,$

(9) $$\frac{d^n \operatorname{Ei}(-x)}{dx^n} = (-)^{n-1} (n-1)!\, x^{-n} e^{-x} e_{n-1}(x) \qquad n = 1, 2, \ldots,$$

(10) $$\frac{d^n [e^x \operatorname{Ei}(-x)]}{dx^n} = e^x \operatorname{Ei}(-x) + \sum_{m=0}^{n-1} \frac{(-)^m m!}{x^{m+1}} \qquad n = 1, 2, \ldots,$$

(11) $$\int_0^\infty e^{-st} t^{\beta-1} \operatorname{Ei}(-t)\, dt = -\frac{\Gamma(\beta)}{\beta(1+s)^\beta}\, {}_2F_1\left(1, \beta; \beta+1; \frac{s}{1+s}\right)$$

$\operatorname{Re} \beta > 0, \quad \operatorname{Re} s > -½.$

To these we add Raabe's integrals

(12) $$\int_0^\infty \frac{\sin(xt)}{a^2+t^2} dt = \frac{1}{2a} [e^{ax} E_1(ax) + e^{-ax} E^*(ax)]$$

$a > 0, \quad x > 0,$

(13) $$\int_0^\infty \frac{t \cos(xt)}{a^2+t^2} dt = ½ [e^{ax} E_1(ax) - e^{-ax} E^*(ax)]$$

$a > 0, \quad x > 0,$

both of which may be deduced from (1) and (2), and

(14) $$\int_a^\infty (b+t)^{-1} e^{-ct} dt = e^{bc} E_1[(a+b)c] \qquad \operatorname{Re} c > 0,$$

(15) $$\int_1^\infty e^{-xt} \log t\, dt = x^{-1} E_1(x) \qquad \operatorname{Re} x > 0,$$

(16) $$\int_x^\infty t^{\alpha-1} E_1(t)\, dt = \alpha^{-1}[\Gamma(\alpha, x) - x^\alpha E_1(x)] \qquad \operatorname{Re} x > 0, \quad \alpha \neq 0.$$

For other integrals see Nielsen(1906, especially Chapters II and IV), Le Caine (1948), Busbridge (1950).

From 9.4(5) we have

(17) $$E_1(x) = e^{-x} \sum_{n=0}^{\infty} \frac{L_n(x)}{n+1} \qquad x > 0,$$

and from 9.4 (2)

$$(18)\quad E_1(x+y) = E_1(x) + e^{-x} \sum_{n=0}^{\infty} n!\,(-x)^{-n-1}[1 - e^{-y} e_n(y)] \qquad |y| < |x|.$$

The formulas for li (x) may be derived from those for $E_1(x)$.

Certain generalizations of the exponential integral function occur in the investigation of wave propagation in a dissipative medium. A typical example is

$$\int_0^x e^{-u} u^{-1}\, dt \quad \text{where} \quad u = (a^2 + t^2)^{1/2}.$$

For this and related functions see Harvard University (1949b).

9.8. Sine and cosine integrals

The definitions used in modern tables are

$$(1)\quad \operatorname{si} x = \int_\infty^x \frac{\sin t}{t}\, dt = \frac{1}{2i}[\operatorname{Ei}(ix) - \operatorname{Ei}(-ix)],$$

$$(2)\quad \operatorname{Si} x = \int_0^x \frac{\sin t}{t} dt = \frac{\pi}{2} + \operatorname{si} x,$$

$$(3)\quad \operatorname{Ci} x = \int_\infty^x \frac{\cos t}{t}\, dt = \frac{1}{2}[\operatorname{Ei}(ix) + \operatorname{Ei}(-ix)],$$

$$(4)\quad \operatorname{Ei}(\pm ix) = \operatorname{Ci} x \pm i \operatorname{si} x.$$

Here $\pm i = \exp(\pm \tfrac{1}{2} i\pi)$. Nielsen (1906) uses the same definition of si, and writes ci instead of Ci. Some authors define the symbols Ci, Si, somewhat differently.

Si x and also si x are entire functions of x,

$$(5)\quad \operatorname{Si}(-x) = -\operatorname{Si}(x), \qquad \operatorname{si}(-x) = -\pi - \operatorname{si} x.$$

Ci x is a many-valued function, with a logarithmic branch-point at $x = 0$. However,

$$(6)\quad \operatorname{Ci} x = \gamma + \log x - \int_0^x \frac{1 - \cos t}{t}\, dt,$$

so that Ci x − log x is an even entire function of x. In particular, we have

$$(7)\quad \operatorname{Ci}(xe^{\pm i\pi}) = \operatorname{Ci} x \pm i\pi, \qquad x > 0.$$

The following formulas, and many others, are obtained by straightforward manipulation of the definitions or of results in the earlier parts of this chapter:

(8) $$\operatorname{Si} x = \tfrac{1}{2}\pi + \operatorname{si} x = \sum_{n=0}^{\infty} \frac{(-)^n x^{2n+1}}{(2n+1)!\,(2n+1)},$$

(9) $$\operatorname{Ci} x = \gamma + \log x + \sum_{n=1}^{\infty} \frac{(-)^n x^{2n}}{(2n)!\,(2n)},$$

(10) $$\operatorname{si} x = -\cos x \left[\sum_{m=0}^{M-1} \frac{(-)^m (2m)!}{x^{2m+1}} + O(|x|^{-2M-1}) \right] + \sin x \left[\sum_{m=1}^{N-1} \frac{(-)^m (2m-1)!}{x^{2m}} + O(|x|^{-2N}) \right]$$
$$-\pi < \arg x < \pi, \qquad M, N = 1, 2, \ldots,$$

(11) $$\operatorname{Ci} x = \cos x \left[\sum_{m=1}^{N-1} \frac{(-)^m (2m-1)!}{x^{2m}} + O(|x|^{-2N}) \right] + \sin x \left[\sum_{m=0}^{M-1} \frac{(-)^m (2m)!}{x^{2m+1}} + O(|x|^{-2M-1}) \right]$$
$$-\pi < \arg x < \pi, \qquad M, N = 1, 2, \ldots,$$

(12) $$\int_0^\infty e^{-st} \operatorname{Ci}(t)\, dt = -\frac{1}{2s} \log(1+s^2) \qquad \operatorname{Re} s > 0,$$

(13) $$\int_0^\infty e^{-st} \operatorname{si}(t)\, dt = -\frac{1}{s} \tan^{-1} s, \qquad \operatorname{Re} s > 0,$$

(14) $$\int_0^\infty e^{-st} t^{-1} \log(1+t^2)\, dt = [\operatorname{Ci}(s)]^2 + [\operatorname{si}(s)]^2 \qquad \operatorname{Re} s > 0,$$

(15) $$\int_0^\infty \sin x \operatorname{si} x\, dx = \int_0^\infty \cos x \operatorname{Ci} x\, dx = -\frac{\pi}{4},$$

(16) $$\int_0^\infty \operatorname{si} x \operatorname{Ci} x\, dx = -\log 2, \qquad \int_0^\infty (\operatorname{si} x)^2\, dx = \int_0^\infty (\operatorname{Ci} x)^2\, dx = \tfrac{1}{2}\pi.$$

For other integrals see Nielsen (1906b, especially Chap. IV).

The notations

(17) $$\operatorname{Shi} x = \int_0^x \sinh t \, \frac{dt}{t} = -i \operatorname{Si}(ix),$$

(18) $$\operatorname{Chi} x = \gamma + \log x + \int_0^x \frac{\cosh t - 1}{t}\, dt = \operatorname{Ci}(ix) - \tfrac{1}{2} i\pi$$

are also used. The generalizations

(19) $\int_0^x \sin u \dfrac{dt}{u}, \quad u = (a^2 + t^2)^{1/2}$

and other similar generalizations have been discussed (Harvard University 1949 a).

9.9. The error functions

The principal functions in this group are

(1) $\operatorname{Erf} x = \int_0^x e^{-t^2}\, dt = \tfrac{1}{2}\gamma(\tfrac{1}{2}, x^2) = x\, \Phi(1/2, 3/2; -x^2)$

$= xe^{-x^2}\, \Phi(1, 3/2; x^2),$

(2) $\operatorname{Erfc} x = \int_x^\infty e^{-t^2}\, dt = \tfrac{1}{2}\pi^{1/2} - \operatorname{Erf} x = \tfrac{1}{2}\Gamma(\tfrac{1}{2}, x^2) = \tfrac{1}{2}e^{-x^2}\, \Psi(\tfrac{1}{2}, \tfrac{1}{2}; x^2),$

(3) $\operatorname{Erfi} x = -i \operatorname{Erf}(ix) = \int_0^x e^{t^2}\, dt = x\, \Phi(1/2, 3/2; x^2),$

(4) $H(x) = 2\pi^{-1/2} \int_0^x e^{-t^2}\, dt = 2\pi^{-1/2} \operatorname{Erf} x = 1 - 2\pi^{-1/2} \operatorname{Erf} x,$

(5) $\alpha(x) = (2/\pi)^{1/2} \int_0^x e^{-\frac{1}{2}t^2}\, dt = 2\pi^{-1/2} \operatorname{Erf}(2^{-1/2} x).$

The first three are the most convenient for mathematical work, and (2) is the function, although not the notation, originally introduced by Kramp (1799). The function (4) is more convenient for numerical work, and (5) arises in statistics where it is frequently used. There is a great variety of notations.

All the error functions are entire functions; Erf x and Erfi x are odd functions of x. Most of the following formulas are either straighforward deductions from the definitions or else specializations of earlier results of this section:

(6) $\operatorname{Erf} x = \sum_{n=0}^{\infty} \dfrac{(-)^n x^{2n+1}}{n!(2n+1)} = e^{-x^2} \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(3/2)_n},$

(7) $\operatorname{Erfi} x = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{n!\,(2n+1)} = e^{x^2} \sum_{n=0}^{\infty} \dfrac{(-)^n x^{2n+1}}{(3/2)_n},$

(8) $\operatorname{Erfc} x = \tfrac{1}{2} e^{-x^2} \left[\sum_{m=0}^{M-1} \dfrac{(-)^m (\tfrac{1}{2})_m}{x^{2m+1}} + O(|x|^{-2M-1}) \right]$

$\operatorname{Re} x > 0, \quad x \to \infty, \quad M = 1, 2, \ldots,$

(9) $\operatorname{Erfi} x = -\tfrac{1}{2} i \pi^{1/2} + \tfrac{1}{2} e^{x^2} \left[\sum_{m=0}^{M-1} \dfrac{(\tfrac{1}{2})_m}{x^{2m+1}} + O(|x|^{-2M-1}) \right]$

$x > 0, \quad x \to \infty, \quad M = 1, 2, \ldots,$

(10) $\int_0^\infty e^{-a^2t^2-bt}\,dt = a^{-1}\exp\left(\frac{b^2}{4a^2}\right)\operatorname{Erfc}\left(\frac{b}{2a}\right)$ $\qquad \operatorname{Re} a > 0,$

(11) $\int_0^\infty \operatorname{Erf}(at)\, e^{-st}\,dt = s^{-1}\exp\left(\frac{s^2}{4a^2}\right)\operatorname{Erfc}\left(\frac{s}{2a}\right)$

$|\arg a| < \tfrac14\pi, \quad \operatorname{Re} s > 0,$

(12) $\int_0^\infty \operatorname{Erf}(at)^{1/2}\, e^{-st}\,dt = \tfrac12(a\pi)^{1/2}\, s^{-1}(a+s)^{-1/2}$

$\operatorname{Re} s > 0, \quad \operatorname{Re}(a+s) > 0,$

(13) $\int_0^\infty \operatorname{Erfc}(at^{-1/2})\, e^{-st}\,dt = \tfrac12\pi^{1/2}\, s^{-1}\, e^{-2as^{1/2}}$ $\qquad |\arg a| < \tfrac14\pi, \quad \operatorname{Re} s > 0,$

(14) $\int_0^\infty \operatorname{Erfi}(at)\, e^{-a^2t^2-st}\,dt = \frac{-1}{4a}\exp\left(\frac{s^2}{4a^2}\right)\operatorname{Ei}\left(-\frac{s^2}{4a^2}\right)$

$\operatorname{Re} s > 0, \quad |\arg a| < \tfrac14\pi,$

(15) $\int_0^1 e^{-a^2t^2}\frac{dt}{1+t^2} = e^{a^2}\left[\tfrac14\pi - (\operatorname{Erf} a)^2\right]$ $\qquad \operatorname{Re} a > 0,$

(16) $\int_0^x \operatorname{Erf} t\,dt = x\operatorname{Erf} x - \tfrac12(1-e^{-x^2}),$

(17) $\frac{d^{n+1}\operatorname{Erf} x}{dx^{n+1}} = (-)^n e^{-x^2} H_n(x)$ $\qquad n = 0, 1, 2, \dots,$

where H_n is the Hermite polynomial of Chap. X.

A series of Nielsen's type is given below:

(18) $\operatorname{Erf}[(x+y)^{1/2}]$

$$= \operatorname{Erf}(x^{1/2}) + \frac{e^{-x}}{2x^{1/2}}\sum_{n=0}^\infty (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\,\frac{\gamma(n+1,y)}{x^n}$$

See errata! $\qquad |y| < |x|.$

Expansions in series of Bessel functions (Tricomi, 1951) follow:

(19) $\operatorname{Erf}(x) = \tfrac12(\pi x)^{1/2}\, e^{-x^2}\sum_{n=0}^\infty e_n(-1)\; x^n I_{n+1/2}(2x),$

(20) $\operatorname{Erf}(x^{1/2}) = (\tfrac12\pi)^{1/2}\sum_{n=0}^\infty (-1)^{[n/2]} I_{n-1/2}(x),$

(31) $\operatorname{Erfi}(x^{1/2}) = (\tfrac12\pi)^{1/2}\sum_{n=0}^\infty (-1)^{[n/2]} I_{n+1/2}(x)$

The first of these expansions is a particular case of 9.4 (3), the other two can be verified by means of the Laplace transformation.

The most recent monograph on error functions is that by Rosser (1948) who discusses the double integral

(22) $\int_0^z e^{-p^2 y^2} dy \int_0^y e^{-x^2} dx \qquad n = 1, 2, \ldots.$

as a function of the complex variables p, z and also other related integrals. Repeated integrals of the error function have been investigated by Hartree (1936) who puts

(23) $i^0 \operatorname{erfc} x = 2\pi^{-1/2} \operatorname{Erfc} x, \quad i^n \operatorname{erfc} x = \int_x^\infty i^{n-1} \operatorname{erfc} t \, dt,$

9.10. Fresnel integrals and generalizations

Fresnel's integrals are

$$C(x) = (2\pi)^{-1/2} \int_0^x t^{-1/2} \cos t \, dt,$$

$$S(x) = (2\pi)^{-1/2} \int_0^x t^{-1/2} \sin t \, dt.$$

Instead of these, we shall consider the more general integrals introduced by Böhmer (1939)

(1) $C(x, a) = \int_x^\infty t^{a-1} \cos t \, dt$

$= \tfrac{1}{2} e^{-\frac{1}{2} i\pi a} \Gamma(a, ix) + \tfrac{1}{2} e^{+\frac{1}{2} i\pi a} \Gamma(a, -ix),$

(2) $S(x, a) = \int_0^\infty t^{a-1} \sin t \, dt$

$= \frac{1}{2i} e^{\frac{1}{2} i\pi a} \Gamma(a, -ix) - \frac{1}{2i} e^{-\frac{1}{2} i\pi a} \Gamma(a, ix).$ See errata!

The same functions, with a different notation, have been discussed by Bateman (1946). Clearly we have

(3) $\Gamma(a, ix) = e^{\frac{1}{2} i\pi a} [C(x, a) - i\, S(x, a)].$

Fresnel's integrals are:

(4) $C(x) = (2/\pi)^{1/2} \int_0^{x^{1/2}} \cos(t^2) \, dt = \tfrac{1}{2} - (2\pi)^{-1/2} C(x, \tfrac{1}{2})$

$= (2\pi)^{-1/2} [e^{-\frac{1}{4} i\pi} \operatorname{Erf}(e^{+\frac{1}{4} i\pi} x^{1/2}) + e^{\frac{1}{4} i\pi} \operatorname{Erf}(e^{-\frac{1}{4} i\pi} x^{1/2})],$

(5) $S(x) = (2/\pi)^{1/2} \int_0^{x^{1/2}} \sin(t^2) \, dt = \tfrac{1}{2} - (2\pi)^{-1/2} S(x, \tfrac{1}{2})$

$= i(2\pi)^{-1/2} [e^{-\frac{1}{4} i\pi} \operatorname{Erf}(e^{\frac{1}{4} i\pi} x^{1/2}) - e^{\frac{1}{4} i\pi} \operatorname{Erf}(e^{-\frac{1}{4} i\pi} x^{1/2})].$

There follows a brief collection of formulas:

(6) $$C(x, a) = \Gamma(a)\cos(\tfrac{1}{2}a\pi) - \sum_{m=0}^{\infty} \frac{(-)^m x^{2m+a}}{(2m)!\,(2m+a)},$$

(7) $$S(x, a) = \Gamma(a)\sin(\tfrac{1}{2}a\pi) - \sum_{m=0}^{\infty} \frac{(-)^m x^{2m+1+a}}{(2m+1)!\,(2m+1+a)},$$

(8) $$C(x, a) = -x^{a}[P(x)\sin x + Q(x)\cos x],$$

(9) $$S(x, a) = x^{a}[P(x)\cos x - Q(x)\sin x],$$

where

(10) $$P(x) = \sum_{m=0}^{M-1} \frac{(-)^m (1-a)_{2m}}{x^{2m+1}} + O(|x|^{-2M-1})$$

and

$$Q(x) = \sum_{m=1}^{M} \frac{(-)^m (1-a)_{2m-1}}{x^{2m}} + O(|x|^{-2M-2})$$

$$x \to \infty, \quad -\pi < \arg x < \pi, \quad M = 1, 2, \ldots,$$

(11) $$\int_0^\infty e^{-st} C(t, a)\,dt = s^{-1}\Gamma(a)[\cos(\tfrac{1}{2}a\pi) - \tfrac{1}{2}(s+i)^{-a} - \tfrac{1}{2}(s-i)^{-a}]$$

$$\operatorname{Re} s > 0, \quad -1 < \operatorname{Re} a,$$

(12) $$\int_0^\infty e^{-st} S(t, a)\,dt = s^{-1}\Gamma(a)[\sin(\tfrac{1}{2}a\pi) - \tfrac{1}{2}i(s+i)^{-a} + \tfrac{1}{2}i(s-i)^{-a}]$$

$$\operatorname{Re} s > 0, \quad -1 < \operatorname{Re} a,$$

(13) $$\int_0^\infty t^{\beta-1} C(t, a)\,dt = \beta^{-1}\Gamma(a+\beta)\cos[\tfrac{1}{2}(a+\beta)\pi]$$

$$\operatorname{Re}\beta > 0, \quad 0 < \operatorname{Re}(a+\beta) < 1,$$

(14) $$\int_0^\infty t^{\beta-1} S(t, a)\,dt = \beta^{-1}\Gamma(a+\beta)\sin[\tfrac{1}{2}(a+\beta)\pi]$$

$$\operatorname{Re}\beta > 0, \quad 0 < \operatorname{Re}(a+\beta) < 1,$$

(15) $$C(x) = J_{1/2}(x) + J_{5/2}(x) + J_{9/2}(x) + \cdots,$$

(16) $$S(x) = J_{3/2}(x) + J_{7/2}(x) + J_{11/2}(x) + \cdots.$$

An integral representation of

$$[C(x, a)]^2 + [S(x, a)]^2$$

follows from 9.3 (6).

The curve represented parametrically by

(17) $\xi = C(t, a), \quad \eta = S(t, a) \qquad t \geq 0$

for a fixed a, $0 < a < 1$, is a spiral and has been investigated by Böhmer (1939). It reduces to Cornu's spiral when $a = \frac{1}{2}$. It may be of interest to note that this spiral has a simple "intrinsic equation"

(18) $\rho = (a s)^{1-1/a}$

where ρ is the radius of curvature and s is the arc length.

REFERENCES

Bateman, Harry, 1946: *Proc. Nat. Acad. Sci.* 32, 70-72.

Böhmer, Eugen, 1939: *Differenzengleichungen und bestimmte Integrale*, Leipzig.

Busbridge, I. W., 1950: *Quart. J. Math. Oxford Ser.* (2) 1, 176-184.

Furch, R., 1939: *Z. Physik* 112, 92-95.

Hartree, D. R., 1936: *Manchester Memoirs* 80, 85-102.

Harvard University, 1949a: *Annals of the Computation Laboratory*, Vols. XVIII *and* XIX. *Generalized sine - and cosine - integral functions.* Parts I and II. Harvard University Press, Cambridge, Mass.

Harvard University, 1949b: *Annals of the Computation Laboratory*, Vol. XXI. *Tables of the generalized exponential - integral functions*, Harvard University Press, Cambridge, Mass.

Jahnke, Eugen and Fritz Emde, 1945: *Tables of functions with formulas and curves*, Dover Publications, New York.

Kramp, Christian, 1799: *Analyse des Réfractions*, Strasbourg and Leipzig.

Le Caine, J., 1948: *National Research Council of Canada, Division of Atomic Energy*, Document No. MT-131 (NRC 1553), 45 pp.

Legendre, A. M.. 1811: *Exercises de calcul intégral*, Paris.

Nielsen, Niels, 1906a: *Handbuch der Theorie der Gammafunktion*, Leipzig, 326 pp.

Nielsen, Niels, 1906b: *Theorie des Integrallogarithmus und verwandter Transcendenten*, 106 pp., B. G. Teubner, Leipzig.

Nielsen, Niels, 1906c: *Monatsch. Math. Phys.* 17, 47-58.

Placzek, George, 1946: *National Research Council of Canada, Division of Atomic Energy*, Document No. MT-1, 39 pp.

Prym, F. E., 1877: *J. Math.* 82, 165-172.

Rosser, J. B., 1948: *Theory and application of*

$$\int_0^z e^{-x^2}dx \text{ and } \int_0^z e^{-p^2y^2}\,dy \int_0^y e^{-x^2}\,dx$$

Mapleton House, Brooklyn, New York.

Schlömilch, Oskar, 1871: *Z. Math. Phys.* 16, 261-262.

Tannery, Jules, 1882: *Comptes Rendus* 94, 1698-1701, 95, 75.

Tricomi, F. G., 1950a: *Boll. Un. Mat. Ital.* (3) 4, 341-344.

Tricomi, F. G., 1950b: *Z. Math.* 53, 136-148.

Tricomi, F. G., 1951: *Ann. Mat. Pura Appl.* (4) 28, 263-289.

Tricomi, F. G., 1951: *J. D'Analyse Math.* 1, 209-231.

CHAPTER X

ORTHOGONAL POLYNOMIALS

The standard textbook on this subject is the book by Szegö (1939) to which we shall refer frequently. There is a systematic bibliography up to 1938, by Shohat, Hille, and Walsh (1940). Although the present chapter is concerned with orthogonal *polynomials* only, in the introductory sections we consider more generally systems of orthogonal functions. For further information on this latter topic the reader may be referred to books by Kaczmarz and Steinhaus (1935) and by Tricomi (1948), and by Vitali and Sansone (1946).

10.1. Systems of orthogonal functions

With an interval (a, b) and a *weight function* $w(x)$ which is *non-negative* there, we may associate the scalar product

(1) $(\varphi_1, \varphi_2) \equiv \int_a^b w(x)\, \varphi_1(x)\, \varphi_2(x)\, dx$

which is defined for all functions φ for which $w^{1/2}\varphi$ is quadratically integrable in (a, b). More generally, a scalar product may be defined by a Stieltjes integral

(2) $(\varphi_1, \varphi_2) \equiv \int_a^b \varphi_1(x)\, \varphi_2(x)\, d\,\alpha(x)$

where $\alpha(x)$ is a non-decreasing function. If $\alpha(x)$ is absolutely continuous, (2) reduces to (1) with $w(x) = \alpha'(x)$. On the other hand, if $\alpha(x)$ is a jump function, that is constant except for jumps of the magnitude w_i at $x = x_i$, then (2) reduces to a sum

(3) $(\varphi_1, \varphi_2) = \sum_i w_i\, \varphi_1(x_i)\, \varphi_2(x_i)$

which is the appropriate definition for functions of a discrete variable.

The above definitions refer to real functions of a real variable, and to this case we shall restrict ourselves throughout this chapter. If the functions in question are complex-valued, or else if the domain of integration is an arc in the complex plane rather than a segment of the real axis, then $\varphi_2(x)$ in all these definitions must be replaced by the conjugate complex quantity.

Except in the last few sections (where we use definition (3)), we shall use definition (1) mostly, and shall assume moreover that $w(x)$ is positive almost everywhere and integrable. It should be mentioned, however, that many of the results of the introductory sections hold for the definition (2), and therefore also for the definition (3), of a scalar product.

Two functions are said to be *orthogonal* if their scalar product vanishes. A family of functions is an *orthogonal system*, on the interval (a, b) and with the weight function $w(x)$ (or distribution $a(x)$), if for any two *distinct* members of the family, $(\varphi_1, \varphi_2) = 0$. Since the space of quadratically integrable functions is *separable*, it follows that an orthogonal system consists either of a finite number or at most of a denumerable infinity of elements. Thus an orthogonal system can always be written as a (finite or infinite) *sequence*, $\varphi_0(x)$, $\varphi_1(x)$, ... or shortly $\{\varphi_n(x)\}$, and the orthogonal property is then expressed as

$$(4) \quad (\varphi_h, \varphi_k) = 0 \qquad h \neq k.$$

We shall assume that $\{\varphi_n(x)\}$ does not contain any null function, i.e., that (φ_h, φ_h) is positive for all h. It is then easy to see that the functions of any finite subset of an orthogonal system are linearly independent, that is that a relation of the form

$$(5) \quad c_0\varphi_0(x) + c_1\varphi_1(x) + \cdots + c_k\varphi_k(x) = 0$$

cannot be valid almost everywhere in (a, b), except when $c_0 = c_1 = \cdots = c_k = 0$. (Form the scalar product with $\varphi_h(x)$ for $h = 0, 1, \ldots, k$.)

The functions $\{\varphi_n(x)\}$ form an *orthonormal system* if

$$(6) \quad (\varphi_h, \varphi_k) = \begin{cases} 0 & \text{if} \quad h \neq k, \\ 1 & \text{if} \quad h = k. \end{cases}$$

Every orthogonal system can be *normalized* by replacing $\varphi_h(x)$ by

$$(\varphi_h, \varphi_h)^{-1/2}\,\varphi_h(x).$$

A (finite or infinite) sequence $\{\psi_n(x)\}$ of linearly independent functions can be *orthogonalized* with respect to the scalar product (2) by the formation of suitable linear combinations. For instance we may put recurrently

$$(7) \quad \begin{aligned} \varphi_0(x) &= \psi_0(x) \\ \varphi_1(x) &= \mu_{10}\varphi_0(x) + \psi_1(x) \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \\ \varphi_n(x) &= \mu_{n0}\varphi_0(x) + \mu_{n1}\varphi_1(x) + \cdots + \mu_{n,n-1}\varphi_{n-1}(x) + \psi_n(x) \end{aligned}$$

and see that $\{\varphi_n(x)\}$ is an orthogonal system if we take

(8) $\mu_{nm} = -(\psi_n, \varphi_m)/(\varphi_m, \varphi_m) \qquad m = 0, 1, \dots, n-1.$

Alternatively, we may put

(9) $\varphi_n(x) = \lambda_{n0}\psi_0(x) + \lambda_{n1}\psi_1(x) + \cdots + \lambda_{nn}\psi_n(x) \qquad \lambda_{nn} \neq 0$

and determine the λ's so that $\{\varphi_n(x)\}$ is an orthogonal system. One possible determination leads to

$$(10) \quad \phi_n(x) = \begin{vmatrix} (\psi_0, \psi_0) & (\psi_0, \psi_1) & \cdots & (\psi_0, \psi_n) \\ (\psi_1, \psi_0) & (\psi_1, \psi_1) & \cdots & (\psi_1, \psi_n) \\ \cdots & \cdots & \cdots & \cdots \\ (\psi_{n-1}, \psi_0) & (\psi_{n-1}, \psi_1) & \cdots & (\psi_{n-1}, \psi_n) \\ \psi_0(x) & \psi_1(x) & & \psi_n(x) \end{vmatrix}$$

It is clear that $\{\phi_n(x)\}$ is an orthogonal system, for (10) is orthogonal to $\psi_0(x)$, $\psi_1(x)$, ... , $\psi_{n-1}(x)$ and hence to $\phi_m(x)$ for all $m < n$. Moreover, any orthogonal system of the form (9) is a constant multiple of $\{\phi_n(x)\}$.

In order to normalize the system (9), we introduce *Gram's determinant* G_n which is the cofactor of $\psi_{n+1}(x)$ in the expression(10)for $\phi_{n+1}(x)$. G_n is also the discriminant of the positive definite quadratic form

$$\int_a^b [\xi_0\,\psi_0(x) + \cdots + \xi_n\,\psi_n(x)]^2\, w(x)\, dx$$

in ξ_0, ... , ξ_n, and hence positive. We also put $G_{-1} = 1$. The orthonormal system of the form (9) with $\lambda_{nn} > 0$ is then uniquely determined as

(11) $\varphi_n(x) = (G_{n-1}\, G_n)^{-1/2}\, \phi_n(x).$

Furthermore the following integral representation can be established

$$(12) \quad \phi_n(x) = [(n-1)!]^{-1} \int^{(n)} \Psi_{n-1}(\xi_0, \dots, \xi_{n-1})\, \Psi_n(\xi_0, \dots, \xi_{n-1}, x) \times w(\xi_0) \cdots w(\xi_{n-1})\, d\xi_0 \cdots d\xi_{n-1} \qquad n = 1, 2, \dots$$

where the integral is an n-tuple integral over (a, b) and

$$(13) \quad \Psi_n(x_0, \dots, x_n) = \begin{vmatrix} \psi_0(x_0) & \psi_1(x_0) & \cdots & \psi_n(x_0) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_0(x_n) & \psi_1(x_n) & \cdots & \psi_n(x_n) \end{vmatrix}.$$

(See Szegö, 1939, sec. 2.1.)

In this chapter we shall be concerned with the orthogonalization, in the form (9), of the functions $\psi_n(x) = x^n$. Thus we obtain a sequence of *orthogonal polynomials* $\{p_n(x)\}$, $n = 0, 1, 2, \dots$ where $p_k(x)$ is a polynomial in x of exact degree k, and $(p_h, p_k) = 0$ for $h, k = 0, 1, 2, \dots$ and $h \neq k$.

The interval and the weight function (or distribution) determine the system of orthogonal polynomials up to an arbitrary constant factor in each $p_n(x)$. The polynomials may be standardized by the adoption of additional requirements. The three most frequently used additional requirements are: (i) $\{p_n(x)\}$ shall be an orthonormal system and the coefficient of x^n in $p_n(x)$ shall be positive; (ii) the coefficient of x^n in $p_n(x)$ shall have a prescribed value (usually unity); (iii) for a given x_0 (for instance $x_0 = a$), $p_n(x_0)$ shall have a prescribed value.

10.2. The approximation problem

Let L^2_w be the class of all functions $f(x)$ for which the (Lebesgue) integral

$$\int_a^b w(x)\,[f(x)]^2\,dx$$

exists and is finite, and let $\{\varphi_n(x)\}$ be an orthonormal system in L^2_w. In approximating any function $f(x)$ of L^2_w by a linear combination

$$c_0\,\varphi_0(x) + \cdots + c_n\,\varphi_n(x),$$

we regard

$$(1)\qquad I_n(c_h) = \int_a^b w(x)\,[f(x) - c_0\,\varphi_0(x) - \cdots - c_n\,\varphi_n(x)]^2\,dx$$

as the measure of accuracy of this approximation. It is easy to see that the best possible choice for c_h is that of the Fourier coefficients

$$(2)\qquad a_h = (f, \varphi_h).$$

In fact, expanding $[\cdots]^2$ in (1) we find

$$I_n(c_h) = \int_a^b w(x)\,[f(x)]^2\,dx + \sum_{h=0}^{n} c_h^2 - 2\sum_{h=0}^{n} a_h c_h$$

$$= \int_a^b w(x)\,[f(x)]^2\,dx - \sum_{h=0}^{n} a_h^2 + \sum_{h=0}^{n} (c_h - a_h)^2,$$

that is the best approximation is the $(n+1)$st partial sum of the (generalized) *Fourier series*

$$(3)\qquad a_0\,\varphi_0(x) + a_1\,\varphi_1(x) + \cdots$$

of $f(x)$, and the measure of the accuracy of this approximation is

$$(4)\qquad I_n(a_h) = \int_a^b w(x)\,[f(x)]^2\,dx - \sum_{h=0}^{n} a_h^2.$$

Since $I_n(a_h) \geq 0$, it follows that $\Sigma\, a_h^2$ is convergent and we have *Bessel's inequality*

$$(5) \quad \sum_{h=0}^{\infty} a_h^2 \leq \int_a^b w(x)\,[f(x)]^2\,dx.$$

It may happen that *Parseval's formula*

$$(6) \quad \sum_{h=0}^{\infty} a_h^2 = \int_a^b w(x)\,[f(x)]^2\,dx$$

holds for every function $f(x)$ of L_w^2. Then the orthonormal system $\{\varphi_n(x)\}$ is said to be *closed* in L_w^2. In this case clearly

$$(7) \quad \int_a^b w(x)\,[f(x) - \sum_{h=1}^{n} a_h\,\varphi_h(x)]^2\,dx \to 0 \quad \text{as} \quad n \to \infty,$$

and we say that the partial sums of the Fourier series (3) *converge in the mean* to $f(x)$. In L_w^2, every closed orthogonal system is also *complete*, i.e., if $(f, \varphi_h) = 0$ for all h, then $f(x)$ vanishes almost everywhere. This is a consequence of the Riesz-Fischer theorem (cf. for instance Kaczmarz and Steinhaus, 1935, or Tricomi, 1948, sec. 3.3).

For a finite interval (a, b) every function of L_w^2 can be approximated arbitrarily closely, in the mean, by a continuous function, and by the theorem of Weierstrass the continuous function can be approximated by a polynomial. Thus for a finite interval and $\psi_n(x) = x^n$, or $\varphi_n(x) = p_n(x)$, we may make $I_n(a_h)$ arbitrarily small by making n sufficiently large. In other words, any system of orthogonal polynomials for a *finite interval* is *closed*. This need no longer be true if the interval (a, b) is of infinite length (Szegö 1939, sec. 3.1).

10.3. General properties of orthogonal polynomials

A weight function $w(x)$ on an interval (a, b) determines a system of orthogonal polynomials $\{p_n(x)\}$ uniquely apart from a constant factor in each polynomial. The numbers

$$(1) \quad c_n = \int_a^b w(x)\,x^n\,dx$$

are the *moments* of the weight function, and with $\psi_n(x) = x^n$ we have

$$(2) \quad (\psi_m, \psi_n) = c_{m+n}.$$

In the notation of sec. 10.1 we then have

(3) $$G_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix}, \quad \Psi_n = \begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & \cdots & x_n^n \end{vmatrix} = \prod_{r>s} (x_r - x_s).$$

If the (undetermined) coefficient of x^n in $p_n(x)$ is denoted by k_n, we have

(4) $$p_n(x) = \frac{k_n}{G_{n-1}} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

(5) $$p_n(x) = \frac{k_n}{n!\,G_{n-1}} \int^{(n)} \prod_{r>s} (\xi_r - \xi_s)^2 \prod_{\nu=1}^{n} [(x-\xi_\nu)\, w(\xi_\nu)\, d\xi_\nu].$$

Since $1, x, \ldots, x^{n-1}$ are orthogonal to $p_n(x)$, we have

(6) $$h_n = (p_n, p_n) = k_n^2 \frac{G_n}{G_{n-1}} .$$

For the *normalized* polynomials $k_n = (G_{n-1}/G_n)^{1/2}$, but we shall not standardize our polynomials at this stage.

Any polynomial of degree $m < n$ is a linear combination of $p_0(x)$, $p_1(x), \ldots, p_m(x)$ and hence orthogonal to $p_n(x)$. This leads to a simple proof of the following theorem on the zeros of orthogonal polynomials. *All zeros of* $p_n(x)$ *are simple, and located in the interior of the interval* (a, b). For if $p_n(x)$ changed its sign in (a, b) only at $m < n$ points, we could construct a polynomial $\pi_m(x)$ of degree m so that $p_n(x)\,\pi_m(x) \geq 0$ in (a, b), and this contradicts $(p_n, \pi_m) = 0$. It can also be shown that *between two consecutive zeros of* $p_n(x)$ *there is exactly one zero of* $p_{n+1}(x)$, *and at least one zero of* $p_m(x)$ *for each* $m > n$ (Szegö, 1939, sec. 3.3).

Any three consecutive polynomials are connected by a linear relation. We use the following notations: k_n is the coefficient of x^n, and k'_n the coefficient of x^{n-1}, in $p_n(x)$; $r_n = k'_n/k_n$, and $h_n = (p_n, p_n)$. We shall then prove the *recurrence formula*

(7) $$p_{n+1}(x) = (A_n x + B_n)\, p_n(x) - C_n\, p_{n-1}(x) \qquad n = 1, 2, 3, \ldots$$

in which

(8) $A_n = k_{n+1}/k_n, \quad B_n = A_n(r_{n+1} - r_n),$

$$C_n = A_n h_n/(A_{n-1} h_{n-1}) = k_{n+1} k_{n-1} h_n/(k_n^2 h_{n-1}).$$

To prove (7), we remark that with the value (8) of A_n, the expression $p_{n+1}(x) - A_n x p_n(x)$ is a polynomial of degree n or less, and consequently of the form

$$\gamma_0 p_n(x) + \gamma_1 p_{n-1}(x) + \cdots + \gamma_n p_0(x).$$

From the orthogonal property of the $p_n(x)$, we find that $\gamma_2 = \gamma_3 = \cdots = = \gamma_n = 0$, and

$$-A_n(p_n, xp_{n-1}) = \gamma_1 (p_{n-1}, p_{n-1}).$$

Now, $xp_{n-1}(x) - (k_{n-1}/k_n) p_n(x)$ is a polynomial of degree $n - 1$ or less, and hence

$$-A_n h_n k_{n-1}/k_n = \gamma_1 h_{n-1}$$

or $\gamma_1 = C_n$. Lastly, the value of B_n follows on comparing coefficients of x^n on both sides of (7). The recurrence formula (7) remains valid for $n = 0$ if we put

(9) $p_{-1}(x) = 0.$

This convention will be retained throughout this chapter.

It may be noted that conversely, a system of polynomials satisfying a recurrence relation (7) with positive A_n and C_n, is an orthogonal system.

From (7) we easily obtain the *Christoffel-Darboux formula*

(10) $$\sum_{\nu=0}^{n} h_\nu^{-1} p_\nu(x) p_\nu(y) = \frac{k_n}{k_{n+1} h_n} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y}$$

and for $y \to x$,

(11) $$\sum_{\nu=0}^{n} h_\nu^{-1} [p_\nu(x)]^2 = \frac{k_n}{k_{n+1} h_n} [p_n(x) p'_{n+1}(x) - p'_n(x) p_{n+1}(x)].$$

Let $\{p_n(x)\}$ be the system of orthogonal polynomials for the weight function $w(x)$, and let $\rho(x)$ be a polynomial of degree l which is non-negative in (a, b) and has simple zeros at $x_1, x_2, \ldots, x_l$. The orthogonal polynomials $q_n(x)$, belonging to the weight function $\rho(x) w(x)$ are then given by *Christoffel's formula*

$$(12)\quad c_n\,\rho(x)q_n(x) = \begin{vmatrix} p_n(x) & p_{n+1}(x) & \cdots & p_{n+l}(x) \\ p_n(x_1) & p_{n+1}(x_1) & \cdots & p_{n+l}(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ p_n(x_l) & p_{n+1}(x_l) & \cdots & p_{n+l}(x_l) \end{vmatrix},$$

in which c_n is an arbitrary constant factor (Szegö, 1939, sec. 2.5). If some of the zeros of $\rho(x)$ are multiple zeros, (12) must be replaced by a confluent form.

Orthogonal polynomials have some important extremum properties. The first of these can be derived from the result at the beginning of sec. 10.2 and reads: *The integral*

$$(13)\quad \int_a^b |\pi_n(x)|^2\, w(x)\, dx$$

in which $\pi_n(x)$ *denotes any polynomial of degree n with the leading term* x^n *becomes a minimum if and only if* $\pi_n(x) = \epsilon k_n^{-1} p_n(x)$ *where* ϵ *is a constant and* $|\epsilon| = 1$. The second property involves the polynomials

$$(14)\quad K_n(x, y) = \sum_{m=1}^{n} h_m^{-1}\, p_m(\bar{x})\, p_m(y)$$

which are defined for complex x, y ($\bar{x}$ is the conjugate complex of x). We may remark here that for finite x_0, a and for $x_0 \le a$, the polynomials $K_n(x_0, x)$ are orthogonal with respect to the weight function $(x-x_0)w(x)$ (cf. (10) and (11)). The extremum property in question may be formulated as follows (Szegö, 1939, theorem 3.1.3). *Let* $\pi_n(x)$ *be an arbitrary polynomial of degree n with complex coefficients such that the integral* (13) *is equal to unity. For any fixed (possibly complex)* x_0 *the maximum of* $|\pi_n(x_0)|^2$ *is reached if and only if*

$$\pi_n(x) = \epsilon\,[K_n(x_0, x_0)]^{-1/2}\, K_n(x_0, x)$$

where $|\epsilon| = 1$. *The maximum itself is* $K_n(x_0, x_0)$.

10.4. Mechanical quadrature

Many interesting properties of orthogonal polynomials depend on their connection with problems of interpolation and mechanical quadrature. In this section we can give no more than a brief description of some of the basic results, and refer to Szegö's book (1939, sec. 3.4, chapters XIV, XV) for further information.

Let $x_1, x_2, \ldots, x_n$ be n distinct points of the interval (a, b) and let

$$(1)\quad \pi_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n),$$

$$l_\nu(x) = (x - x_\nu)^{-1}\,\pi_n(x)/\pi_n'(x_\nu) \qquad \nu = 1, \ldots, n.$$

The $l_\nu(x)$ are the fundamental polynomials associated with the abscissae $x_1, \ldots, x_n$ in the Lagrangean interpolation

$$(2)\quad L(x) = \sum_{\nu=1}^{n} f(x_\nu)\, l_\nu(x)$$

of the function $f(x)$.

If the integral

$$(3)\quad I = \int_a^b w(x)\, f(x)\, dx$$

is to be computed for a function whose values at the x_ν are given, it seems natural to use (2) and compute

$$(4)\quad J = \int_a^b w(x)\, L(x)\, dx = \sum_{\nu=1}^{n} f(x_\nu) \int_a^b w(x)\, l_\nu(x)\, dx$$

in the expectation that J will be an approximation to I. Actually, for any $x_1, \ldots, x_n$, we have $I = J$ for all polynomials $f(x)$ of degree $\leq n - 1$. However, if we choose the x_ν to be the n zeros of $p_n(x)$, the orthogonal polynomial of degree n associated with the weight function $w(x)$, then $I = J$ for all polynomials $f(x)$ of degree $\leq 2n - 1$. For in this case $f(x) - L(x)$ is a polynomial of degree $\leq 2n - 1$ vanishing at all the zeros of $p_n(x)$ and hence of the form $p_n(x)\,\pi_{n-1}(x)$ where $\pi_{n-1}(x)$ is a polynomial of degree $\leq n - 1$. Then

$$I - J = \int_a^b w(x)\,[f(x) - L(x)]\, dx = (p_n, \pi_{n-1}) = 0.$$

It is customary to write

$$(5)\quad J = \int_a^b w(x)\, L(x)\, dx = \sum_{\nu=1}^{n} \lambda_{\nu n}\, f(x_\nu)$$

where the $\lambda_{\nu n}$ are called the *Christoffel numbers*. They are connected with the moments of $w(x)$ by the relations

$$(6)\quad \sum_{\nu=1}^{n} x_\nu^h\, \lambda_{\nu n} = c_h \qquad h = 0, 1, \ldots, n - 1$$

obtained by choosing $f(x) = x^h$. The Christoffel numbers are positive, and the following formulas hold:

$$(7)\quad \lambda_{\nu n} = \int_a^b \frac{w(x)\, p_n(x)}{p_n'(x_\nu)(x - x_\nu)}\, dx = \int_a^b w(x) \left[\frac{p_n(x)}{p_n'(x_\nu)(x - x_\nu)}\right]^2 dx$$

(8) $$\lambda_{\nu n} = -\frac{k_{n+1}\, h_n/k_n}{p_n'(x_\nu)\, p_{n+1}(x_\nu)} = \frac{1}{K(x_\nu, x_\nu)}\,.$$

If we denote by $x_{1n}, x_{2n}, \dots, x_{nn}$ the n zeros of $p_n(x)$ and by $y_{1n}, \dots, y_{nn}$ the n numbers in (a, b) defined by

(9) $$\int_a^{y_{\nu n}} w(x)\, dx = \lambda_{1n} + \dots + \lambda_{\nu n} = \Lambda_{\nu n}$$

then we have a number of *separation theorems*

(10) $$x_{\nu-1,n} < x_{\nu,n+1} < x_{\nu,n}$$

(11) $$y_{\nu-1,n} < y_{\nu,n+1} < y_{\nu,n}$$

(12) $$x_{\nu,n} < y_{\nu,n} < x_{\nu+1,n}$$

(13) $$\Lambda_{\nu-1,n} < \Lambda_{\nu,n+1} < \Lambda_{\nu,n}.$$

10.5. Continued fractions

The recurrence formula 10.3 (7) suggests the *continued fraction*

(1) $$\frac{1|}{|A_0x + B_0} - \frac{C_1|}{|A_1x + B_1} - \frac{C_2|}{|A_2x + B_2} - \cdots,$$

where A_n, B_n, C_n are given by 10.3 (8). The n^{th} *convergent* R_n/S_n is defined as the finite fraction obtained by stopping at the term $A_{n-1}x + B_{n-1}$ in (1) so that

(2) $$R_0 = 0, \quad S_0 = 1; \qquad R_1 = 1, \quad S_1 = A_0x + B_0 = p_1(x)/p_0(x).$$

Both R_n and S_n satisfy the recurrence relation

(3) $$X_{n+1} = (A_n x + B_n) X_n - C_n X_{n-1}\,.$$

The initial conditions are

(4) for $R_n: X_0 = 0,\ X_1 = 1;$ for $S_n: X_0 = 1,\ X_1 = p_1(x)/p_0(x).$

Referring to 10.3 (7) it is seen that

(5) $$S_n = p_n(x)/p_0(x) = k_0^{-1}\, p_n(x).$$

In order to express also R_n, we introduce the associated polynomial

(6) $$q_n(x) = \int_a^b \frac{p_n(x) - p_n(t)}{x - t}\, w(t)\, dt$$

which is a polynomial of degree $n-1$. From 10.3 (7)

$$q_{n+1}(x)-(A_n x+B_n)\,q_n(x)+C_n q_{n-1}(x)$$
$$=-A_n\int_a^b p_n(t)\,w(t)\,dt=0 \qquad n=1,2,\ldots.$$

Moreover, $q_0(x)=0$, $q_1(x)=\int_a^b k_1\,w(t)\,dt=k_1\,c_0$, and hence

(7) $R_n=(k_1\,c_0)^{-1}\,q_n(x).$

We thus see that R_n/S_n is a rational function of x with simple poles at $x=x_{\nu n}$. The residues at these poles can be computed from

$$\lim_{x\to x_{\nu n}}(x-x_\nu)\frac{q_n(x)}{p_n(x)}=\frac{1}{p_n'(x_\nu)}\int_a^b\frac{p_n(t)}{t-x_{\nu n}}\,w(t)\,dt=\lambda_{\nu n},$$

see 10.4 (7), and we have the decomposition in partial fractions

(8) $$\frac{R_n}{S_n}=\frac{k_0}{k_1\,c_0}\sum_{\nu=1}^{n}\frac{\lambda_{\nu n}}{x-x_{\nu n}}.$$

On expansion of the sum in descending powers of x the relation 10.4 (6) shows that the first $2n$ coefficients are the moments c_h. Hence we obtain formally

(9) $$\lim_{n\to\infty}\frac{R_n}{S_n}=\frac{k_0}{k_1\,c_0}\sum_{h=0}^{\infty}\frac{c_h}{x^{h+1}}.$$

For a finite interval (a, b), and for any x in the complex plane cut along the segment (a, b) of the real axis, Markoff proved that $\lim R_n/S_n$ exists and (9) is valid. Moreover,

(10) $$\lim_{n\to\infty}\frac{R_n}{S_n}=\frac{k_0}{k_1\,c_0}\int_a^b\frac{w(t)}{x-t}\,dt$$

in this case (Szegö, 1939, sec. 3.5). Intervals of infinite length present formidable difficulties which are discussed in the theory of (Stieltjes and Hamburger) *moment problems*. For these see Shohat and Tamarkin (1943).

10.6. The classical polynomials

The orthogonal polynomials belonging to the intervals and weight functions listed in the following table arise very frequently and have been studied in great detail. They are known as the *classical orthogonal polynomials*.

CLASSICAL ORTHOGONAL POLYNOMIALS

a	b	$w(x)$	NAME
-1	1	1	Legendre or spherical
-1	1	$(1-x^2)^{\lambda-\frac{1}{2}}$	Gegenbauer or ultraspherical
-1	1	$(1-x)^{\alpha}(1+x)^{\beta}$	Jacobi or hypergeometric
$-\infty$	∞	$\exp(-x^2)$	Hermite
0	∞	$x^{\alpha} e^{-x}$	(generalized) Laguerre.

All these polynomials have a number of properties in common of which the three most important ones are:

(i) $\{p_n'(x)\}$ is a system of orthogonal polynomials;

(ii) $p_n(x)$ satisfies a differential equation of the form

$$A(x)y'' + B(x)y' + \lambda_n y = 0$$

where $A(x)$ and $B(x)$ are independent of n, and λ_n is independent of x;

(iii) there is a generalized Rodrigues' formula

$$(1) \quad p_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n} [w(x) X^n]$$

where K_n is a constant and X is a polynomial in x whose coefficients are independent of n.

Conversely, any of these three properties *characterizes* the classical orthogonal polynomials in the sense that any system of orthogonal polynomials which has one of these properties can be reduced to a classical system. For (i), this has been proved by Hahn (1935) and Krall (1936); for (ii) by Bochner (1939)((in this case there are some trivial exceptions); and for (iii) by Tricomi (1948a). We shall briefly indicate the argument in this last case.

Let $\{p_n(x)\}$ be a sequence of polynomials, $p_n(x)$ of exact degree n, for which (1) holds for every $n = 0, 1, 2, \dots$, the polynomial X being of degree k. Note that it is not necessary to assume that the $p_n(x)$ are orthogonal polynomials or that $w(x)$ is a weight function. From (1), with $n = 1$, we have

$$(2) \quad K_1 p_1(x) = X' + X w'(x)/w(x).$$

First let $k = 0$. Then X is a constant and w'/w is a linear function of x. By a linear change of the independent variable we may make $w'/w = -2x$, hence $w = \exp(-x^2)$, and the polynomials are the Hermite polynomials, see 10.13(7). Next let $k = 1$. Then a linear change of x brings

(3) $$\frac{w'(x)}{w(x)} = \frac{K_1 p_1(x) - X'}{X}$$

into the form $w'/w = -1 + a/x$, so that $X = x$, $w = x^a e^{-x}$, and we have the Laguerre polynomials, see 10.12(5).

We now discuss $k \geq 2$. In this case we may take

(4) $$X = \prod_{r=1}^{k} (x - a_r)$$

and at first we assume all the a_r different from each other. From (3)

$$\frac{w'(x)}{w(x)} = \sum_{r=1}^{k} \frac{a_r}{x - a_r}$$

so that (1) becomes

$$p_n(x) = K_n \prod_{r=1}^{k} (x - a_r)^{-a_r} \frac{d^n}{dx^n} [\prod_{r=1}^{k} (x - a_r)^{n+a_r}],$$

and for $n = 2$ this fails to be a quadratic polynomial except when $k = 2$. The case of repeated factors in (4) can be excluded by a similar consideration, so that in (4) we must have $k = 2$, $a_1 \neq a_2$. By a linear change of x we may make $a_1 = -1$, $a_2 = 1$, and write

$$X = (1 - x)^2, \quad w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$$

so that this case leads to Jacobi polynomials, see 10.8(10).

It may be mentioned that Hahn (1949) has extended these results considerably. He replaced the differential operator $df(x)/dx$ by the more general linear operator

$$Lf(x) = \frac{f(qx + \omega) - f(x)}{(q - 1)x + \omega}$$

and showed that in this more general case each of conditions (i), (ii), (iii), and of two further conditions, characterizes the same family of orthogonal polynomials. The classical polynomials are limiting cases of Hahn's polynomials, and so are the polynomials of sections 22-25.

10.7. General properties of the classical orthogonal polynomials

Many important properties of the classical orthogonal polynomials follow easily from the generalized Rodrigues' formula 10.6(1). We assume $\alpha > -1$ in the Laguerre case and $\alpha > -1$, $\beta > -1$ in the Jacobi case.

In each case we have in 10.6(1) a $w(x)$ which is non-negative and integrable in (a, b). Moreover, since all derivatives up to and including the $(n-1)$st of $w(x) X^n$ vanish at a and b, we may integrate by parts n times in

$$(f, p_n) = K_n^{-1} \int_a^b f(x) \frac{d^n}{dx^n} [w(x) X^n] \, dx,$$

obtain

$$(f, p_n) = (-1)^n K_n^{-1} \int_a^b f^{(n)}(x) \, w(x) \, X^n \, dx$$

and hence $(f, p_n) = 0$ if f is a polynomial of degree $< n$. In other words, the polynomials 10.6(1) form an orthogonal system in the interval (a, b) with the weight function $w(x)$, and all the results of the previous sections are valid for these functions. In particular, we have the recurrence formula 10.3(7) with the notation 10.3(8) which we shall use again in the present section.

In deriving the differential equation from 10.6(1) we shall write D instead of d/dx. From 10.6(1) and from Leibniz' formula of the differentiation of a product we have

$$D^{n+1} [XD(wX^n)] = K_n [XD^2(wp_n) + (n+1) X' D(wp_n) + \tfrac{1}{2} n(n+1) X'' wp_n].$$

On the other hand, using 10.6(3),

$$D^{n+1} [XD(wX^n)] = D^{n+1} \{[K_1 p_1 + (n-1) X'] wX^n\}$$
$$= K_n \{[K_1 p_1 + (n-1) X'] D(wp_n) + (n+1)[K_1 p_1' + (n-1) X''] wp_n\}$$

since $K_1 p_1 + (n-1) X'$ is at most a linear function of x. Comparison of the two results yields the differential equation

$$(1) \qquad X \frac{d^2 y}{dx^2} + K_1 p_1(x) \frac{dy}{dx} + \lambda_n y = 0$$

for $y = p_n(x)$, where

$$(2) \qquad \lambda_n = -n [k_1 K_1 + \tfrac{1}{2}(n-1) X''].$$

The self-adjoint form of the differential equation is

$$(3)\quad \frac{d}{dx}\left[Xw(x)\frac{dy}{dx}\right] + \lambda_n w(x)\, y = 0.$$

For the details of the proof see Tricomi (1948a, p. 210-212). Since X is at most a quadratic polynomial, and $p_1(x)$ is a linear polynomial, the differential equation (1) can be reduced to the hypergeometric equation or to one of its special or limiting cases.

For the classical polynomials we also have the *differentiation formula*

$$(4)\quad X\frac{dp_n(x)}{dx} = (a_n + \tfrac{1}{2}nX''x)\, p_n(x) + \beta_n p_{n-1}(x)$$

where

$$(5)\quad a_n = nX'(0) - \tfrac{1}{2}X''r_n, \qquad A_n\beta_n = -C_n[k_1K_1 + (n-\tfrac{1}{2})X''],$$

and A_n, C_n, k_n, r_n have the same meaning as in sec. 10.3. By means of 10.3(7), the right-hand side of (4) can be expressed in terms of p_n and p_{n+1}.

The proof of (4) in Tricomi (1948a, p. 212-215) is based on the fact that

$$Xp_n'(x) - \tfrac{1}{2}nX''xp_n(x)$$

is a polynomial of degree $\le n$ and hence of the form

$$a_n p_n(x) + \beta_n p_{n-1}(x) + \gamma_2 p_{n-2}(x) + \cdots + \gamma_n p_0(x).$$

The coefficients $a_n, \ldots, \gamma_n$ are then determined by the orthogonal property. In the determination of β_n the differential equation (3) is also used.

Finally we note that by n successive integrations by parts as at the beginning of this section,

$$(6)\quad h_n = (p_n, p_n) = (-1)^n k_n\, n!\, K_n^{-1}\int_a^b X^n w(x)\, dx,$$

from 10.4(8), 10.3(7), and (4)

$$(7)\quad \lambda_{\nu,n} = A_{n-1}h_{n-1}[X(x_{\nu,n})/\beta_n]\,[p_{n-1}(x_{\nu,n})]^{-2}$$
$$= A_{n-1}h_{n-1}[\beta_n/X(x_{\nu,n})]\,[p_n'(x_{\nu,n})]^{-2},$$

and from (6)

$$(8)\quad (-1)^n k_n K_n > 0.$$

Each of the following six sections is devoted to one of the principal families of classical orthogonal polynomials. Each of these six sections is organized on the following plan:

(i) Standardization of the polynomials.
(ii) Computation of the ten constants

(9) $h_n, k_n, r_n, A_n, B_n, C_n, K_n, \lambda_n, a_n, \beta_n$

given by 10.7(6), 10.3(8), 10.7(2), 10.7(5).
(iii) Statement of the recurrence relation, differential equation, and other relations, except that whenever these relations are cumbersome, it will be left to the reader to substitute the values of the ten constants (9) into the general formulas of this and the previous sections.
(iv) Connection with functions of the hypergeometric type and complete integration of the differential equation.
(v) Generating function or functions.
(vi) Integral representations.
(vii) Addition theorems, series expansions, and miscellaneous results.

Asymptotic properties, zeros, expansion problems will be discussed in later sections.

We shall use the notation

(10) $D = \dfrac{d}{dx}$

and shall put

(11) $(a)_0 = 1, \quad (a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \cdots (a+n-1).$

Accounts of the classical orthogonal polynomials are given in the works referred to in the introduction, and also in the book by Magnus and Oberhettinger (1948, Chap. V).

10.8. Jacobi polynomials

We shall use Szegö's notation $P_n^{(\alpha,\beta)}(x)$ for the suitably standardized orthogonal polynomials associated with

(1) $a = -1, \quad b = 1, \quad w(x) = (1-x)^\alpha (1+x)^\beta, \quad X = 1 - x^2.$

In order to make the weight function non-negative and integrable, we assume

(2) $\alpha > -1, \quad \beta > -1.$

Many of the formal relations are valid without this restriction.

(i) *Standardization.*

(3) $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}.$

(ii) *Constants.*

(4) $(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)\,h_n = 2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)$

(5) $k_n = 2^{-n}\binom{2n+\alpha+\beta}{n}, \quad r_n = \frac{n(\alpha-\beta)}{2n+\alpha+\beta}$

(6) $2(n+1)(n+\alpha+\beta+1)\,A_n = (2n+\alpha+\beta+1)(2n+\alpha+\beta+2)$

(7) $2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\,B_n = (\alpha^2-\beta^2)(2n+\alpha+\beta+1)$

(8) $(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\,C_n = (n+\alpha)(n+\beta)(2n+\alpha+\beta+2)$

(9) $K_n = (-2)^n\,n!, \quad \lambda_n = n(n+\alpha+\beta+1), \quad \alpha_n = r_n,$

$(2n+\alpha+\beta)\,\beta_n = 2(n+\alpha)(n+\beta)$

(iii) *Rodrigues' formula.*

(10) $2^n\,n!\,P_n^{(\alpha,\beta)}(x) = (-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}D^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$

Recurrence formula:

(11) $2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\,P_{n+1}^{(\alpha,\beta)}(x)$

$= (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+\alpha^2-\beta^2]\,P_n^{(\alpha,\beta)}(x)$

$-\,2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\,P_{n-1}^{(\alpha,\beta)}(x).$

From (10) we obtain the explicit expression

(12) $P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{m=0}^{n}\binom{n+\alpha}{m}\binom{n+\beta}{n-m}(x-1)^{n-m}(x+1)^m$

which shows that

(13) $P_n^{(\alpha,\beta)}(-x) = (-1)^n\,P_n^{(\beta,\alpha)}(x).$

Differential equation:

(14) $(1-x^2)\,y'' + [\beta-\alpha-(\alpha+\beta+2)\,x]\,y' + n(n+\alpha+\beta+1)\,y = 0.$

Differentiation formula:

$$(15)\quad (2n+\alpha+\beta)(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)$$

$$= n[(\alpha-\beta)-(2n+\alpha+\beta)x]P_n^{(\alpha,\beta)}(x)+2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x).$$

(iv) *Hypergeometric functions*. Equation (14) can be reduced to the hypergeometric differential equation 2.1(1), and the Jacobi polynomial is that solution of (14) which is regular and has the value (3) at $x = 1$. From the formulas of sec. 2.9,

$$(16)\quad P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}-\tfrac{1}{2}x)$$

$$= (-1)^n \binom{n+\beta}{n} F(-n, n+\alpha+\beta+1; \beta+1; \tfrac{1}{2}+\tfrac{1}{2}x)$$

$$= \binom{n+\alpha}{n} (\tfrac{1}{2}+\tfrac{1}{2}x)^n F\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right)$$

$$= \binom{n+\beta}{n} (\tfrac{1}{2}x-\tfrac{1}{2})^n F\left(-n, -n-\alpha; \beta+1; \frac{x+1}{x-1}\right).$$

From this we find the further differentiation formula

$$(17)\quad 2^m D^m P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)_m P_{n-m}^{(\alpha+m,\beta+m)}(x) \qquad m = 1, 2, \dots, n$$

which confirms statement (i) of sec. 10.6.

It follows from 2.9(14) that the function $Q_n^{(\alpha,\beta)}(x)$ defined by

$$(18)\quad \Gamma(2n+\alpha+\beta+2)\, Q_n^{(\alpha,\beta)}(x) = \frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(x-1)^{n+\alpha+1}(x+1)^{\beta}}$$

$$\times F[n+1, n+\alpha+1; 2n+\alpha+\beta+2; 2(1-x)^{-1}]$$

is a second solution of (14). It is known as the *Jacobi function of the second kind*. This function is not a polynomial, but it satisfies the same recurrence formula (11), and the differentiation formula (15), as the Jacobi polynomial (except that $n = 0$ is not admissible with the Q); it vanishes at infinity when $\mathrm{Re}(\alpha+\beta) > -n-1$. For the various transformations of the hypergeometric series in (18), and for its analytic continuation, see sec. 2.1.4.

Jacobi polynomials and Jacobi functions of the second kind are connected by several relations. From the connection between various solutions of the hypergeometric equation, see sec. 2.9, we have

(19) $$Q_n^{(\alpha,\beta)}(x) = -\tfrac{1}{2}\pi \operatorname{cosec}(\alpha\pi)\, P_n^{(\alpha,\beta)}(x) + 2^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} (x-1)^{-\alpha}(x+1)^{-\beta} \times F(n+1, -n-\alpha-\beta; 1-\alpha; \tfrac{1}{2}-\tfrac{1}{2}x).$$

There is also the integral relation

(20) $$Q_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(x-1)^{-\alpha}(x+1)^{-\beta} \int_{-1}^{1} (x-t)^{-1}(1-t)^{\alpha}(1+t)^{\beta} P_n^{(\alpha,\beta)}(t)\,dt$$

valid for all points x in the complex plane cut along the segment $(-1, 1)$. This segment is a branchcut, and $Q_n^{(\alpha,\beta)}$ assumes different values according as x approaches a point ξ on the branchcut from the upper half-plane $(\xi+i0)$ or from the lower half-plane $(\xi-i0)$. The values of $Q_n^{(\alpha,\beta)}(\xi \pm i0)$ may be computed from (19), taking $\arg(x-1) = \pi$ for $x = \xi + i0$, and $\arg(x-1) = -\pi$ for $x = \xi - i0$. In particular,

(21) $$Q_n^{(\alpha,\beta)}(\xi+i0) - Q_n^{(\alpha,\beta)}(\xi-i0) = -i\, 2^{\alpha+\beta} \sin(\alpha\pi) \frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} (1-\xi)^{-\alpha}(1+\xi)^{-\beta} \times F(n+1, -n-\alpha-\beta; 1-\alpha; \tfrac{1}{2}-\tfrac{1}{2}\xi) \qquad -1 < \xi < 1.$$

On the cut itself, one may use the function

(22) $$\mathrm{Q}_n^{(\alpha,\beta)}(\xi) = \tfrac{1}{2}[Q_n^{(\alpha,\beta)}(\xi+i0) + Q_n^{(\alpha,\beta)}(\xi-i0)] \qquad -1 < \xi < 1$$

which is real when α and β are real. From (19)

(23) $$\mathrm{Q}_n^{(\alpha,\beta)}(\xi) = -\tfrac{1}{2}\pi \operatorname{cosec}(\alpha\pi)\, P_n^{(\alpha,\beta)}(\xi) + 2^{\alpha+\beta-1}\cos(\alpha\pi) \frac{\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} (1-\xi)^{-\alpha}(1+\xi)^{-\beta} \times F(n+1, -n-\alpha-\beta; 1-\alpha; \tfrac{1}{2}-\tfrac{1}{2}\xi) \qquad -1 < \xi < 1.$$

Jacobi functions of the second kind are also connected with the polynomials

(24) $$q_n^{(\alpha,\beta)}(x) = \int_{-1}^{1} (t-x)^{-1}(1-t)^{\alpha}(1+t)^{\beta} [P_n^{(\alpha,\beta)}(t) - P_n^{(\alpha,\beta)}(x)]\,dt$$

associated with Jacobi polynomials according to 10.5(6), for clearly (20) may be rewritten as

(25) $$Q_n^{(\alpha,\beta)}(x) = -\tfrac{1}{2}(x-1)^{-\alpha}(x+1)^{-\beta} q_n^{(\alpha,\beta)}(x) + Q_0^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x).$$

Other relations connecting the P and Q are

$$(26)\quad P_n^{(\alpha,\beta)}(x)\, Q_{n-1}^{(\alpha,\beta)}(x) - P_{n-1}^{(\alpha,\beta)}(x)\, Q_n^{(\alpha,\beta)}(x)$$

$$= 2^{\alpha+\beta-1}\,(2n+\alpha+\beta)\frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)}{n!\,\Gamma(n+\alpha+\beta+1)}\,(x-1)^{-\alpha}\,(x+1)^{-\beta}$$

$$(27)\quad P_n^{(\alpha,\beta)}(x)\frac{d}{dx}\,Q_n^{(\alpha,\beta)}(x) - Q_n^{(\alpha,\beta)}(x)\,\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x)$$

$$= -2^{\alpha+\beta}\,\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)}\,(x-1)^{-\alpha-1}\,(x+1)^{-\beta-1},$$

and from these it follows that $Q_n^{(\alpha,\beta)}$ satisfies the same differentiation formula (15) as $P_n^{(\alpha,\beta)}$.

From the theory of hypergeometric functions one obtains *integral representations* for $Q_n^{(\alpha,\beta)}$. The simplest of these is

$$(28)\quad Q_n^{(\alpha,\beta)}(x) = 2^{-n-1}(x-1)^{-\alpha}(x+1)^{-\beta}$$

$$\times \int_{-1}^{1}(x-t)^{-n-1}\,(1-t)^{n+\alpha}\,(1+t)^{n+\beta}\,dt,$$

valid when x is in the complex plane cut along the segment $(-1, 1)$.

(v) *Generating function.*

$$(29)\quad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, z^n = 2^{\alpha+\beta} R^{-1}(1-z+R)^{-\alpha}\,(1+z+R)^{-\beta} \qquad |z|<1$$

where

$$(30)\quad R = (1-2xz+z^2)^{1/2}$$

and $R = 1$ when $z = 0$. For several ways of proving (29) see Szegö (1939, sec. 4.4). For particular values of α, β there are other generating functions.

(vi) *Integral representations.* From Rodrigues' formula (10), we have

$$(31)\quad P_n^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\int_{(x+)}\left(\frac{1}{2}\,\frac{t^2-1}{t-x}\right)^n \left(\frac{1-t}{1-x}\right)^{\alpha}\left(\frac{1+t}{1+x}\right)^{\beta} dt$$

where $x \neq \pm 1$, the contour of integration is a simple closed contour, in the positive sense, around $t = x$. The points $t = \pm 1$ are outside the contour, and $[(1-t)/(1-x)]^{\alpha}$ and $[(1+t)/(1+x)]^{\beta}$ are to be taken as unity when $t = x$.

Further integral representations may be obtained, from integrals representing hypergeometric functions, by means of (16).

(vii) *Miscellaneous results*. We may apply Christoffel's formula 10.3(12) to $w(x) = (1-x)^\alpha (1+x)^\beta$, $\rho(x) = (1-x)$. In virtue of (3) we obtain

$$(32)\quad (n + \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + 1)(1-x) P_n^{(\alpha+1,\beta)}(x) = (n+\alpha+1) P_n^{(\alpha,\beta)}(x) - (n+1) P_{n+1}^{(\alpha,\beta)}(x)$$

and similarly

$$(33)\quad (n + \tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + 1)(1+x) P_n^{(\alpha,\beta+1)}(x) = (n+\beta+1) P_n^{(\alpha,\beta)}(x) + (n+1) P_{n+1}^{(\alpha,\beta)}(x).$$

These are examples of relations between contiguous hypergeometric functions (see 2.8(31) to 2.8(45)): other relations of this nature are

$$(34)\quad (1-x) P_n^{(\alpha+1,\beta)}(x) + (1+x) P_n^{(\alpha,\beta+1)}(x) = 2 P_n^{(\alpha,\beta)}(x)$$

$$(35)\quad (2n+\alpha+\beta) P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta) P_n^{(\alpha,\beta)}(x) - (n+\beta) P_{n-1}^{(\alpha,\beta)}(x)$$

$$(36)\quad (2n+\alpha+\beta) P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta) P_n^{(\alpha,\beta)}(x) + (n+\alpha) P_{n-1}^{(\alpha,\beta)}(x)$$

$$(37)\quad P_n^{(\alpha,\beta-1)}(x) - P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x).$$

Repeated application of these formulas results in the expression of $P_n^{(\alpha+h,\beta+k)}(x)$ for any integers h, k in terms of $P_m^{(\alpha,\beta)}(x)$.

From Rodrigues' formula (10) we have

$$(38)\quad 2n \int_0^x (1-y)^\alpha (1+y)^\beta P_n^{(\alpha,\beta)}(y)\, dy = P_{n-1}^{(\alpha+1,\beta+1)}(0) - (1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

Toscano (1949) found a counterpart of Rodrigues' formula in terms of finite differences. We define the difference operator by

$$(39)\quad \Delta_\alpha F(\alpha) = F(\alpha+1) - F(\alpha), \qquad \Delta_\alpha^n F = \Delta_\alpha(\Delta_\alpha^{n-1} F)$$

and write Toscano's result in the form

$$(40)\quad n!\,\Gamma(\alpha+\beta+n+1) P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n \Gamma(\alpha+n+1)}{(\tfrac{1}{2}-\tfrac{1}{2}x)^{\alpha+1}}\, \Delta_\alpha^n \left[\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+1)} (\tfrac{1}{2}-\tfrac{1}{2}x)^{\alpha+1}\right].$$

Lastly we quote the important limit

$$(41)\quad \lim_{n\to\infty} \left[n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right)\right] = \lim_{n\to\infty} \left[n^{-\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{z^2}{2n^2}\right)\right] = (\tfrac{1}{2}z)^{-\alpha} J_\alpha(z)$$

where J_α is the Bessel function of the first kind. This formula holds for arbitrary α and β, uniformly in any bounded region of the complex z-plane.

10.9. Gegenbauer polynomials

We use Gegenbauer's notation $C_n^\lambda(x)$ for the suitably standardized polynomials associated with

(1) $a = -1, \quad b = 1, \quad w(x) = (1 - x^2)^{\lambda - \frac{1}{2}}, \quad X = 1 - x^2.$

These polynomials are also known as *ultraspherical* polynomials and are often denoted by $P_n^{(\lambda)}(x)$. Clearly, Gegenbauer polynomials are constant multiples of Jacobi polynomials with $\alpha = \beta = \lambda - \frac{1}{2}$. In order to have a real and integrable weight function we assume

(2) $\lambda > -\frac{1}{2}$,

although many of the formal relations are valid without this restriction. For these polynomials see also sec. 3.15.

(i) *Standardization.*

(3) $$C_n^\lambda(1) = \binom{n + 2\lambda - 1}{n} = \frac{(2\lambda)_n}{n!}.$$

By comparison with 10.8(3)

(4) $$(\lambda + \tfrac{1}{2})_n \, C_n^\lambda(x) = (2\lambda)_n \, P_n^{(\alpha, \alpha)}(x) \qquad \alpha = \lambda - \tfrac{1}{2}.$$

The standardization (3) fails when 2λ is zero or a negative integer. The only exception in the range (2) is $\lambda = 0$ and for this we standardize according to

(5) $$C_0^0(1) = 1, \quad C_n^0(1) = \frac{2}{n} \qquad n = 1, 2, \dots,$$

and have

(6) $$C_n^0(x) = \lim_{\lambda \to \infty} \lambda^{-1} C_n^\lambda(x) = 2\frac{(n-1)!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x).$$

In many formulas of this section $\lambda = 0$ must be excluded. This case will be considered in sec. 10.10.

(ii) *Constants.*

(7) $$(n + \lambda)\, n!\, \Gamma(\lambda)\, h_n = \pi^{\frac{1}{2}} (2\lambda)_n \, \Gamma(\lambda + \tfrac{1}{2})$$

(8) $$n!\, k_n = 2^n (\lambda)_n, \quad r_n = 0, \quad (2\lambda)_n K_n = (-2)^n (\lambda + \tfrac{1}{2})_n$$

(9) $$(n + 1) A_n = 2(n + \lambda), \quad B_n = 0, \quad (n + 1) C_n = n + 2\lambda - 1$$

(10) $\lambda_n = n(n+2\lambda), \quad \alpha_n = 0, \quad \beta_n = n + 2\lambda - 1.$

(iii) *Rodrigues' formula.*

(11) $2^n n!(\lambda+\tfrac{1}{2})_n (1-x^2)^{\lambda-\frac{1}{2}} C_n^\lambda(x) = (-1)^n (2\lambda)_n D^n[(1-x^2)^{n+\lambda-\frac{1}{2}}]$

(12) $C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x.$

Recurrence formula

(13) $(n+1)\, C_{n+1}^\lambda(x) = 2(n+\lambda)\, x C_n^\lambda(x) - (n+2\lambda-1)\, C_{n-1}^\lambda(x).$

Differential equation

(14) $(1-x^2)\, y'' - (2\lambda+1)\, xy' + n(n+2\lambda)\, y = 0.$

Differentiation formula

(15) $$(1-x^2)\frac{d}{dx} C_n^\lambda(x) = -nx\, C_n^\lambda(x) + (n+2\lambda-1)\, C_{n-1}^\lambda(x)$$
$$= (n+2\lambda)\, x C_n^\lambda(x) - (n+1)\, C_{n+1}^\lambda(x).$$

Parity

(16) $C_n^\lambda(-x) = (-1)^n C_n^\lambda(x).$

Explicit representations

(17) $$C_n^\lambda(\cos\theta) = \sum_{m=0}^{n} \frac{(\lambda)_m (\lambda)_{n-m}}{m!(n-m)!} \cos(n-2m)\theta$$

(18) $$C_n^\lambda(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m}}{m!\,(n-2m)!} (2x)^{n-2m}$$

(19) $$C_n^\lambda(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^m (\lambda)_m/m! & \text{if } n = 2m \text{ is even.} \end{cases}$$

(iv) *Hypergeometric functions.* The differential equation (14) can be reduced to the hypergeometric equation, and $C_n^\lambda(x)$ is that solution which is regular at $x = 1$ and has the value (3) there. Moreover, in the case of Gegenbauer polynomials the hypergeometric series in question admit of quadratic transformation, see sec. 2.1.5, and we obtain the following representations:

(20) $n!\, C_n^{\lambda}(x) = (2\lambda)_n\, F\,(-n,\, n+2\lambda;\, \lambda+\tfrac{1}{2};\, \tfrac{1}{2}-\tfrac{1}{2}x)$

$$= (-1)^n (2\lambda)_n\, F(-n,\, n+2\lambda;\, \lambda+\tfrac{1}{2};\, \tfrac{1}{2}+\tfrac{1}{2}x)$$

$$= 2^n (\lambda)_n (x-1)^n F\left(-n,\, -n-\lambda+\frac{1}{2};\, -2n-2\lambda+1;\, \frac{2}{1-x}\right)$$

$$= (2\lambda)_n \left(\frac{1}{2}+\frac{1}{2}x\right)^n F\left(-n,\, -n-\lambda+\frac{1}{2};\, \lambda+\frac{1}{2};\frac{x-1}{x+1}\right)$$

(21) $C_{2m}^{\lambda}(x) = (-1)^m \dfrac{(\lambda)_m}{m!}\, F(-m,\, m+\lambda;\, \tfrac{1}{2};\, x^2)$

$$=\frac{(2\lambda)_{2m}}{(2m)!}\, F(-m,\, m+\lambda;\, \lambda+\tfrac{1}{2};\, 1-x^2)$$

$$=\frac{(\lambda)_m}{(\tfrac{1}{2})_m}\, P_m^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

(22) $C_{2m+1}^{\lambda}(x) = (-1)^m \dfrac{(\lambda)_{m+1}}{m!}\, 2x\, F\left(-m,\, m+\lambda+1;\dfrac{3}{2};\, x^2\right)$

$$=\frac{(2\lambda)_{2m+1}}{(2m+1)!}\, x\, F(-m,\, m+\lambda+1;\, \lambda+\tfrac{1}{2};\, 1-x^2)$$

$$=\frac{(\lambda)_{m+1}}{(\tfrac{1}{2})_{m+1}}\, x\, P_m^{(\lambda-\frac{1}{2},\frac{1}{2})}(2x^2-1).$$

From these representations in conjunction with (13) and (19) one obtains

(23) $D^m\, C_n^{\lambda}(x) = 2^m\, (\lambda)_m\, C_{n-m}^{\lambda+m}(x)$ $\qquad m = 1, 2, \ldots, n$

(24) $D\,C_{n-1}^{\lambda}(x) = x\,D\,C_n^{\lambda}(x) - n\,C_n^{\lambda}(x)$

(25) $D\,C_{n+1}^{\lambda}(x) = x\,D\,C_n^{\lambda}(x) + (n+2\lambda)\,C_n^{\lambda}(x)$

(26) $2(n+\lambda)\int C_n^{\lambda}(x)\,dx = C_{n+1}^{\lambda}(x) - C_{n-1}^{\lambda}(x)$

(27) $D\,C_n^{\lambda}(0) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2(-1)^m\,(\lambda)_{m+1}/m! & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$

A second solution of the differential equation (14) can be obtained from the work of 10.8(iv) by means of the connection, (4), (6), (21), or (22), between Gegenbauer and Jacobi polynomials. No generally accepted notation or standardization seems to exist in this case.

(v) *Generating functions.* From 10.8(29),

$$(28)\quad \sum_{n=0}^{\infty} \frac{(\lambda+\tfrac{1}{2})_n}{(2\lambda)_n} C_n^{\lambda}(x)\, z^n = 2^{\lambda-\frac{1}{2}} R^{-1}(1-xz+R)^{\frac{1}{2}-\lambda}$$

$$|z|<1, \quad R=(1-2xz+z^2)^{\frac{1}{2}}, \quad R=1 \text{ when } z=0;$$

but in this case there is a simpler generating function, viz.

$$(29)\quad \sum_{n=0}^{\infty} C_n^{\lambda}(x)\, z^n = (1-2xz+z^2)^{-\lambda} \qquad |z|<1$$

which can be verified by putting $x = \cos\theta$, factorizing the right-hand side as $(1-e^{i\theta}z)^{-\lambda}(1-e^{-i\theta}z)^{-\lambda}$, expanding in the binomial series, and using (17). A third generating function

$$(30)\quad \sum_{n=0}^{\infty} C_n^{\lambda}(x) \frac{z^n}{(2\lambda)_n} = \Gamma(\lambda+\tfrac{1}{2})\, e^{z\cos\theta}(\tfrac{1}{2}z\sin\theta)^{\frac{1}{2}-\lambda} J_{\lambda-\frac{1}{2}}(z\sin\theta)$$

is connected with (29) by means of the Laplace transformation.

(vi) *Integral representations.* Each of the generating functions leads to a contour integral representation of Gegenbauer polynomials. In addition, we have the real integrals

$$(31)\quad C_n^{\lambda}(x) = \frac{2^{1-2\lambda}\Gamma(2\lambda+n)}{n!\,[\Gamma(\lambda)]^2} \int_0^{\pi} [x+(x^2-1)^{\frac{1}{2}}\cos\varphi]^n (\sin\varphi)^{2\lambda-1}\, d\varphi$$

$$(32)\quad C_n^{\lambda}(\cos\theta) = \frac{2^{\lambda}\Gamma(\lambda+\tfrac{1}{2})(2\lambda)_n}{\pi^{\frac{1}{2}}\, n!\,\Gamma(\lambda)} (\sin\theta)^{1-2\lambda} \int_0^{\theta} \frac{\cos(n+\lambda)\varphi}{(\cos\varphi-\cos\theta)^{1-\lambda}}\, d\varphi,$$

both for $\lambda > 0$. For (31) see 3.15(22) and Seidel and Szász (1950). Equation (32) is *Mehler's integral* 3.15(23); there is a second integral obtained by replacing ϕ and θ by $\pi-\phi$ and $\pi-\theta$ respectively. Mehler's integral suggests a functional transformation which will carry ultraspherical polynomials into powers.

(vii) *Miscellaneous results.* From the connection with Legendre functions,

$$(33)\quad n!\, C_n^{\lambda}(x) = \Gamma(\lambda+\tfrac{1}{2})(2\lambda)_n\, [\tfrac{1}{4}(x^2-1)]^{\frac{1}{4}-\frac{1}{2}\lambda} P_{n+\lambda-\frac{1}{2}}^{\frac{1}{2}-\lambda}(x),$$

we have the addition theorem

(34) $C_n^\lambda(\cos\theta\cos\psi+\sin\theta\sin\psi\cos\varphi)$

$$=\sum_{m=0}^{n} 2^m(2\lambda+2m-1)(n-m)!\,\frac{[(\lambda)_m]^2}{(2\lambda-1)_{n+m+1}}$$

$$\times(\sin\theta)^m\,C_{n-m}^{\lambda+m}(\cos\theta)\,(\sin\psi)^m\,C_{n-m}^{\lambda+m}(\cos\psi)\,C_m^{\lambda-\frac12}(\cos\varphi).$$

Relations between contiguous hypergeometric functions are

(35) $2\lambda(1-x^2)\,C_{n-1}^{\lambda+1}(x)=(2\lambda+n-1)\,C_{n-1}^{\lambda}(x)-nx\,C_n^{\lambda}(x)$

$$=(n+2\lambda)x\,C_n^{\lambda}(x)-(n+1)\,C_{n+1}^{\lambda}(x)$$

(36) $(n+\lambda)\,C_{n+1}^{\lambda-1}(x)=(\lambda-1)\,[C_{n+1}^{\lambda}(x)-C_{n-1}^{\lambda}(x)].$

The differentiation formula

(37) $(x^2-1)^{\lambda+\frac12 n}\,D^n[(x^2-1)^{-\lambda}]=(-1)^n\,n!\,C_n^{\lambda}[x(x^2-1)^{-\frac12}]$

follows from (11) and a linear transformation of the hypergeometric series in (21) and (22). It is due to Tricomi (1949). We note also Gegenbauer's integral

(38) $n!\int_0^{\pi} e^{iz\cos\theta}\,C_n^{\lambda}(\cos\theta)(\sin\theta)^{2\lambda}d\theta$

$$=2^{\lambda}\pi^{\frac12}\,\Gamma(\lambda+\tfrac12)(2\lambda)_n\,i^n\,z^{-\lambda}J_{\lambda+n}(z)$$

and the expansion in a trigonometric series

(39) $$\Gamma(\lambda)C_n^{\lambda}(\cos\theta)=2\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}\,\frac{\Gamma(n+m+2\lambda)}{\Gamma(n+m+\lambda+1)}\cos[(n+2m+2\lambda)\theta-\lambda\pi]$$

$$0<\lambda<1,\qquad 0<\theta<\pi$$

(Şzegö, 1939, p. 95).

10.10. Legendre polynomials

Legendre polynomials $P_n(x)$ are the suitably standardized polynomials associated with

(1) $a=-1,\quad b=1,\quad w(x)=1,\quad X=1-x^2.$

These polynomials are also known as *spherical polynomials*. Clearly they are Jacobi polynomials with $\alpha=\beta=0$, and also Gegenbauer polynomials with $\lambda=\frac12$. Legendre polynomials, and more generally Legendre functions have been studied extensively (cf. chapter III).

(i) *Standardization.*

(2) $P_n(1) = 1.$

Hence

(3) $P_n(x) = C_n^{1/2}(x) = P_n^{(0,0)}(x).$

(ii) *Constants.*

(4) $h_n = (n + \tfrac{1}{2})^{-1}, \quad k_n = 2^n g_n = 2^n \dfrac{(\tfrac{1}{2})_n}{n!}, \quad r_n = 0$

(5) $K_n = (-2)^n n!, \quad (n+1) A_n = 2n + 1, \quad B_n = 0, \quad (n+1) C_n = -n$

(6) $\lambda_n = n(n+1), \quad \alpha_n = 0, \quad \beta_n = n.$

(iii) *Rodrigues' formula.*

(7) $2^n n! P_n(x) = D^n[(x^2 - 1)^n]$

(8) $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \dfrac{3}{2} x^2 - \dfrac{1}{2}.$

Recurrence formula

(9) $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x).$

Christoffel-Darboux formula

(10) $\sum_{m=0}^{n} (2m+1) P_m(x) P_m(y) = \dfrac{n+1}{x-y} [P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)].$

Differential equation

(11) $(1 - x^2) y'' - 2xy' + n(n+1) y = 0.$

Differentiation and integration formulas

(12) $(1 - x^2) P_n'(x) = n [P_{n-1}(x) - x P_n(x)] = (n+1) [x P_n(x) - P_{n+1}(x)]$

(13) $x P_n'(x) - P_{n-1}'(x) = n P_n(x)$

(14) $P_{n+1}'(x) - x P_n'(x) = (n+1) P_n(x)$

(15) $(2n+1) \int P_n(x)\, dx = P_{n+1}(x) - P_{n-1}(x).$

In these formulas $P_n'(x) = dP_n(x)/dx.$

Explicit representations, parity, special values

(16) $P_n(x) = 2^{-n} \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{m} \binom{2n-2m}{n} x^{n-2m}$

(17) $P_n(\cos\theta) = \sum_{m=0}^{n} g_m\, g_{n-m} \cos(n-2m)\theta$

(18) $P_n(-x) = (-1)^n P_n(x), \quad P_n(\pm 1) = (\pm 1)^n$

(19) $P_{2m}(0) = (-1)^m g_m, \quad P_{2m+1}(0) = 0$

(20) $P'_{2m}(0) = 0, \quad P'_{2m+1}(0) = (-1)^m (2m+1) g_m$.

Here

(21) $g_m = \frac{(\frac{1}{2})_m}{m!} = 2^{-2m} \binom{2m}{m}$.

(iv) *Hypergeometric functions*. See also 10.9(iv).

(22) $P_n(x) = F(-n, n+1; 1; \frac{1}{2} - \frac{1}{2}x)$
$= 2^n g_n x^n F(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; \frac{1}{2}; x^{-2})$

(23) $P_n(\cos\theta) = F(-n, n+1; 1; \sin^2 \frac{1}{2}\theta) = (-1)^n F(-n, n+1; 1; \cos^2 \frac{1}{2}\theta)$

(24) $P_{2m}(x) = (-1)^m g_m F(-m, m+\frac{1}{2}; \frac{1}{2}; x^2)$

(25) $P_{2m+1}(x) = (-1)^m (2m+1) g_m\, x F\left(-m, m+\frac{3}{2}; \frac{3}{2}; x^2\right)$

(26) $\frac{d^m}{dx^m} P_n(x) = 2^m\, m!\, g_m\, C_{n-m}^{m+\frac{1}{2}}(x) \qquad n \geq m.$

Information about a second solution of Legendre's differential equation (11) may be obtained from 10.8(iv). Such a second solution is the *Legendre function of the second kind*

(27) $Q_n(x) = Q_n^{(0,0)}(x)$.

In the complex x-plane cut along the segment $(-1, 1)$

(28) $2^{-n}(2n+1)!\,(n!)^{-2} Q_n(x)$
$= (x-1)^{-n-1} F[n+1, n+1; 2n+2; 2(1-x)^{-1}]$
$= (x+1)^{-n-1} F[n+1, n+1; 2n+2; 2(1+x)^{-1}]$
$= x^{-n-1} F(1/2 + n/2, 1 + n/2; 3/2 + n; x^{-2})$.

The Legendre function of the second kind is not a polynomial: it satisfies the same recurrence relation (9) and the same differentiation formulas (12) - (15) as the Legendre polynomials, except that $n = 0$ is inadmissible in these formulas when written for Q.

(29) $Q_n(-x) = (-1)^{n+1} Q_n(x)$

(30) $Q_0(x) = \frac{1}{2} \log \frac{x+1}{x-1}, \quad Q_1(x) = \frac{1}{2} x \log \frac{x+1}{x-1} - 1$

(31) $Q_n(x) = 2^{-n-1} \int_{-1}^{1} (1-t^2)^n (x-t)^{-n-1}\, dt$

(32) $Q_n(x) = \int_0^\infty [x + (x^2-1)^{1/2} \cosh t]^{-n-1}\, dt$

(33) $Q_n(\cosh \zeta) = \int_\zeta^\infty [2(\cosh z - \cosh \zeta)]^{-1/2} e^{-(n+1/2)z}\, dz$

$\operatorname{Re} z \geq \operatorname{Re} \zeta, \quad \operatorname{Im} z = \operatorname{Im} \zeta$

(34) $Q_n(x) = \frac{1}{2} \int_{-1}^{1} (x-t)^{-1} P_n(t)\, dt.$

(35) $Q_n(x) = Q_0(x) P_n(x) - \sum_{k=1}^{[(n+1)/2]} \frac{2n-4k+3}{(2k-1)(n-k+1)} P_{n-2k+1}(x).$

The last formula is equivalent to the special case $\alpha = \beta = 0$ of 10.8.25; for the proof in the form (35) see Hobson (1931, pp. 53-54). The point at infinity is a zero of multiplicity $n + 1$ of $Q_n(x)$; and this function has no other zero in the cut x-plane.

The segment of the real axis from -1 to 1 is a branchcut of $Q_n(x)$, and

(36) $Q_n(\xi + i0) - Q_n(\xi - i0) = -\pi i P_n(\xi) \qquad -1 < \xi < 1.$

On the branchcut we may define a second solution of Legendre's equation by

(37) $\mathrm{Q}_n(\xi) = \frac{1}{2} Q_n(\xi + i0) + \frac{1}{2} Q_n(\xi - i0) \qquad -1 < \xi < 1.$

We then have

(38) $\mathrm{Q}_n(\xi) = \frac{1}{2} ⨍_{-1}^{1} (\xi - t)^{-1} P_n(t)\, dt \qquad -1 < \xi < 1$

where the integral is a Cauchy principal value, that is

$\lim \left(\int_{-1}^{\xi-\epsilon} + \int_{\xi+\epsilon}^{1}\right) \quad \text{as} \quad \epsilon > 0, \ \epsilon \to 0.$

(v) *Generating functions.*

(39) $$\sum_{n=0}^{\infty} P_n(x)\, z^n = (1-2xz+z^2)^{-\frac{1}{2}} \qquad -1 < x < 1, \quad |z| < 1$$

(40) $$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(\cos\theta)\, z^n = e^{z\cos\theta} J_0(z \sin\theta)$$

(41) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+\frac{1}{2}} P_n(\cos\theta)\, x^{2n+1} = F(\sin \tfrac{1}{2}\theta, \varphi)$$
$$x = \tan \tfrac{1}{2}\varphi, \quad 0 < \varphi < \tfrac{1}{2}\pi, \quad 0 < \theta < \pi.$$

The first two formulas are particular cases of 10.9(29) and 10.9(30). The last formula may be derived from (39), and $F(k, \varphi)$ denotes Legendre's incomplete elliptic integral of the first kind with modulus k

(vi) *Integral representations.*

(42) $$P_n(\cos\theta) = \pi^{-1} \int_0^{\pi} (\cos\theta + i \sin\theta \cos\varphi)^n \, d\varphi$$
$$= \pi^{-1} \int_0^{\pi} (\cos\theta + i\sin\theta\cos\varphi)^{-n-1} \, d\varphi$$

(43) $$P_n(\cos\theta) = 2^{\frac{1}{2}} \pi^{-1} \int_0^{\theta} (\cos\varphi - \cos\theta)^{-\frac{1}{2}} \cos(n+\tfrac{1}{2})\varphi \, d\varphi \qquad 0 < \theta < \pi$$

(44) $$P_n(x) = (2\pi i)^{-1} \int^{(0+)} (1-2xz+z^2)^{-\frac{1}{2}} z^{-n-1} \, dz$$

(45) $$P_n(x) = (-2)^{-n} (2\pi i)^{-1} \int^{(x+)} (1-z^2)^n (z-x)^{-n-1} \, dz.$$

Equation (44) follows from (39), and (45) from Rodrigues' formula. The integral in (45) is known as Schläfli's integral. Laplace's first and second integral, (42), may be deduced from (45) when the contour of integration is taken to be the circle

$$z = x + (x^2-1)^{\frac{1}{2}} e^{i\varphi} \qquad -\pi \leq \varphi < \pi,$$

and Mehler's integral (43) may be deduced from Laplace's integral (Whittaker and Watson 1940, sections 15.23 and 15.231).

(vii) *Miscellaneous results.* With the notation

(46) $$\mathrm{P}_n^m(\cos\theta) = (-2)^m\, m!\, g_m\, (\sin\theta)^m\, C_{n-m}^{m+\frac{1}{2}}(\cos\theta)$$

for the *associated Legendre function of the first kind* [see 3.4(1) and 3.15(4)] we have from 10.9(34) the *addition theorem* of Legendre polynomials

$$(47)\quad P_n(\cos\theta\cos\psi+\sin\theta\sin\psi\cos\varphi)=P_n(\cos\theta)\,P_n(\cos\psi)$$
$$+\,2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\,P_n^m(\cos\theta)\,P_n^m(\cos\psi)\cos n\varphi.$$

We note the expansion in a trigonometric series

$$(48)\quad P_{n-1}(\cos\theta)=\frac{2}{\pi n g_n}\sum_{m=0}^{\infty}\frac{(n)_m\,g_m}{(n+\tfrac{1}{2})_m}\sin[(n+2m)\theta]\qquad n=2,3,\ldots$$

and the integral formulas

$$(49)\quad \int_{-1}^{1}(1-x)^{-\frac{1}{2}}\,P_n(x)\,dx=\frac{2^{3/2}}{2n+1}$$

$$(50)\quad \int_0^{\pi}P_{2m}(\cos\theta)\,d\theta=\pi g_m^2,\qquad \int_0^{\pi}P_{2m+1}(\cos\theta)\cos\theta\,d\theta=\pi g_m\,g_{m+1}$$

$$(51)\quad \int_0^1 x^{\lambda}\,P_{2m}(x)\,dx=\frac{(-1)^m(-\tfrac{1}{2}\lambda)_m}{2(\tfrac{1}{2}+\tfrac{1}{2}\lambda)_{m+1}}\qquad \mathrm{Re}\,\lambda>-1$$

$$(52)\quad \int_0^1 x^{\lambda}P_{2m+1}(x)\,dx=\frac{(-1)^m(\tfrac{1}{2}-\tfrac{1}{2}\lambda)_m}{2(1+\tfrac{1}{2}\lambda)_{m+1}}\qquad \mathrm{Re}\,\lambda>-2$$

and the bilinear expansion

$$(53)\quad \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\,P_n(x)\,P_n(y)=2\log 2-1-\log[(1-x)(1+y)]$$
$$-1<x\le y<1.$$

10.11. Tchebichef polynomials

Sometimes (especially in the French literature) orthogonal polynomials in general are called Tchebichef polynomials. There are also several special systems of orthogonal polynomials called Tchebichef polynomials. In this chapter we shall reserve the name Tchebichef polynomials of the first and second kind for the suitably standardized orthogonal polynomials associated with

$$(1)\quad a=-1,\qquad b=1,\qquad w(x)=(1-x^2)^{\mp\frac{1}{2}},\qquad X=1-x^2.$$

Clearly these polynomials are multiples of Jacobi polynomials with $\alpha=\beta=-\tfrac{1}{2}$ for the polynomials of the first kind $T_n(x)$ and with $\alpha=\beta=\tfrac{1}{2}$ for the polynomials of the second kind $U_n(x)$. Also, the Jacobi polynomials in question are ultraspherical polynomials with $\lambda=0$ for the polynomials of the first kind and with $\lambda = 1$ for the polynomials of the second kind.

The orthogonal relationship for Tchebichef polynomials of the first kind reads

$$\int_{-1}^{1} T_m(x)\, T_n(x)\, (1 - x^2)^{-\frac{1}{2}}\, dx = 0 \qquad m \neq n.$$

If we substitute $x = \cos\theta$ and note that $\cos n\theta$ is a polynomial of exact degree n in $\cos\theta$, we see that $T_n(x)$ must be a constant multiple of $\cos n\theta$; and we show in a similar manner that $U_n(x)$ is a constant multiple of $\csc\theta \sin(n+1)\theta$. We standardize our polynomials by putting

$$(2) \quad T_n(\cos\theta) = \cos n\theta, \qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Many identities involving Tchebichef polynomials are paraphrases of well-known trigonometric identities. As an example, we mention the connection between the two kinds of Tchebichef polynomials,

$$(3) \quad T_n(x) = U_n(x) - x\, U_{n-1}(x)$$

$$(4) \quad (1 - x^2)\, U_{n-1}(x) = x\, T_n(x) - T_{n+1}(x).$$

Tchebichef polynomials are ultraspherical polynomials with $\lambda = 0, 1$. From 10.9(23) it is seen that $C_n^{\lambda}(x)$ can be expressed as a derivative of a Tchebichef polynomial whenever λ is a positive integer.

(i) *Standardization*. This is given by (2). It follows that

$$(5) \quad T_n(x) = \tfrac{1}{2} n\, C_n^0(x) = (g_n)^{-1} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) \qquad n = 1, 2, \ldots$$

$$(6) \quad U_n(x) = C_n^1(x) = (2g_{n+1})^{-1} P_n^{(\frac{1}{2}, \frac{1}{2})}(x) \qquad n = 0, 1, \ldots$$

where C_n^0 is defined by 10.9(6) and g_n by 10.10(21).

(ii) *Constants*. For $T_n(x)$

$$(7) \quad h_0 = \pi, \quad h_n = \tfrac{1}{2}\pi \qquad n = 1, 2, \ldots$$

$$(8) \quad k_n = 2^{n-1}, \quad r_n = 0, \quad K_n = (-1)^n\, 2^n\, n!\, g_n$$

$$(9) \quad A_n = 2, \quad B_n = 0, \quad C_n = 1$$

$$(10) \quad \lambda_n = n^2, \quad a_n = 0, \quad \beta_n = n.$$

For $U_n(x)$

(11) $h_n = \frac{1}{2}\pi, \quad k_n = 2^n, \quad r_n = 0, \quad K_n = (-1)^n 2^{n+1} n! g_{n+1}$

(12) $A_n = 2, \quad B_n = 0, \quad C_n = 1$

(13) $\lambda_n = n(n+2), \quad \alpha_n = 0, \quad \beta_n = n+1.$

(iii) *Rodrigues' formulas.*

(14) $2^n (\frac{1}{2})_n T_n(x) = (-1)^n (1-x^2)^{\frac{1}{2}} D^n[(1-x^2)^{n-\frac{1}{2}}]$

(15) $2^{n+1} (\frac{1}{2})_{n+1} U_n(x) = (-1)^n (n+1)(1-x^2)^{-\frac{1}{2}} D^n[(1-x^2)^{n+\frac{1}{2}}]$.

Recurrence formula [$z_n(x)$ is either $T_n(x)$ or $U_n(x)$]

(16) $z_{n+1}(x) = 2xz_n(x) - z_{n-1}(x)$.

Christoffel-Darboux formula.

(17) $$\sum_{m=0}^{n}{}' z_m(x) z_m(y) = (x-y)^{-1} [z_{m+1}(x) z_m(y) - z_m(x) z_{m+1}(y)]$$

where z_n is either T_n or U_n, but in the case of T_n, the first term ($m = 0$) of the sum must be halved.

Differential equations.

(18) $(1-x)y'' - xy' + n^2 y = 0$ for $y = T_n(x)$

(19) $(1-x)y'' - 3xy' + n(n+2)y = 0$ for $y = U_n(x)$

Differentiation formulas (primes denote differentiation with respect to x)

(20) $(1-x^2) T_n'(x) = n[T_{n-1}(x) - x T_n(x)]$

(21) $(1-x^2) U_n'(x) = (n+1) U_{n-1}(x) - nx U_n(x)$.

Explicit representations

(22) $$T_n(x) = \frac{n}{2} \sum_{m=0}^{[n/2]} \frac{(-1)^m (n-m-1)!}{m!(n-2m)!} (2x)^{n-2m} \qquad n = 1, 2, \ldots$$

(23) $$U_n(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (n-m)!}{m!(n-2m)!} (2x)^{n-2m}.$$

(iv) *Hypergeometric functions.*

(24) $T_n(x) = F(-n, n; \tfrac{1}{2}; \tfrac{1}{2} - \tfrac{1}{2}x)$

(25) $U_n(x) = (n+1)\, F\left(-n, n+1; \dfrac{3}{2}; \dfrac{1}{2} - \dfrac{x}{2}\right).$

From these relations and from 10.9(iv)

(26) $D^m\, T_n(x) = 2^{m-1}(m-1)!\, n\, C^m_{n-m}(x)$ $\qquad n \geq m$

(27) $D^m\, U_n(x) = 2^m\, m!\, C^{m+1}_{n-m}(x)$ $\qquad n \geq m$

(28) $T_n'(x) = n\, U_{n-1}(x).$

(v) *Generating functions.*

(29) $1 + 2\sum_{n=1}^{\infty} T_n(x)\, z^n = \dfrac{1 - z^2}{1 - 2xz + z^2}$

(30) $1 + 2\sum_{n=1}^{\infty} n^{-1}\, T_n(x)\, z^n = -\log(1 - 2xz + z^2)$

(31) $\sum_{n=0}^{\infty} U_n(x)\, z^n = (1 - 2xz + z^2)^{-1}$

(32) $\sum_{n=0}^{\infty} g_n\, T_n(x)\, z^n = 2^{-1/2} R^{-1} (1 - xz + R)^{1/2}$

(33) $\sum_{n=0}^{\infty} g_{n+1}\, U_n(x)\, z^n = 2^{-1/2} R^{-1} (1 - xz + R)^{-1/2}.$

In all five formulas

$$-1 < x < 1, \qquad |z| < 1.$$

In the last two formulas

$$R = (1 - 2xz + z^2)^{1/2}.$$

Equation (31) is a special case of 10.9(29), and (30) is a limiting case of the same relation: (29) can be derived from (30). $R = 1$ and $\log R^2 = 0$ when $z = 0$. Formulas (32) and (33) are special cases of 10.9(28).

(vi) *Integral representations.* Contour integrals which represent Tchebichef polynomials follow from any of the generating functions.

(vii) *Miscellaneous results.*

(34) $2T_m(x)\,T_n(x) = T_{n+m}(x) - T_{n-m}(x)$ $\quad n \geq m$

(35) $2(x^2-1)\,U_{m-1}(x)\,U_{n-1}(x) = T_{n+m}(x) - T_{n-m}(x)$ $\quad n \geq m$

(36) $2T_m(x)\,U_{n-1}(x) = U_{n+m-1}(x) + U_{n-m-1}(x)$ $\quad n > m$

(37) $2T_n(x)\,U_{m-1}(x) = U_{n+m-1}(x) - U_{n-m-1}(x)$ $\quad n > m$

(38) $2[T_n(x)]^2 = 1 + 2\,T_{2n}(x), \qquad 2T_n(x)\,U_{n-1}(x) = U_{2n-1}(x)$

(39) $2(1-x^2)\,[U_{n-1}(x)]^2 = 1 - 2\,T_{2n}(x)$

(40) $\sum_{m=0}^{n} T_{2m}(x) = \tfrac{1}{2} + \tfrac{1}{2}U_{2n}(x), \qquad \sum_{m=0}^{n-1} T_{2m+1}(x) = \tfrac{1}{2}U_{2n-1}(x)$

(41) $2(1-x^2)\sum_{m=0}^{n} U_{2m}(x) = 1 - T_{2n+2}(x)$

(42) $2(1-x^2)\sum_{m=0}^{n-1} U_{2m+1}(x) = x - T_{2n+1}(x).$

All these formulas are paraphrases of trigonometric identities.

Mehler's integral 10.10(43) may be interpreted as a relation between Legendre and Tchebichef polynomials. Inverting this relationship, Tricomi (1935) found

(43) $(n+\tfrac{1}{2})(1+x)^{\frac{1}{2}} \int_{-1}^{x} (x-t)^{-\frac{1}{2}}\,P_n(t)\,dt = T_n(x) + T_{n+1}(x)$

(44) $(n+\tfrac{1}{2})(1-x)^{\frac{1}{2}} \int_{x}^{1} (t-x)^{-\frac{1}{2}}\,P_n(t)\,dt = T_n(x) - T_{n+1}(x).$

From 10.9(21) and 10.9(22) we obtain

(45) $P_n^{(\frac{1}{2},-\frac{1}{2})}(2x^2-1) = g_n\,U_{2n}(x)$

(46) $x\,P_n^{(-\frac{1}{2},\frac{1}{2})}(2x^2-1) = g_n\,T_{2n+1}(x).$

Finally we note the principal value integrals

(47) $⨍_{-1}^{1} (y-x)^{-1}\,(1-y^2)^{-\frac{1}{2}}\,T_n(y)\,dy = \pi\,U_{n-1}(x)$

(48) $⨍_{-1}^{1} (y-x)^{-1}\,(1-y^2)^{\frac{1}{2}}\,U_{n-1}(y)\,dy = -\pi\,T_n(x)$ $\quad n = 1, 2, \ldots$

which are paraphrases of trigonometric integrals and are of importance in the theory of the integral equation sometimes called the airfoil equation.

10.12. Laguerre polynomials

The polynomials $L_n^\alpha(x)$ are the suitably standardized orthogonal polynomials associated with

(1) $a = 0, \quad b = \infty, \quad w(x) = e^{-x} x^\alpha, \quad X = x \qquad \alpha > -1.$

Instead of $L_n^0(x)$ it is usual to write $L_n(x)$. This is the polynomial introduced by Laguerre. The $L_n^\alpha(x)$ are often called generalized Laguerre polynomials, but we shall call them Laguerre polynomials simply. Equivalent polynomials have also been discussed by Sonine (1880, p. 41).

(i) *Standardization*. We shall adopt the standardization $k_n = (-1)^n/n!$. The standardizations $k_n = (-1)^n$ and, less frequently, $k_n = 1$ are sometimes used.

(ii) *Constants*.

(2) $n!\,h_n = \Gamma(\alpha + n + 1), \quad n!\,k_n = (-1)^n,$

$nr_n = -(n + \alpha), \quad K_n = n!$

(3) $(n+1)A_n = -1, \quad (n+1)B_n = 2n + \alpha + 1, \quad (n+1)C_n = n + \alpha$

(4) $\lambda_n = n, \quad a_n = n, \quad \beta_n = -(n + \alpha).$

(iii) *Relationships*.

(5) $n!\,L_n^\alpha(x) = e^x x^{-\alpha} D^n(e^{-x} x^{n+\alpha})$

(6) $L_0^\alpha(x) = 1, \quad L_1^\alpha(x) = \alpha + 1 - x$

(7) $$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

(8) $(n+1)L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x)L_n^\alpha(x) + (n + \alpha)L_{n-1}^\alpha(x) = 0$

(9) $$\sum_{m=0}^{n} \frac{m!}{\Gamma(m + \alpha + 1)} L_m^\alpha(x) L_m^\alpha(y)$$

$$= \frac{(n+1)!}{\Gamma(n + \alpha + 1)} \frac{1}{x - y} [L_n^\alpha(x) L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x) L_n^\alpha(y)]$$

(10) $xy'' + (\alpha + 1 - x)y' + ny = 0, \quad y = L_n^\alpha(x)$

(11) $$(xz')' + \left(n + \frac{\alpha+1}{2} - \frac{x}{4} - \frac{\alpha^2}{4x}\right)z = 0, \quad z = e^{-\frac{1}{2}x} x^{\frac{1}{2}\alpha} L_n^\alpha(x)$$

$$(12)\quad x\frac{d}{dx}L_n^\alpha(x) = n L_n^\alpha(x) - (n+\alpha) L_{n-1}^\alpha(x)$$

$$= (n+1) L_{n+1}^\alpha(x) - (n+\alpha+1-x) L_n^\alpha(x)$$

$$(13)\quad L_n^\alpha(0) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}.$$

(iv) *Hypergeometric functions.* Laguerre polynomials are connected with the confluent hypergeometric functions of Chapter VI. From the explicit representation (7)

$$(14)\quad L_n^\alpha(x) = \binom{n+\alpha}{n}\,\Phi(-n, \alpha+1; x)$$

$$= \frac{(-1)^n}{n!}\Psi(-n-\alpha, 1-\alpha; x).$$

From this we have

$$(15)\quad \frac{d}{dx}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x)$$

confirming statement (i) of sec. 10.6,

$$(16)\quad \frac{d}{dx}[L_n^\alpha(x) - L_{n+1}^\alpha(x)] = L_n^\alpha(x),$$

and many other formulas which are instances of relations between contiguous confluent hypergeometric functions.

The general solution of Laguerre's differential equation (10) may be obtained from the theory of confluent hypergeometric functions.

(v) *Generating functions.*

$$(17)\quad \sum_{n=0}^{\infty} L_n^\alpha(x) z^n = (1-z)^{-\alpha-1}\exp\frac{xz}{z-1} \qquad |z|<1$$

$$(18)\quad \sum_{n=0}^{\infty}[\Gamma(n+\alpha+1)]^{-1} L_n^\alpha(x) z^n = (xz)^{-\frac{1}{2}\alpha} e^z J_\alpha[2(xz)^{\frac{1}{2}}]$$

$$(19)\quad \sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n = e^{-xz}(1+z)^\alpha \qquad |z|<1$$

$$(20)\quad \sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\alpha+1)} L_n^\alpha(x) L_n^\alpha(y) z^n$$

$$= (1-z)^{-1}\exp\left(-z\frac{x+y}{1-z}\right)(xyz)^{-\frac{1}{2}\alpha} I_\alpha\left[2\frac{(xyz)^{\frac{1}{2}}}{1-z}\right] \qquad |z|<1.$$

The function on the right-hand side of (17) is the most common generating function and may be established by means of (7). Equation (18) is due to Doetsch and follows from (17) by means of the Laplace transformation. Equation (19) follows from (7) and is due to Erdélyi. Equation (20) is a bilinear generating function and is known as the Hille-Hardy formula (see also Myller-Lebedeff, 1907).

(vi) *Integral representations*. Contour integrals representing Laguerre polynomials may be obtained from the Rodrigues formula (5) and from any of the generating functions in an obvious manner. In addition, the connection (14) with confluent hypergeometric functions may be exploited (cf. sec. 6.11). We mention only the following integrals

$$(21)\quad n!\,L_n^{\alpha}(x) = e^{x}\,x^{-\frac{1}{2}\alpha}\int_0^{\infty} e^{-t}\,t^{n+\frac{1}{2}\alpha}\,J_{\alpha}[2(tx)^{\frac{1}{2}}]\,dt$$

$$(22)\quad 2\pi i\,2^{\alpha}\,L_n^{\alpha}(x) = (-1)^n\,e^{\frac{1}{2}x}\int^{(1+)} e^{-\frac{1}{2}xz}\left(\frac{1+z}{1-z}\right)^{k}(1-z^2)^{\frac{1}{2}\alpha-\frac{1}{2}}\,dz$$

$$k = n+\tfrac{1}{2}\alpha+\tfrac{1}{2}.$$

The first of these is a consequence of 6.11(5); and the second is due to Tricomi.

(vii) *Miscellaneous results*. The number of results under this heading is tremendous, and many of them were discovered several times. We give a small selection only and do not attempt to credit the formulas to their original discoverers.

Contiguous polynomials. In addition to the recurrence formula (8) we have

$$(23)\quad x\,L_n^{\alpha+1}(x) = (n+\alpha+1)\,L_n^{\alpha}(x) - (n+1)\,L_{n+1}^{\alpha}(x)$$

$$= (n+\alpha)\,L_{n-1}^{\alpha}(x) - (n-x)\,L_n^{\alpha}(x)$$

$$(24)\quad L_n^{\alpha-1}(x) = L_n^{\alpha}(x) - L_{n-1}^{\alpha}(x)$$

$$(25)\quad (n+\alpha)\,L_n^{\alpha-1}(x) = (n+1)\,L_{n+1}^{\alpha}(x) - (n+1-x)\,L_n^{\alpha}(x).$$

Differentiation formulas and indefinite integrals. In addition to (5) and (12) we have

$$(26)\quad D^n[x^{-\alpha-1}\exp(-x^{-1})] = (-1)^n\,n!\,x^{-\alpha-n-1}\,L_n^{\alpha}(x^{-1})\exp(-x^{-1})$$

$$(27)\quad D^m[x^{\alpha}L_n^{\alpha}(x)] = (n-m+\alpha+1)_m\,x^{\alpha-m}\,L_n^{\alpha-m}(x)$$

$$(28)\quad n!\,D^m[e^{-x}\,x^{\alpha}L_n^{\alpha}(x)] = (m+n)!\,e^{-x}\,x^{\alpha-m}\,L_{m+n}^{\alpha-m}(x)$$

(29) $\int_x^\infty e^{-y} L_n^\alpha(y)\, dy = e^{-x} [L_n^\alpha(x) - L_{n-1}^\alpha(x)]$

(30) $\Gamma(\alpha+\beta+n+1) \int_0^x (x-y)^{\beta-1} y^\alpha L_n^\alpha(y)\, dy$

$$= \Gamma(\alpha+n+1)\, \Gamma(\beta)\, x^{\alpha+\beta} L_n^{\alpha+\beta}(x) \qquad \text{Re}\ \alpha > -1, \quad \text{Re}\ \beta > 0$$

(31) $\int_0^x L_m(y)\, L_n(x-y)\, dy = \int_0^x L_{m+n}(y)\, dy = L_{m+n}(x) - L_{m+n+1}(x).$

Further indefinite integrals follow from the product theorem of Laplace transforms.

Laplace integrals. With the notation

$$\mathcal{L}[F(t)] = \int_0^\infty e^{-st} F(t)\, dt$$

we have

(32) $\mathcal{L}[t^\alpha L_n^\alpha(t)] = \dfrac{\Gamma(\alpha+n+1)(s-1)^n}{n!\, s^{\alpha+n+1}} \qquad \text{Re}\ \alpha > -1, \quad \text{Re}\ s > 0$

(33) $n!\, \Gamma(\alpha+1)\, \mathcal{L}[t^\beta L_n^\alpha(t)]$

$$= \Gamma(\beta+1)\, \Gamma(\alpha+n+1)\, s^{-\beta-1} F(-n, \beta+1; \alpha+1; s^{-1})$$

$$\text{Re}\ \beta > -1, \quad \text{Re}\ s > 0$$

(34) $\mathcal{L}[t^{\frac{1}{2}\alpha+n} J_\alpha(2(kt)^{\frac{1}{2}})] = n!\, k^{\frac{1}{2}\alpha} s^{-\alpha-n-1} e^{-k/s} L_n^\alpha(k/s).$

Limit formulas.

(35) $L_n^\alpha(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1 - 2x/\beta)$

(36) $\lim_{n\to\infty} [n^{-\alpha} L_n^\alpha(x/n)] = x^{-\frac{1}{2}\alpha} J_\alpha(2x^{\frac{1}{2}}).$

Finite difference formula. With the notation

$$\Delta_\alpha f(\alpha) = f(\alpha+1) - f(\alpha), \quad \Delta_\alpha^{n+1} f(\alpha) = \Delta_\alpha(\Delta_\alpha^n f(\alpha)) \qquad n = 1, 2, \ldots$$

we have

$$\Delta_\alpha^n f(\alpha) = \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} f(\alpha+m) \qquad n = 1, 2, \ldots$$

and hence

(37) $L_n^\alpha(x) = (-1)^n \dfrac{\Gamma(\alpha+n+1)}{n!\, x^\alpha}\, \Delta_\alpha^n \dfrac{x^\alpha}{\Gamma(\alpha+1)}.$

Finite sums. In addition to those already recorded we have

(38) $\sum_{m=0}^{n} L_m^{\alpha}(x) = L_n^{\alpha+1}(x) = x^{-1}[(x-n)L_n^{\alpha}(x) + (\alpha+n)L_{n-1}^{\alpha}(x)]$

(39) $L_n^{\alpha}(x) = \sum_{m=0}^{n} (m!)^{-1}(\alpha-\beta)_m L_{n-m}^{\beta}(x)$

(40) $L_n^{\alpha}(\lambda x) = \sum_{m=0}^{n} \binom{n+\alpha}{m} \lambda^{n-m}(1-\lambda)^m L_{n-m}^{\alpha}(x)$

(41) $\sum_{m=0}^{n} L_m^{\alpha}(x) L_{n-m}^{\beta}(y) = L_n^{\alpha+\beta+1}(x+y)$

(42) $n!\, L_n^{\alpha}(x) L_n^{\alpha}(y)$

$$= \Gamma(\alpha+n+1) \sum_{m=0}^{n} [m!\,\Gamma(\alpha+m+1)]^{-1} (xy)^m L_{n-m}^{\alpha+2m}(x+y).$$

Infinite series: generating functions have already been given [(17) to (20)], Bessel function expansions are in sec. 10.15, and other examples of infinite series involving Laguerre polynomials are in sec. 10.20.

10.13. Hermite polynomials

Hermite polynomials are orthogonal polynomials associated with the interval $(-\infty, \infty)$ and an exponential weight function. Unfortunately, the notations adopted by different authors vary a great deal. The simplest form of the exponential weight function seems to be $\exp(-x^2)$, but for applications in mathematical statistics $\exp(-\frac{1}{2}x^2)$ is preferable. Ofthe most important books on the subject, Courant-Hilbert, Doetsch, Sansone, Szegö use $\exp(-x^2)$, and Appell and Kampé de Fériet, Jahnke-Emde, Magnus-Oberhettinger, Pólya-Szegö, and Tricomi use the weight function $\exp(-\frac{1}{2}x^2)$. In this chapter we shall adopt Szegö's (1939) notation and regard Hermite polynomials, $H_n(x)$, as the suitably standardized orthogonal polynomials associated with

(1) $a = -\infty, \quad b = \infty, \quad w(x) = \exp(-x^2), \quad X = 1.$

The orthogonal polynomials associated with the weight function $\exp(-\frac{1}{2}x^2)$ may be denoted by $He_n(x)$. These polynomials may also be expressed in terms of parabolic cylinder functions [see 8.2(9)].

(i) *Standardization.* We shall adopt the standardization $K_n = (-1)^n$. This agrees with the standardization adopted, among other authors, by Courant-Hilbert, Feldheim, Hille, and Szegö. The standardization $K_n = 1$ is used by Doetsch, Erdélyi, Sansone, and others.

Hermite polynomials, so standardized, have been expressed by Szegö and Koschmieder in terms of Laguerre polynomials.

(2) $H_{2m}(x) = (-1)^m\, 2^{2m}\, m!\, L_m^{-\frac{1}{2}}(x^2)$

(3) $H_{2m+1}(x) = (-1)^m\, 2^{2m+1}\, m!\, x\, L_m^{\frac{1}{2}}(x^2)$.

These expressions show that $H_n(x)$ is an even or an odd function of x according as n is even or odd. These formulas are analogous to (and are in fact limiting cases of) 10.9(21) and 10.9(22).

(ii) *Constants.*

(4) $h_n = \pi^{\frac{1}{2}}\, 2^n\, n!, \quad k_n = 2^n, \quad r_n = 0$

(5) $K_n = (-1)^n, \quad A_n = 2, \quad B_n = 0, \quad C_n = 2n$

(6) $\lambda_n = 2n, \quad a_n = 0, \quad \beta_n = 2n$.

(iii) *Relationships.*

(7) $H_n(x) = (-1)^n\, e^{x^2}\, D^n\, e^{-x^2}$

(8) $H_0(x) = 1, \quad H_1(x) = 2x$

(9) $$H_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m\, (2x)^{n-2m}}{m!\,(n-2m)!}.$$

Here $[n/2]$ is $n/2$ or $(n-1)/2$ according as n is even or odd.

(10) $H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$

(11) $$\sum_{m=0}^{n} \frac{H_m(x)\, H_m(y)}{2^m\, m!} = \frac{H_{n+1}(x)\, H_n(y) - H_n(x)\, H_{n+1}(y)}{2^{n+1}\, n!\,(x-y)}$$

(12) $y'' - 2xy' + 2ny = 0, \quad y = H_n(x)$

(13) $z'' + (2n+1-x^2)z = 0, \quad z = \exp(-\frac{1}{2}x^2)\, H_n(x)$

(14) $H_n(-x) = (-1)^n\, H_n(x), \quad H_n'(x) = 2n\, H_{n-1}(x)$

(15) $H_{2m}(0) = (-1)^m\, (2m)!/m!, \quad H_{2m+1}(0) = 0.$

(iv) *Hypergeometric functions*. Hermite polynomials are connected with parabolic cylinder functions which are special confluent hypergeometric functions

(16) $H_n(x) = 2^{\frac{1}{2}n} \exp(\frac{1}{2}x^2) D_n(2^{\frac{1}{2}} x) = 2^n \Psi(-\frac{1}{2}n, \frac{1}{2}; x^2)$

(17) $m! H_{2m}(x) = (-1)^m (2m)! \Phi(-m, \frac{1}{2}; x^2)$

(18) $m! H_{2m+1}(x) = (-1)^m (2m+1)! 2x \Phi(-m, 3/2; x^2)$.

The general solution of Hermite's differential equation (12), or of the self-adjoint form (13) (which is virtually Weber's equation), may be obtained from the theory of parabolic cylinder functions.

(v) *Generating functions*.

(19) $\sum_{n=0}^{\infty} H_n(x) z^n/n! = \exp(2xz - z^2)$

(20) $\sum_{m=0}^{\infty} (-1)^m H_{2m}(x) z^{2m}/(2m)! = \exp(z^2) \cos(2^{\frac{1}{2}} xz)$

(21) $\sum_{m=0}^{\infty} (-1)^m H_{2m+1}(x) z^{2m+1}/(2m+1)! = \exp(z^2) \sin(2^{\frac{1}{2}} xz)$

(22) $$\sum_{n=0}^{\infty} \frac{(\frac{1}{2}z)^n}{n!} H_n(x) H_n(y) = (1-z^2)^{-\frac{1}{2}} \exp\left\{\frac{2xyz - (x^2+y^2)z^2}{1-z^2}\right\}.$$

Equation (19) is the well-known generating function, (20) and (21) may be derived from (19), and (22) is Mehler's formula.

(vi) *Integral representations*. Contour integrals follow in the usual manner from (7) or from any of the generating functions. In addition, the connection with parabolic cylinder functions may be exploited (see sec. 8.3). We note explicitly

(23) $e^{-x^2} H_n(x) = 2^{n+1} \pi^{-\frac{1}{2}} \int_0^\infty e^{-t^2} t^n \cos(2xt - \frac{1}{2}n\pi)\, dt$.

(vii) *Miscellaneous results*. See remarks under 10.12 (vii).

Limits.

(24) $$\lim_{m\to\infty} \left[\frac{(-1)^m m^{\frac{1}{2}}}{2^{2m} m!} H_{2m}\left(\frac{x}{2m^{\frac{1}{2}}}\right)\right] = \pi^{-\frac{1}{2}} \cos x$$

(25) $$\lim_{m\to\infty} \left[\frac{(-1)^m}{2^{2m} m!} H_{2m+1}\left(\frac{x}{2m^{\frac{1}{2}}}\right)\right] = 2\pi^{-\frac{1}{2}} \sin x.$$

Integrals.

(26) $\int_0^x e^{-y^2} H_n(y)\, dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$

(27) $\int_0^x H_n(y)\, dy = [2(n+1)]^{-1} [H_{n+1}(x) - H_{n+1}(0)]$

(28) $\int_{-\infty}^{\infty} e^{-y^2} H_{2m}(xy)\, dy = \pi^{1/2} \frac{(2m)!}{m!} (x^2 - 1)^m$

$\int_{-\infty}^{\infty} e^{-y^2} y H_{2m+1}(xy)\, dy = \pi^{1/2} \frac{(2m+1)!}{m!} x (x^2 - 1)^m$

(29) $\int_{-\infty}^{\infty} e^{-y^2} y^n H_n(xy)\, dy = \pi^{1/2} n!\, P_n(x).$

Here $P_n(x)$ is the Legendre polynomial.

Gauss transforms.

$$\mathcal{G}_x^u[F(y)] = (2\pi u)^{-1/2} \int_{-\infty}^{\infty} F(y) \exp[-(x-y)^2/(2u)]\, dy$$

is the Gauss transform (with parameter u). We have

(30) $\mathcal{G}_x^u[H_n(y)] = (1-2u)^{1/2 n} H_n[(1-2u)^{-1/2} x]$ $\qquad 0 \le u < 1/2$

(31) $\mathcal{G}_x^{1/2}[H_n(y)] = (2x)^n, \quad \mathcal{G}_x^{1/2}[y^n] = (2i)^{-n} H_n(ix).$

Connection with Laguerre polynomials. In addition to (2) and (3) we have

(32) $\sum_{k=0}^{n} \binom{n}{k} H_{2k}(x) H_{2n-2k}(y) = (-1)^n n!\, L_n(x^2 + y^2)$

(33) $\int_0^{\infty} e^{-y^2} [H_n(y)]^2 \cos(2^{1/2} xy)\, dy = \pi^{1/2} 2^{n-1} n!\, L_n(x^2)$

(34) $\Gamma(n + \alpha + 1) \int_{-1}^{1} (1 - t^2)^{\alpha - 1/2} H_{2n}(x^{1/2} t)\, dt = (-1)^n \pi^{1/2} (2n)!\, \Gamma(\alpha + 1/2) L_n^{\alpha}(x)$

$\mathrm{Re}\, \alpha > -1/2.$

The first two of these formulas are due to Feldheim, the last one to Uspensky (1927).

Finite sums. In addition to those already given we have, among others, the following relations.

(35) $\sum_{m=0}^{n} (2^m m!)^{-1} [H_m(x)]^2 = (2^{n+1} n!)^{-1} \{[H_{n+1}(x)]^2 - H_n(x) H_{n+2}(x)\}$

(36) $\sum_{k=0}^{\min(m,n)} (-2)^k k! \binom{m}{k} \binom{n}{k} H_{m-k}(x) H_{n-k}(x) = H_{m+n}(x)$

(37) $\sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(x) = H_m(x) H_n(x)$

(38) $\sum_{k=0}^{m} \binom{m}{k} H_k(2^{1/2} x) H_{m-k}(2^{1/2} y) = 2^{1/2 m} H_m(x + y)$

(39) $$\sum_{k=0}^{m} \binom{2m}{2k} H_{2k}(2^{1/2} x) H_{2m-2k}(2^{1/2} y)$$

$$= 2^{m-1}[H_{2m}(x+y) + H_{2m}(x-y)]$$

(40) $$\sum_{m_1+\cdots+m_r=n} \frac{a_1^{m_1}}{m_1!} \cdots \frac{a_r^{m_r}}{m_r!} H_{m_1}(x_1) \cdots H_{m_r}(x_r)$$

$$= \frac{(a_1^2+\cdots+a_r^2)^{\frac{1}{2}n}}{n!} H_n\left[\frac{a_1x_1+\cdots+a_rx_r}{(a_1^2+\cdots+a_r^2)^{1/2}}\right].$$

Equation (35) is due to Demir and Hsü. The last three formulas are addition theorems and can easily be proved by means of the generating function (19). In (40) the sum is extended over all non-negative integers $m_1, \ldots, m_r$ whose sum is n.

Infinite series. Generating functions have already been given [(19) to (22)]. For expansions in spherical Bessel functions see sec. 10.15, and for other infinite series involving Hermite polynomials see sec. 10.20.

10.14. Asymptotic behavior of Jacobi, Gegenbauer and Legendre polynomials

The behavior of Jacobi polynomials as $n \to \infty$ and at the same time $x \to 1$ in a suitable manner, is given by 10.8(41). The corresponding behavior as $x \to -1$ follows from 10.8(13), and the behavior of Gegenbauer and Legendre polynomials may be obtained by means of 10.9(4) and 10.10(3). The behavior of Jacobi polynomials as $\beta \to \infty$ and $x \to 1$ in a suitable manner is given by 10.12(35).

For the investigation of the convergence of infinite series of Jacobi polynomials, and for many other purposes, it is desirable to determine the behavior of Jacobi polynomials for fixed α, β, x, and for $n \to \infty$. The example of Tchebichef polynomials

$$T_n(\cos\theta) = \cos n\theta, \quad x = \cos\theta$$

suggests that the asymptotic behavior will be different according as x is on the interval $(-1, 1)$ (θ real) or outside it (θ complex). Also the end-points of the interval have to be considered separately. In the present section we shall entirely omit the case in which x is outside $(-1, 1)$ and refer the reader in this respect to Szegö (1939, Chapter VIII). We shall give certain results for $-1 < x < 1$; and all estimates given in this section hold uniformly in any interval $-1+\epsilon \le x \le 1-\epsilon$ ($\epsilon > 0$). We shall also give important results for the case that x is in the neighborhood of ± 1.

Proofs of the results to be given are based either on explicit series or integral representations (with integral representations the method of steepest descent is frequently used), on generating functions (Darboux's method), or else on the differential equations (Liouville's method and its later developements).

Darboux has proved (from the generating function)

$$(1)\quad P_n(\cos\theta) = 2g_n \sum_{m=0}^{M-1} \frac{g_m\,(\tfrac12)_m}{(n-m+\tfrac12)_m}\;\frac{\cos[(n-m+\tfrac12)\theta-(\tfrac12 m+\tfrac14)\pi]}{(2\sin\theta)^{m+\frac12}}$$

$$+\,O(n^{-M-\frac12}) \qquad 0<\theta<\pi$$

where g_n is defined by 10.10(21).

A similar formula was obtained by Stieltjes in a manner permitting to estimate the remainder.

$$(2)\quad P_n(\cos\theta) = \frac{2}{\pi}\sum_{m=0}^{M-1} \frac{n!\,g_m}{(m+\tfrac12)_{n+1}}\;\frac{\cos[(n+m+\tfrac12)\theta-(\tfrac12 m+\tfrac14)\pi]}{(2\sin\theta)^{m+\frac12}} + R_M(\theta)$$

$$0<\theta<\pi$$

where

$$(3)\quad |R_M(\theta)| \le \frac{2}{\pi}\;\frac{n!\,g_M}{(M+\tfrac12)_{n+1}}\;\frac{A}{(2\sin\theta)^{M+\frac12}}$$

and

$$(4)\quad A = 2\sin\theta \quad \text{if} \quad \sin^2\theta \ge \tfrac12$$

$$A = |\cos\theta|^{-1} \quad \text{if} \quad \sin^2\theta \le \tfrac12$$

so that in any event $1 \le A \le 2$.

If $2\sin\theta > 1$, i.e., for $\pi/6 < \theta < 5\pi/6$, we may make $M \to \infty$ in both (1) and (2) and obtain convergent trigonometric expansions of Legendre polynomials.

For the neighborhood of $x = 1$ we have Hilb's formula

$$(5)\quad P_n(\cos\theta) = (\theta\csc\theta)^{\frac12}\, J_0[(n+\tfrac12)\theta] + O(n^{-3/2})$$

valid uniformly for $0 \le \theta \le \pi - \epsilon$ ($\epsilon > 0$). For more precise bounds for the error term see Szegö (1939, p. 189). For an expansion of Legendre polynomials in series of Bessel functions see Szegö (1933). If 10.14(11) is specialized for $\alpha = \beta = 0$ and the confluent hypergeometric function there is expanded in a series of Bessel functions by means of 6.12(6), the result is

(6) $P_n(x) = [4/(x+3)]^{n+1} e^{-\xi/(2n+1)} [J_0(2\xi^{1/2}) + \frac{\xi}{8n^2} J_2(2\xi^{1/2}) + O(n^{-3})]$

where

$$2(x+3)\xi = (1-x)(2n+1)^2.$$

Some of these results can be extended to Gegenbauer polynomials and to a lesser extent also to Jacobi polynomials.

(7) $$C_n^\lambda(\cos\theta) = 2\frac{(\lambda)_n}{n!}\sum_{m=0}^{M-1}\frac{(\lambda)_m(1-\lambda)_m}{(n-m+\lambda)_m\, m!} \times \frac{\cos[(n-m+\lambda)\theta - \frac{1}{2}(m+\lambda)\pi]}{(2\sin\theta)^{\lambda+m}} + O(n^{-M-\frac{1}{2}})$$

$$\lambda \neq 0, -1, -2, \ldots, \quad 0<\theta<\pi$$

(8) $$C_n^\lambda(\cos\theta) = 2\frac{\Gamma(2\lambda+n)}{[\Gamma(\lambda)]^2}\sum_{m=0}^{M-1}\frac{(1-\lambda)_m}{(m+\lambda)_{n+1}\, m!} \times \frac{\cos[(n+m+\lambda)\theta - \frac{1}{2}(m+\lambda)\pi]}{(2\sin\theta)^{\lambda+m}} + R_M(\theta) \quad 0<\lambda<1, \quad 0<\theta<\pi$$

(9) $$|R_M(\theta)| \le 2\frac{\Gamma(n+2\lambda)}{[\Gamma(\lambda)]^2}\frac{(1-\lambda)_M}{(M+\lambda)_{n+1}\, M!}\frac{A}{(2\sin\theta)^{\lambda+M}}$$

and A is given by (4).

(10) $$P_n^{(\alpha,\beta)}(\cos\theta) = \frac{\cos\{[n+\frac{1}{2}(\alpha+\beta+1)]\theta - (\frac{1}{2}\alpha+\frac{1}{4})\pi\}}{(\pi n)^{1/2}(\sin\frac{1}{2}\theta)^{\alpha+1/2}(\cos\frac{1}{2}\theta)^{\beta+1/2}} + O(n^{-3/2})$$

$$\alpha, \beta \text{ real}, \quad 0<\theta<\pi.$$

A formula of Hilb's type for Jacobi polynomials has been given by Szegö and by Rau: see Szegö (1939, p. 191). Tricomi (1950a) obtained the expansion

(11) $$P_n^{(\alpha,\beta)}(x) = e^{-z}\left(\frac{1+x}{2}\right)^{-N}\sum_{m=0}^{\infty}\frac{\Gamma(N+m)\Gamma(\alpha+n+1)}{n!\,\Gamma(N)\,\Gamma(\alpha+m+1)} \times A_m(k, \tfrac{1}{2}\alpha+\tfrac{1}{2})[z/(2k)]^m\,\Phi(-n-\beta, \alpha+m+1; z)$$

where

(12) $k = n+\frac{1}{2}\alpha+\frac{1}{2}, \quad N = n+\alpha+\beta+1, \quad x = 1-4z/(2k+z) \qquad |z|<2|k|,$

Φ is the confluent hypergeometric series and the A_m are the coefficients defined in sec. 6.12. Using the expansion 6.12(6), a Bessel function expansion may be obtained for Jacobi polynomials. In the particular case $a = \beta = 0$ it leads to (6).

10.15. Asymptotic behavior of Laguerre and Hermite polynomials

The general remarks of the preceding section apply here too, but the situation is more involved on account of the infinite interval. The polynomials are oscillatory in part of the interval, and monotonic outside this part.

The asymptotic behavior of Laguerre and Hermite polynomials as $n \to \infty$ and at the same time $x \to 0$ in a suitable manner is given by 10.12 (36), 10.13(24), and 10.13(25).

For real a and fixed $x > 0$, or uniformly in $0 < \epsilon \leq x \leq \omega < \infty$, we have Fejér's formula

(1) $$L_n^a(x) = \pi^{-1/2} e^{1/2 x} x^{-1/2 a - 1/4} n^{1/2 a - 1/4} \cos[2(nx)^{1/2} - \tfrac{1}{2} a\pi - \tfrac{1}{4}\pi] + O(n^{1/2 a - 3/4})$$

which has been generalized by Perron (see Szegö, 1939, p. 192). Sansone (1950) gave a two-term approximation with an estimate of the error. This formula fails for small x, but there is a formula of Hilb's type

(2) $$e^{-1/2 x} x^{1/2 a} L_n^a(x) = \frac{\Gamma(n+a+1)}{(\frac{1}{4}\nu)^{1/2 a}\, n!} J_a[(\nu x)^{1/2}] + O(n^{1/2 a - 3/4})$$

valid for $a > -1$ uniformly in $0 < x \leq \omega < \infty$. The notation

(3) $$\nu = 4n + 2a + 2$$

is used in (2), and will be retained throughout this section.

The behavior of Laguerre polynomials when $n \to \infty$ and x is unrestricted has been investigated by several authors (see sec. 6.13). We shall confine ourselves to a brief survey here, based on a memoir by Tricomi (1949). Tricomi distinguishes four cases according as x is near 0, in the oscillatory region, near ν, or in the monotonic region.

The expansion

(4) $$n!\, e^{-1/2 x} L_n^a(x) = \Gamma(a+n+1)(\nu x/4)^{-1/2 a} \sum_{m=0}^{\infty} A_m^*(x/\nu)^{1/2 m} J_{a+m}([\nu x]^{1/2})$$

which is a particular case of 6.12(11) and in which

(5) $$A_0^* = 1, \quad A_1^* = 0, \quad A_2^* = \tfrac{1}{2} a + \tfrac{1}{2}$$

$$(m+2) A_{m+2}^* = (m+a+1) A_m^* - \tfrac{1}{2}\nu A_{m-1}^* \qquad m = 1, 2, \dots,$$

converges uniformly in any bounded region of the complex variable x. By considering the order of magnitude of successive terms, one sees that (4) has the character of an asymptotic expansion as $n \to \infty$ provided that $x = O(n^{\lambda})$ with $\lambda < 1/3$. This establishes the behavior of $L_n^{\alpha}(x)$ "near" the origin.

A similar expansion,

(6) $$n!\,(ux)^{\frac{1}{2}\alpha} e^{-hx} L_n^{\alpha}(x) = \Gamma(\alpha+n+1) \sum_{m=0}^{\infty} A_m(h)(x/u)^{\frac{1}{2}m} J_{\alpha+m}(2[ux]^{\frac{1}{2}})$$

with appropriate coefficients is due to Toscano (1949), in case $u = n$ to Tricomi (1941).

In the oscillatory region, $0 < x < \nu$, Tricomi puts

(7) $$x = \nu \cos^2\theta, \quad 0 < \theta < \tfrac{1}{2}\pi, \quad 4\Theta = \nu(2\theta - \sin 2\theta) + \pi$$

and proves that for a fixed θ

(8) $$e^{-\frac{1}{2}x} L_n^{\alpha}(x) = 2(-1)^n (2\cos\theta)^{-\alpha} (\pi\nu \sin 2\theta)^{-\frac{1}{2}} \times \Big[\sum_{m=0}^{M-1} A_m^{(\alpha)}(\theta)(\tfrac{1}{4}\nu \sin 2\theta)^{-m} \sin(\Theta + 3m\pi/2) + O(n^{-M})\Big]$$

where

(9) $$A_0^{(\alpha)}(\theta) = 1, \quad A_1^{(\alpha)}(\theta) = \frac{1}{12}\left[\frac{5}{4\sin^2\theta} - (1 - 3\alpha^2)\sin^2\theta - 1\right].$$

For the general expression for the $A_m^{(\alpha)}$ see Tricomi (1949).

Near the transition point ν,

(10) $$e^{-\frac{1}{2}x} L_n^{\alpha}(x) = \gamma_1 \left\{ A(t) + \left(\frac{4}{3\nu^2}\right)^{1/3} \left[\frac{t^2}{5} A'(t) + \frac{3+5\alpha}{10}\left(t - \frac{\Gamma(1/3)}{2\Gamma(2/3)}\right) A(t)\right] + O(n^{-5/3})\right\}$$

where

(11) $$t = (4\nu/3)^{-1/3}(\nu - x),$$

(12) $$\pi\gamma_1 = (-1)^n 2^{-\alpha}\left[6^{1/3}\nu^{-1/3} + \frac{3+5\alpha}{10}\frac{\Gamma(1/3)}{\Gamma(2/3)}\nu^{-1} + O(n^{-5/3})\right],$$

(13) $$A(t) = (\pi/3)(t/3)^{\frac{1}{2}} \{J_{-1/3}[2(t/3)^{3/2}] + J_{1/3}[2(t/3)^{3/2}]\}$$

is the Airy function, and $A'(t)$ is the derivative of $A(t)$.

Finally, in the monotonic region

(14) $x = \nu \cosh^2\theta, \quad \theta > 0, \quad 4\Theta = \nu(\sinh 2\theta - 2\theta)$

(15) $e^{-\frac{1}{2}x} L_n^{\alpha}(x) = (-1)^n e^{-\Theta}(2\cosh\theta)^{-\alpha}(\pi\nu \sinh 2\theta)^{-\frac{1}{2}}$

$$\times \left[\sum_{m=0}^{M-1} (-1)^m A_m^{[\alpha]}(\theta)\left(\tfrac{1}{4}\nu \sinh 2\theta\right)^{-m} + O(n^{-M})\right]$$

where

(16) $A_0^{[\alpha]}(\theta) = 1, \quad A_1^{[\alpha]}(\theta) = \dfrac{1}{12}\left[\dfrac{5}{4\sinh^2\theta} - (1 - 3\alpha^2)\sinh^2\theta + 1\right].$

In the following summary of the corresponding results for Hermite polynomials we use the abbreviations

(17) $N = 2n + 1, \quad m = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even,} \\ \frac{1}{2}n - \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$

For a fixed real x (or uniformly in any bounded interval)

(18) $\Gamma(\tfrac{1}{2}n + 1)\exp(-\tfrac{1}{2}x^2) H_n(x)$

$$= \Gamma(n+1)\left[\cos(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi) + O(n^{-\frac{1}{2}})\right].$$

Szegö (1939, p. 194) gives a second term explicitly, and also the general form of the asymptotic expansion.

For the behavior of Hermite polynomials for $n \to \infty$ and unrestricted x we have the Plancherel-Rotach formulas (Szegö 1939, p. 195). Tricomi's work covers this case too if use is made of 10.13(2) and 10.13(3). The Bessel functions involved in (4) when $\alpha = \pm\frac{1}{2}$ are so-called spherical Bessel functions and can be expressed in closed form. They serve as asymptotic expansions provided that for some $\lambda < 1/3$ the quantity $n^{-\lambda}x$ is bounded as $n \to \infty$.

The oscillatory region is $0 < |x| < 2m^{\frac{1}{2}}$, and here the expansion (8), with $\alpha = \pm\frac{1}{2}$, may be used. In the neighborhood of the transition points $x = \pm 2m^{\frac{1}{2}}$ we have (10), and in the monotonic region $|x| > 2m^{\frac{1}{2}}$ we have (15).

The basic expansions in series of spherical Bessel functions are particular cases of the more general expansions given by Tricomi (1941):

(19) $e^{-hx^2} H_{2m}(x) = (-1)^m 2^{2m+1} (\tfrac{1}{2})_m x^2 \sum_{r=0}^{\infty} (2m)^{1-r} C_r' \mathcal{G}_{r-1}(2m^{\frac{1}{2}}x)$

(20) $e^{-hx^2} H_{2m+1}(x) = (-1)^m 2^{2m+1} (3/2)_m x \sum_{r=0}^{\infty} (2m)^{-r} C_r'' \mathcal{G}_r(2m^{\frac{1}{2}}x)$

where

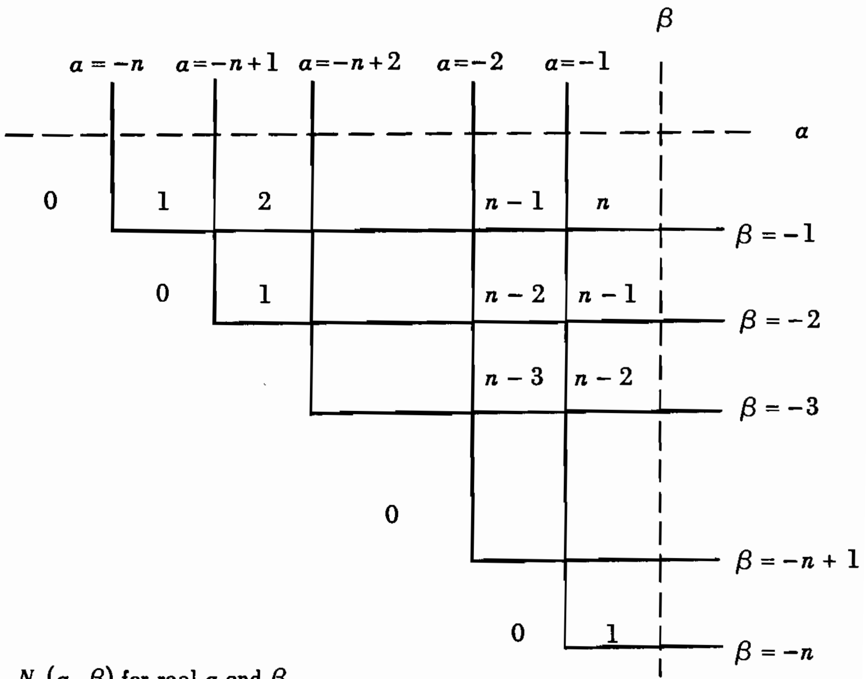
(21) $\mathcal{G}_0(z) = z^{-1} \sin z, \quad \mathcal{G}_{-1}(z) = z^{-2} \cos z$

(22) $\mathcal{G}_{r+1}(z) = (2r+1)\,\mathcal{G}_r(z) - z^2\,\mathcal{G}_{r-1}(z) \qquad r = 0, 1, 2, \dots,$

and the coefficients C_r also satisfy certain recurrence relations. The expansions (19) and (20) are convergent. They can also be used as asymptotic representations as $m \to \infty$, and for this purpose it is convenient to take $h = \frac{1}{2}$.

10.16. Zeros of Jacobi and related polynomials

Let us define Jacobi polynomials for all values of α, β, x by 10.8(12), and let us denote by $N_1(\alpha, \beta)$ the number of zeros of $P_n^{(\alpha,\beta)}(x)$ in the interval $(-1, 1)$. If $\alpha > -1$ and $\beta > -1$, Jacobi polynomials are orthogonal polynomials associated with the weight function 10.8(1), and by sec. 10.3 all their zeros are simple and located in $(-1, 1)$. For other real values of α and β the number of zeros in $(-1, 1)$ is indicated in the figure



$N_1(\alpha, \beta)$ for real α and β.

We see from 10.8(12) that for negative integer a, $P_n^{(\alpha,\beta)}(x)$ has a zero of order $|\alpha|$ at $x = 1$, and for negative integer β it has a zero of order $|\beta|$ at $x = -1$. In the interval $(-\infty, -1)$ there are $N_1(1-\alpha-\beta-2n, \beta)$ zeros, in the interval $(1, \infty)$ there are $N_1(1-\alpha-\beta-2n, \alpha)$ zeros. All zeros not accounted for in this enumeration occur in conjugate complex pairs.

Gegenbauer polynomials are defined by 10.9(18) for all values of λ, x. They are orthogonal polynomials, and all their zeros are simple and in the interval $(-1, 1)$, if $\lambda > -\frac{1}{2}$. For other real values of λ, the number of their zeros can be deduced from the result on Jacobi polynomials by means of 10.9(4).

The location of zeros of orthogonal Jacobi polynomials, and of their particular cases, in $(-1, 1)$ has been investigated by many authors. We refer the reader to Szegö (1939, Chapter VI), and for more recent work in particular to papers by Gatteschi; Geronimus; Lowan, Davids, and Levenson; and Tricomi listed at the end of this chapter.

We assume

$$(1)\quad \alpha > -1, \quad \beta > -1, \quad \lambda > -\tfrac{1}{2}, \quad x = \cos\theta \qquad 0 < \theta < \pi,$$

and arrange the zeros in a monotonic sequence,

$$(2)\quad P_n^{(\alpha,\beta)}(\cos\theta_m) = 0, \qquad 0 < \theta_1 < \theta_2 < \cdots < \theta_n < \pi.$$

$$(3)\quad P_n^{(\alpha,\beta)}(x_m) = 0, \quad -1 < x_n < x_{n-1} < \cdots < x_1 < 1, \quad x_m = \cos\theta_m.$$

For ultraspherical polynomials

$$(4)\quad x_m + x_{n-m} = 0$$

and hence it is sufficient to investigate the positive zeros $(1 \le m \le \frac{1}{2}n)$. For Jacobi polynomials

$$x_m = x_m(\alpha, \beta, n)$$

and for Gegenbauer polynmials

$$x_m = x_m(\lambda, n) = x_m(\lambda - \tfrac{1}{2}, \lambda - \tfrac{1}{2}, n).$$

If m and n (and in the case of Jacobi polynomials also one of the parameters α, β) are fixed, we have the following monotonic properties:

$$(5)\quad x_m(\alpha, \beta, n) \downarrow -1 \quad \text{as} \quad \alpha \to \infty, \quad \uparrow 1 \quad \text{as} \quad \beta \to \infty \qquad m = 1, \ldots, n$$

$$(6)\quad x_m(\lambda, n) \downarrow 0 \quad \text{as} \quad \lambda \to \infty \qquad m = 1, \ldots, [\tfrac{1}{2}n].$$

The last of these relations, for instance, means that the m-th (positive) zero of a Gegenbauer polynomial is a strictly decreasing function of λ (for $\lambda > -\frac{1}{2}$) and tends to 0 as $\lambda \to \infty$. From (5) and (6) there follow corresponding statements for θ_m. Since we know θ_m $(\pm\frac{1}{2}, \pm\frac{1}{2}, n)$ from 10.11(5) and 10.11(6), we have the following inequalities

(7) $(2m-1)\pi \leq (2n+1)\,\theta_m(a, \beta, n) \leq 2m\pi$

$$-\tfrac{1}{2} \leq a, \quad \beta \leq \tfrac{1}{2}, \quad 1 \leq m \leq n$$

(8) $(m-\frac{1}{2})\pi/n < \theta_m(\lambda, n) < m\pi/(n+1) \qquad 0 \leq \lambda \leq 1, \quad 1 \leq m \leq \frac{1}{2}n,$

For further results see Szegö (1939, Chapter VI). Tricomi (1947) pointed out that the asymptotic behavior of the zeros of any function can be deduced from the asymptotic behavior of the function itself and has applied this principle to many functions, among them orthogonal polynomials (see Tricomi 1950, Gatteschi 1949, 1949a). It transpires that the asymptotic distribution of the zeros towards the middle of the interval depends on the zeros of trigonometric functions [see 1.14(5)] and the zeros near the end-points depend on the zeros of Bessel functions [see the remark following 10.14(12)].

Asymptotic formulas for the Christoffel numbers may be derived from asymptotic formulas for the zeros by means of 10.7(7).

For numerical values of the zeros and Christoffel numbers of Legendre polynomials see Lowan, Davids, and Levenson (1942, 1943).

10.17. Zeros of Laguerre and Hermite polynomials

The polynomial defined by 10.12(7) for all values of a and x, has n positive zeros if $a > -1$, $[n + a]$ positive zeros if $-n < a \leq -1$, no positive zeros if $a \leq -n$; it has a zero of order k at $x = 0$ when $a = -k$, $k = 1, 2, \ldots, n$; and it has one negative zero if $(a+1)_n < 0$. All the zeros not accounted for in this enumeration occur in conjugate complex pairs. The Hermite polynomial of degree n has n real zeros which are situated symmetrically around the origin.

For detailed information on the location of the zeros of orthogonal Laguerre polynomials (i.e. for $a > -1$) and of Hermite polynomials we refer to Szegö (1939, Chapter VI), and to papers by Greenwood and Miller (1948), W. Hahn (1934), Salzer and Zucker (1949), Spencer (1937), and Tricomi.

We assume

(1) $a > -1, \quad x > 0$

and arrange the zeros of $L_n^\alpha(x)$ in increasing order so that

(2) $L_n^\alpha(x_m) = 0, \quad 0 < x_1 < x_2 < \cdots < x_n, \quad x_m = x_m(\alpha, n).$

For fixed m, n we find again that x_m is an increasing function of α. For bounds for the zeros see Szegö (1939, Chapter VI), and W. Hahn (1934). The asymptotic representations of Laguerre and Hermite polynomials can be used to find approximations for the zeros (Tricomi 1949). It is clear from sec. 10.15 that we have to distinguish three cases. The "first" zeros are those for which m remains bounded while $n \to \infty$: these are investigated by means of 10.15(2). The "middle" zeros are those for which $|m - \frac{1}{2}n|$ remains bounded while $n \to \infty$: these are deduced from 10.15(8). The "last" zeros, for which $n - m$ remains bounded as $n \to \infty$, are deduced from 10.15(10). The resulting approximations give satisfactory numerical results even for moderate values of n, for instance $n = 10$.

Asymptotic formulas for Christoffel numbers may be derived from 10.7(7).

For numerical values of the zeros and Christoffel numbers of Laguerre polynomials, $L_n(x)$, see Salzer and Zucker (1949).

10.18. Inequalities for the classical polynomials

For inequalities for general orthogonal polynomials and for their application to the classical polynomials, see Szegö (1939, Chapter VII).

In the notation of sec. 10.3, there is the following result for monotonic weight functions (Szegö 1939, Theorem 7.2). If $w(x)$ is non-decreasing [non-increasing] and $b\,[a]$ is finite, then $[w(x)]^{\frac{1}{2}}\,|p_n(x)|$ attains its maximum in (a, b) at $b\,[a]$.

Application of this to those of the classical orthogonal polynomials whose weight function is monotonic, at once leads to the inequalities

(1) $|P_n(x)| \le 1 \qquad -1 \le x \le 1$

(2) $[(1-x)/2]^{\frac{1}{2}\alpha+\frac{1}{4}}\,|P_n^{(\alpha,0)}(x)| \le 1 \qquad -1 \le x \le 1, \quad \alpha \ge -\frac{1}{2}$

(3) $e^{-\frac{1}{2}x}\,|L_n(x)| \le 1 \qquad x \ge 0.$

Another fruitful source of inequalities is the Sonine-Pólya theorem (Szegö 1939, Theorem 7.31.1 and footnote). If in the differential equation

(4) $[k(x)\,y']' + \phi(x)\,y = 0$

$k(x)$ and $\phi(x)$ are positive and continuously differentiable, and if $k(x)\,\phi(x)$ is monotonic, then the successive (relative) maxima of $|y|$ form an increasing or decreasing sequence according as $k(x)\;\phi(x)$ is decreasing or increasing.

The following results can be proved by constructing the differential equation satisfied by the functions involved, and applying the Sonine-Pólya theorem.

The successive maxima of $|P_n(x)|$, $n \geq 2$, as x increases from 0 to 1 form an increasing sequence. [This confirms (1).] The successive maxima of $(\sin\theta)^{1/2}\ |P_n(\cos\theta)|$, $n \geq 2$, as θ increases from 0 to $\frac{1}{2}\pi$ form an increasing sequence. As an application, it can be proved that

(5) $$(\sin\theta)^{1/2}\ |P_n(\cos\theta)| < (\tfrac{1}{2}\pi n)^{-1} \qquad 0 \leq \theta \leq \pi.$$

Furthermore,

(6) $$|P_n'(x)| \leq \tfrac{1}{2}n(n+1) \qquad -1 \leq x \leq 1.$$

For Gegenbauer polynomials

(7) $$\max_{-1\leq x\leq 1} |C_n^\lambda(x)| = C_n^\lambda(1) = \frac{(2\lambda)_n}{n!} \qquad \lambda > 0$$

(8) $$\max_{-1\leq x\leq 1} |C_{2m}^\lambda(x)| = |C_{2m}^\lambda(0)| = \left|\frac{(\lambda)_m}{m!}\right| \qquad -m < \lambda < 0, \quad \lambda \text{ not integer}$$

(9) $$\max_{-1\leq x\leq 1} |C_{2m+1}^\lambda(x)| < 2[(2m+1)(2\lambda+2m+1)]^{-1/2}\ |(\lambda)_{m+1}|/m! \qquad -m-\tfrac{1}{2} < \lambda < 0, \quad \lambda \text{ not integer.}$$

(10) $$(\sin\theta)^\lambda |C_n^\lambda(\cos\theta)| < (\tfrac{1}{2}n)^{\lambda-1}\ [\Gamma(\lambda)]^{-1} \qquad 0 < \lambda < 1, \quad 0 \leq \theta \leq \pi.$$

For Jacobi polynomials we put

(11) $$q = \max(\alpha, \beta)$$

and obtain

(12) $$\max_{-1\leq x\leq 1} |P_n^{(\alpha,\beta)}(x)| = \max P_n^{(\alpha,\beta)}(\pm 1) = \binom{n+q-1}{n} \qquad \alpha > -1, \quad \beta > -1, \quad q \geq -\tfrac{1}{2}.$$

If $-1 < \alpha, \beta < -\frac{1}{2}$, the largest maximum of $|P_n^{(\alpha,\beta)}(x)|$ is one of the two nearest to $x_0 = (\beta - \alpha)/(\alpha + \beta + 1)$, and this maximum is of the order of $n^{-1/2}$ as $n \to \infty$. Among the numerous estimates for large n we mention only

(13) $$\frac{d^m}{dx^m} P_n^{(\alpha,\beta)}(x) = O(n^q), \qquad q = \max(2m+\alpha,\ 2m+\beta,\ m-\tfrac{1}{2}) \qquad n \to \infty.$$

For the special Laguerre polynomial L_n^0 we already have (3). Bounds for L_n^α may be obtained from this by using 10.12(39) with $\beta = 0$. The result is

(14) $|L_n^\alpha(x)| \leq (\alpha+1)_n (n!)^{-1} e^{\frac{1}{2}x}$ $\qquad \alpha \geq 0$

(15) $|L_n^\alpha(x)| \leq [2-(\alpha+1)_n (n!)^{-1}]\, e^{\frac{1}{2}x}$ $\qquad -1 < \alpha < 0.$

The following results can be proved by applying the Sonine-Pólya theorem to the differential equation satisfied by the functions involved.

For any real α, the successive maxima of

$$e^{-\frac{1}{2}x} x^{\frac{1}{2}\alpha+\frac{1}{2}} |L_n^\alpha(x)|$$

form an increasing sequence provided that $2n+\alpha+1>1$ and

$$x > \max\{0, (\alpha^2-1)/(2n+\alpha+1)\}.$$

The successive maxima of

$$e^{-\frac{1}{2}x} x^{\frac{1}{2}\alpha+\frac{1}{4}} |L_n^\alpha(x)|$$

form an increasing sequence provided that $x > 0$ and

$$x^2 > \max(0, \alpha^2 - \tfrac{1}{4}).$$

The successive maxima of

$$e^{-\frac{1}{2}x} |L_n^\alpha(x)|$$

form a decreasing sequence when $\alpha > -1$,

$$0 \leq x < (2\alpha+1)(2n+\alpha+1)/(\alpha+1)$$

and an increasing sequence when $\alpha > -1$,

$$x > (2\alpha+1)(2n+\alpha+1)/(\alpha+1).$$

The successive maxima of

$$e^{-\frac{1}{2}x} x^{\frac{1}{2}\alpha} L_n^\alpha(x)$$

form a decreasing sequence if

$$0 < x < 2n+\alpha+1,$$

and an increasing sequence when

$$x > 2n+\alpha+1 > 0.$$

All these statements are contained in the following more general result. For real α and β, the successive maxima for $x > 0$ of

$$e^{-\frac{1}{2}x} x^{\beta} |L_n^\alpha(x)|$$

form an increasing or decreasing sequence according as

$$4\beta(\beta-a)(a-2\beta)+(2n+a+1)(2a-4\beta+1)x-(a-2\beta+1)x^2$$

is negative or positive.

For an asymptotic estimate see Szegö (1939, Theorem 7.6.4); improvements of this estimate may be derived from Tricomi's expansion 10.15(4).

Bounds for Hermite polynomials may be derived from (14) and (15) by means of 10.13(2) and 10.13(4). See also Sansone (1950a).

$$(16)\quad \exp(-\tfrac{1}{2}x^2)\,|H_{2m}(x)| \le 2^{2m}\, m!(2-g_m)$$

$$(17)\quad x^{-1}\exp(-\tfrac{1}{2}x^2)\,|H_{2m+1}(x)| \le 2^{2m+2}\,(m+1)!\,g_{m+1}$$

where

$$(18)\quad g_n=(\tfrac{1}{2})_n/n! = (\pi n)^{-\frac{1}{2}} + O(n^{-3/2}).$$

H. Cramér has proved

$$(19)\quad \exp(-\tfrac{1}{2}x^2)\,|H_n(x)| < k\,2^{\frac{1}{2}n}(n!)^{\frac{1}{2}}$$

where k is a constant for which Charlier (1931) gave the approximation 1.086435. Sansone (1950) gave bounds valid for complex values of the variable.

From the Sonine-Pólya theorem it may be proved that the successive maxima of $|H_n(x)|$, and likewise those of $\exp(-\frac{1}{2}x^2)\,|H_n(x)|$, for $x \ge 0$ form an increasing sequence.

Let $\mu_{r,n}$ be the r-th (relative) maximum of $f(x)\,|p_n(x)|$, where $f(x)$ is a fixed non-negative function and $\{p_n(x)\}$ is a sequence of orthogonal polynomials. The results derived from the Sonine-Pólya theorem state monotonic properties of $\mu_{r,n}$ as r increases while n is fixed. The study of numerical tables led John Todd to some conjectures about monotonic properties of $\mu_{r,n}$ for fixed r and increasing n. The following results were subsequently proved. For

$$f(x)=1, \quad p_n(x)=P_n(x),$$

and counting maxima from $x = 1$ (to the left), Cooper (1950) proved that $\mu_{r,n}$ is a decreasing function of n for sufficiently large n, and Szegö (1950) proved that this true for all $n \ge r+1$. For

$$f(x)=1, \quad p_n(x)=C_n^{\lambda}(x),$$

Szász (1950) proved that $n!\,\mu_{r,n}/\Gamma(n+2\lambda)$ is a decreasing function of n. For

$$f(x)=e^{-\frac{1}{2}x}, \quad p_n(x)=L_n(x).$$

J. Todd (1950) proved that $\mu_{r,n}$ is an increasing or decreasing function of n as r is odd or even.

P. Turán observed that

$$u_n = P_n(x) \qquad -1 \leq x \leq 1$$

satisfies the inequality

$$(20) \quad u_n^2 - u_{n-1}\, u_{n+1} \geq 0.$$

Szegö (1948) gave several proofs of this inequality and showed that it is also satisfied by

$$u_n = C_n^\lambda(x)/C_n^\lambda(1) = n!\, C_n^\lambda(x)/(2\lambda)_n \qquad -1 \leq x \leq 1$$

$$u_n = L_n^\alpha(x)/L_n^\alpha(0) = n!\, L_n^\alpha(x)/(\alpha+1)_n \qquad x \geq 0$$

$$u_n = H_n(x).$$

These results have been reproved, refined, and generalized; determinants whose elements are orthogonal polynomials have been considered, and other related investigations have been carried out by Madhava Rao and Thiruvenkatachar (1949), Sansone (1949), Szász (1950a, 1951), Beckenbach, Seidel, and Szász (1951), Forsythe (1951). See also J. L. Burchnall (1951, 1952).

10.19. Expansion problems

The expansion of a given, "arbitrary" or analytic function in a series or orthogonal polynomials has been discussed often and in great detail. The subject is somewhat outside the scope of the present survey, and a brief indication of some of the more important results must suffice. For further information see Szegö (1939, especially Chapter IX), Kaczmarz and Steinhaus (1935).

Let $\{p_n(x)\}$ be a system of orthogonal polynomials belonging to the weight function $w(x)$ on the interval (a, b). We assume that the assumptions of sections 10.1 and 10.2 are satisfied, and denote by L_w^p, $p \geq 1$ the class of functions $f(x)$ for which the (Lebesgue) integral

$$\int_a^b |f(x)|^p\, w(x)\, dx$$

exists and is finite. We put

$$(1) \quad h_n = \int_a^b [p_n(x)]^2\, w(x)\, dx$$

and call

(2) $a_n = h_n^{-1} \int_a^b f(x)\, p_n(x)\, dx$

the Fourier coefficients,

(3) $\Sigma\, a_n\, p_n(x)$

the (generalized) Fourier series of $f(x)$ with respect to the system $\{p_n(x)\}$ of orthogonal polynomials. We shall say that the series (3) *converges in* L_w^2 to $f(x)$ if

(4) $\int_a^b |f(x) - s_n(x)|^p\, w(x)\, dx \to 0 \quad \text{as} \quad n \to \infty,$

where $s_n(x)$ is the n-th partial sum of (3).

Approximation in L_w^2 has already been discussed in sec. 10.2, and from the results described there it follows that in case of a *finite* interval (a, b) for any function $f(x)$ of L_w^2, (3) converges in L_w^2 to $f(x)$. Convergence in L_w^p has been investigated by Pollard (1946, 1947, 1948, 1949) and Wing (1950). For Jacobi polynomials, given by 10.8(1), Pollard proved convergence in L_w^p when

(5) $\alpha \geq -\tfrac{1}{2}, \quad \beta \geq -\tfrac{1}{2}$

and

(6) $4 \max\left(\dfrac{\alpha+1}{2\alpha+3}, \dfrac{\beta+1}{2\beta+3}\right) < p < 4 \min\left(\dfrac{\alpha+1}{2\alpha+1}, \dfrac{\beta+1}{2\beta+1}\right)$.

For Gegenbauer polynomials we have 10.9(1) and convergence in L_w^p when

(7) $\lambda > 0, \quad \dfrac{2\lambda+1}{\lambda+1} < p < \dfrac{2\lambda+1}{\lambda}.$

Lastly, for Legendre polynomials $w(x) = 1$ and we have convergence in L_w^p when

(8) $\dfrac{4}{3} < p < 4.$

It has been pointed out in sec. 10.2 that infinite intervals present additional difficulties. Nevertheless, convergence in L_w^p for Laguerre polynomials with $\alpha > -1$, and for Hermite polynomials, has been proved when $p = 2$.

The series (3) is said to converge to $f(x)$ for a fixed x, or in an interval, if for that x, or for all x in that interval

$$s_n(x) \to f(x) \quad \text{as} \quad n \to \infty,$$

where $s_n(x)$ is again the n-th partial sum of (3). This type of convergence (sometimes called "point-wise convergence") requires much more restrictive assumptions on $f(x)$ than convergence in L_w^p.

Rau (1950) has investigated the convergence of the expansion of a function $f(x)$ in a series of Jacobi polynomials with $\alpha > -1$, $\beta > -1$. Assuming that $f(x)$ is continuous and has a piece-wise continuous derivative, he proved that the expansion converges to $f(x)$ uniformly in every interval $-1+\epsilon \le x \le 1-\epsilon$, $\epsilon > 0$.

The Abel summability of series of Laguerre polynomials was investigated by Caton and Hille (1945) by means of Laplace integrals.

Asymptotic formulas such as 10.14(1), (7), (10), and 10.15(1), (18) suggest a connection between the convergence of orthogonal expansions and that of certain related Fourier series. This is the source of the so-called equiconvergence theorems. As a sample, we shall give an equiconvergence theorem for Legendre polynomials (Haar, 1918).

Let $|f(x)|^2$ be integrable in $(-1, 1)$; let $s_n(x)$ be the n-th partial sum of the expansion of $f(x)$ in Legendre polynomials, and let $\sigma_n(\theta)$ be the the n-th partial sum of the Fourier cosine expansion of $f(\cos\theta)$. Then

$$s_n(\cos\theta) - \sigma_n(\theta) \to 0 \quad \text{as} \quad n \to \infty, \qquad 0 < \theta < \pi.$$

Such equiconvergence theorems, in combination with conditions for the convergence of Fourier series, enable one to discuss the convergence of orthogonal expansions. Equiconvergence theorems for Jacobi, Laguerre, and Hermite polynomials were given by Szegö (1939, Chapter IX). Szegö also gives some results regarding the behavior of such series at the endpoints of the basic interval.

The expansion of analytic functions presents rather different problems. A series of Jacobi polynomials converges in an ellipse whose foci are at ± 1, and every function which is analytic in such an ellipse may be expanded in a series of Jacobi polynomials (α, $\beta > -1$) there. A function which is analytic outside such an ellipse, and vanishes at infinity, may be expanded in a series of Jacobi functions of the second kind, $Q_n^{(\alpha,\beta)}$, there (α, $\beta > -1$, see Szegö sec. 9.2).

In the case of Laguerre polynomials the region of convergence is a parabola around the positive real axis, with its focus at the origin: in the case of Hermite polynomials the region of convergence is a strip whose central line is the real axis. In both cases the region of convergence is unbounded and an analytic function which is to be expanded in a series of Laguerre or Hermite polynomials must satisfy certain growth conditions in addition to being analytic in an appropriate region. Expansions in series of Laguerre polynomials were investigated by Pollard

(1947a), series of Hermite polynomials by Giuliotto (1939), and Hille (1939, 1939a, 1940).

10.20. Examples of expansions

In this section we list some series of orthogonal polynomials whose sum canbe given in closed form. Except in the case of Legendre, Hermite, and Laguerre polynomials, not many such series are known, and some of the following examples have been developed by Tricomi to fill this gap. The computation of the coefficients of such an expansion is based on 10.19(2), where one may often take advantage of Rodrigues' formula (or its generalizations) to simplify the integral by integration by parts in the manner explained in the second paragraph of sec. 10.7.

In the following formulas we shall freely use the notations for confluent hypergeometric and related functions which have been introduced in Chapters VI, VIII, IX.

SERIES OF JACOBI POLYNOMIALS

Notations as in sec. 10.8. We always assume $\alpha, \beta > -1$, and use h_n as defined in 10.8(4).

$$(1) \quad \operatorname{sgn} x = c_0 + \sum_{n=1}^{\infty} \frac{1}{nh_n} P_{n-1}^{(\alpha+1,\beta+1)}(0)\, P_n^{(\alpha,\beta)}(x) \qquad -1 < x < 1.$$

Here

$$(2) \quad \operatorname{sgn} x = \begin{cases} 1 & \text{when} \quad x > 0, \\ -1 & \text{when} \quad x < 0, \end{cases}$$

and

$$c_0 = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\, 2^{-\alpha-\beta-1} \int_0^1 [(1-x)^\alpha (1+x)^\beta - (1+x)^\alpha (1-x)^\beta]\, dx.$$

Note that only the terms corresponding to odd values of n actually occur in the summation in (1).

$$(3) \quad (1-x)^\rho = 2^\rho\, \Gamma(\alpha+\rho+1) \times \sum_{n=0}^{\infty} \frac{\Gamma(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,(-\rho)_n}{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+\beta+\rho+2)} P_n^{(\alpha,\beta)}(x)$$

$$-\rho < \min(\alpha+1, \tfrac{1}{2}\alpha + \tfrac{3}{4}), \quad -1 < x < 1$$

(4) $$e^{ixy} = (2iy)^{-\frac{1}{2}(\alpha+\beta)-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)} M_{k,m}(2iy) P_n^{(\alpha,\beta)}(x)$$

$$-1 < x < 1$$

where

$$k = \tfrac{1}{2}(\alpha - \beta), \qquad m = n + \tfrac{1}{2}(\alpha + \beta + 1).$$

For a generating function see 10.8(29); for a bilinear generating function see Watson (1934), Erdélyi (1937a) and Bailey; for another expansion in products of Jacobi polynomials see Bateman (1904, 1905).

SERIES OF GEGENBAUER POLYNOMIALS

Notation as in sec. 10.9. The constant h_n is defined by 10.9(7).

(5) $$\operatorname{sgn} x = 4 \sum_{m=1}^{\infty} \frac{(-1)^m (\lambda)_{m+1}}{(2m+1)(2m+2\lambda+1)\, m!\, h_{2m+1}} C_{2m+1}^{\lambda}(x)$$

$$\lambda > -\tfrac{1}{2}, \qquad -1 < x < 1.$$

(6) $$(1-x)^{\rho} = 2^{2\lambda+\rho} \pi^{-\frac{1}{2}} \Gamma(\lambda) \Gamma(\lambda + \rho + \tfrac{1}{2}) \times \sum_{n=0}^{\infty} \frac{(n+\lambda)\, (-\rho)_n}{\Gamma(n+2\lambda+\rho+1)} C_n^{\lambda}(x)$$

$$-1 < x < 1, \quad -\rho < \tfrac{1}{2}(\lambda+1) \text{ if } \lambda \geq 0, \quad -\rho < \tfrac{1}{2} + \lambda \text{ if } -\tfrac{1}{2} < \lambda \leq 0.$$

(7) $$e^{ixy} = \Gamma(\lambda)(\tfrac{1}{2}y)^{-\lambda} \sum_{n=0}^{\infty} i^n (n+\lambda)\, J_{n+\lambda}(y)\, C_n^{\lambda}(x)$$

$$-1 < x < 1, \qquad \lambda > 0$$

(8) $$(y \sin\phi \sin\theta)^{\frac{1}{2}-\lambda} J_{\lambda-\frac{1}{2}}(y \sin\phi \sin\theta)\, e^{iy\cos\phi\cos\theta} = 2^{\frac{1}{2}} y^{-\lambda} \Gamma(\lambda) \sum_{n=0}^{\infty} i^n \frac{n!\,(n+\lambda)}{(2\lambda)_n \Gamma(2n+\lambda)} \times J_{n+\lambda}(y)\, C_n^{\lambda}(\cos\phi)\, C_n^{\lambda}(\cos\theta)$$

$$0 < \phi,\ \theta < \pi, \qquad \lambda > 0.$$

(9) $$\omega^{-\lambda} \mathrm{C}_{\lambda}(\omega) = 2^{\lambda} \Gamma(\lambda) \sum_{n=0}^{\infty} (n+\lambda)\, z^{-\lambda} Z^{-\lambda} \times J_{n+\lambda}(z)\, \mathrm{C}_{n+\lambda}(Z)\, C_n^{\lambda}(\cos\phi)$$

where

$$|ze^{\pm i\phi}| < |Z|, \qquad \omega^2 = z^2 + Z^2 - 2zZ\cos\phi,$$

and

$$C_\lambda(\omega) = c_1\, J_\lambda(\omega) + c_2\, J_{-\lambda}(\omega)$$

is any cylinder function in the sense of Sonine and Watson (Watson, 1922, sec. 3.9). In the case $c_2 = 0$ the restrictions on z, Z may be omitted.

Some expansions in series of Gegenbauer polynomials have been noted in sec. 10.9: for a bilinear generating function see Watson (1933b).

SERIES OF LEGENDRE POLYNOMIALS

Notations as in sec. 10.10. The constant g_n is defined in 10.10(4). All expansions valid for $-1 < x < 1$, or $0 < \theta < \pi$, respectively, unless stated differently.

$$(10)\quad |x|^\rho = \sum_{m=0}^{\infty} (-1)^m \frac{(2m+\tfrac{1}{2})(-\tfrac{1}{2}\rho)_m}{(\tfrac{1}{2}\rho+\tfrac{1}{2})_{m+1}} P_{2m}(x) \qquad \rho > -1$$

$$(11)\quad |x|^\rho \operatorname{sgn} x = \sum_{m=0}^{\infty} (-1)^m (2m+\tfrac{3}{4}) \frac{(\tfrac{1}{2}-\tfrac{1}{2}\rho)_m}{(1+\tfrac{1}{2}\rho)_{m+1}} P_{2m+1}(x) \qquad \rho > -1$$

see errata!

$$(12)\quad (1-x)^\rho = 2^\rho \sum_{n=0}^{\infty} \frac{2n+1}{n+\rho+1} \frac{(-\rho)_n}{(1+\rho)_n} P_n(x) \qquad \rho > -\tfrac{3}{4}$$

$$(13)\quad (1-x^2)^{\frac{1}{2}} = \tfrac{1}{2}\pi \left[\tfrac{1}{2} - \sum_{m=1}^{\infty} \frac{4m+1}{(2m-1)(2m+2)} g_m^2\, P_{2m}(x) \right]$$

$$(14)\quad \tfrac{1}{2} e^{-\frac{1}{2}\phi i} [\cos^2(\tfrac{1}{2}\phi) - \cos^2(\tfrac{1}{2}\theta)]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} e^{in\phi} P_n(\cos\theta)$$

$$0 \le \phi < \theta < \pi$$

$$(15)\quad \log[1 + \csc(\tfrac{1}{2}\theta)] = \sum_{n=0}^{\infty} (n+1)^{-1} P_n(\cos\theta).$$

Series involving Bessel functions may be derived from (7), (8), and (9) by putting $\lambda = \frac{1}{2}$. Generating functions are listed in 10.10(v) and (viii).

SERIES OF LAGUERRE POLYNOMIALS

Notations as in sec. 10.12. We always assume $\alpha > -1$, $x > 0$.

$$(16)\quad x^\rho = \Gamma(\alpha+\rho+1) \sum_{n=0}^{\infty} \frac{(-\rho)_n}{\Gamma(\alpha+n+1)} L_n^\alpha(x) \qquad -\rho < 1 + \min(\alpha, \tfrac{1}{2}\alpha - \tfrac{1}{4})$$

(17) $\psi(\alpha+1) - \log x = \Gamma(\alpha+1) \sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(\alpha+n+1)} L_n^{\alpha}(x)$

(18) $-e^{x+y} \operatorname{Ei}[-\max(x, y)] = \sum_{n=0}^{\infty} (n+1)^{-1} L_n(x) L_n(y) \qquad x, y > 0$

(19) $e^x x^{-\alpha} \Gamma(\alpha, x) = \sum_{n=0}^{\infty} (n+1)^{-1} L_n^{\alpha}(x)$

(20) $e^{x+y}(xy)^{-\alpha} \Gamma[\alpha, \max(x, y)]\, \gamma[\alpha, \min(x, y)]$

$$= \sum_{n=0}^{\infty} \frac{n!}{(n+1)(\alpha)_{n+1}} L_n^{\alpha}(x) L_n^{\alpha}(y)$$

(21) $(xy)^{-\alpha} e^{x+y} \{\Gamma[\alpha, \max(x, y)] - \Gamma(\alpha, x)\Gamma(\alpha, y)/\Gamma(\alpha)\}$

$$= \sum_{n=0}^{\infty} \frac{n!}{(n+1)\Gamma(n+\alpha+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) \qquad x, y > 0$$

(22) $e^{\min(x,y)} = 1 + \sum_{n=1}^{\infty} [L_n(x) - L_{n-1}(x)][L_n(y) - L_{n-1}(y)]$

(23) $(xy)^{-\frac{1}{2}\alpha} e^{-\frac{1}{2}(x+y)} e^{-\alpha\pi i} \gamma[\alpha, e^{i\pi} \min(x, y)]$

$$= \sum_{n=0}^{\infty} \frac{n!}{(n+\alpha)\Gamma(n+\alpha+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) \qquad \operatorname{Re} \alpha > 0$$

(24) $\frac{\Gamma(\alpha+1, x)}{\Gamma(\alpha+1)} - H(x-y) = y^{\alpha+1} e^{-y} \sum_{n=1}^{\infty} \frac{(n-1)!}{\Gamma(\alpha+n+1)} L_{n-1}^{\alpha+1}(y) L_n^{\alpha}(x)$

$0 < x, y.$

In (24), $H(z) = 0, \frac{1}{2}, 1$ according as $z < 0, = 0, > 0$.

(25) $x^{\frac{1}{2}(\alpha-\beta)} y^{-\frac{1}{2}(\alpha+\beta)} e^y J_{\alpha+\beta}[2(xy)^{\frac{1}{2}}] = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\beta-n}(y)$

(26) $\Gamma(\alpha)\Psi(\alpha, \alpha+1; x) = \sum_{n=0}^{\infty} (n+\alpha)^{-1} L_n^{\alpha}(x) \qquad \alpha < \frac{1}{2}$

(27) $(1-y)^{-a} \Phi\left(a, c; \frac{xy}{y-1}\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} y^n L_n^{c-1}(x)$

$x > 0, \quad |y| < 1, \quad c > 0.$

Other series of Laguerre polynomials in 10.12(v) and (vii). The expansion 10.14(11) is an expansion in Laguerre polynomials when β is an integer $\geq -n$.

SERIES OF HERMITE POLYNOMIALS

Notations as in sec. 10.13.

$$(28)\quad |x|^{\rho} = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{\pi^{\frac{1}{2}}} \sum_{m=0}^{\infty} (-1)^m \frac{(-\frac{1}{2}\rho)_m}{(2m)!} H_{2m}(x) \qquad \rho > -1$$

$$(29)\quad |x|^{\rho}\,\mathrm{sgn}\,x = \frac{\Gamma(1+\frac{1}{2}\rho)}{\pi^{\frac{1}{2}}} \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{1}{2}-\frac{1}{2}\rho)_m}{(2m+1)!} H_{2m+1}(x) \qquad \rho > -1$$

$$(30)\quad \pi^{\frac{1}{2}}\,\mathrm{Erfi}\,[\min(x,y)] = \sum_{m=0}^{\infty} \frac{H_{2m+1}(x)\,H_{2m+1}(y)}{2^{2m+1}(2m+1)(2m+1)!} \qquad x, y \geq 0$$

$$(31)\quad \exp(\tfrac{1}{4}x^2)\,D_{2\nu}(x) = \frac{2^{\nu}}{\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(-1)^m H_{2m}(2^{-\frac{1}{2}}x)}{(m-\nu)\,2^{2m}\,m!}$$

$$(32)\quad \exp(\tfrac{1}{4}x^2)\,D_{2\nu+1}(x) = \frac{2^{\nu+\frac{1}{2}}}{\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(-1)^m H_{2m+1}(2^{-\frac{1}{2}}x)}{(m-\nu)\,2^{2m+1}\,m!}$$

$$(33)\quad (1+y)^{-a}\,\Phi\left(a, \tfrac{1}{2}; \frac{x^2 y}{1+y}\right) = \sum_{m=0}^{\infty} \frac{(a)_m}{(2m)!}\, y^m H_m(x) \qquad |y| < 1$$

$$(34)\quad 2x(1+y)^{-a}\,\Phi\left(a, \frac{3}{2}; \frac{x^2 y}{1+y}\right) = \sum_{m=0}^{\infty} \frac{(a)_m}{(2m+1)!}\, y^m H_{2m+1}(x).$$

Other series of Hermite polynomials are in 10.13(v).

The following key indicates the derivation of these examples; it also gives references to further material on infinite series of classical polynomials.

Series of Jacobi polynomials. The coefficients were computed by integrations by parts. For other examples see Brafman (1951).

(5) From (1) by 10.9(4).
(6) From (3) by 10.9(4).
(7) From (4) by 10.9(4); Watson (1922, p. 368).
(8) Watson (1922, p. 370).
(9) Watson (1922, p. 365).

Series of Legendre polynomials. Many examples may be obtained from series of Jacobi polynomials or series of Gegenbauer polynomials by means of 10.10(3). Numerous other examples are found in books on Legendre functions. For some examples, see Tricomi (1936, 1939-40).

(16) Tricomi (1948, p. 332).
(17) Toscano (1949).
(18) Neumann (1912).
(19) 9.4(5).
(20) 9.4(4).
(21) Watson, (1938).
(22) Tricomi (1935), Doetsch (1935).
(23) Erdélyi (1936).
(24) Tricomi (1948).
(25) Toscano (1949).
(26) 6.12(3).
(27) 6.12(5).

For some examples of series of Laguerre polynomials see Erdélyi (1937, 1938).

(28), (29) From (16) by 10.13(2) and (3).
(30) From (3) by 10.13(2) and (3).
(31), (32) Tricomi (1950a).
(33), (34) From (27) by 10.13(2) and (3).

10.21. Some classes of orthogonal polynomials

Beside the classical orthogonal polynomials there are other classes of special orthogonal polynomials which have been investigated in detail. In this section we shall describe some of these, mentioning very briefly those discussed in Szegö's book, and giving fuller details about those not otherwise conveniently accessible.

POLYNOMIALS OF S. BERNSTEIN AND G. SZEGÖ

These polynomials belong to the interval $(-1, 1)$ and their weight function $w(x)$ is of one of the forms

$$(1-x^2)^{-1/2}[\rho(x)]^{-1}, \quad (1-x^2)^{1/2}[\rho(x)]^{-1},$$

$$[(1-x)/(1+x)]^{1/2}[\rho(x)]^{-1}$$

where $\rho(x)$ is a polynomial of exact degree l, and positive for $-1 \le x \le 1$. Christoffel's formula 10.3(12) suggests a connection between these polynomials on the one side, and certain Jacobi polynomials on the other side.

The polynomials were encountered by Szegö (1921) and investigated by Bernstein (1930, 1932). See Szegö (1939) sec. 2.6.

POLYNOMIALS OF E. HEINE AND N. ACHYESER

Heine's polynomials belong to the interval $(0, a)$ and to the weight function

(1) $w(x) = [x(a-x)(b-x)]^{-1/2}$ $\qquad 0 < a < b.$

They are related to Jacobian elliptic functions.

Heine (1878-1881, vol. 1, p. 294-296) showed that the polynomial of degree n satisfies a differential equation of the form

(2) $$2\psi(x)(x-\gamma)\frac{d^2y}{dx^2} + [(x-\gamma)\psi'(x) - 2\psi(x)]\frac{dy}{dx} + [\alpha + \beta x - n(2n-1)x^2]y = 0$$

where

$$\psi(x) = x(a-x)(b-x)$$

and α, β, γ are certain constants. This differential equation has four singularities of the regular type and hence is an instance of Heun's equation.

Achyeser (1934) investigated the orthogonal polynomials associated with the interval $(-1, 1)$ and the weight function

$$w(x) = \begin{cases} |c-x|[(1-x^2)(a-x)(b-x)]^{-1/2} & -1 < x < a \text{ or } b < x < 1 \\ 0 & a < x < b. \end{cases}$$

Here $-1 < a < b < 1$, and c depends on a and b. These polynomials are also related to elliptic functions.

POLYNOMIALS OF F. POLLACZEK

Recently, F. Pollaczek defined certain families of orthogonal polynomials which are generalizations of classical orthogonal polynomials. The weight functions associated with Pollaczek's polynomials fail to satisfy certain conditions which it is customary to impose in the general theory (roughly speaking, they vanish too strongly at the end-points of the interval), and thus these polynomials are important, and readily accessible, examples of certain "irregular" phenomena in the general theory of orthogonal polynomials.

Finite interval. Let a, b, λ be real parameters, $a \geq |b|$, $\lambda > -1$. We put

(3) $-1 \leq x = \cos\theta \leq 1$ $\qquad 0 \leq \theta \leq \pi,$

and use the abbreviation

(4) $t = (a\cos\theta + b)/\sin\theta = (ax+b)(1-x^2)^{-1/2}.$

The polynomials $P_n^\lambda(x;\, a,\, b)$ are defined recurrently.

(5) $P_{-1}^\lambda = 0, \quad P_0^\lambda = 1$

(6) $n P_n^\lambda - 2[(n-1+\lambda+a)\, x + b]\, P_{n-1}^\lambda + (n+2\lambda-2)\, P_{n-2}^\lambda = 0$

$n = 1, 2, \ldots .$

These polynomials were defined by Pollaczek (1949a, for $\lambda = \frac{1}{2}$, 1949c, for Re $\lambda > 0$) and studied by Szegö (1950a). Some related polynomials were also studied by Pollaczek (1949b, 1950a).

Multiplying (6) by z^n and adding, one obtains a simple differential equation of the first order for the generating function, and hence

(7) $\sum_{n=0}^{\infty} P_n^\lambda(x;\, a,\, b)\, z^n = (1 - ze^{i\theta})^{-\lambda+it}\,(1 - ze^{-i\theta})^{-\lambda-it} \qquad |z| < 1.$

Comparison with 10.9(29) and 10.10(39) shows the relation to Gegenbauer and Legendre polynomials

(8) $P_n^\lambda(x;\, 0,\, 0) = C_n^\lambda(x), \quad P_n^{1/2}(x;\, 0,\, 0) = P_n(x).$

The polynomials are orthogonal on the interval (4), the weight function being

(9) $w^{(\lambda)}(x;\, a,\, b) = \pi^{-1}\, 2^{2\lambda-1}\, e^{(2\theta-\pi)t}\,(\sin\theta)^{2\lambda-1}\, |\Gamma(\lambda+it)|^2.$

The asymptotic behavior of $P_n^{1/2}(x;\, a,\, b)$ when x is fixed, between -1 and 1, and $n \to \infty$ was investiaged by Szegö.

Either from the generating function (7), or from the recurrence relation (6) it may be proved that

(10) $n!\, P_n^\lambda(x;\, a,\, b) = (2\lambda)_n\, e^{in\theta}\, {}_2F_1(-n,\, \lambda+it;\, 2\lambda;\, 1 - e^{-2i\theta}).$

This expression in terms of hypergeometric polynomials is the source of many further formulas for Pollaczek's polynomials. It should be noted that t depends on x so that P_n^λ does not satisfy any differential equation. Formulas connecting P_n^λ for different values of λ follow from (10) as instances of relations between contiguous hypergeometric series.

In a later paper (1950c), Pollaczek introduced a more general system of polynomials which depends on the real parameters a, b, c, λ where

(11) either $\quad a > |b|, \quad 2\lambda + c > 0, \quad c \geq 0$

or $\quad a > |b|, \quad 2\lambda + c \geq 1, \quad c > -1.$

With the notations (3) and (4), $P_n^\lambda(x;\, a,\, b,\, c)$ satisfies

(12) $P_{-1}^\lambda = 0, \quad P_0^\lambda = 1$

(13) $(n+c)P_n^\lambda - 2[(n-1+\lambda+a+c)x+b]P_{n-1}^\lambda + (n+2\lambda+c-2)P_{n-2}^\lambda = 0$

$n = 1, 2, \ldots$.

The generating function of these polynomials has been obtained by Pollaczek, who also proved that these polynomials are orthogonal on the interval (4), the weight function being

(14) $$w^{(\lambda)}(x;\, a,\, b,\, c) = \frac{(2\sin\theta)^{2\lambda-1}\, e^{(2\theta-\pi)t}}{2\pi\,\Gamma(2\lambda+c)\,\Gamma(c+1)}$$

$$\times\, |\Gamma(\lambda+c+it)|^2\, |{}_2F_1(1-\lambda+it, c; c+\lambda+it;\, e^{2i\theta})|^{-2}\,.$$

The recurrence relation (13) is a difference equation for P as a function of n. This equation serves to express $P_n^\lambda(x;\, a,\, b,\, c)$ in terms of hypergeometric functions. The expression is fairly complicated and the hypergeometric series appearing in it are no longer polynomials. Putting

$$A_n = \frac{\Gamma(2\lambda+c+n)}{\Gamma(c+n+1)\,\Gamma(2\lambda)}\, e^{i(c+n)\theta}\, {}_2F_1(-c-n, \lambda+it;\, 2\lambda;\, 1-e^{-2i\theta}),$$

$$B_n = \frac{\Gamma(1-\lambda+it)\,\Gamma(1-\lambda-it)}{\Gamma(2-2\lambda)}\,(2\sin\theta)^{1-2\lambda} e^{i(2\lambda+c+n-1)\theta}$$

$$\times\, {}_2F_1(1-2\lambda-c-n,\, 1-\lambda+it;\, 2-2\lambda;\, 1-e^{-2i\theta})$$

the resulting expression is

(15) $$P_n^\lambda(x;\, a,\, b,\, c) = \frac{A_{-1}B_n - A_nB_{-1}}{A_{-1}B_0 - A_0B_{-1}}\,.$$

In this form, it is valid when 2λ is not an integer. An alternative form, valid for integer values of 2λ is available. $A_{-1} = 0$ when $c = 0$, and in this case (15) reduces to (10).

Infinite interval. For the infinite interval $-\infty < x < \infty$, Pollaczek (1950b) has the system of polynomials $P_n^\lambda(x;\, \varphi)$ where

(16) $\lambda > 0, \quad 0 < \varphi < \pi$

are parameters,

(17) $P_{-1}^\lambda = 0, \quad P_0^\lambda = 1,$

(18) $n\,P_n^\lambda - 2[(n-1+\lambda)\cos\varphi + x\sin\varphi]\,P_{n-1}^\lambda + (n-2+2\lambda)\,P_{n-2}^\lambda = 0$

$n = 1, 2, \ldots$.

Clearly, these polynomials may be obtained from those defined by (6) by replacing θ by φ and t by x. The generating function is

(19) $$\sum_{n=0}^{\infty} P_n^{\lambda}(x;\varphi)\, z^n = (1-ze^{i\varphi})^{-\lambda+ix}(1-ze^{-i\varphi})^{-\lambda-ix} \qquad |z|<1,$$

and the weight function is

(20) $$w^{(\lambda)}(x;\varphi) = \pi^{-1}(2\sin\varphi)^{2\lambda-1} e^{-(\pi-2\varphi)x} |\Gamma(\lambda+ix)|^2.$$

These polynomials may also be expressed in terms of hypergeometric series in the form

(21) $$n!\,P_n^{\lambda}(x;\varphi) = (2\lambda)_n e^{in\varphi}\, {}_2F_1(-n, \lambda+ix; 2\lambda; 1-e^{-2i\varphi}).$$

These polynomials were mentioned by Meixner (1934) and by W. Hahn (1949). They have a representation in terms of finite differences, an analogue of Rodrigues' formula (Toscano, 1949). Setting

$$\delta F(x) = F(x+\tfrac{1}{2}i) - F(x-\tfrac{1}{2}i)$$
$$\delta^k F(x) = \delta[\delta^{k-1}F(x)] \qquad k = 2, 3, \ldots,$$

we have

(22) $$P_n^{\lambda}(x;\varphi) = \frac{(-1)^n}{n!}\;\frac{\delta^n G(\lambda+\tfrac{1}{2}n, x)}{G(\lambda, x)}$$

where

$$G(\lambda, x) = \frac{\Gamma(\lambda+ix)}{\Gamma(1-\lambda+ix)}\, e^{2\varphi x}.$$

10.22. Orthogonal polynomials of a discrete variable

In the remaining sections of this chapter we shall briefly list a few systems of orthogonal polynomials for which the distribution function $\alpha(x)$ of sec. 10.1 is a jump function, and the appropriate definition of the scalar product is 10.1(3). The points at which the jumps of $\alpha(x)$ occur are x_i, and we shall use the *jump function* $j(x)$, the jump of $\alpha(x)$ at $x = x_i$ being $j(x_i)$. Thus, the appropriate definition of the scalar product is

(1) $$(\varphi_1, \varphi_2) = \sum_i j(x_i)\,\varphi_1(x_i)\,\varphi_2(x_i),$$

and the jump function corresponds in some measure to the weight function of the earlier sections. We always assume that the jump function is positive and that $\sum_i j(x_i)$ is finite. Many results of the introductory sections of this chapter hold for scalar products of the form 10.1(2) and hence remain valid for the definition (1) of a scalar product.

The x_i will be taken as integers, $a \le x_i \le b$. The intervals and jump functions listed in the table below are those of most frequent occurrence. The orthogonal polynomials associated with them correspond to the classical orthogonal polynomials of a discrete variable, and most of them have been studied in some detail.

POLYNOMIALS OF A DISCRETE VARIABLE

a	b	$j(x)$	NAME
0	$N-1$	1	Tchebichef
0	N	$p^x q^{N-x} \binom{N}{x}$	Krawtchouk
0	∞	$\dfrac{e^{-a} a^x}{\Gamma(x+1)}$	Charlier
0	∞	$c^x \dfrac{(\beta)_x}{x!}$	Meixner
0	∞	$\dfrac{(\beta)_x (\gamma)_x}{x!\,(\delta)_x}$	W. Hahn

All these polynomials have a number of properties in common, among which we mention only the finite difference analogue of Rodrigues' formula

(2) $$p_n(x) = [K_n\, j(x)]^{-1}\, \Delta^n [j(x-n)\, X(x)\, X(x-1) \cdots X(x-n+1)]$$

where K_n is a constant, $X(x)$ is a polynomial in x whose coefficients are independent of n, and Δ is the operator of forward differences,

(3) $$\Delta f(x) = f(x+1) - f(x), \qquad \Delta^{n+1} f(x) = \Delta[\Delta^n f(x)] \qquad n = 1, 2, \dots .$$

Conversely, this property *characterizes* the above orthogonal polynomials in the sense that any system of orthogonal polynomials possessing a Rodrigues' formula can be reduced to one of the systems listed above (Hahn 1949, Weber and Erdélyi 1952). The proof is analogous to that given in sec. 10.6 and will be omitted.

The proof of the orthogonal property of these polynomials may be based on (2) and "summation by parts". Alternatively, the method of generating functions may be used.

10.23. Tchebichef's polynomials of a discrete variable and their generalizations

Tchebichef's polynomials $t_n(x)$ arise in the graduation (fitting) of data by least squares. For an account of their properties see Szegö (1939, sec. 2.8), Jordan (1921 and 1947, Chapter VIII), and the references given in these places.

Definition and orthogonal property.

(1) $$t_n(x) = n!\,\Delta^n \left[\binom{x}{n}\binom{x-N}{n}\right] \qquad n = 0, 1, \dots, N-1$$

(2) $$\sum_{x=0}^{N-1} t_m(x)\,t_n(x) = (2n+1)^{-1} N(N^2-1^2)(N^2-2^2)\cdots(N^2-n^2)\,\delta_{mn}$$
$$m, n = 0, 1, \dots, N-1.$$

Symmetry and "central values".

(3) $$t_n(N-1-x) = (-1)^n t_n(x)$$

(4) $$t_{2m}(\tfrac{1}{2}N - \tfrac{1}{2}) = (-1)^m (2m)! \binom{2m}{m}\binom{\tfrac{1}{2}N - \tfrac{1}{2} + m}{m}$$

$$t_{2m+1}(\tfrac{1}{2}N - \tfrac{1}{2}) = 0.$$

Difference equation

(5) $$(x+2)(x-N+2)\,\Delta^2 t_n(x) + [2x - N + 3 - n(n+1)]\,\Delta t_n(x) - n(n+1)\,t_n(x) = 0.$$

Recurrence formula

(6) $$(n+1)t_{n+1}(x) - (2n+1)(2x-N+1)t_n(x) + n(N^2-n^2)\,t_{n-1}(x) = 0$$
$$n = 1, 2, \dots .$$

Connection with Legendre polynomials.

(7) $$\lim_{N\to\infty} N^{-n} t_n(Nx) = P_n(2x-1).$$

A generalization of Tchebichef's polynomials may be obtained by the definition

(8) $$p_n(x;\beta,\gamma,\delta) = \frac{1}{n!}\,\frac{x!\,(\delta)_x}{(\beta)_x(\gamma)_x}\,\Delta^n\left[\frac{(\beta)_x(\gamma)_x}{(x-n)!\,(\delta)_{x-n}}\right].$$

In particular,

(9) $$p_n(x;\ 1,\ a+1,\ a+1) = \frac{1}{n!}\,\Delta^n\left[\binom{x}{n}\binom{x+a}{n}\right],$$

and in this form it is immediately seen that

(10) $$p_n(x;\ 1,\ 1-N,\ 1-N) = t_n(x).$$

Certain polynomials introduced by Bateman (1933) are also particular cases of (8). The polynomials (8) were introduced by Hahn (1949). They belong to the jump function

(11) $$j(x;\ \beta,\ \gamma,\ \delta) = \frac{(\beta)_x\,(\gamma)_x}{x!\,(\delta)_x}.$$

The explicit formula

(12) $$p_n(x;\ \beta,\ \gamma,\ \delta) = \frac{(\beta)_n\,(\gamma)_n}{n!}\ {}_3F_2(-n,\ -x,\ \beta+\gamma-\delta+n;\ \beta,\ \gamma;\ 1)$$

and a recurrence relation were given by Weber and Erdélyi (1952).

There is a connection with Jacobi polynomials,

(13) $$\lim_{\gamma\to\infty}\gamma^{-n}\,p_n(\gamma x;\ a+1,\ \gamma,\ \gamma-\beta) = \binom{n+a}{a}\,P_n^{(a,\beta)}(2x+1).$$

10.24. Krawtchouk's and related polynomials

The orthogonal polynomials associated with the binomial distribution in probability theory were introduced by Krawtchouk (1929). They were studied by Aitken and Gonin (1935), and an account of their properties is found in Szegö's book (1939, sec. 2.8.2).

We assume

(1) $$p>0,\qquad q>0,\qquad p+q=1,\qquad N \text{ a positive integer.}$$

Definition, jump function, orthogonal property.

(2) $$k_n(x) = \frac{(-1)^n\,x!\,(N-x)!}{n!\,p^x\,q^{N-x}}\,\Delta^n\left[\frac{p^x\,q^{N-x+n}}{(x-n)!\,(N-x)!}\right] \qquad n=0,\ 1,\ \ldots,\ N.$$

(3) $$j(x) = \binom{N}{x}\,p^x\,q^{N-x}$$

(4) $$\sum_{x=0}^{N} j(x)\,k_n(x)\,k_m(x) = \binom{N}{n}\,p^n\,q^n\,\delta_{mn} \qquad m,\ n=0,\ 1,\ \ldots,\ N.$$

Explicit representation, generating function.

$$(5)\quad k_n(x) = q^n \binom{x}{n} F(-n, x-N; x-n; -p/q)$$

$$(6)\quad \sum_{n=0}^{N} k_n(x)\, z^n = (1+qz)^x (1-pz)^{N-x}.$$

The explicit representation shows the connection with the Jacobi polynomials; the (limiting) relations with Hermite polynomials and with the Charlier polynomials are given in Szegö (1939, p. 35, 36).

The special case $p = q = \frac{1}{2}$ has been studied by Gram (1882) and Greenleaf (1932).

The polynomials

$$(7)\quad m_n(x; \beta, c) = \frac{x!}{(\beta)_x} c^{-x-n} \Delta^n \left[\frac{c^x (\beta)_x}{(x-n)!}\right]$$

were investigated by Meixner (1934), Gottlieb (1938, $\beta = 1$), and other authors. (See references in Hahn 1949, p. 32). They are generalizations of Krawtchouk's polynomials.

$$(8)\quad p^n m_n(x; -N, -p/q) = n!\, k_n(x).$$

Explicit representation, jump function, orthogonal property.

$$(9)\quad m_n(x; \beta, c) = (\beta + x)_n F(-n, -x; 1-\beta-n-x; c^{-1})$$
$$= (\beta)_n F(-n, -x; \beta; 1-c^{-1})$$

$$(10)\quad j(x) = c^x (\beta)_x / x!$$

$$(11)\quad \sum_{x=0}^{\infty} j(x)\, m_n(x; \beta, c)\, m_l(x; \beta, c) = n!\,(\beta)_n c^{-n} (1-c)^{-\beta} \delta_{nl}$$
$$\beta > 0, \quad 0 < c < 1.$$

Symmetry, generating function

$$(12)\quad (\beta)_x m_n(x; \beta, c) = (\beta)_n m_x(n; \beta, c)$$

$$(13)\quad \sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{z^n}{n!} = \left(1 - \frac{z}{c}\right)^x (1-z)^{-x-\beta} \qquad |z| < \min(1, |c|).$$

The explicit representation (9) leads to the following connections with Jacobi, Laguerre and Charlier polynomials.

(14) $$m_n(x;\,\beta,\,c) = n!\,P_n^{(\beta-1,\,-\beta-n-x)}\left(\frac{2}{c}-1\right)$$

(15) $$\lim_{c\to 1} m_n\left(\frac{cx}{c-1};\,\beta,\,c\right) = n!\,L_n^{\beta-1}(x)$$

(16) $$\lim_{\beta\to\infty}\left[\left(-\frac{\beta}{a}\right)^n m_n\left(x;\,\beta,\,\frac{a}{\beta}\right)\right] = n!\,L_n^{x-n}(a) = (-a)^n\,c_n(x;\,a).$$

A recurrence relation and a difference equation have been given by Meixner (1934).

10.25. Charlier's polynomials

The polynomials introduced by Charlier are the orthogonal polynomials associated with Poisson's distribution of rare events in probability theory. They have been investigated by several authors among whom we mention Meixner (1934, 1938) and Doetsch (1933). For an account of their properties see Szegö (1939, sec. 2.8.1) and Jordan (1947, sec. 148).

Jump function, definition, orthogonal property.

(1) $$j(x) = e^{-a}\,a^x/x! \qquad a>0, \quad x = 0, 1, 2, \dots$$

(2) $$c_n(x;\,a) = \frac{x!}{a^x}\,\Delta^n\left[\frac{a^{x-n}}{(x-n)!}\right]$$

(3) $$\sum_{x=0}^{\infty} j(x)\,c_m(x;\,a)\,c_n(x;\,a) = a^{-n}\,n!\,\delta_{mn}.$$

Explicit representations, generating function.

(4) $$c_n(x;\,a) = \sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{x}{r}\frac{r!}{a^r}$$

(5) $$c_n(x;\,a) = \frac{x!\,(-a)^{-n}}{(x-n)!}\,\Phi(-n,\,x-n+1;\,a)$$

(6) $$\sum_{n=0}^{\infty} c_n(x;\,a)\,\frac{z^n}{n!} = e^{-z}\left(1-\frac{z}{a}\right)^x \qquad |z|<a.$$ See errata!

A bilinear generating function was given by Meixner (1938).

Symmetry, recurrence relation, difference equation.

(7) $c_n(x;\, a) = c_x(n;\, a)$

(8) $ac_{n+1}(x;\, a) + (x-n-a)\, c_n(x;a) + nc_{n-1}(x;\, a) = 0$

(9) $ac_n(x+1;\, a) + (n-x-a)\, c_n(x;\, a) + xc_n(x-1;\, a) = 0.$

From the explicit representation (5) follows the connection with Laguerre polynomials

(10) $c_n(x;\, a) = (-a)^{-n}\, n!\, L_n^{x-n}(a).$

The connection with Meixner's polynomials has already been given in 10.24(16).

REFERENCES

Achyeser, N., 1934: *Comm. Inst. Sci. Math. Méc. Univ. Kharkoff (Zapiski Inst. Mat. Mech.)* (4) 9, 3-8.

Aitken, A.C. and H. T. Gonin, 1935: *Proc. Roy. Soc. Edinburgh* 55, 114-125.

Bateman, Harry, 1904: *Messenger of Math.* 33, 182-188.

Bateman, Harry, 1905: *Proc. London Math. Soc.* (2) 3, 111-123.

Bateman, Harry, 1933: *Tohoku Math. J.* 37, 24-38.

Beckenbach, E.F., Wladimir Seidel and Otto Szász, 1951: *Duke J.* 18, 1-10.

Bernstein, S., 1930: *Comm. Soc. Math. Kharkoff* (4) 4, 79-93.

Bernstein, S., 1932: *Comm. Soc. Math. Kharkoff* (4) 5, 59-60.

Bochner, Salomon, 1929: *Math. Z.* 29, 730-736.

Brafman, Fred, 1951: *Proc. Amer. Math. Soc.* 2, 942-949.

Burchnall, J.L., 1951: *Proc. London Math. Soc.* (3) 1, 232-240.

Burchnall, J.L., 1952: *Quart. J. Math. Oxford* (2) 3, 151-157.

Caton, W.B. and Einar Hille, 1945: *Duke Math. J.* 12, 217-242.

Charlier, C.V.L., 1931: *Application de la théorie des probabilités a l'astronomie*, Gauthier-Villars.

Cooper, R., 1950: *Proc. Cambridge Philos. Soc.* 46, 549-554.

Doetsch, Gustav, 1933: *Math. Ann.* 109, 257-266.

Doetsch, Gustav, 1935: *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (6) 22, 300-324.

Erdélyi, Arthur, 1936: *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (6) 24, 347-350.

Erdélyi, Arthur, 1937: *Math. Z.* 42, 641-670.

Erdélyi, Arthur, 1937a: *J. London Math. Soc.* 12, 56-57.

Erdélyi, Arthur, 1938: *Akad. Wiss. Wien. S.-B. IIa* 147, 513-520.

Forsythe, G. E., 1951: *Duke J.* 18, 361-371.

Gatteschi, Luigi, 1949: *Boll. Un. Mat. Ital.* (3) 4, 240-250.

Gatteschi, Luigi, 1949a: *Rend. Mat. e applicazioni Roma* (5) 8, 399-411.

Geronimus, J., 1944: *Doklady Akad. Nauk. S.S.S.R.* (N.S.) 44, 355-359.

Giuliotto, Virgilio, 1939: *Ist. Lombardo, Rend.* 72, 37-57.

Gottlieb, M.J., 1938: *Amer. J. Math.* 60, 453-458.

Gram, J.P., 1882: *J. Math.* 114, 41-73

Greenleaf, H.E.H., 1932: *Ann. Math. Statistics* 3, 204-255.

Greenwood, R.E. and J.J. Miller, 1948: *Bull. Amer. Math. Soc.* 54, 765-769.

REFERENCES

Haar, Alfred, 1918: *Math. Ann.* 78, 121-136.

Hahn, Wolfgang, 1934: *Jber. Deutsch. Math. Verein* 44, 215-236.

Hahn, Wolfgang, 1935: *Math. Z.* 39, 634-638.

Hahn, Wolfgang, 1949: *Math. Nachr.* 2, 4-34.

Heine, Emil, 1878-1881: *Handbuch der Kugelfunktionen*, second edition, Riemer, Berlin.

Hille, Einar, 1939: *C. R. Acad. Sci. Paris* 209, 714-716.

Hille, Einar, 1939a: *Duke Math. J.* 5, 875-936.

Hille, Einar, 1940: *Trans. Amer. Math. Soc.* 47, 80-94.

Hobson, E.W., 1931: *The theory of spherical and ellipsoidal harmonics*, Cambridge.

Jordan, Charles, 1921: *Proc. London Math. Soc.* 20, 297-325.

Jordan Charles, 1947: *Calculus of finite differences*, Chelsea Publishing Co.

Kaczmarz, Stefan and Hugo Steinhaus, 1935: *Theorie der Orthogonalreihen*, Warsaw.

Krawtchouk, M., 1929: *C. R. Acad. Sci. Paris* 189, 620-622.

Krall, H.L., 1936: *Bull. Amer. Math. Soc.* 42, 423-428.

Lowan, A.N., Norman Davids and Arthur Levenson, 1942: *Bull. Amer. Math. Soc.* 48, 739-743.

Lowan, A.N.. Norman Davids and Arthur Levenson, 1943: *Bull. Amer. Math. Soc.* 49, 939.

Madhava Rao, B.S. and V.R. Thiruvenkatachar, 1949: *Proc. Indian Acad. Sci. Sect. A* 29, 391-393.

Magnus, Wilhelm and Fritz Oberhettinger,1948: *Formeln und Sätze für die speziellen Funktionen der Mathematischen Physik*, Springer, Berlin.

Meixner, Joseph, 1934: *J. London Math. Soc.* 9, 6-13.

Meixner, Joseph, 1938: *Math. Z.* 44, 531-535.

Myller-Lebedeff, Wera, 1907: *Math. Ann.* 64, 388-416.

Neumann, Richard, 1912: *Die Entwicklung willkürlicher Funktionen noch den Hermiteschen und Laguerreschen Orthogonalfunktionen auf Grund der Theorie der Integralgleichungen*, Breslau.

Pollaczek, Felix, 1949a: *C. R. Acad. Sci. Paris* 228, 1363-1365.

Pollaczek, Felix, 1949b: *C. R. Acad. Sci. Paris* 228, 1553-1556.

Pollaczek, Felix, 1949c: *C. R. Acad. Sci. Paris*, 228, 1998-2000.

Pollaczek, Felix, 1950a: *C. R. Acad. Sci. Paris* 230, 36-37.

Pollaczek, Felix, 1950b: *C. R. Acad. Sci. Paris* 230, 1563-1565.

REFERENCES

Pollaczek, Felix, 1950c: *C. R. Acad. Sci. Paris* 230, 2254-2256.

Pollard, Harry, 1946: *Proc. Nat. Acad. Sci. U.S.A.* 32, 8-10.

Pollard, Harry, 1947: *Trans. Amer. Math. Soc.* 62, 387-403.

Pollard, Harry, 1947a: *Ann. of Math.* (2) 48, 358-365.

Pollard, Harry, 1948: *Trans. Amer. Math. Soc.* 63, 355-367.

Pollard, Harry, 1949: *Duke Math. J.* 16, 189-191.

Rau, Heinz, 1950: *Arch. Math.* 2, 251-257.

Salzer, H.E. and Ruth Zucker, 1949: *Bull. Amer. Math. Soc.* 55, 1004-1012.

Sansone, Giovanni, 1949: *Boll. Un. Mat. Ital.* (3) 4, 221-223 and 339-341.

Sansone, Giovanni, 1950: *Math. Z.* 53, 97-105.

Sansone, Giovanni, 1950a: *Math. Z.* 52, 593-598.

Seidel, Wladimir and Otto Szász, 1951: *J. London Math. Soc.* 26, 36-41.

Shohat, J.A., Einar Hille and J.L. Walsh, 1940: *A bibliography on orthogonal polynomials*, Washington.

Shohat, J.A. and J.D. Tamarkin, 1943: *The problem of moments*, Mathematical Surveys 1, New York.

Sonine, N., 1880: *Math. Ann.* 16, 1-80.

Spencer, V.E., 1937: *Duke Math. J.* 3, 667-675.

Szász, Otto, 1950: *Boll. Un. Mat. Ital.* (3) 5, 125-127.

Szász, Otto, 1950a: *Proc. Amer. Math. Soc.* 1, 256-267.

Szász, Otto, 1951: *J. d'Analyse Math.* 1, 116-134.

Szegö, Gábor, 1921: *Math. Z.* 12, 61-94.

Szegö, Gábor 1933: *Proc. London Math. Soc.* (2) 36, 427-450.

Szegö, Gábor, 1939: *Orthogonal polynomials*, New York.

Szegö, Gábor, 1948: *Bull. Amer. Math. Soc.* 54, 401-405.

Szegö, Gábor, 1950: *Boll. Un. Mat. Ital.* (3) 5, 120-121.

Szegö, Gábor, 1950a: *Proc. Amer. Math. Soc.* 1, 731-737.

Todd, John, 1950: *Boll. Un. Mat. Ital.* (3) 5, 122-125.

Toscano, Letterio, 1949: *Boll. Un. Mat. Ital.* (3) 4, 398-409.

Tricomi, Francesco, 1935: *Boll. Un. Mat. Ital.* 14, 213-218; 277-282.

Tricomi, Francesco, 1935a: *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* (6) 21, 332-335.

REFERENCES

Tricomi, Francesco, 1936: *Boll. Un. Mat. Ital.* 15, 102-105.

Tricomi, Francesco, 1939-40: *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat.* 75, 369-390.

Tricomi, Francesco, 1941: *Giorn. Ist. Ital. Attuari* 12, 14-33.

Tricomi, Francesco, 1947: *Ann. Mat. Pura Appl.* (4) 26, 283-300.

Tricomi, Francesco, 1948: *Serie Ortogonali di Funzioni*, Torino.

Tricomi, Francesco, 1948a: *Equazioni differenziali*, Torino.

Tricomi, Francesco, 1949: *Ann. Mat. Pura Appl.* (4) 28, 263-289.

Tricomi, Francesco, 1950: *Ann. Mat. Pura Appl.* (4) 31, 93-97.

Uspensky, J.V., 1927: *Ann. of Math.* (2) 28, 593-619.

Vitali, Giuseppe and Giovanni Sansone, 1946: *Moderna Teoria delle Funzioni di Variabile Reale*, parte seconda, Bologna.

Watson, G.N., 1922: *A treatise on the theory of Bessel functions*, Cambridge.

Watson, G.N., 1933: *J. London Math. Soc.* 8, 189-192.

Watson, G. N., 1933a: *J. London Math. Soc.* 8, 194-199.

Watson, G. N., 1933b: *J. London Math. Soc.* 8, 289-292.

Watson, G. N., 1934: *J. London Math. Soc.* 9, 22-28.

Watson, G.N., 1938: *Akad. Wiss. Wien. S.-B. IIa* 147, 151-159.

Weber, Maria and Arthur Erdélyi, 1952: *Amer. Math. Monthly*, 59, 163-168.

Whittaker, E.T. and G.N. Watson, 1940: *A course of modern analysis*, Cambridge.

Wing, G. Milton, 1950: *Amer. J. Math.* 72, 792-808.

CHAPTER XI

SPHERICAL AND HYPERSPHERICAL HARMONIC POLYNOMIALS[1]

11.1. Preliminaries

11.1.1. Vectors

We shall define a point in $(p + 2)$-dimensional Euclidean space ($p = 1, 2, 3, \ldots$) by a vector

(1) $\mathfrak{x} = (x_1, x_2, \ldots, x_{p+2})$,

and shall write $u(\mathfrak{x})$ for a function u of $x_1, x_2, \ldots, x_{p+2}$. The length of $\mathfrak{x}$ will be denoted by $||\mathfrak{x}||$ or r. Explicitly, we have

$$||\mathfrak{x}|| = r = (x_1^2 + x_2^2 + \cdots + x_{p+2}^2)^{1/2}.$$

In sec. 11.7 we have both vectors with three and vectors with four components. We then shall write $||\mathfrak{x}||_3$, $||\mathfrak{y}||_4$ to indicate the number of components of $\mathfrak{x}$, $\mathfrak{y}$, respectively.

A point on the unit-hypersphere Ω, i.e., on the hypersurface $r = 1$, in $(p + 2)$-dimensional space can be defined by a unit vector

(2) $\xi = r^{-1}\,\mathfrak{x} = (\xi_1, \xi_2, \ldots, \xi_{p+2})$,

we shall reserve the letters ξ, η, ζ for unit vectors of $p + 2$ components.

If $\mathfrak{y} = (y_1, y_2, \ldots, y_{p+2})$ is a second vector, the *inner product* of $\mathfrak{x}$ and $\mathfrak{y}$ is denoted by

(3) $(\mathfrak{x}, \mathfrak{y}) = x_1 y_1 + x_2 y_2 + \cdots + x_{p+2}\, y_{p+2}$.

For unit-vectors ξ, η making an angle θ we have $(\xi, \eta) = \cos\theta$.

We shall encounter matrices (i.e., linear operators which are applied to vectors). For full definitions and an outline of the theory, see Birkhoff and MacLane (1947). Only square matrices will occur. If M is a matrix with the general element μ_{jk} $(j, k = 1, 2, \ldots, p + 2)$ the determinant of M will be denoted by

$$\det M = \det \mu_{jk}.$$

(1) In preparing this chapter, the unpublished notes of a course given by G. Herglotz have been used. The idea and the arrangement of many of the proofs are due to him.

The *identity* or *unit-matrix* will be denoted by I; a matrix O will be called *orthogonal*, if

(4) $O' \, O = I,$

where O' denotes the transposed matrix of O. From this it follows that $O\,O'$ is also the identity. The vector resulting from the application of a matrix O or M to a vector $\mathfrak{x}$ will be denoted by $O\,\mathfrak{x}$, $M\,\mathfrak{x}$. A matrix O is orthogonal if and only if for all $\mathfrak{x}$

(5) $(O\mathfrak{x}, O\mathfrak{x}) = (\mathfrak{x}, \mathfrak{x}).$

I can be defined by the property that $I\mathfrak{x} = \mathfrak{x}$ for all $\mathfrak{x}$.

A function of $x_1, x_2, \ldots, x_{p+2}$ will be called a function of $\mathfrak{x}$ and will be denoted by $f(\mathfrak{x})$. (A function of two or more vectors is defined in an analogous manner.)

A function $f(\mathfrak{x})$ will be called an *orthogonal invariant* if for all $\mathfrak{x}$ and for all orthogonal matrices O

(6) $f(O\mathfrak{x}) = f(\mathfrak{x}).$

Similarly, a function of two variable vectors is an orthogonal invariant if $f(O\mathfrak{x}, O\mathfrak{y}) = f(\mathfrak{x}, \mathfrak{y})$ for all $\mathfrak{x}$, $\mathfrak{y}$ and for all orthogonal matrices O.

Sometimes we shall use hyperspherical polar coordinates

$$r, \theta_1, \ldots, \theta_p, \phi,$$

defined by

$$x_1 = r \cos \theta_1,$$

$$x_2 = r \sin \theta_1 \cos \theta_2,$$

$$x_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3,$$

(7) $\cdots \qquad \cdots \qquad \cdots$

$$x_p = r \sin \theta_1 \sin \theta_2, \ldots, \sin \theta_{p-1} \cos \theta_p,$$

$$x_{p+1} = r \sin \theta_1 \sin \theta_2, \ldots, \sin \theta_p \cos \phi,$$

$$x_{p+2} = r \sin \theta_1 \sin \theta_2, \ldots, \sin \theta_p \sin \phi,$$

where $r \geq 0$

(8) $0 \leq \theta_j \leq \pi \; (j = 1, 2, \ldots, p), \quad 0 \leq \phi \leq 2\pi.$

In these coordinates, the $(p+2)$-dimensional volume element is given by

(9) $dV = r^{p+1} (\sin \theta_1)^p (\sin \theta_2)^{p-1} \cdots (\sin \theta_p) \, dr \, d\theta_1 \cdots d\theta_p \, d\phi,$

and the surface element $d\Omega$ becomes

(10) $d\Omega = (\sin \theta_1)^p (\sin \theta_2)^{p-1} \cdots (\sin \theta_p) \, d\theta_1 \cdots d\theta_p \, d\phi.$

The *total area* ω *of* Ω can be computed either from this or from the remark that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-x_1^2-\cdots-x_{p+2}^2)\,dx_1\cdots dx_{p+2}=\left(\int_{-\infty}^{\infty}e^{-x^2}dx\right)^{p+2}$$

$$=\iiint_V\exp(-r^2)\,dV=\omega\int_0^{\infty}r^{p+1}e^{-r^2}\,dr$$

which gives

$$(11)\quad \omega=\frac{2\pi^{1+\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}.$$

Here and in the whole of this chapter we shall use three, two or one integral signs to denote integrals taken over a $(p+2)$, $(p+1)$ or p-dimensional manifold respectively.

A function which is defined on Ω can be considered as a function $F(\xi)$ of the components of the unit-vector ξ. The expression

$$(12)\quad \int\limits_{\Omega(\xi)}\int F(\xi)\,d\Omega(\xi)$$

denotes the $(p+1)$-tuple integral which will be obtained if we substitute for the components of ξ the expressions in terms of $\theta_1,\ldots,\theta_p,\phi$, from (2), (7) and for $d\Omega(\xi)$ the corresponding expression from (10).

If $F_1(\xi)$, $F_2(\xi)$ are two functions which are defined on Ω, and if

$$\int\limits_{\Omega}\int F_1(\xi)F_2(\xi)\,d\Omega(\xi)$$

exists and is zero, we shall say that $F_1(\xi)$, $F_2(\xi)$ are *orthogonal on* $\Omega(\xi)$. We shall write Ω instead of $\Omega(\xi)$ if the context indicates which is the variable vector.

If not stated otherwise, Laplace's operator Δ will refer to the components of $\mathfrak{x}$, i.e.,

$$(13)\quad \Delta\equiv\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_{p+2}^2}.$$

We have

$$(14)\quad \Delta[r^l(\mathfrak{x},\mathfrak{y})^m]=\left[\frac{m(m-1)(\mathfrak{y},\mathfrak{y})}{(\mathfrak{x},\mathfrak{y})^2}+\frac{l(l+p+2m)}{r^2}\right]r^l(\mathfrak{x},\mathfrak{y})^m.$$

The operator Δ is invariant under orthogonal transformations, i.e.,

$$\sum_{k=1}^{p+2}\frac{\partial^2}{\partial x_k^2}=\sum_{k=1}^{p+2}\frac{\partial^2}{\partial y_k^2}\qquad \mathfrak{y}=O\mathfrak{x}$$

where O denotes an orthogonal matrix.

In polar coordinates (3) we have

$$(15)\quad \Delta u = r^{-p-1}\frac{\partial}{\partial r}\left(r^{p+1}\frac{\partial}{\partial r}u\right) + r^{-2}(\sin\theta_1)^{-p}\frac{\partial}{\partial\theta_1}\left[(\sin\theta_1)^p\frac{\partial}{\partial\theta_1}u\right]$$

$$+r^{-2}(\sin\theta_1)^{-2}(\sin\theta_2)^{1-p}\frac{\partial}{\partial\theta_2}\left[(\sin\theta_2)^{p-1}\frac{\partial}{\partial\theta_2}u\right]$$

$$+r^{-2}(\sin\theta_1\sin\theta_2)^{-2}(\sin\theta_3)^{2-p}\frac{\partial}{\partial\theta_3}\left[(\sin\theta_3)^{p-2}\frac{\partial}{\partial\theta_3}u\right]+\cdots$$

$$+r^{-2}(\sin\theta_1\cdots\sin\theta_{p-1})^{-2}(\sin\theta_p)^{-1}\frac{\partial}{\partial\theta_p}\left[(\sin\theta_p)^1\frac{\partial}{\partial\theta_p}u\right]$$

$$+r^{-2}(\sin\theta_1\cdots\sin\theta_p)^{-2}\frac{\partial^2}{\partial\phi^2}u.$$

11.1.2. Gegenbauer polynomials

The polynomial $C_n^\nu(x)$ of degree n which is defined by the generating function

$$(16)\quad (1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^\nu(x)\,t^n \qquad \nu\neq 0$$

is called the *Gegenbauer polynomial* or the ultraspherical polynomial of degree n and order ν. Szegö (1939) denotes it by $P_n^{(\nu)}(x)$. Gegenbauer (1877, 1884, 1890, 1891, 1893) has investigated these polynomials for arbitrary values of ν. An account of his theory is given in sec. 3.15. We shall need here only the case where 2ν is an integer, $2\nu = p = 1, 2, 3, \ldots$. In this case we have

$$(17)\quad C_n^{l+\frac12}(x)=\frac{2^l l!}{(2l)!}\frac{d^l}{dx^l}P_{n+l}(x)=\frac{2^{-n}l!}{(n+l)!(2l)!}\frac{d^{n+2l}}{dx^{n+2l}}(x^2-1)^{n+l}$$

$$(18)\quad C_n^{l+1}(x)=\frac{2^{-l}}{l!(n+l+1)}\frac{d^{l+1}}{dx^{l+1}}T_{n+l+1}(x)$$

where $l = 0, 1, 2, \ldots$,

$$(19)\quad P_n(x)=\frac{2^{-n}}{n!}\frac{d^n}{dx^n}(x^2-1)^n = {}_2F_1(-n, n+1; 1; \tfrac12-\tfrac12 x)$$

is the Legendre polynomial of degree n, and

$$(20)\quad T_n(x)=\tfrac12\{[x+i(1-x^2)^{\frac12}]^n+[x-i(1-x^2)^{\frac12}]^n\}$$

$$(21)\quad = {}_2F_1(-n, n; \tfrac12, \tfrac12-\tfrac12 x)$$

$$(22)\quad = \cos(n\cos^{-1}x)$$

is the Tchebichef polynomial of degree n. Tchebichef polynomials take the place of the ultraspherical polynomials for $\nu = 0$; their generating function is

$$(23)\quad -\tfrac{1}{2}\log(1-2tx+t^2) = \sum_{n=0}^{\infty} (n+1)^{-1}\, T_{n+1}(x)\, t^{n+1}.$$

From (20) we have for $n = 0, 1, 2, \dots,$

$$(24)\quad [x + i(1-x^2)^{\frac{1}{2}}]^{n+1} = T_{n+1}(x) + i(n+1)^{-1}(1-x^2)^{\frac{1}{2}} \frac{d}{dx} T_{n+1}(x).$$

We also have for an arbitrary $\nu \neq 0$:

$$(25)\quad C_n^{\nu}(x) = (-2)^{-n}(1-x^2)^{-\nu+\frac{1}{2}} \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n\, n!} \frac{d^n}{dx^n}(1-x^2)^{n+\nu-\frac{1}{2}}.$$

Here

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \qquad n = 1, 2, \dots.$$

Equation (25) is a consequence of 3.15(3) and 2.8(23).

Between the numbers ω in (6), $h(n, p)$ in 11.2(2), the square of the normalization factor for the Gegenbauer polynomial

$$(26)\quad N = \int_{-1}^{+1} [C_n^{\frac{1}{2}p}(x)]^2 (1-x^2)^{(p-1)/2}\, dx = \frac{2^{2-p}\pi\, \Gamma(n+p)}{n!\,(p+2n)\,[\Gamma(\frac{1}{2}p)]^2},$$

the total area of the surface of the unit-sphere in $(p + 1)$-dimensional space

$$(27)\quad \omega' = \frac{2\pi^{\frac{1}{2}+\frac{1}{2}p}}{\Gamma(\frac{1}{2}+\frac{1}{2}p)},$$

and the value

$$(28)\quad C_n^{\frac{1}{2}p}(1) = \frac{(n+p-1)!}{n!\,(p-1)!} = \frac{(p)_n}{n!} = (-1)^n \binom{-p}{n},$$

there exist the relations

$$(29)\quad \frac{\omega' N}{C_n^{\frac{1}{2}p}(1)} = \frac{\omega\, C_n^{\frac{1}{2}p}(1)}{h(n, p)} = \frac{4\pi^{1+\frac{1}{2}p}}{(2n+p)\,\Gamma(\frac{1}{2}p)}.$$

Proofs for the formulas in this section are given in Appell-Kampé de Fériet (1926).

For an investigation of the $C_n^{\frac{1}{2}p}$ which starts from particular solutions of $\Delta u + k^2 u = 0$ [waves in $(p + 2)$-dimensional space] see A. Sommerfeld, (1943) and also W. Magnus (1949).

11.2. Harmonic polynomials

A polynomial $H_n(\mathfrak{x})$ of degree n in $x_1, x_2, \ldots, x_{p+2}$ which is homogeneous of degree n, so that $H_n(\lambda \mathfrak{x}) = \lambda^n H_n(\mathfrak{x})$, and satisfies Laplace's equation $\Delta H_n(\mathfrak{x}) = 0$, is known as a *harmonic polynomial* of degree n. Clearly, $r^{-n} H_n(\mathfrak{x}) = H_n(\xi)$ is a one-valued continuous function on the hypersphere Ω, or $r = 1$, and can also be expressed as a trigonometric polynomial in $\theta_1, \ldots, \theta_p, \phi$. For the notations see sec. 11.1.

A partial differential equation of the form $\Delta u + f(r)\, u = 0$, where $f(r)$ is a given analytic function of r only, and $u = u(\mathfrak{x})$, has solutions of the form $u = R_n(r)\, H_n(\xi)$, where $H_n(\mathfrak{x})$ is an arbitrary harmonic polynomial of degree n, and $R_n(r)$ is a solution of the ordinary differential equation

$$(1)\quad \frac{d^2R}{dr^2} + \frac{p+1}{r}\frac{dR}{dr} + [f(r) - n(n+p)\, r^{-2}]\, R = 0.$$

We shall now show that there are

$$(2)\quad h(n, p) = (2n+p)\, \frac{(n+p-1)\,!}{p\,!\,n\,!}$$

linearly independent harmonic polynomials of degree n of the $p + 2$ variables $x_1, x_2, \ldots, x_{p+2}$.

To prove this, we compute first the number $g(n, p)$ of linearly independent homogeneous polynomials of degree n of $p + 2$ variables. Clearly,

$$(3)\quad g(n, p) = g(n, p-1) + g(n-1, p-1) + \cdots + g(0, p-1),$$

$$(4)\quad g(n, 0) = n + 1,$$

and $g(n, p)$ is uniquely determined by (3) and (4).

$$(5)\quad g(n, p) = \frac{(n+p+1)\,!}{n!\,(p+1)\,!} = \binom{p+n+1}{n}.$$

Now, Laplace's equation imposes conditions upon the coefficients in H_n, since ΔH_n is a homogeneous polynomial of degree $n - 2$, there are at most $g(n-2, p)$ independent conditions and

$$(6)\quad h(n, p) \geq g(n, p) - g(n-2, p).$$

On the other hand, the $g(n-2, p)$ linearly independent polynomials

$$x_1^2\, P(x_1, \ldots, x_{p+2}),$$

where P denotes any homogeneous polynomial of degree $n - 2$, do not satisfy Laplace's equation, so that

$$(7)\quad h(n, p) \leq g(n, p) - g(n-2, p),$$

and this proves (2).

Except for $n = 0$, there is no harmonic polynomial which is invariant under all orthogonal transformations.(1) But there exists an $H_n(\mathfrak{x})$ which is invariant under all those orthogonal transformations which leave one point of the unit-sphere fixed. Since $(O\mathfrak{x}, \eta) = (\mathfrak{x}, \eta)$ for all orthogonal transformations which leave η invariant, it is sufficient to prove

LEMMA 1. *For each unit vector, η, there exists one and only one harmonic polynomial $H_n(\mathfrak{x})$ such that*

(i) *$H_n(\mathfrak{x})$ depends only on r and $(\mathfrak{x}, \eta)$;*

(ii) *$H_n(\eta) = 1$.*

This polynomial is given by

$$(8) \quad H_n(\mathfrak{x}) = r^n \frac{C_n^{\frac{1}{2}p}[(\xi, \eta)]}{C_n^{\frac{1}{2}p}(1)},$$

where $\xi = \mathfrak{x}/r$ and where $C_n^{\frac{1}{2}p}$ is given by 11.1(16).

Since $C_n^{\frac{1}{2}p}(x)$ can be expressed in terms of even or odd powers of x according as n is even or odd, the right-hand side of (8) is a polynomial of $x_1, \ldots, x_{p+2}$, although r^n is not necessarily one. Since $C_n^{\frac{1}{2}p}(1) \neq 0$, (8) satisfies (ii). Therefore we have to show now that (i) determines $H_n(\mathfrak{x})$ apart from a constant factor. Since $H_n(\mathfrak{x})$ is homogeneous and of degree n it must be of the form

$$C_0(\mathfrak{x}, \eta)^n + c_1 r(\mathfrak{x}, \eta)^{n-1} + \cdots + c_n r^n,$$

where $C_0, \ldots, c_n$ are constants.

Since $\Delta H_n = 0$, we find from 11.1(14) the relations

$$(9) \quad (n-m)(n-m-1)\, c_m + (m+2)(2n-m-2+p)\, c_{m+2} = 0$$

for $m = 0, 1, 2, \ldots$, and

$$(10) \quad c_1 = 0.$$

Therefore H_n is uniquely determined by c_0 and $c_1 = c_3 = \cdots = 0$. To construct H_n, we observe that we have from 11.1(14) $\Delta r^{-p} = 0$ and therefore

$$(11) \quad \Delta\, ||r\eta - \mathfrak{x}||^{-p} = \Delta\, [\sum_{k=0}^{p+2} (r\eta_k - x_k)^2]^{-\frac{1}{2}p} = 0$$

for all values of r. With $r = t^{-1}$, we find that the coefficient of t^n in the expansion of

$$(12) \quad [1 - 2(\xi, \eta)\, rt + r^2 t^2]^{-\frac{1}{2}p}$$

satisfies Laplace's equation. This completes the proof of Lemma 1 for

(1) G. Pólya and B. Meyer (1950) have investigated the harmonic polynomials of three variables which are invariant under any given finite subgroup of the orthogonal group.

$p = 1, 2, \ldots$. In the case $p = 0$ we can start with 11.1(23) instead of 11.1(16) and we find instead of (8) that for $p = 0$

(13) $r^n T_n[(\xi, \eta)]$

is the polynomial whose existence is stated in Lemma 1.

We can now construct a complete set of linearly independent harmonic polynomials of degree n. Let

(14) $H_{m,k}(x_k, x_{k+1}, \ldots, x_{p+2})$

denote any homogeneous harmonic polynomial of degree m which is independent of $x_1, \ldots, x_{k-1}$. It can be verified that

(15) $\Delta[(1 - 2x_1 t + r^2 t^2)^{-m-\frac{1}{2}p} H_{m,2}] = 0$

for all values of the parameter t, and this enables us to find all homogeneous harmonic polynomials of $p + 2$ variables if those of $p + 1$ variables are known already. From the $h(m, p + 1)$ linearly independent polynomials $H_{m,2}$ we obtain $h(m, p - 1)$ linearly independent polynomials $H_n(\mathfrak{x})$ which are of degree $n - m$ with respect to x_1, namely,

(16) $r^{n-m} C_{n-m}^{m+\frac{1}{2}p}(x_1/r) H_{m,2}$,

where $m = 0, 1, \ldots, n$. Since it follows from (3) and

$$h(n, p) = g(n, p) - g(n - 2, p)$$

that

(17) $h(n, p) = h(n, p - 1) + h(n - 1, p - 1) + \cdots + h(0, p - 1)$,

we obtain all the $H_n(\mathfrak{x})$ from (16).

Since

(18) $(x_{p+1} \pm ix_{p+2})^m$

form a complete set of linearly independent $H_{m,p}$, we obtain by induction

THEOREM 1. *Let $m_0, \ldots, m_p$ be integers such that*

(19) $n = m_0 \geq m_1 \geq \cdots \geq m_p \geq 0$,

and let r_k be defined by

(20) $r_k = (x_{k+1}^2 + x_{k+2}^2 + \cdots + x_{p+2}^2)^{\frac{1}{2}}$

where $k = 0, 1, \ldots, p$ and $r_0 = r$. Then

(21) $H(m_k, \pm; \mathfrak{x}) \equiv H(n, m_1, \ldots, m_{p-1}, \pm m_p; x_1, \ldots, x_{p+2})$

$$= \left(\frac{x_{p+1}}{r_p} + i\,\frac{x_{p+2}}{r_p}\right)^{\pm m_p} r_p^{m_p} \prod_{k=0}^{p-1} r_k^{m_k - m_{k+1}} C_{m_k - m_{k+1}}^{m_{k+1}+\frac{1}{2}p-\frac{1}{2}k}(x_{k+1}/r_k)$$

form a complete set of $h(n, p)$ linearly independent harmonic polynomials of degree n. Of course, $H(m_k, +; \mathfrak{x}) = H(m_k, -; \mathfrak{x})$ if $m_p = 0$.

Corollary. In hyperspherical polar coordinates 11.1 (7) we have

(22) $H(m_k, \pm; \mathfrak{x}) = r^n Y(m_k; \theta_k, \pm\phi)$

where

(23) $Y(m_k; \theta_k, \pm\phi) = e^{\pm i m_p \phi} \prod_{k=0}^{p-1} (\sin\theta_{k+1})^{m_{k+1}} C_{m_k - m_{k+1}}^{m_{k+1} + \frac{1}{2}p - \frac{1}{2}k} (\cos\theta_{k+1}).$

11.3. Surface harmonics

If $H_n(\mathfrak{x})$ is a homogeneous harmonic polynomial of degree n, we call

(1) $r^{-n} H_n(\mathfrak{x}) = H_n(\xi) = Y_n(\theta, \phi)$

a *surface harmonic* of degree n. Here θ stands for $\theta_1, \ldots, \theta_p$ and ξ denotes again $\mathfrak{x}/r$. The surface harmonics are one-valued continuous functions on Ω (the unit-hypersphere $r = 1$). In particular, we have from 11.2 (22) and 11.2 (23) the surface harmonics of degree $n = m_0$

(2) $r^{-n} H(n, m_1, \ldots, \pm m_p; x_1, \ldots, x_{p+2}) = r^{-n} H(m_k, \pm; \mathfrak{x})$

$= H(n, m_1, \ldots, \pm m_p; \xi_1, \ldots, \xi_{p+2}) = H(m_k, \pm; \xi)$

(3) $= Y(n, m_1, \ldots, m_p; \theta_1, \ldots, \theta_p, \pm\phi) = Y(m_k; \theta, \pm\phi).$

We shall now state the orthogonal property (compare sec. 11.1 for the definition) of the functions (2), (3). With the notations

(4) $E_k(l, m)$

$$= \frac{\pi\, 2^{k-2m-p}\, \Gamma(l + m + p + 1 - k)}{(l + \frac{1}{2}p + \frac{1}{2} - \frac{1}{2}k)\,(l - m)!\,[\Gamma(m + \frac{1}{2}p + \frac{1}{2} - \frac{1}{2}k)]^2}$$

for any integers l, m where $l \geq m \geq 0$, and

(5) $N(m_0, m_1, \ldots, m_p) = 2\pi \prod_{k=1}^{p} E_k(m_{k-1}, m_k)$

where $m_0, m_1, \ldots, m_p$ satisfy 11.2 (19), we have:

THEOREM 2. *Any two distinct functions in* (2) *or* (3) *are orthogonal on Ω unless they are conjugate complex. In the case of conjugate complex functions* [*or in the case of the square of a real function* (2) *or* (3)] *we have:*

(6) $\int_\Omega \int |H(m_k, \pm; \xi)|^2 \, d\Omega = \int_\Omega \int |Y(m_k; \theta, \pm\phi)|^2 \, d\Omega$

$$= N(m_0, m_1, \ldots, m_p) \equiv N(m_k).$$

In particular, any two surface harmonics of different degrees are orthogonal on the unit-hypersphere.

The functions in (2) or (3) form a *complete* set of orthogonal functions on Ω. We shall prove:

THEOREM 3. *A function $f(\xi)$ which is continuous everywhere on Ω and is orthogonal, on Ω, to all the functions $H(m_k, \pm; \xi)$ vanishes identically on Ω.*

To prove this we assume that $f(\eta) = 2a > 0$, where η is a fixed unit-vector (i.e., a point on Ω). Since $f(\xi)$ is continuous, we may assume that $f(\xi) \geq a$ for all ξ satisfying $||\xi - \eta|| \leq \delta$ where δ is a sufficiently small positive number, or $f(\xi) \geq a$ if $1 - (\xi, \eta) \leq \frac{1}{2}\delta^2$. According to Weierstrass' theorem on polynomial approximation (cf. Widder, 1947, p. 355) applied to the function

$$\begin{aligned} \phi(x) &= 1 - (1 - x)/(\tfrac{1}{2}\delta^2) && 1 - x \leq \tfrac{1}{2}\delta^2, \\ &= 0 && 1 - x \geq \tfrac{1}{2}\delta^2, \end{aligned}$$

we have that given any $\epsilon > 0$, there exists a positive integer n and a polynomial $F_n(x)$ of degree n such that

$$|F_n(x) - \phi(x)| \leq \epsilon \qquad -1 \leq x \leq 1.$$

Then

$$\int_\Omega\int f(\xi)\, \phi[(\xi, \eta)]\, d\Omega \geq a^* > 0,$$

where a^* is a positive number depending on a and δ but not on n and ϵ, and hence

$$(7) \qquad \lim_{\epsilon \to 0} \int_\Omega\int f(\xi)\, F_n[(\xi, \eta)]\, d\Omega = a^*.$$

Since $f(\xi)$ is orthogonal to all functions in (2) or (3) and, according to theorem 1, $C_k^{\frac{1}{2}p}[(\xi, \eta)]$ is a linear combination of these functions, $f(\xi)$ must be orthogonal to $C_k^{\frac{1}{2}p}[(\xi, \eta)]$ for each k. Moreover, since $C_k^{\frac{1}{2}p}(z)$ is precisely of degree k in z, $F_n(z)$ is a linear combination of the $C_k^{\frac{1}{2}p}(z)$, $k = 0, 1, \ldots, n$. Hence $f(\xi)$ is orthogonal to $F_n[(\xi, \eta)]$ and this contradicts (7) and proves theorem 3.

From the proof of theorem 3, we can obtain a statement about the approximation of a special class of continuous functions by surface harmonics. We have:

LEMMA 2. *Let $F(x)$ be a function of the real variable x which is continuous for $-1 \leq x \leq 1$. We define for $n = 0, 1, 2, \ldots,$*

(8) $\phi_n[(\xi, \eta)] = \sum_{m=0}^{n} a_m C_m^{\frac{1}{2}p}[(\xi, \eta)]$

where

(9) $C_n^{\frac{1}{2}p}(1)\, A(n, p)\, a_n = \int\int_{\Omega} C_n^{\frac{1}{2}p}[(\xi, \eta)]\, F[(\xi\, \eta)]\, d\Omega(\xi),$

and

(10) $A(n, p)\, C_n^{\frac{1}{2}p}(1) = \int\int_{\Omega} \{C_n^{\frac{1}{2}p}[(\xi, \eta)]\}^2\, d\Omega(\xi).$

Then $F[(\xi, \eta)]$, *which is a continuous function of* ξ *on* Ω *will be approximated by the* ϕ_n *in such a way that*

(11) $\lim_{n\to\infty} \int\int_{\Omega} |F[(\xi, \eta)] - \phi_n[(\xi, \eta)]|^2\, d\Omega = 0.$

Incidentally, the $A(n, p)$ do not depend on the fixed unit-vector η; their values are given in 11.4 (13).

To prove this lemma we choose in (10) the coefficients a_n so as to minimize the integral in (11). Since $C_k^{\frac{1}{2}p}[(\xi, \eta)]$ and $C_m^{\frac{1}{2}p}[(\xi, \eta)]$ are orthogonal on Ω when $k \neq m$ (cf. the remark after theorem 2), we find precisely the values (9) for the a_n. On the other hand we know from Weierstrass' theorem on polynomial approximation that for a suitable choice of the a_n and a sufficiently large n the integrand in (11) can be made arbitrarily small. Therefore the minimum of the integral in (11) must tend to zero as $n \to \infty$.

The problem of the expansion of a function which is given on Ω in a series of surface harmonics has been investigated by several authors. For $p = 1$ see Hobson (1931), where many references are given. The case $p = 2$ has been investigated by Kogbetliantz (1924), Koschmieder (1929), and the case of an arbitrary p has been treated by Koschmieder (1931). The expansion of a function in a series of surface harmonics is sometimes called its *Laplace-series*. In general, one does not know much about the convergence of the Laplace series of a continuous function but its Césaro-summability (of a sufficiently high order) can be proved.

11.4. The addition theorem

For a fixed η, the surface harmonic $C_n^{\frac{1}{2}p}[(\xi, \eta)]$ can be expressed in terms of the $S(m_k, \pm; \xi)$ where $m_0 = n$. More generally we have:

THEOREM 4. *Let* $S_n^l(\xi)$, $l = 1, 2, \ldots, h$ *be* $h = h(n, p)$ *linearly independent real surface harmonics of degree* n, *and let the* S_n^l *be orthonormal on* Ω *so that, for* $l, m = 1, 2, \ldots, h$,

(1) $$\int_{\Omega}\int S_n^l(\xi)\, S_n^m(\xi)\, d\Omega = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

then for any fixed unit-vector η

(2) $$\frac{C_n^{\frac{1}{2}p}[(\xi, \eta)]}{C_p^{\frac{1}{2}p}(1)} = (\omega/h) \sum_{l=1}^{h} S_n^l(\xi)\, S_n^l(\eta)$$

For notations compare 11.1(11), 11.1(12), 11.2(2), 11.1(16).

As a special case of (2) we have from theorem 2:

(3) $$C_n^{\frac{1}{2}p}[(\xi, \eta)]$$
$$= \frac{1}{2}\, \frac{p\,\omega}{(2n+p)} \sum \frac{\epsilon(m_p)}{N(m_k)} [H(m_k, +;\xi)\, H(m_k, -;\eta)$$
$$+ H(m_k, -;\xi)\, H(m_k, +;\eta)]$$

where the sum is to be taken over all integer values of m_k such that $n = m_0 \geq m_1 \geq \cdots \geq m_p \geq 0$ and where

(4) $\epsilon(0) = 1, \quad \epsilon(m) = 2 \qquad\qquad m > 0.$

From 11.2(21) we find that $S(m_k, \pm; \xi)$ vanishes if the last $p + 2 - l$ components of ξ vanish, i.e., if

$$\xi_{l+1} = \xi_{l+2} = \cdots = \xi_{p+2} = 0$$

except when

$$m_l = m_{l+1} = \cdots = m_p = 0.$$

Therefore, if we put

$$\xi = (\cos\rho, \sin\rho, 0, \ldots, 0)$$
$$\eta = (\cos\sigma, \sin\sigma, 0, \ldots, 0)$$

(3) becomes for $p > 1$

(5) $$C_n^{\frac{1}{2}p}(\cos\rho\cos\sigma + \sin\rho\sin\sigma) = C_n^{\frac{1}{2}p}[\cos(\rho - \sigma)]$$
$$= \frac{\Gamma(p-1)\, C_n^{\frac{1}{2}p-\frac{1}{2}}(1)}{\Gamma(\frac{1}{2}p)\;\Gamma(\frac{1}{2}p)} \sum_{m=0}^{n} B_{n,m} (\sin\rho)^m\, C_{n-m}^{m+\frac{1}{2}p}(\cos\rho)$$
$$\times (\sin\sigma)^m\, C_{n-m}^{m+\frac{1}{2}p}(\cos\sigma)$$

where

(6) $$B_{n,m} = 2^{2m}(n-m)!\,(p + 2m - 1)\,[\Gamma(m + \tfrac{1}{2}p)]^2\,[\Gamma(p + n + m)]^{-1}.$$

If we put in (3)

$$\xi = (\cos a, \ \sin a \cos \rho, \ \sin a \sin \rho, \ 0, \ \ldots, \ 0),$$
$$\eta = (\cos \beta, \ \sin \beta \cos \sigma, \ \sin \beta \sin \sigma, \ 0, \ \ldots, \ 0),$$

we obtain from (5) with $\rho - \sigma = \phi$ for $p > 1$

$$(7)\quad C_n^{\frac{1}{2}p}(\cos a \cos \beta + \sin a \sin \beta \cos \phi) = \frac{\Gamma(p-1)}{[\Gamma(\frac{1}{2}p)]^2} \sum_{m=0}^{n} B_{n,m} (\sin a)^m \times C_{n-m}^{m+\frac{1}{2}p}(\cos a)\,(\sin \beta)^m\, C_{n-m}^{m+\frac{1}{2}p}(\sin a)\, C_m^{\frac{1}{2}p-\frac{1}{2}}(\cos \phi)$$

where $B_{n,m}$ is given by (6). For $p = 1$ we find

$$(8)\quad P_n(\cos a \cos \beta + \sin a \sin \beta \cos \phi) = P_n(\cos a)\, P_n(\cos \beta) + 2 \sum_{m=0}^{n} \frac{(n-m)!}{(n+m)!} \mathrm{P}_n^m(\cos a)\, \mathrm{P}_n^m(\cos \beta) \cos m\phi,$$

where

$$(9)\quad P_n(x) = C_n^{\frac{1}{2}}(x)$$

is Legendre's polynomial and

$$(10)\quad \mathrm{P}_n^m(x) = (-1)^m\, \pi^{-\frac{1}{2}}\, \Gamma(m + \tfrac{1}{2})\, 2^m\, (1 - x^2)^{\frac{1}{2}m}\, C_{n-m}^{m+\frac{1}{2}}(x)$$

is an *associated Legendre function.*

Usually, (7) or, in the case $p = 1$, (8) are called the *addition theorem of ultraspherical polynomials.* We can obtain (3) (but not the whole theorem 4) by a repeated application of (7) and (8). In a modified form, (7) and (8) are also valid for a general C_n^{ν} where 2ν is not necessarily an integer; for this see 3.15(19) and 3.11(2).

The proof of theorem 4 will be based upon the fact that $C_n^{\frac{1}{2}p}[(\xi, \eta)]$ is an orthogonal invariant of ξ, η (see sec. 11.1.1 for the definition). We shall show first that apart from a constant factor $C_n^{\frac{1}{2}p}[(\xi, \eta)]$ is the only orthogonal invariant which is a surface harmonic of degree n. To do this we need

LEMMA 3. *Let $F(\mathfrak{x}, \mathfrak{y})$ be a polynomial in the components of $\mathfrak{x}$ and $\mathfrak{y}$ and let*

$$(11)\quad F(O\mathfrak{x}, O\mathfrak{y}) = F(\mathfrak{x}, \mathfrak{y})$$

for all orthogonal transformations O (compare sec. 11.1.1). Then there exists a polynomial $\Phi(u, v, w)$ in three variables u, v, w, such that

(12) $F(\mathfrak{x}, \mathfrak{y}) = \Phi[(\mathfrak{x}, \mathfrak{x}), (\mathfrak{x}, \mathfrak{y}), (\mathfrak{y}, \mathfrak{y})]$

identically in the components of $\mathfrak{x}$ *and* $\mathfrak{y}$.

Proof: If $\mathfrak{x}$, $\mathfrak{y}$ are fixed, we can find an orthogonal coordinate system such that

$$\mathfrak{x} = (\alpha, 0, 0, \ldots, 0), \quad \mathfrak{y} = (\beta, \gamma, 0, \ldots, 0),$$
$$(\mathfrak{x}, \mathfrak{x}) = \alpha^2, \quad (\mathfrak{x}, \mathfrak{y}) = \alpha\beta, \quad (\mathfrak{y}, \mathfrak{y}) = \beta^2 + \gamma^2,$$

and therefore

$$\alpha = u^{1/2}, \quad \beta = v/u^{1/2}, \quad \gamma = (uw - v^2)^{1/2}/u.$$

Since F is an orthogonal invariant this shows that it can be written as a polynomial

$$F = F^*(\alpha, \beta, \gamma) = F^*[u^{1/2}, v/u^{1/2}, (uw - v^2)^{1/2}/u]$$

in α, β, γ. Since there exist orthogonal transformations which have the effect that

$$\alpha \to -\alpha, \quad \beta \to -\beta, \quad \gamma \to \gamma$$

or

$$\alpha \to \alpha, \quad \beta \to \beta, \quad \gamma \to -\gamma,$$

we find that F^* is a polynomial in γ^2, α^2, β^2, $\alpha\beta$ and that we can write F^* in the form

(13) $F^* = u^{-m}\,\Phi^*(u, v, w),$

where m is an integer and Φ^* is a polynomial of u, v, w.

Interchanging the role of $\mathfrak{x}$ and $\mathfrak{y}$

(14) $w^{-k}\,\Psi(u, v, w) = u^{-m}\,\Phi^*(u, v, w),$

where k is an integer and Ψ is a polynomial. Since u, v, w are algebraically independent, we can conclude from (14) that $u^{-m}\,\Phi^*$ is a polynomial and this completes the proof of lemma 3.

LEMMA 4. *Let* ξ, η, ζ *be arbitrary unit-vectors in the* $(p+2)$*-dimensional space. Then*

(15) $$\int\int_{\Omega(\eta)} C_n^{\frac{1}{2}p}[(\xi, \eta)]\, C_n^{\frac{1}{2}p}[(\eta, \zeta)]\, d\Omega(\eta) = A(n, p)\, C_n^{\frac{1}{2}p}[(\xi, \zeta)],$$

where

(16) $$A(n, p) = C_n^{\frac{1}{2}p}(1)\frac{\omega}{h(n, p)} = \frac{2\pi^{1+\frac{1}{2}p}}{(n + \frac{1}{2}p)\,\Gamma(\frac{1}{2}p)}.$$

Lemma 4 is of the nature of a convolution theorem for the basic surface harmonic $C_n^{\frac{1}{2}p}[(\xi, \eta)]$.

To prove this lemma, let $\mathfrak{x}$ and $\mathfrak{z}$ be any two vectors, $\xi = \mathfrak{x}/||\mathfrak{x}||$, $\zeta = \mathfrak{z}/||\mathfrak{z}||$. Since

$$||\mathfrak{x}||^n C_n^{\frac{1}{2}p}[(\xi, \eta)], \qquad ||\mathfrak{z}||^n C_n^{\frac{1}{2}p}[(\eta, \zeta)]$$

are harmonic polynomials in the components of $\mathfrak{x}$ and $\mathfrak{z}$ respectively, we see that $||\mathfrak{x}||^n \; ||\mathfrak{z}||^n$ times the left-hand side of (15) is a harmonic polynomial both in $\mathfrak{x}$ and $\mathfrak{z}$, of degree n in each set of variables. Moreover, this harmonic polynomial is an orthogonal invariant in $\mathfrak{x}$ and $\mathfrak{z}$, for it remains unchanged if any orthogonal transformation is applied simultaneously to $\mathfrak{x}$, $\mathfrak{z}$ and η (and therefore to ξ, ζ and η) and the integral remains unchanged if any orthogonal transformation is applied to η. Thus by lemma 3, our harmonic polynomial is a polynomial in $||\mathfrak{x}||^2$, $||\mathfrak{z}||^2$, and $(\mathfrak{x}, \mathfrak{z}) = ||\mathfrak{x}|| \; ||\mathfrak{z}|| \; (\xi, \zeta)$. Therefore we find from lemma 1 that it is a multiple of

$$||\mathfrak{x}||^n \; ||\mathfrak{z}||^n \; C_n^{\frac{1}{2}p}[(\xi, \zeta)],$$

and this proves lemma 4. We can determine the factor $A(n, p)$ by putting

$$\xi = \zeta = (1, 0, \dots, 0)$$

which gives

$$(17) \quad A(n, p)\, C_n^{\frac{1}{2}p}(1) = \omega' \int_{-1}^{+1} [C_n^{\frac{1}{2}p}(x)]^2 (1 - x^2)^{\frac{1}{2}p - \frac{1}{2}} \, dx,$$

where ω' denotes the area of the hypersphere in the $(p + 1)$-dimensional space. From 3.15 (17), 11.1 (26), 11.1 (29) and 11.2 (2) we obtain (16).

Now we can describe the effect on the surface harmonics of an orthogonal transformation of ξ.

LEMMA 5. *Let $S_n^l(\xi)$, $l = 1, 2, \dots, h$ be a complete set of orthonormal surface harmonics of degree n, so that* (1) *holds, and let O be an orthogonal transformation of the* $(p + 2)$*-dimensional space. Then*

$$(18) \quad S_n^l(O\xi) = \sum_{k=1}^{h} g_{lk}\, S_n^k(\xi),$$

where the matrix G of the h^2 elements g_{lk} is an orthogonal matrix of $h = h(n, p)$ rows and columns, i.e.,

$$(19) \quad G'G = GG' = I.$$

Here G' is the transposed matrix of G, and I is the unit-matrix of $h(n, p)$ rows and columns.

Proof: Since Laplace's operator is invariant under orthogonal transformations (compare sec. 11.1), $S_n^l(O\xi)$ is a surface harmonic of degree n, and so can be expressed, in the form (15), in terms of the complete system $S_n^k(\xi)$.

Since the integrals in (1) remain unchanged if ξ is replaced by $O\xi$, it follows that also the $S_n^l(O\xi)$ form an orthonormal system, and this gives $GG' = I$. But it is well-known that from this we also have $G'G = I$ (see, e.g., Birkhoff and MacLane 1947, Chapter IX).

Now we can prove theorem 4 by showing that

$$(20) \quad \sum_{l=1}^{h} S_n^l(\xi)\, S_n^l(\eta) = \sum_{l=1}^{h} S_n^l(O\xi)\, S_n^l(O\eta)$$

is an orthogonal invariant of ξ and η. This follows from Lemma 5 and in particular $GG' = I$. From the proof of Lemma 4 we see that (20) must be a multiple of $C_n^{\frac{1}{2}p}[(\xi,\ \eta)]$. The constant factor can be determined by integrating the square of (20) with respect to η over the whole area Ω. On account of (1) this gives

$$(21) \quad \sum_{l=1}^{h} [S_n^l(\xi)]^2.$$

On the other hand we can see that it must be a certain multiple of $C_n^{\frac{1}{2}p}(1)$ by making $\xi = \eta$ in (12). By integrating (21) over $\Omega(\xi)$ we obtain h because of (1), and this leads to (2) in theorem 4.

From theorem 4 we have that for every surface harmonic $S_m(\xi)$ of degree m

$$(22) \quad \int\!\!\int_{\Omega(\xi)} C_n^{\frac{1}{2}p}[(\xi,\ \eta)]\, S_m(\xi)\, d\Omega(\xi) = \begin{cases} 0 & n \neq m, \\ (\omega/h)\, C_n^{\frac{1}{2}p}(1)\, S_p(\eta) & n = m. \end{cases}$$

From Lemma 2, in particular from 11.3 (8), 11.3 (11), we find by an application of Schwarz's inequality:

$$\lim_{n\to\infty} \int\!\!\int_{\Omega(\xi)} \{F[(\xi,\ \eta)] - \phi_n[(\xi,\ \eta)]\}\, S_n(\xi)\, d\Omega(\xi) = 0,$$

where F, ϕ_n are defined in 11.3, 11.3 (8). If we combine this with (22) we obtain (cf. Funk, Hecke, 1916, 1918):

FUNK-HECKE THEOREM: *Let $F(x)$ be a function of the real variable x which is continuous for $-1 \leq x \leq 1$ and let $S_n(\xi)$ be any surface harmonic of degree n. Then for any unit-vector η*

$$(23) \quad \int\!\!\int_{\Omega(\xi)} F[(\xi,\ \eta)]\, S_n(\xi)\, d\Omega(\xi) = \lambda_n\, S_n(\eta),$$

where the integral in (23) *is taken over the whole area of the unit-hypersphere Ω, and where*

(24) $\lambda_n = \dfrac{\omega'}{C_n^{\frac{1}{2}p}(1)} \int_{-1}^{1} F(x)\, C_n^{\frac{1}{2}p}(x)\,(1-x^2)^{\frac{1}{2}p-\frac{1}{2}}\,dx.$

Here ω' denotes the total area of the unit-hypersphere in the $(p+1)$-dimensional space,

$$\omega' = \frac{2\pi^{\frac{1}{2}p+\frac{1}{2}}}{\Gamma(\frac{1}{2}p+\frac{1}{2})}, \quad \frac{\omega'}{C_n^{\frac{1}{2}p}(1)} = \frac{(4\pi)^{\frac{1}{2}p}\, n!\,\Gamma(\frac{1}{2}p)}{(n+p-1)!}.$$

Erdélyi (1938) has shown that it is sufficient to assume that $|F(x)|$ and $|F(x)|^2$ are Lebesgue-integrable for $-1 \leq x \leq 1$, and he also has shown that

$$\lambda_n = i^n (2\pi)^{1+\frac{1}{2}p} \int_{-\infty}^{\infty} t^{-\frac{1}{2}p} J_{n+\frac{1}{2}p}(t)\, f(t)\, dt,$$

where

$$f(t) = (2\pi)^{-1} \int_{-1}^{1} e^{-ixt} F(x)\, dx.$$

Here J denotes a Bessel function. Note that

$$t^{-\frac{1}{2}p} J_{n+\frac{1}{2}p}(t) = t^n\, 2^{-n-\frac{1}{2}p} \sum_{m=0}^{\infty} \frac{(-t^2/4)^m}{m!\,\Gamma(n+m+1+\frac{1}{2}p)}$$

is a one-valued function of t.

11.5. The case $p = 1$, $h(n, p) = 2n + 1$

11.5.1. A generating function for surface harmonics in the three-dimensional case

(1) $\mathfrak{x} = (x_1, x_2, x_3)$

denotes a vector with three components. We define the polynomials $H_n^m(\mathfrak{x})$ by

(2) $[x_2 + ix_3 - 2x_1 t - (x_2 - ix_3)t^2]^n = t^n \sum_{m=-n}^{n} H_n^m(\mathfrak{x})\, t^m.$

If we substitute $-\tau^{-1}$ for t we find

(3) $\overline{H}_n^m = (-1)^m H_n^{-m},$

where a bar denotes the conjugate complex polynomial. The left-hand side of (2) can be written in the form $(\mathfrak{u}, \mathfrak{x})^n$ where

(4) $\mathfrak{u} = (-2t,\ 1-t^2,\ i+it^2).$

From $(\mathfrak{u}, \mathfrak{u}) = 0$ and from 11.1(14) we find that both sides in (2) satisfy

Laplace's equation for all t, i.e., $H_n^m(\mathfrak{x})$ is a homogeneous harmonic polynomial of degree n. The linear independence of H_n^m follows from the algebraic independence of

$$x_2 + ix_3, \quad -2x_1, \quad -(x_2 - ix_3).$$

With $r = ||\mathfrak{x}||$, $\xi = \mathfrak{x}/r$, the functions

$$(5) \quad r^{-n} H_n^m(\mathfrak{x}) = S_n^m(\xi) \qquad m = 0, \pm 1, \ldots, \pm n$$

form a complete set of linearly independent surface harmonics of degree n. From (3) we have

$$(6) \quad S_n^{-m}(\xi) = (-1)^m \bar{S}_n^m(\xi).$$

The orthogonality relations

$$(7) \quad \iint_\Omega S_n^m(\xi)\, \bar{S}_n^{m'}(\xi)\, d\Omega = \begin{cases} 0 & m \neq m' \\ 2\pi \dfrac{\Gamma(\frac{1}{2})\,\Gamma(n+1)}{\Gamma(n+3/2)} \dbinom{2n}{m+n} & m = m' \end{cases}$$

$$m, m' = 0, \pm 1, \ldots, \pm n,$$

in which the integral is to be taken over the whole area of the unit-sphere Ω, can be proved by introducing

$$(8) \quad \mathfrak{v} = (-2s,\ 1 - s^2,\ i + is^2)$$

and considering

$$(9) \quad \iint_\Omega (\mathfrak{u}, \xi)^n (\bar{\mathfrak{v}}, \xi)^n \, d\Omega(\xi)$$

which is an orthogonal invariant of $\mathfrak{u}$ and $\mathfrak{v}$ (cf. the proof of Lemma 4 in sec. 11.4). According to Lemma 2 it must be a polynomial in $(\mathfrak{u}, \mathfrak{u})$, $(\bar{\mathfrak{v}}, \bar{\mathfrak{v}})$, $(\mathfrak{u}, \bar{\mathfrak{v}})$, and since $(\mathfrak{u}, \mathfrak{u}) = (\bar{\mathfrak{v}}, \bar{\mathfrak{v}}) = 0$, (9) must be a multiple of $(\mathfrak{u}, \bar{\mathfrak{v}})^n$. If we introduce (2) [and the corresponding expansion of $(\bar{\mathfrak{v}}, \xi)^n$ into (9)] we find

$$(10) \quad (ts)^n \sum_{l, m = -n}^{n} t^l s^m \iint_\Omega S_n^l(\xi)\, \bar{S}_n^m(\xi)\, d\Omega$$

$$= \mu (\mathfrak{u}, \bar{\mathfrak{v}})^n = \mu\, 2^n (1 + st)^{2n}$$

and here we can compute μ by putting $s = t = 0$ and

$$(11) \quad \xi = (\cos\theta,\ \sin\theta \cos\phi,\ \sin\theta \sin\phi), \qquad d\Omega = \sin\theta\, d\theta\, d\phi,$$

which gives

$$(12) \quad 2^n \mu = \int_0^{2\pi} d\phi \int_0^\pi d\theta (\sin\theta)^{2n+1} = 2\pi\, \Gamma(\tfrac{1}{2})\, n!/\Gamma(n + 3/2).$$

By comparing the coefficients of $t^l s^m$ on both sides of (10) we obtain (7).

To obtain an explicit expression for $S_n^m(\xi)$ we apply Cauchy's formula to (2) and obtain

$$(13)\quad H_n^m(\mathfrak{x}) = \frac{1}{2\pi i} \int^{(0+)} (\mathfrak{u}, \mathfrak{x})^n t^{-n-m-1}\, dt$$

$$= (2\pi i)^{-1}(-1)^n (x_2 - ix_3)^n \int^{(0+)} \{[t + x_1/(x_2 - ix_3)]^2 - r^2(x_2 - ix_3)^{-2}\}^n t^{-n-m-1}\, dt.$$

If we put

$$t + x_1/(x_2 - ix_3) = \tau, \qquad x_1/(x_2 - ix_3) = \sigma,$$

this gives

$$(14)\quad H_n^m(\mathfrak{x}) = (2\pi i)^{-1}(ix_3 - x_2)^n \int^{(\sigma+)} [\tau^2 - r^2(x_2 - ix_3)^{-2}]^n (\tau - \sigma)^{-n-m-1}\, d\tau$$

$$(15)\quad = \frac{(-1)^m}{(n+m)!} (x_2 - ix_3)^n \frac{d^{n+m}}{d\tau^{n+m}} \left[\tau^2 - \left(\frac{r\sigma}{x_1}\right)^2 \right]^n$$

$$(16)\quad = \frac{r^n}{(n+m)!} \left(\frac{x_2 - ix_3}{r}\right)^m \frac{d^{n+m}}{d\xi_1^{n+m}} (1 - \xi_1^2)^n \qquad \xi_1 = x_1/r.$$

If we define the associated Legendre's functions $P_n^m(x)$ by

$$(17)\quad P_n^m(x) = (-1)^{n+m}\, 2^{-n}(n!)^{-1}(1 - x^2)^{\frac{1}{2}m} \frac{d^{n+m}}{dx^{n+m}} (1 - x^2)^n$$

$$m = 0, \pm 1, \ldots, \pm n,$$

we find that

$$(18)\quad S_n^m(\xi) = r^{-n} H_n^m(\mathfrak{x})$$

$$= (-1)^{n+m} \frac{2^n\, n!}{(n+m)!} (\xi_2 - i\xi_2)^m (1 - \xi_1^2)^{-\frac{1}{2}m} P_n^m(\xi_1),$$

and for the corresponding functions in spherical polar coordinates (see sec. 11.3),

$$(19)\quad Y_n^m(\theta, \phi) = S_n^m(\xi) = (-1)^{n+m} \frac{2^n\, n!}{(n+m)!} e^{-im\phi} P_n^m(\cos\theta).$$

According to (3) and (18) we have

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x).$$

The addition-theorem has been stated as equation 11.4 (8).

The orthogonality relations (7) give

$$(20)\quad \int_{-1}^{+1} [P_n^m(x)]^2\, dx = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}.$$

From (2) we obtain the generating function

$$(21)\quad [1 - st\cos\theta - \tfrac{1}{2}(1-t^2)\sin\theta]^{-1}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{2n} (n!/k!)\, P_n^{k-n}(\cos\theta)\, s^n t^k.$$

For other properties of the P_n^m see sec. 3.6.1.

11.5.2. Maxwell's theory of poles

Let x_1, x_2, x_3, be independent variables, let $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and define the differential operator D_k by

$$(22)\quad D_k = \frac{\partial}{\partial x_k} \qquad k = 1, 2, 3.$$

Since

$$(23)\quad \Delta r^{-1} = (D_1^2 + D_2^2 + D_3^2)\, r^{-1} = 0,$$

clearly $D_1^a\, D_2^b\, D_3^c\, r^{-1}$ satisfies Laplace's equation. Moreover this is clearly of the form of a homogeneous polynomial of degree $n = a + b + c$ multiplied by r^{-2n-1}. Lastly, it can be verified that for every homogeneous polynomial H_n of degree n, the statements

$$\Delta H_n = 0 \quad \text{and} \quad \Delta H_n\, r^{-2n-1} = 0$$

are equivalent. Thus we find

$$(24)\quad D_1^a D_2^b D_3^c\, r^{-1} = H_n(x_1, x_2, x_3)\, r^{-2n-1} \qquad n = a + b + c.$$

It is a consequence of this observation that to every homogeneous polynomial of degree n of three quantities D_1, D_2, D_3 for which

$$(25)\quad D_1^2 + D_2^2 + D_3^2 = 0$$

there corresponds a harmonic polynomial of x_1, x_2, x_3 of degree n. Comparing this with the remarks after 11.7(12), it seems plausible that we can obtain all harmonic polynomials from (24). Actually, it can be shown that (see Hobson, 1931, Chap. 4, Nos. 85-92)

$$(26)\quad D_1^{n-m}(D_2 \pm iD_3)^m\, \frac{1}{r} = \frac{(-1)^{n-m}(n-m)!}{r^{n+1}}\, e^{\pm im\phi}\, P_n^m(\cos\theta)$$

$$m = 0, 1, \ldots, n,$$

and

(27) $x_1 = r\cos\theta, \quad x_2 = r\sin\theta\cos\phi, \quad x_3 = r\sin\theta\sin\phi.$

According to (19) this shows that all spherical harmonics can be obtained from (24).

For geometrical reasons, the surface harmonics in (26) are called *zonal* if $m = 0$, *sectorial* if $m = n$ and *tesseral* if $1 \le m \le n-1$. For this and for the following remarks on Maxwell's results see Hobson (1931) and Maxwell (1873, 1892).

Let

(28) $\eta_k = (\alpha_k, \beta_k, \gamma_k) \qquad k = 1, 2, \dots, n$

be unit-vectors which therefore define points on the unit-sphere. These points will be called *poles*. Then the surface harmonic of degree n with the poles η_k is defined by

(29) $S_n(\eta_k) = (-1)^n r^{n+1} [\prod_{k=1}^{n} (\alpha_k D_1 + \beta_k D_2 + \gamma_k D_3)] r^{-1}.$

Introducing n parameters, $t_1, \dots, t_n$, we find that this is the coefficient of $t_1 t_2 \cdots t_n$ in the expansion of

(30) $$\frac{1}{n!} T^n P_n \left[\frac{\Sigma\, t_k(\xi, \eta_k)}{T}\right]$$

where

(31) $$T^2 = \sum_{k,l=1}^{n} t_k t_l (\eta_k, \eta_l), \qquad \xi = \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}\right)$$

and where the sum in (30) is to be taken over $k = 1, 2, \dots, n$. This is a function of the cosines of the angles between the vectors $\xi, \eta_1, \dots, \eta_k$. The standard surface harmonics (26) are obtained when the vectors η_k coincide with some of the axes of the coordinate system.

Van der Pol (1936) and Erdélyi (1937) have extended (26) to solutions of the wave equation $\Delta u + k^2 u = 0$ by showing that

(32) $$i^{n-m}\left(\frac{\pi}{2r}\right)^{1/2} J_{n+1/2}(kr)\, P_n^m(\cos\theta)\, e^{im\phi}$$
$$= k^{-m}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3}\right)^m P_n^{(m)}\left(\frac{-i}{k}\frac{\partial}{\partial x_1}\right)\frac{\sin kr}{kr},$$

where $P_n^{(m)}$ denotes the m-th derivative of the Legendre polynomial P_n, where P_n^m is defined by (17), $J_{n+1/2}$ denotes the Bessel function of the first kind and of order $n + 1/2$ and $r, \theta, \phi, x_1, x_2, x_3$ are connected by (27).

11.6. The case $p = 2$, $h(n, p) = (n+1)^2$

From now on let $\mathfrak{y}$ be a vector with four components

(1) $\mathfrak{y} = (y_1, y_2, y_3, y_4)$,

and let

(2) $\eta = \mathfrak{y}/\rho, \quad \rho = ||\mathfrak{y}||$.

We introduce the vectors

(3) $\mathfrak{u} = (i - its, \, -it - is, \, -t + s, \, 1 + ts)$

(4) $\mathfrak{v} = (i - i\tau\sigma, \, -i\tau - i\sigma, \, -\tau + \sigma, \, 1 + \tau\sigma)$

for which we have

(5) $(\mathfrak{u}, \mathfrak{u}) = (\mathfrak{v}, \mathfrak{v}) = 0, \quad (\mathfrak{u}, \overline{\mathfrak{v}}) = 2(1 + t\tau)(1 + s\sigma)$.

From (5) we find again as in sec. 11.5.1 that the $(n+1)^2$ polynomials $H_n^{k,\,l}(\mathfrak{y})$ defined by

(6) $$(\mathfrak{u}, \mathfrak{y})^n = \sum_{k,\, l=0}^{n} \binom{n}{k} H_n^{k,\,l}(\mathfrak{y})\, t^k s^l$$

are harmonic polynomials of degree n.

By the same argument as in sec. 11.5.2 we find that

(7) $$\int\limits_{\Omega(\eta)}\int (\mathfrak{u}, \eta)^n (\overline{\mathfrak{v}}, \eta)^n \, d\Omega(\eta) = \frac{2^{1-n}\pi^2}{n+1} (\mathfrak{u}, \overline{\mathfrak{v}})^n,$$

and therefore the surface harmonics

(8) $$S_n^{k,\,l}(\eta) = \rho^{-n} H_n^{k,\,l}(\mathfrak{y})$$

form an orthogonal set of $h(n, 2) = (n+1)^2$ linearly independent surface harmonics where

(9) $$\int\limits_{\Omega}\int S_n^{k,\,l}(\eta)\, \overline{S}_n^{k',\,l'}(\eta)\, d\Omega = \begin{cases} 0 & k \neq k' \quad \text{or} \quad l \neq l', \\ \dfrac{2\pi^2}{n+1} \dbinom{n}{l} \Big/ \dbinom{n}{k} & k = k', \quad l = l'. \end{cases}$$

From (6) we also have

(10) $\overline{S}_n^{k,\,l}(\eta) = (-1)^{k+l} S_n^{n-k,\, n-l}(\eta)$.

In order to find explicit expressions for the $S_n^{k,\,l}$ we introduce

(11) $a = y_4 + iy_1, \quad b = y_3 - iy_2, \quad c = -y_3 - iy_2, \quad d = y_4 - iy_1.$

Then

(12) $\rho = ||\mathfrak{y}|| = (ad - bc)^{1/2}, \quad (\mathfrak{u}, \mathfrak{y}) = a + bs + (c + ds)\, t,$

and we obtain from (6) that

$$(13) \quad \sum_{l=0}^{n} H_n^{k,l}(\mathfrak{y})\, s^l = (a + bs)^{n-k} (c + ds)^k,$$

$$(14) \quad H_n^{k,l}(\mathfrak{y}) = \frac{1}{2\pi i} \int^{(0+)} (a + bs)^{n-k} (c + ds)^k s^{-l-1}\, ds.$$

Putting

(15) $\sigma = -s(bc - ad)/bd,$

(16) $\sigma_0 = ad/(ad - bc) = (y_1^2 + y_4^2)/(y_1^2 + y_2^2 + y_3^2 + y_4^2),$

and expressing a, b, c, d in terms of the y_i,

$$(17) \quad H_n^{k,l}(\mathfrak{y}) = \frac{(-1)^k}{2\pi i} \rho^n (d/\rho)^{k+l-n} (b/\rho)^{l-k} \int^{(\sigma_0+)} \sigma^{n-k} (1-\sigma)^k \frac{d\sigma}{(\sigma-\sigma_0)^{l+1}}$$

$$(18) \quad = \frac{(-1)^k}{l!} \rho^n \left(\frac{y_4 - iy_1}{\rho}\right)^{k+l-n} \left(\frac{y_3 - iy_2}{\rho}\right)^{l-k} \frac{d^l}{d\sigma_0^l} \sigma_0^{n-k} (1 - \sigma_0)^k$$

where σ_0 is given by (16). Here the l-th derivative can be expressed by a hypergeometric function (which is a Jacobi polynomial) and our final result is [cf. 2.8(27), 2.1(2), (1) and (2)] as follows.

If $n \geq k + l$

$$(19) \quad S_n^{k,l}(\eta) = \rho^{-n} H_n^{k,l}(\mathfrak{y})$$
$$= (-1)^k \binom{n-k}{l} (\eta_4 + i\eta_1)^{n-k-l} (\eta_3 + i\eta_2)^{k-l}$$
$$\times {}_2F_1(-l, n - l + 1; n - k - l + 1; \eta_4^2 + \eta_1^2)$$

$$(20) \quad = (-1)^k (\eta_4 + i\eta_1)^{n-k-l} (\eta_3 + i\eta_2)^{k-l}$$
$$\times P_l^{(n-k-l,\, k-l)} (\eta_3^2 + \eta_2^2 - \eta_4^2 - \eta_1^2).$$

If $n < k + l$

$$(21) \quad S_n^{k,l}(\eta) = \rho^{-n} H_n^{k,l}(\mathfrak{y})$$

$$= (-1)^{n-l} \binom{k}{n-l} (\eta_4 - i\eta_1)^{k+l-n} (\eta_3 - i\eta_2)^{l-k}$$
$$\times {}_2F_1(l - n, l + 1; l + k - n + 1; \eta_4^2 + \eta_1^2)$$

$$(22)\quad S_n^{k,\,l}(\eta) = \rho^{-n} H_n^{k,\,l}(\mathfrak{y})$$
$$= (-1)^{n-l}(\eta_4 - i\eta_1)^{k+l-n}(\eta_3 - i\eta_2)^{l-k}$$
$$\times \mathrm{P}_{n-l}^{(l+k-n,\, l-k)}(\eta_3^2 + \eta_2^2 - \eta_4^2 - \eta_1^2),$$

where $\mathrm{P}_n^{(\alpha,\beta)}$ denotes a Jacobi-polynomial (see Chap. 10).

If we introduce polar coordinates, the expressions (20), (22) for the $S_n^{k,\,l}$ became rather complicated and it is better to use the functions 11.2(23) (in the special case $p = 2$) for this purpose. But for the transformation of spherical harmonics the $S_n^{k,\,l}$ (with an even value of n) are very useful; they also satisfy some relations which do not have an analogue in the cases where $p \neq 2$. These relations (which will be proved in sec. 11.7) are the following ones (written in terms of the $H_{2n}^{k,\,l}$ instead of the $S_{2n}^{k,\,l}$).

Let $\mathfrak{y}$, $\mathfrak{z}$ be two vectors with four components each, and let $\mathfrak{w}$ be a vector with the components

$$(23)\quad \begin{aligned} w_1 &= y_1 z_4 + y_4 z_1 - y_2 z_3 + y_3 z_2 \\ w_2 &= y_2 z_4 + y_4 z_2 - y_3 z_1 + y_1 z_3 \\ w_3 &= y_3 z_4 + y_4 z_3 - y_1 z_2 + y_2 z_1 \\ w_4 &= y_4 z_4 - y_1 z_1 - y_2 z_2 - y_3 z_3. \end{aligned}$$

If we introduce quaternions(see Birkhoff and MacLane, 1947, Chap. VIII, 5) this can be written in the form

$$(24)\quad w_4 + iw_2 + jw_3 + kw_1$$
$$= (z_4 + iz_2 + jz_3 + kz_1)(y_4 + iy_2 + jy_3 + ky_1),$$

where 1, i, j, k are the fundamental units. Then we have the addition theorem:

$$(25)\quad H_{2n}^{k,\,l}(\mathfrak{w}) = \sum_{m=0}^{2n} H_{2n}^{k,\,m}(\mathfrak{z})\, H_{2n}^{m,\,l}(\mathfrak{y}).$$

The matrix

$$(26)\quad [H_{2n}^{k,\,l}(\mathfrak{y})] \qquad k,\, l = 0,\, 1,\, \dots,\, 2n$$

where k denotes the rows and l denotes the columns has the determinant

$$(27)\quad (y_1^2 + y_2^2 + y_3^2 + y_4^2)^{n(2n+1)},$$

the characteristic roots

$$(28)\quad \lambda_1^m \lambda_2^{2n-m} \qquad m = 0,\, 1,\, \dots,\, 2n$$

where λ_1, λ_2 are the roots of the equation [cf. (4)]

$$(29)\quad \begin{vmatrix} a-\lambda, & b \\ c, & d-\lambda \end{vmatrix} = 0,$$

and the *trace*

$$(30)\quad \sum_{l=0}^{2n} H_{2n}^{l,\,l}(\mathfrak{y}) = \frac{\rho^{2n}}{2n+1}\, T'_{2n+1}(y_4/\rho)$$

where T'_{2n+1} denotes the derivative of the Tchebichef-polynomial 11.1(20).

11.7. The transformation formula for spherical harmonics

Let $\mathfrak{x}$ be a vector with three components and $\mathfrak{y}$ be a vector with four components. We use the notations

$$(1)\quad \|\mathfrak{x}\|_3 = r, \quad \|\mathfrak{y}\|_4 = \rho, \quad \xi = \mathfrak{x}/r, \quad \eta = \mathfrak{y}/\rho.$$

We shall now show that every orthogonal transformation O of $\mathfrak{x}$ with the determinant +1 can be uniquely described by a unit-vector η. If $\det O = +1$, there exists a vector $\mathfrak{x}_0 \neq 0$ (the axis of rotation) such that

$$(2)\quad \mathfrak{x}_0 = O\,\mathfrak{x}_0 .$$

The transformation O is completely defined if $\mathfrak{x}_0$ and the angle of rotation ψ are given. Since $-\mathfrak{x}_0$ is also an axis of rotation we can choose $\mathfrak{x}_0$ in such a way that $0 \le \psi \le \pi$. If ψ is zero, every vector $\mathfrak{x}_0$ is an axis of rotation, and in this case we put $\mathfrak{x}_0 = 0$. We may assume therefore that

$$(3)\quad \|\mathfrak{x}_0\|_3 = \sin \tfrac{1}{2}\psi \qquad 0 \le \psi \le \pi$$

which means that the components $x_{0,1}$, $x_{0,2}$, $x_{0,3}$ of $\mathfrak{x}_0$ are given by

$$x_{0,l} = \cos a_l \sin \tfrac{1}{2}\psi \qquad l = 1, 2, 3,$$

where a_l is the angle between the axis of rotation and the x_l axis.

Now we define the four-dimensional unit-vector

$$(4)\quad \eta = (\cos a_1 \sin \tfrac{1}{2}\psi,\ \cos a_2 \sin \tfrac{1}{2}\psi,\ \cos a_3 \sin \tfrac{1}{2}\psi,\ \cos \tfrac{1}{2}\psi)$$

and put $\mathfrak{y} = \rho\eta$. Then the orthogonal matrix O can be written in the form

$$(5)\quad O = (y_4 I - A)(y_4 I + A)^{-1} = (1/\rho^2)(\rho^2 I - 2y_4 A + 2A^2)$$

$$(6)\quad = \frac{1}{\rho^2}\begin{pmatrix} y_4^2+y_1^2-y_2^2-y_3^2, & 2y_1y_2-2y_3y_4, & 2y_1y_3+2y_2y_4 \\ 2y_1y_2+2y_3y_4, & y_4^2+y_2^2-y_1^2-y_3^2, & 2y_2y_3-2y_1y_4 \\ 2y_1y_3-2y_2y_4, & 2y_2y_3+2y_1y_4, & y_4^2+y_3^2-y_1^2-y_2^2 \end{pmatrix}$$

where

$$(7)\quad I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad A=\begin{pmatrix}0&y_3&-y_2\\-y_3&0&y_1\\y_2&-y_1&0\end{pmatrix}.$$

This is Cayley's representation of the orthogonal group (see H. Weyl (1939), p. 169ff). In the form (6) it is valid without exceptions, i.e., even if the determinant of $y_4 I+A$ vanishes.

With the notations (1), (2), (4), (5), 11.5(18), 11.5(19), 11.6(8) we have the

TRANSFORMATION FORMULA OF SPHERICAL HARMONICS

$$(8)\quad S_n^k(O\xi)=\sum_{l=-n}^{n}(-1)^{k+l}\binom{2n}{n+k}\Big/\binom{2n}{n+l}\,S_{2n}^{n+k,\,n+l}(\eta)\,S_n^l(\xi).$$

This formula shows the effect of an orthogonal transformation O of the three-dimensional space upon the surface harmonics on the sphere, and it gives the coefficients of the linear transformation of S_n^k in terms of surface harmonics in four-dimensional space with Cayley's parameters of O as variables.

A formula equivalent to (8) has been proved by Adam Schmidt (1899) (see also Hoenl, 1934). In an unpublished note left by Bateman, it is shown that the coefficients of the S_n^l in (8) can be expressed by a hypergeometric series. In its present form, (8) is due to Herglotz, whose proof will be given here.

In order to prove (8), we show:

(i) We can map the harmonic polynomials $H_n^m(\mathfrak{x})$ upon the product of the powers of two variables w_1, w_2 by putting

$$(9)\quad x_1=w_1w_2,\quad x_2+ix_3=w_1^2,\quad -x_2+ix_3=w_2^2$$

because then 11.5(2) becomes

$$(10)\quad (w_1^2-2w_1w_2t+w_2^2t^2)^n=\sum_{m=-n}^{n}H_n^m(\mathfrak{x})\,t^{n+m}$$

and therefore

$$(11)\quad H_n^m(\mathfrak{x})=(-1)^m\binom{2n}{n+m}w_1^{n-m}w_2^{n+m}.$$

Although this implies a relation between x_1, x_2, x_3, namely,

$$(12)\quad x_1^2+x_2^2+x_3^2=0,$$

[cf. 11.5(25)], we see from (11) that the complete set $H_n^m(\mathfrak{x})$ of linearly independent harmonic polynomials is mapped upon the set of linearly independent products of powers of w_1 and w_2.

(ii) If we define a, b, c, d by 11.6(11), then the linear substitution

(13) $w_1' = aw_1 + bw_2, \quad w_2' = cw_1 + dw_2$

leads to the substitution for $w_1 w_2$, w_1^2, w_2^2 given by

(14) $$\begin{aligned} w_1' w_2' &= (ad+bc)\, w_1 w_2 + ac\, w_1^2 + bd\, w_2^2 \\ w_1'^2 &= 2ab\, w_1 w_2 \qquad + a^2 w_1^2 + b^2 w_2^2 \\ w_2'^2 &= 2cd\, w_1 w_2 \qquad + c^2 w_1^2 + d^2 w_2^2 \end{aligned}$$

and if we put $w_1' w_2' = x_1'$, $w_1'^2 = x_2' + ix_3'$, $w_2'^2 = -x_2' + ix_3'$ and assume that

$$ad - bc = y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1,$$

(14) is precisely the linear substitution

(15) $\mathfrak{x}' = O\mathfrak{x}, \qquad \mathfrak{x}' = (x_1', x_2', x_3')$

where O is given by (6). This is the representation of the ternary orthogonal group by unitary binary substitutions (cf. Van der Waerden, 1932, Chap. III, 16).

(iii) With the expressions in 11.6(11) for a, b, c, d and with $s = w_2/w_1$ we obtain from 11.6(13)

(16) $$\sum_{l=0}^{2n} H_{2n}^{k,l}(\mathfrak{y})\, w_1^{2n-l} w_2^{l} = (aw_1 + bw_2)^{2n-k}(cw_1 + dw_2)^k.$$

If $||\mathfrak{y}|| = 1$ we obtain the transformation formula (8) from (11), (13), (14), (15), (16) and (6).

The formulas 11.6(25) to 11.6(30) are consequences of the fact that (8) can be considered as a representation of the orthogonal group (cf. H. Weyl, 1939, for the concepts used here). In particular, 11.6(30) follows from the fact that the characteristic roots of an orthogonal matrix O for which det $O = 1$ are completely determined by the angle of rotation, i.e., by y_4/ρ. Since the characteristic roots of a matrix U corresponding to O in a representation of the orthogonal group depend only on the characteristic roots of O, the trace of U (which is the negative sum of the characteristic roots of U) must depend on y_4/ρ only. According to lemma 1, the expression on the right-hand side of 11.6(30) and its multiples are the only surface harmonics satisfying this condition.

Y. Satô (1950) expressed the transformation O as a product of three simple transformations, proved equation (8) for these transformations and gave a table of the coefficients in (8) for $n \leq 7$.

11.8. The polynomials of Hermite - Kampé de Fériet

A different approach to the investigation of surface harmonics has been made by Hermite, Didon and Kampé de Fériet. The far reaching and important theory as developed by these authors has been fully presented in the second part of the book by Appell and Kampé de Fériet (1926). Rather than giving all the results obtained there in detail we shall confine ourselves to a short indication of what can be found there and refer the reader to the book itself for a full account of the theory.

Generalizing Maxwell's construction of surface harmonics in the three-dimensional space we define the following functions of the $p+2$ components of a vector $\mathfrak{x}$,

$$(1)\quad w_{m_1,\ldots,m_p}(\mathfrak{x}) = \frac{(-1)^n}{m_1!\cdots m_p!}\,\frac{\partial^n}{\partial x_1^{m_1}\cdots\partial x_p^{m_p}}\,(r^{-p}),$$

where $r = ||\mathfrak{x}||$ and where the non-negative integers $m_1, \ldots, m_p$ satisfy

$$(2)\quad m_1 + m_2 + \cdots + m_p = n.$$

The function on the left-hand side of (1) satisfies Laplace's equation; it is the coefficient of

$$(3)\quad a_1^{m_1}\, a_2^{m_2} \ldots a_p^{m_p}$$

in the expansion of

$$(4)\quad [(x_1 - a_1)^2 + \cdots + (x_p - a_p)^2 + x_{p+1}^2 + x_{p+2}^2]^{-\frac{1}{2}p}$$

into a series of products of powers of $a_1, \ldots, a_p$.

Then

$$(5)\quad V_{m_1,\ldots,m_p}(\xi_1, \ldots, \xi_p) = r^{n+p}\, w_{m_1,\ldots,m_p}(\mathfrak{x})$$

is a surface harmonic of degree n which depends on the first p components of $\mathfrak{x}/r$. As a generating function, we have

$$(6)\quad (1 - 2a_1\xi_1 - \cdots - 2a_p\xi_p + a_1^2 + \cdots + a_p^2)^{-\frac{1}{2}p} = \sum a_1^{m_1} \ldots a_p^{m_p}\, V_{m_1,\ldots,m_p}(\xi_1, \ldots, \xi_p)$$

where the sum is to be taken over all non-negative integers $m_1, \ldots, m_p$. Explicit expressions and expressions in terms of hypergeometric functions of p variables for the functions V have been given by Appell-Kampé de Fériet (1926). The connection with the ultraspherical polynomials is given by

$$(7)\quad \sum a_1^{m_1} \cdots a_p^{m_p} V_{m_1,\dots,m_p}(\xi_1,\dots,\xi_p)$$

$$= (a_1^2 + \cdots + a_p^2)^{\frac{1}{2}n} C_n^{\frac{1}{2}p} \left[\frac{a_1\xi_1 + \cdots + a_p\xi_p}{(a_1^2 + \cdots + a_p^2)^{\frac{1}{2}}} \right]$$

where the sum is taken over all non-negative integers $m_1, \dots, m_p$ satisfying (2). From this, recurrence formulas can be obtained.

With the definition

$$(8)\quad V^{(s)}_{m_1,\dots,m_q}(\xi_1,\dots,\xi_q)$$

$$= V_{m_1,\dots,m_q,0,\dots,0}(\xi_1,\dots,\xi_q,\dots,\xi_{q+s-1})$$

where $s, q = 1, 2, 3, \dots$, it is found that the functions

$$(9)\quad (1-\xi_1^2 - \cdots - \xi_p^2)^{\frac{1}{2}l} e^{\pm il\phi}\, V^{(2l+1)}_{l_1,\dots,l_p}(\xi_1,\dots,\xi_p)$$

form a complete set of linearly independent surface harmonics of degree n, if the non-negative integers $l, l_1, \dots, l_p$ satisfy

$$(10)\quad l + l_1 + \cdots + l_p = n$$

and

$$(11)\quad e^{i\phi} = (\xi_{p+1} + i\xi_{p+2})(1-\xi_1^2 - \cdots - \xi_p^2)^{-\frac{1}{2}}$$

$$= (\xi_{p+1} + i\xi_{p+2})(\xi_{p+1}^2 + \xi_{p+2}^2)^{-\frac{1}{2}}.$$

The functions in (11) do not form an orthogonal set on the unit-sphere; the integral

$$\int_\Omega \int (1-\xi_1^2 - \cdots - \xi_p^2)^l V^{(2l+1)}_{l_1,\dots,l_p} V^{(2l+1)}_{m_1,\dots,m_p}\, d\Omega$$

vanishes only if either

$$l_1 + \cdots + l_p \neq m_1 + \cdots + m_p,$$

or all the differences $l_1 - m_1, \dots, l_p - m_p$ are odd numbers. For this reason, a second set of functions U is introduced by means of the generating function

$$(12)\quad \Sigma\, a_1^{m_1} \cdots a_p^{m_p} U^{(l)}_{m_1,\dots,m_p}(\xi_1,\dots,\xi_p)$$

$$= [(a_1\xi_1 + \cdots + a_n\xi_n - 1)^2 + (a_1^2 + \cdots + a_p^2)(1-\xi_1^2 - \cdots - \xi_p^2)]^{\frac{1}{2}l}.$$

These functions are surface harmonics in $p + l + 1$ dimensional space, and the U and V together form a *biorthogonal system* so that

$$(13)\quad \int_{\Omega}\int (1 - \xi_1^2 - \cdots - \xi_p^2)^{\frac{1}{2}l-\frac{1}{2}} V^{(l)}_{l_1, \ldots, l_p} U^{(l)}_{m_1, \ldots, m_p} d\Omega = 0$$

unless $m_1 = l_1$, $m_2 = l_2$, $\ldots$, $m_p = l_p$. Thus the functions U can be used to determine the coefficients in the expansion of a function on the hypersphere, and in particular of a hypersurface-harmonic of given degree, in terms of the functions (11).

For many other results about the functions U and V, in particular for partial differential equations, expressions in terms of Lauricella's generalized hypergeometric series and expansion of arbitrary functions in terms of the U and V compare Appell-Kampé de Fériet (1926). A generalization of the $V^{(l)}_{m_1, \ldots, m_p}$ for values of l which are not a positive integer, see A. Angelescu.

Generalizations of surface harmonics connected with operators other than Laplace's operator have been investigated by M. H. Protter.

REFERENCES

Angelescu, Aurel, 1916: *Sur les polynomes généralisant les polynomes de Legendre et d'Hermite et sur le calcul approché des integrales multiples.* Thèse no. 1579, Paris.

Appell, Paul and J. Kampé de Fériet, 1926: *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite,* Gauthier-Villars.

Birkhoff, Garrett and Saunders MacLane, 1947: *A survey of modern algebra,* New York.

Erdélyi, Arthur, 1937: *Physica* 4, 107-120.

Erdélyi, Arthur, 1938: *Math. Ann.* 115, 456-465.

Funk, Paul, 1916: *Math. Ann.* 77, 136-152.

Gegenbauer, Leopold, 1877: *Akad. Wiss. Wien., S.-B.* IIa, 75, 891-905.

Gegenbauer, Leopold, 1884: *Denkschriften Akad. Wiss. Wien. Math. Naturw. Kl.* 48, 293-316.

Gegenbauer, Leopold, 1888: *Akad. Wiss. Wien., S.-B.* IIa, 97, 259-270.

Gegenbauer, Leopold, 1890: *Denkschriften Akad. Wiss. Wien. Math. Naturw. Kl.* 57, 425-480.

Gegenbauer, Leopold, 1891: *Akad. Wiss. Wien., S.-B.* IIa, 100, 225-244.

Gegenbauer, Leopold, 1893: *Akad. Wiss. Wien., S.-B.* IIa, 102, 942-950.

Hecke, Erich, 1918: *Math. Ann.* 78, 398-404.

Hobson, E. W., 1931: *The theory of spherical and ellipsoidal harmonics,* Cambridge.

Hoenl, H., 1934: *Z. Physik* 89, 244-253.

Kogbetliantz, Ervand, 1924: *J. Math. Pures Appli.,* IX Ser., 3, 107-187.

Koschmieder, Lothar, 1929: *Math. Ann.* 101, 120-125.

Koschmieder, Lothar, 1931: *Math. Ann.* 104, 387-402.

Magnus, Wilhelm, 1949: *Abh. Math. Sem. Univ. Hamburg* 16, 77-94.

Maxwell, J. C., 1873, 1892: *A treatise on electricity and magnetism,* Vol. 1, Chapter 9, Oxford, Third edition 1892.

Nielsen, Niels, 1911: *Théorie des Fonctions Métasphériques,* Gauthier-Villars.

Pólya, George and Burnett Meyer, 1950: *C. R. Acad. Sci. Paris,* 228, 28-30, 1083-1084.

Protter, M. H., 1949: *Trans. Amer. Math. Soc.* 63, 314-341.

Satô, Yasuo, 1950: *Bull. Earthquake Res. Inst. Tokyo* 28, 1-22, 175-217.

Schmidt, Adam, 1899: *Z. Math. Phys.* 44, 327-338.

Sommerfeld, Arnold, 1943: *Math. Ann.* 119, 1-20.

Van der Pol, Balthasar, 1936: *Physica* 3, 385-392.

REFERENCES

Van der Waerden, B. L., 1932: *Die gruppentheoretische Methode in der Quantenmechanik,* Berlin.

Weyl, Hermann, 1939: *The classical groups,* Princeton University Press, Princeton, New Jersey.

Widder, D. V., 1947: *Advanced calculus,* New York.

CHAPTER XII

ORTHOGONAL POLYNOMIALS IN SEVERAL VARIABLES

12.1. Introduction

Let R be a region in n-dimensional Euclidean space in which $x_1, \ldots, x_n$ are Cartesian coordinates, and let $w(x) = w(x_1, \ldots, x_n)$ be a non-negative weight function defined in R. For any two functions $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ we put

$$(1) \quad (f, g) = \int_R \cdots \int f(x_1, \ldots, x_n)\, g(x_1, \ldots, x_n)\, w(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

and call this the *scalar product* of f and g: it is defined whenever f and g are defined in R and the integral exists. Two functions are called *orthogonal* (with respect to the weight function w) if their scalar product vanishes.

Given a weight function and any sequence of linearly independent functions $\psi_1, \psi_2, \ldots$ for which all scalar products (ψ_i, ψ_j) are defined, the process of *orthogonalization* described in sec. 10.1 may be carried out with respect to the scalar product (1), and leads to an orthogonal system which is determined uniquely up to a constant factor in each function. This is no longer true of a *multiple sequence* of functions. Before proceeding to orthogonalize a multiple sequence, it is necessary to rearrange it as a simple sequence. To every possible rearrangement there corresponds an orthogonal system, and in general different rearrangements will lead to different orthogonal systems. Thus, a multiple sequence does not, in general, determine an orthogonal system (essentially) uniquely; moreover, in most cases, the rearrangement destroys the symmetry of the multiple sequence. For these reasons it is often preferable, in the case of a given multiple sequence

$$\{\psi_{m_1, \ldots, m_n}(x_1, \ldots, x_n)\}$$

of linearly independent functions, to construct two multiple sequences

$$\{\phi_{m_1, \ldots, m_n}(x_1, \ldots, x_n)\} \quad \text{and} \quad \{\chi_{m_1, \ldots, m_n}(x_1, \ldots, x_n)\}$$

which form a *biorthogonal system*, i.e., for which the integral

$$(\phi_{m_1, \ldots, m_n}, \chi_{m'_1, \ldots, m'_n})$$

vanishes except in case $m_1 = m'_1$, $m_2 = m'_2$, ... , $m_n = m'_n$. Biorthogonal systems give a greater freedom of choice which may be utilized to preserve symmetry.

These remarks are pertinent when dealing with *orthogonal polynomials*. In order to orthogonalize the multiple sequence of *monomials*

$$(2) \quad x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \qquad m_1, m_2, \ldots, m_n = 0, 1, \ldots,$$

it is necessary to *order* monomials in a simple sequence. Except in the case of very special regions and weight functions, there is no (essentially) unique system of orthogonal polynomials, and any system of orthogonal polynomials obtained by an ordering of the monomials (2) is necessarily unsymmetric in the x_1, ... , x_n. The equal standing of the variables may be preserved by adopting a biorthogonal system of polynomials.

There does not seem to be an extensive general theory of orthogonal polynomials in several variables. Special biorthogonal systems, corresponding to the classical orthogonal polynomials in one variable, are known, and have been investigated in some detail. The book by Appell and Kampé de Fériet gives a comprehensive account, and an extensive bibliography, of these investigations up to about 1925.

In the present chapter we shall give a brief account of the general properties of orthogonal polynomials in two variables, and then discuss in somewhat greater detail those systems of biorthogonal polynomials in two and more variables which correspond to, and are generalizations of, the classical systems of orthogonal polynomials in one variable. There are many points of contact with Chapters 10 and 11.

12.2 General properties of orthogonal polynomials in two variables

The general properties of orthogonal polynomials in two variables have been investigated by Jackson (1937) who also considered orthogonal polynomials in three, and in two complex, variables (Jackson, 1938, 1938a). In this section, and in sec. 12.3, we restrict ourselves to the case of two (real) variables. The corresponding properties for orthogonal polynomials of n variables will suggest themselves to the reader.

Given a region R in the x, y-plane and a non-negative weight function $w(x, y)$, both fixed, we shall assume in the case of a bounded region that w is integrable over R, and in the case of an unbounded region R that all integrals

$$(1) \quad \int_R \int w(x, y)\, x^m\, y^n\, dx\, dy \qquad m, n = 0, 1, \dots$$

converge. Orthogonal property, normalization, etc. will be understood to refer to the scalar product

$$(2) \quad (f, g) = \int_R \int f(x, y)\, g(x, y)\, w(x, y)\, dx\, dy.$$

Since f and g will be polynomials, the integral in (2) certainly exists.

The monomials $x^m\, y^n$ will be *ordered* as follows:

(3) $x^m\, y^n$ *is higher than* $x^k\, y^l$ *if*

either $m + n > k + l$

or $m + n = k + l$ *and* $m > k$.

The ordered sequence of monomials is

$$(4) \quad 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2 y,\ \dots .$$

The ordering (3) induces a partial ordering of the polynomials in x, y. A polynomial $q(x, y)$ will be said to be higher than $p(x, y)$ if the highest monomial (with non-zero coefficient) in q is higher than any monomial (with non-zero coefficient) in p.

It is to be noted that the ordering (3) is arbitrary, and is not symmetric in x and y. The orthogonal polynomials to be described below will be based on (3): in general, a different ordering will result in a different system of orthogonal polynomials.

Applying the process of orthogonalization described in sec. 10.1 to the sequence (4), the scalar product being determined by (2), we obtain a sequence of orthonormal polynomials which will be written as

$$(5) \quad q_{00},\ \ q_{10},\ q_{11},\ q_{20},\ q_{21},\ q_{22},\ q_{30},\ q_{31},\ \dots$$

so that $q_{nm}(x, y)$ is of degree n in x and y, and of degree m in y alone, $n = 0, 1, 2, \dots, m = 0, 1, \dots, n$. The orthonormal property is

$$(6) \quad (q_{nm}, q_{kl}) = \delta_{kn}\, \delta_{lm}$$

where $\delta_{rs} = 0$ if $r \neq s$, and $= 1$ if $r = s$; and q_{nm} is higher than q_{kl} if either $n > k$ or $n = k$ and $m > l$.

There are $n + 1$ polynomials of degree n in x and y, viz.,

$$q_{n0}, q_{n1}, \cdots, q_{nn}.$$

Any polynomial of degree n which is orthogonal to all polynomials of lower degree is a linear combination of $q_{n0}, \cdots, q_{nn}$. Note that such a polynomial is not necessarily orthogonal to all lower polynomials [lower, that is to say, in the sense defined in (3)].

With any real orthogonal constant matrix $[c_{ij}]$, where

$$\sum_{j=0}^{n} c_{ij} c_{kj} = \delta_{ik} \qquad i, k = 0, 1, \cdots, n, \tag{7}$$

the polynomials

$$p_{ni}(x, y) = \sum_{j=0}^{n} c_{ij} q_{nj}(x, y) \qquad i = 0, 1, \cdots, n \tag{8}$$

are orthogonal to each other, normalized, and orthogonal to all polynomials of lower degree (but not to all lower polynomials). Conversely, any $n + 1$ mutually orthogonal, normalized polynomials which are orthogonal to all polynomials of lower degree, may be represented in the form (8) where the c_{ij} satisfy (7). Note that in $p_{ni}(x, y)$, the subscript n indicates the degree in x and y, but the subscript i does not indicate the degree in y.

Suppose there is an affine tranformation

$$x' = \alpha x + \beta y, \quad y' = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma = 1 \tag{9}$$

which maps R onto itself, and leaves the weight function invariant. For each n,

$$p_{n0}(x', y'), p_{n1}(x', y'), \ldots, p_{nn}(x', y')$$

form a system of $n + 1$ mutually orthogonal and normalized polynomials which are orthogonal to all polynomials of lower degree. Thus, the $p_{ni}(\alpha x + \beta y, \gamma x + \delta y)$ may be obtained by a real orthogonal transformation of the $q_{ni}(x, y)$ and hence of the $p_{ni}(x, y)$. *An affine transformation* (9) *under which R and w are invariant induces, for each n, an orthogonal transformation of $p_{n0}, \cdots, p_{nn}$.* Different systems of p_{ni} (for the same R, w, n and α, β, γ, δ) undergo similar transformations; to a group of affine transformations (9) which leave R and w invariant there corresponds, foreach n, a group of orthogonal transformations. For further details and for a reference to work by A. Sobczyk, see Jackson (1937).

If R is a rectangle,

(10) $a \leq x \leq b, \quad c \leq y \leq d,$

and $w(x, y) = u(x)\, v(y)$, then we may take

(11) $p_{ni}(x, y) = p_{n-i}(x)\, q_i(y) \qquad i = 0, 1, \dots, n; \quad n = 0, 1, \dots$

where $\{p_n\}$ is the system of orthogonal polynomials associated with the weight function u on the interval (a, b) and $\{q_n\}$ the system of orthogonal polynomials associated with the weight function v on the interval (c, d).

12.3. Further properties of orthogonal polynomials in two variables

Let $\{p_{ni}(x, y)\}$ be a system, of the form 12.2(8), of orthonormal polynomials for the weight function w on the region R. For each i, $p_{ni}(x, y)$ is a polynomial of degree n in x and y, and any polynomial of degree n may be expressed as a linear combination of the $p_{mi}(x, y)$, $0 \leq i \leq m$, $0 \leq m \leq n$. Several of the general properties of orthogonal polynomials in one variable (see sec. 10.3) have their analogues in two variables, although the corresponding formulas are less simple.

First, we shall prove the existence of a *recurrence relation*, expressing $(ax + by)\, p_{ni}(x, y)$ as a linear combination of polynomials of degree $n + 1$, n, and $n - 1$. The proof is analogous to the proof of 10.3(7). For fixed n, i, the product

$$(ax + by)\, p_{ni}(x, y)$$

is a polynomial of degree $n + 1$, and hence of the form

(1) $$(ax + by)\, p_{ni}(x, y) = \sum_{m=0}^{n+1} \sum_{j=0}^{m} \gamma_{mj}\, p_{mj}(x, y),$$

(2) $$\gamma_{mj} = \iint_R (ax + by)\, p_{ni}(x, y)\, p_{mj}(x, y)\, w(x, y)\, dx\, dy.$$

Since $(ax + by)\, p_{mj}(x, y)$ is a polynomial of degree $m + 1$, and p_{ni} is orthogonal to all polynomials of degree less than n, we see that

(3) $$\gamma_{mj} = 0 \qquad m = 0, 1, \dots, n - 2.$$

Thus, in (1), only terms corresponding to $m = n - 1, n, n + 1$ actually occur.

It does not seem to be known whether the p_{ni}, that is to say the c_{ij} in 12.2(8), may be chosen so as to result in simple recurrence relations; nor does it seem to be known under what conditions a system of polynomials satisfying a recurrence relation of the kind described here, is a system of orthogonal polynomials corresponding to a non-negative weight function [compare the remark following 10.3(9)].

As in the case of one variable, the recurrence relation may be used to derive a relation which corresponds to the Christoffel-Darboux formula. With the p_{ni} as in 12.2(8), we form

(4) $$K_n(x, y, u, v) = \sum_{k=0}^{n} \sum_{i=0}^{k} p_{ki}(x, y)\, p_{ki}(u, v)$$

(5) $$L_n(x, y, u, v) = K_n(x, y, u, v) - K_{n-1}(x, y, u, v)$$
$$= \sum_{i=0}^{n} p_{ni}(x, y)\, p_{ni}(u, v)$$

(6) $$M_n(x, y, u, v, r, s) = L_{n+1}(u, v, r, s)\, L_n(x, y, r, s)$$
$$- L_n(u, v, r, s)\, L_{n+1}(x, y, r, s).$$

Note that although the p_{ni} are arbitrary to the extent of an orthogonal transformation for each i, the polynomials defined by (4) to (6) are uniquely determined by the weight function $w(x, y)$ and the region R. The "Christoffel-Darboux formula" is

(7) $$[(au + bv) - (ax + by)]\, K_n(x, y, u, v)$$
$$= \int\int_R (ar + bs)\, M_n(x, y, u, v, r, s)\, w(r, s)\, dr\, ds\, .$$

For the proof see Jackson (1937).

For the minimum properties of orthogonal polynomials in two variables see Gröbner (1948).

ORTHOGONAL POLYNOMIALS IN THE TRIANGLE

12.4. Appell's polynomials

Let T be the triangle

(1) $$x > 0, \quad y > 0, \quad x + y < 1,$$

and

(2) $$t(x) = x^{\gamma-1}\, y^{\gamma'-1}(1 - x - y)^{\alpha-\gamma-\gamma'}$$

the corresponding weight function. The weight function is integrable if

(3) $$\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \gamma' > 0, \quad \operatorname{Re} \alpha > \operatorname{Re}(\gamma + \gamma') - 1,$$

but many of the formal results are valid without this restriction.

Appell (1881) introduced the polynomials

$$(4)\quad \mathfrak{F}_{mn}(a, \gamma, \gamma', x, y) = (1-x-y)^{\gamma+\gamma'-a} \frac{x^{1-\gamma}\, y^{1-\gamma'}}{(\gamma)_m\, (\gamma')_n}$$

$$\times \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} [x^{\gamma+m-1}\, y^{\gamma'+n-1}\, (1-x-y)^{a+m+n-\gamma-\gamma'}]$$

which are analogous to Jacobi polynomials [cf. 10.8(10)]. Here, and throughout this chapter,

$$(5)\quad (a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \qquad n = 1, 2, \ldots$$

$$(a)_\nu = \Gamma(a+\nu)/\Gamma(a).$$

For a detailed study of these polynomials, and for references to the literature, see Appell and Kampé de Fériet (1926, Chapter VI and the bibliography).

From equation (4) it is seen that $\mathfrak{F}_{mn}$ is a polynomial of degree $m+n$ in x and y. The expression of $\mathfrak{F}_{mn}$ in terms of Appell's hypergeometric series F_2 is given in 5.13(1).

Adopting the region (1) and the weight function (2) in the definition of the scalar product 12.1(1), we see that

$$(\gamma)_m\, (\gamma')_n\, (P, \mathfrak{F}_{mn})$$

$$= \int\int_T P(x, y)\, \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} [x^{\gamma+m-1}\, y^{\gamma'+n-1} (1-x-y)^{a+m+n-\gamma-\gamma'}]\, dx\, dy$$

and repeated integration by parts shows that $\mathfrak{F}_{mn}$ is orthogonal to all polynomials of degree $< m+n$. In particular,

$$(6)\quad (\mathfrak{F}_{mn}, \mathfrak{F}_{kl}) = 0 \qquad m+n \neq k+l.$$

On the other hand, by repeated integrations by parts

$$(7)\quad (\mathfrak{F}_{mn}, \mathfrak{F}_{kl}) = \frac{(-1)^{m+n}}{(\gamma)_m\, (\gamma')_n} \frac{\partial^{m+n}\mathfrak{F}_{kl}}{\partial x^m\, \partial y^n}$$

$$\times \int\int_T x^{\gamma+m-1}\, y^{\gamma'+n-1}\, (1-x-y)^{a+m+n-\gamma-\gamma'}\, dx\, dy$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\gamma')\,\Gamma(a+m+n+1-\gamma-\gamma')}{\Gamma(a+2m+2n+1)} (-1)^{m+n} \frac{\partial^{m+n}\mathfrak{F}_{kl}}{\partial x^m\, \partial y^n}$$

$$m+n = k+l,$$

and since this does not vanish, the polynomials $\mathfrak{Z}_{mn}$ do not form an orthogonal system. No orthogonal or biorthogonal system of polynomials seems to be known for the weight function (2).

The system of partial differential equations satisfied by

$$(1-x-y)^{a-\gamma-\gamma'}\mathfrak{Z}_{mn}(a, \gamma, \gamma'; x, y)$$

may be derived by means of 5.13 (1), 5.11 (8), 5.9 (10). With the notations

$$(8)\quad p=\frac{\partial z}{\partial x},\quad q=\frac{\partial z}{\partial y},\quad r=\frac{\partial^2 z}{\partial x^2},\quad s=\frac{\partial^2 z}{\partial x\,\partial y},\quad t=\frac{\partial^2 z}{\partial y^2}$$

it reads

$$(9)\quad x(1-x)r-xys+[\gamma-(2\gamma+\gamma'-a-n+1)x]p \\ -(\gamma+m)yq-(\gamma+m)(\gamma+\gamma'-a-m-n)z=0$$

$$y(1-y)t-xys+[\gamma'-(\gamma+2\gamma'-a-m+1)y]q \\ -(\gamma'+n)xp-(\gamma'+n)(\gamma+\gamma'-a-m-n)z=0.$$

When $a=\gamma+\gamma'$, the weight function (2) simplifies to

$$(10)\quad t_0(x)=x^{\gamma-1}y^{\gamma'-1} \qquad \text{Re}\,\gamma,\ \text{Re}\,\gamma'>0.$$

For this weight function Appell (1882) considers two systems of polynomials

$$(11)\quad F_{mn}(\gamma, \gamma'; x, y)=\mathfrak{Z}_{mn}(\gamma+\gamma', \gamma, \gamma', x, y)$$

$$=\frac{x^{1-\gamma}y^{1-\gamma'}}{(\gamma)_m(\gamma')_n}\frac{\partial^{m+n}}{\partial x^m\,\partial y^n}[x^{\gamma+m-1}y^{\gamma'+n-1}(1-x-y)^{m+n}]$$

$$=F_2(-m-n, \gamma+m, \gamma'+n, \gamma, \gamma'; x, y)$$

$$(12)\quad E_{mn}(\gamma, \gamma'; x, y)=F_2(\gamma+\gamma'+m+n, -m, -n, \gamma, \gamma'; x, y)$$

where F_2 is the series defined in 5.7 (7). The partial differential equations satisfied by F_{mn} and E_{mn} may be derived by means of 5.9 (10). They are

$$(13)\quad x(1-x)r-xys+[\gamma-(\gamma-n+1)x]p-(\gamma+m)yq \\ +(m+n)(\gamma+m)z=0$$

$$y(1-y)t-xys+[\gamma'-(\gamma'-m+1)y]q-(\gamma'+n)xp \\ +(m+n)(\gamma'+n)z=0 \qquad F_{mn}$$

$$(14)\quad \begin{aligned} &x(1-x)r - xys + [\gamma - (\gamma+\gamma'+n+1)x]p \\ &\quad + my q + m(\gamma+\gamma'+m+n)z = 0 \\ &y(1-y)t - xys + [\gamma' - (\gamma+\gamma'+m+1)y]q \\ &\quad + nxp + n(\gamma+\gamma'+m+n)z = 0 \end{aligned} \qquad E_{mn}.$$

Adding each of these two pairs, it is seen that both F_{mn} and E_{mn} satisfy the partial differential equation

$$(15)\quad \begin{aligned} &x(1-x)r - 2xys + y(1-y)t + [\gamma - (\gamma+\gamma'+1)x]p \\ &\quad + [\gamma' - (\gamma+\gamma'+1)y]q + (m+n)(\gamma+\gamma'+m+n)z = 0, \end{aligned}$$

and this partial differential equation may be used to prove that

$$(16)\quad \iint_T x^{\gamma-1} y^{\gamma'-1} F_{mn}(\gamma, \gamma', x, y)\, E_{kl}(\gamma, \gamma', x, y)\, dx\, dy$$

vanishes except when $m = k$ and $n = l$. This shows that the two systems of polynomials (11) and (12) form a *biorthogonal system* for the region (1) and the weight function (10).

The formula

$$(17)\quad \iint_T x^{\gamma-1} y^{\gamma'-1} F_{mn}(\gamma, \gamma', x, y)\, E_{kl}(\gamma, \gamma', x, y)\, dx\, dy$$

$$= \frac{\delta_{mk}\,\delta_{nl}}{\gamma+\gamma'+2m+2n}\;\frac{m!\,n!\,(m+n)!\;\Gamma(\gamma)\;\Gamma(\gamma')}{(\gamma)_m\,(\gamma')_n\,\Gamma(\gamma+\gamma'+m+n)}$$

is proved in Appell and Kampé de Fériet (1926, p. 110, 111). It may be used to compute coefficients in the expansion of an arbitrary function in a series of the F_{mn}, or in a series of the E_{mn}. Two examples of such expansions are

$$(18)\quad F_{mn}(\gamma, \gamma', x, y) = \sum_{k+l=m+n} \frac{(k+l)!\,(\gamma+m)_k\,(\gamma'+n)_l}{k!\,l!\,(\gamma+\gamma'+k+l)_{k+l}} E_{kl}(\gamma, \gamma', x, y)$$

$$(19)\quad (1-x-y)^{\lambda-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{m+n}(\gamma+\gamma'+2m+2n)$$

$$\times \frac{(1-\lambda)_{m+n}\,(\gamma)_m\,(\gamma')_n\,\Gamma(\lambda)\,\Gamma(\gamma+\gamma'+m+n)}{m!\,n!\,(m+n)!\,\Gamma(\gamma+\gamma'+\lambda+m+n)} E_{mn}(\gamma, \gamma', x, y)$$

(Appell and Kampé de Fériet, 1926, p. 112, 113). In (18) summation is extended over all non-negative integers k and l for which $k + l = m + n$.

For the case $\gamma = \gamma' = 1$, $\alpha = 2$, when the weight function is constant, see Gröbner (1948, sec. 5).

ORTHOGONAL POLYNOMIALS IN CIRCLE AND SPHERE

12.5. The polynomials V

In this section and in the following section we shall use notations similar to those of Chapter XI.

(1) $\mathfrak{x} = (x_1, \dots, x_n)$

will be a vector, with (real) components $x_1, \dots, x_n$ in n-dimensional (real) Euclidean space, and

(2) $||\mathfrak{x}|| = r = (x_1^2 + \cdots + x_n^2)^{1/2}$

will be the length of this vector. With two vectors

(3) $\mathfrak{a} = (a_1, \dots, a_n), \quad \mathfrak{x} = (x_1, \dots, x_n)$

we associate the scalar product

(4) $(\mathfrak{a}, \mathfrak{x}) = a_1 x_1 + \cdots + a_n x_n$

and the angle θ, where

$$\cos\theta = \frac{(\mathfrak{a}, \mathfrak{x})}{||\mathfrak{a}||\ ||\mathfrak{x}||}.$$

[The scalar product (4) of two vectors is to be distinguished from the scalar product of two functions occurring in (17), 12.6(4), and similar relations.] The unit sphere, $||\mathfrak{x}|| < 1$, in our space will be denoted by S, the element of volume by dx, so that

$$\int_S f(\mathfrak{x})\, dx$$

will be written for

$$\int \cdots \int_{x_1^2 + \cdots + x_n^2 \leq 1} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n.$$

We shall consider orthogonal polynomials in the region S with the weight function

$$(5)\quad (1-r^2)^{\frac{1}{2}s-\frac{1}{2}} = (1-x_1^2-\cdots-x_n^2)^{\frac{1}{2}s-\frac{1}{2}}.$$

For $n = 2$, the region is a circle in the plane, for $n = 3$, a sphere in three-dimensional space, and for $n > 3$, a hypersphere.

Polynomials

$$(6)\quad V_m^s(\mathfrak{x}) = V^s_{m_1, m_2, \dots, m_n}(x_1, x_2, \dots, x_n)$$

will be defined by the *generating function*

$$(7)\quad [1-2(\mathfrak{a}, \mathfrak{x}) + \|\mathfrak{a}\|^2]^{-\frac{1}{2}n-\frac{1}{2}s+\frac{1}{2}}$$
$$= \sum a_1^{m_1} \cdots a_n^{m_n}\, V^s_{m_1, \dots, m_n}(x_1, \dots, x_n).$$

In this sum, and in all similar sums, summation will be understood to take place over all non-negative integers $m_1, \dots, m_n$. Clearly, $V_m^s(\mathfrak{x})$ is a polynomial of degree m_k in x_k, being an even or odd polynomial in x_k according as m_k is even or odd; and

$$(8)\quad m = m_1 + \cdots + m_n$$

is the *degree* of this polynomial.

For $n = 1$, a comparison of (7) and 10.9(29) shows that

$$(9)\quad V_m^s(x) = C_m^{\frac{1}{2}s}(x) \qquad n = 1.$$

For $n = 2$ and $s = 0, 2$, the polynomials (6) were introduced by Hermite (1865, 1865 a), for any n by Didon (1868). There is a detailed presentation of these polynomials and of related matters in Part Two of the book by Appell and Kampé de Fériet (1926) where there is also an extensive bibliography. Additional references are listed at the end of this chapter under Angelescu, Appell, Brinkman and Zernike, Caccioppoli, Chen, Dinghas, Erdélyi, Koschmieder, Orloff, and Schmeidler.

The expansion in powers of $a_1, \dots, a_m$ of the generating function (7), by the multinomial theorem, leads at once to the explicit representation

$$(10)\quad V^s_{m_1, \dots, m_n}(x_1, \dots, x_n) = \left(\frac{n+s-1}{2}\right)_m \frac{2^m x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}$$
$$\times F_B\left(-\frac{m_1}{2}, \dots, -\frac{m_n}{2}, \frac{1-m_1}{2}, \dots, \frac{1-m_n}{2},\right.$$
$$\left. -m-\frac{n+s-3}{2};\ \frac{1}{x_1^2}, \dots, \frac{1}{x_n^2}\right)$$

where

(11) $F_B(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma; z_1, \ldots, z_n)$

$$= \sum \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{m_1! \cdots m_n! \, (\gamma)_{m_1 + \cdots + m_n}} z_1^{m_1} \cdots z_n^{m_n}$$

is one of Lauricella's hypergeometric series of n variables (Appell and Kampé de Fériet, 1926, Chapter VII). There are also representations of V_m^s in hypergeometric series of ascending (rather than descending) powers of the x_k, these representations being different according to the parities of the m_k [see also 10.9(21) and 10.9(22)].

If one puts $a_k = tb_k$ in (7), and compares coefficients of t^m on both sides, the relation

(12) $$\|\mathfrak{b}\|^m C_m^{\frac{1}{2}n+\frac{1}{2}s-\frac{1}{2}} \left[\frac{(\mathfrak{b}, \mathfrak{x})}{\|\mathfrak{b}\|} \right]$$

$$= \sum_{m_1 + \cdots + m_n = m} b_1^{m_1} \cdots b_n^{m_n} V_{m_1, \ldots, m_n}^s (x_1, \ldots, x_n)$$

is obtained.

It may be verified from the explicit formula that the polynomial defined by (10) satisfies the following (hypergeometric) system of partial differential equations

(13) $$\frac{\partial}{\partial x_j} \left\{ \frac{\partial V}{\partial x_j} - x_j \left[(m+n+s-1) V + \sum_{k=1}^{n} x_k \frac{\partial V}{\partial x_k} \right] \right\}$$

$$+ (m_j + 1) \left[(m+n+s-1) V + \sum_{k=1}^{n} x_k \frac{\partial V}{\partial x_k} \right] = 0$$

$$j = 1, \ldots, n,$$

where m is the degree given by (8). Adding these n equations, we see that all polynomials of degree m satisfy the partial differential equation

(14) $$(m+n)(m+s-1) V$$

$$+ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left\{ \frac{\partial V}{\partial x_j} - x_j \left[(s-1) V + \sum_{k=1}^{n} x_k \frac{\partial V}{\partial x_k} \right] \right\} = 0.$$

There is a remarkable symbolic representation of our polynomials,

$$(15)\quad V_m^s(\mathfrak{x}) = \frac{2^m (\tfrac{1}{2}n + \tfrac{1}{2}s - \tfrac{1}{2})_m}{m_1! \cdots m_n!} \times {}_0F_1(-n/2 - s/2 + 3/2 + m; \Delta^2/4)(x_1^{m_1} \cdots x_n^{m_n})$$

where ${}_0F_1$ is a generalized hypergeometric series [see 4.1(1)] and

$$(16)\quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is Laplace's operator. This representation is derived by means of the connection between the polynomials V_m^s and hyperspherical harmonics (see sec. 11.8). The same connection may be used to show that the integral

$$(17)\quad \int_S (1 - r^2)^{\frac{1}{2}s - \frac{1}{2}}\, V_m^s(\mathfrak{x})\, V_{m'}^s(\mathfrak{x})\, dx$$

vanishes if $m \neq m'$, and also if $m = m'$ and some of the differences $m_i - m_i'$ are odd numbers. Since the integral does not vanish when $m = m'$ and all differences $m_i - m_i'$ are even numbers, the V_m^s do not form an orthogonal system of polynomials.

The formula corresponding to Rodrigues' formula [equation 10.9(11)] is

$$(18)\quad m_1! \cdots m_n!\, (1 - r^2)^{\frac{1}{2}(m+n+s-1)} V^s_{m_1, \ldots, m_n}(x_1, \ldots, x_n) = (-1)^m \frac{\partial^m}{\partial y_1^{m_1} \cdots \partial y_n^{m_n}} (1 - r^2)^{\frac{1}{2}(n+s-1)}$$

where, on the right-hand side,

$$(19)\quad y_i = x_i (1 - r^2)^{-\frac{1}{2}} \qquad i = 1, \ldots n$$

are the independent variables, and

$$(20)\quad 1 - r^2 = (1 + \|\mathfrak{y}\|^2)^{-1}.$$

The formula may be derived from the generating function (7) by the substitution (19) and upon replacing a_i by $a_i(1 - r^2)^{\frac{1}{2}}$.

The generating function is also the source of the integral representation

$$(21)\quad \pi^{\frac{1}{2}n}\, m_1! \cdots m_n!\, \Gamma(\tfrac{1}{2}s)\, V_m^s(\mathfrak{z}) = i^m (n+s-1)_m\, \Gamma(\tfrac{1}{2}n + \tfrac{1}{2}s)$$

$$\times \int_S x_1^{m_1} \cdots x_n^{m_n} (1-r^2)^{\frac{1}{2}s-1} [\,||\mathfrak{z}|| + i(\mathfrak{x}, \mathfrak{z})]^{-m-n-s+1}\, dx.$$

For other integrals see Dinghas (1950).

Recurrence relations, differentiation formulas, and similar relations also follow from the generating function and are recorded in Appell and Kampé de Fériet (1926, sec. LXXVI).

12.6. The polynomials U

A second system of polynomials,

$$(1)\quad U_m^s(\mathfrak{x}) = U^s_{m_1, \ldots, m_n}(x_1, \ldots, x_n)$$

will be defined by the generating function

$$(2)\quad \{[(\mathfrak{a}, \mathfrak{x}) - 1]^2 + ||\mathfrak{a}||^2 (1 - ||\mathfrak{x}||^2)\}^{-\frac{1}{2}s}$$

$$= \Sigma\, a_1^{m_1} \cdots a_n^{m_n}\; U^s_{m_1, \ldots, m_n}(x_1, \ldots, x_n).$$

For $n = 1$, we have

$$(3)\quad U_m^s(x) = C_m^{\frac{1}{2}s}(x) \qquad n = 1.$$

For $n = 2$, $s = 1, 2$, these polynomials were introduced by Hermite; for any n see the literature quoted in sec. 12.5.

The most important property of these polynomials is the *biorthogonal property* which connects them with the V_n^s. The integral

$$(4)\quad \int_S (1-r^2)^{\frac{1}{2}s-\frac{1}{2}}\, V_m^s(\mathfrak{x})\, U_l^s(\mathfrak{x})\, dx$$

vanishes, except when $m_1 = l_1, \ldots, m_n = l_n$; and

$$(5)\quad \int_S (1-r^2)^{\frac{1}{2}s-\frac{1}{2}}\, V_m^s(\mathfrak{x})\, U_m^s(\mathfrak{x})\, dx = h_m^s$$

$$= \frac{2\pi^{\frac{1}{2}n}}{2m+n+s-1}\; \frac{\Gamma(\tfrac{1}{2}s+1)}{\Gamma(\tfrac{1}{2}n + \tfrac{1}{2}s - \tfrac{1}{2})}\; \frac{(s)_m}{m_1! \cdots m_n!}\,.$$

This biorthogonal property may be proved from the generating functions (see the corresponding proof for Hermite polynomials in sec. 12.9). Conversely, Kampé de Fériet (1915) postulated the biorthogonal property and deduced the generating function from it.

The theory of the polynomials U resembles that of the polynomials V and we shall simply list some of the relevant formulas.

Explicit representation

$$(6)\quad U^s_{m_1,\ldots,m_n}(x_1,\ldots,x_n)=\frac{(s)_m\, x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!}$$
$$\times F_B\left(-\frac{m_1}{2},\ldots,-\frac{m_n}{2},1-\frac{m_1}{2},\ldots,1-\frac{m_n}{2},\ \frac{s+1}{2};\right.$$
$$\left.-\frac{1-r^2}{x_1^2},\ldots,-\frac{1-r^2}{x_n^2}\right)$$

with corresponding series in ascending powers of $x_1,\ldots,x_n$.

$$(7)\quad [(\mathfrak{b},\mathfrak{x})^2+\|\mathfrak{b}\|^2(1-r^2)]^{\frac{1}{2}m}\, C^{\frac{1}{2}s}_m\left(\frac{(\mathfrak{b},\mathfrak{x})}{[(\mathfrak{b},\mathfrak{x})^2+\|\mathfrak{b}\|^2(1-r^2)]^{\frac{1}{2}}}\right)$$
$$=\sum_{m_1+\cdots+m_n=m} b_1^{m_1}\cdots b_n^{m_n}\, U^s_{m_1,\ldots,m_n}(x_1,\ldots,x_n).$$

The polynomial U^s_m satisfies the system of partial differential equations

$$(8)\quad (1-r^2)\frac{\partial}{\partial x_j}\left[\frac{\partial U}{\partial x_j}+x_j\left(mU-\sum_{k=1}^{n}x_k\frac{\partial U}{\partial x_k}\right)\right]$$
$$+m_j(1-r^2)\left(mU-\sum_{k=1}^{n}x_k\frac{\partial U}{\partial x_k}\right)$$
$$-(s-1)\left[x_j\frac{\partial U}{\partial x_j}+x_j^2\left(mU-\sum_{k=1}^{n}x_k\frac{\partial U}{\partial x_k}\right)-m_jU\right]=0$$
$$j=1,\ldots,n.$$

All polynomials of degree m satisfy the partial differential equation

$$(9)\quad (m+n)(m+s-1)U+\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left\{\frac{\partial U}{\partial x_j}\right.$$
$$\left.-x_j\left[(s-1)U+\sum_{k=1}^{n}x_k\frac{\partial U}{\partial x_k}\right]\right\}=0$$

which is obtained by adding the n equations (8), and is identical with the corresponding equation 12.5 (14) for the V_n^s.

The symbolic representation may be written in the form

$$(10)\quad U_m^s(\mathfrak{x}) = \frac{(s)_m}{m_1! \cdots m_n!}\ {}_0F_1[\tfrac{1}{2}s + \tfrac{1}{2}; -\tfrac{1}{4}(1-r^2)\Delta^2](x_1^{m_1} \cdots x_n^{m_n}),$$

where the k-th power of $(1-r^2)\Delta^2$ is to be taken as $(1-r^2)^k \Delta^{2k}$. There is also a relation corresponding to 12.5 (17) but it is of little importance.

The analogue of Rodrigues' formula is simpler in this case than in the case of V_m^s.

$$(11)\quad 2^m \left(\frac{s+1}{2}\right)_m m_1! \cdots m_n! (1-r^2)^{\frac{1}{2}s-\frac{1}{2}} U^s_{m_1, \ldots, m_n}(x_1, \ldots, x_n)$$

$$= (-1)^m (s)_m \frac{\partial^m}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} (1-r^2)^{m+\frac{1}{2}s-\frac{1}{2}}.$$

Koschmieder (1925) obtained expressions for the U_m^s in terms of partial derivatives with respect to x_i^2.

The integral representation corresponding to 12.5 (21) is

$$(12)\quad \pi^{\frac{1}{2}n} m_1! \cdots m_n! \Gamma(\tfrac{1}{2}s - \tfrac{1}{2}n + \tfrac{1}{2}) U_m^s(\mathfrak{z})$$

$$= (s)_m \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}) \int_S (1-r^2)^{\frac{1}{2}s-\frac{1}{2}n-\frac{1}{2}}$$

$$\times [z_1 + ix_1(1-||\mathfrak{z}||^2)^{\frac{1}{2}}]^{m_1} \cdots [z_n + ix_n(1-||\mathfrak{z}||^2)^{\frac{1}{2}}]^{m_n}\, dx.$$

The two systems of polynomials, U_m^s and V_m^s are connected: the connection may be expressed in two equivalent forms.

$$(13)\quad (2-2m-n-s)_m (r^2-1)^{\frac{1}{2}m} V_m^s\left[\frac{\mathfrak{x}}{(r^2-1)^{\frac{1}{2}}}\right]$$

$$= 2^m \left(\frac{n+s-1}{2}\right)_m U_m^{2-2m-n-s}(\mathfrak{x})$$

$$(14)\quad 2^m \left(-m - \frac{s-1}{2}\right)_m (r^2-1)^{\frac{1}{2}m} U_m^s\left[\frac{\mathfrak{x}}{(r^2-1)^{\frac{1}{2}}}\right]$$

$$= (s)_m V_m^{2-2m-n-s}(\mathfrak{x}).$$

The biorthogonal property has already been stated in (4) and (5). Another connection, closely related to the biorthogonal property, is given by the circumstance that the system of partial differential equations satisfied by

$$R^s_m(\mathfrak{x}) = (1 - r^2)^{\frac{1}{2}s - \frac{1}{2}}\, U^s_m(\mathfrak{x}),$$

which can be derived from (8) and is

$$(15)\quad \frac{\partial}{\partial x_j}\left\{\frac{\partial R}{\partial x_j} + x_j\left[(m+s-1)R - \sum_{k=1}^{n} x_k \frac{\partial R}{\partial x_k}\right]\right\}$$

$$+ m_j\left[(m+s-1)R - \sum_{k=1}^{n} x_k \frac{\partial R}{\partial x_k}\right] = 0 \qquad j = 1, 2, \ldots, n,$$

is easily seen to be *adjoint* to the system 12.5(13) of partial differential equation satisfied by $V^s_m(\mathfrak{x})$.

12.7. Expansion problems and further investigations

The biorthogonal property of the U and V suggests the expansion of an "arbitrary" function $f(\mathfrak{x})$ in either of the two series

$$(1)\quad \Sigma\, a^s_m\, U^s_m(\mathfrak{x})$$

$$(2)\quad \Sigma\, b^s_m\, V^s_m(\mathfrak{x}).$$

From 12.6(4) and (5) one obtains the expressions

$$(3)\quad h^s_m\, a^s_m = \int_S (1 - r^2)^{\frac{1}{2}s - \frac{1}{2}} f(\mathfrak{x})\, V^s_m(\mathfrak{x})\, dx$$

$$(4)\quad h^s_m\, b^s_m = \int_S (1 - r^2)^{\frac{1}{2}s - \frac{1}{2}} f(\mathfrak{x})\, U^s_m(\mathfrak{x})\, dx.$$

A general discussion of such expansions is contained in the book by Appell and Kampé de Fériet (1926, Part II, Chapter V). More precise results were obtained by later writers.

In studying the expansion problem it is usually assumed that s is a positive integer in (1) and (2). Koschmieder calls (1) and (2) Appell series if $s \geq 2$, Didon series if $s = 1$, and shows that an Appell series in n variables may be reduced to a Didon series in $n + s - 1$ variables. Moreover, it is usual to rearrange the multiple series (1) and (2) as a simple series, by grouping together all the terms of equal degree. Thus, (1) is interpreted as

(5) $$\sum_{m=0}^{\infty} \left[\sum_{m_1+\cdots+m_n=m} a^s_{m_1,\ldots,m_n} U^s_{m_1,\ldots,m_n}(x_1,\ldots,x_n) \right],$$

and there is a similar interpretation of (2). The rearranged series may then be related to the Laplace expansion of a function on the surface of the unit hypersphere in $n+s+1$ dimensions, and this connection has often been used.

Convergence of the series (1) and (2), rearranged as described above, has been investigated for $n = 2$, $s = 1$ by Caccioppoli (1932), and by Koschmieder (1933). Caccioppoli summed the series and discussed its convergence by means of a singular integral, proving convergence for continuously differentiable functions. Koschmieder used the theory of integral equations and proved absolute convergence for twice continuously differentiable functions.

The case of general n and (positive integer) s was investigated by Koschmieder (1934). Adopting the interpretation (5) for (1), and a corresponding interpretation of (2), Koschmieder showed that these series are equiconvergent with certain expansions in Gegenbauer polynomials. Koschmieder (1934a) also obtained an equiconvergence theorem for Laplace's expansion, with a Fourier series as a comparison series.

The Cesàro summability of Laplace' series has been discussed by Chen (1928) and Koschmieder (1929). The results have been applied to Appell's series by Koschmieder (1931).

The Appell series of a function $f(\mathfrak{x})$ which is integrable in S, is (C, δ) summable to $f(\mathfrak{x})$ almost everyhwere in S, and certainly on the Lebesgue set of f in S, when

(6) $$\delta \geq n + s - 1.$$

Moreover, the Appell series is (C, δ) summable also for

(7) $$\tfrac{1}{2}(n+s-1) < \delta < n+s-1$$

at all those points $\mathfrak{y}$ for which

$$||\mathfrak{x} - \mathfrak{y}||^{-\frac{1}{2}(n+s-1)} |f(\mathfrak{x})|$$

is an integrable function of $\mathfrak{x}$ in S.

The following examples of expansions are taken from Appell and Kampé de Fériet (1926, sections LXXXVIII and XCI).

$$(8)\quad (\mathfrak{a}, \mathfrak{x})^k = \sum \frac{(-1)^m (-k)_m (\tfrac{1}{2})_{\frac{1}{2}k - \frac{1}{2}m}}{(\tfrac{1}{2}n + \tfrac{1}{2}s + \tfrac{1}{2})_{\frac{1}{2}k + \frac{1}{2}m}} \times a_1^{m_1} \cdots a_n^{m_n} \, ||\mathfrak{a}||^{k-m} \, V_m^s(\mathfrak{x}),$$

where k is a positive integer, and summation is over all those values of $m_1, \ldots, m_n$ for which $k - m$ is a positive even integer;

$$(9)\quad \exp[i(\mathfrak{a}, \mathfrak{x})] = 2^{\frac{1}{2}n + \frac{1}{2}s - \frac{1}{2}} \, \Gamma(\tfrac{1}{2}n + \tfrac{1}{2}s - \tfrac{1}{2}) \times \sum i^m \left(m + \frac{n+s-1}{2}\right) a_1^{m_1} \cdots a_n^{m_n} \, ||\mathfrak{a}||^{-m - \frac{1}{2}n - \frac{1}{2}s + \frac{1}{2}} \times J_{m + \frac{1}{2}n + \frac{1}{2}s - \frac{1}{2}}(||\mathfrak{a}||) \, V_m^s(\mathfrak{x})$$

$$(10)\quad \Gamma(s + \tfrac{1}{2}) \exp(\mathfrak{a}, \mathfrak{x}) \, J_{\frac{1}{2}s - \frac{1}{2}}[||\mathfrak{a}|| (1 - r^2)] = [\tfrac{1}{2} ||\mathfrak{a}|| (1 - r^2)]^{\frac{1}{2}s - \frac{1}{2}} \sum \frac{1}{(s)_m} \, a_1^{m_1} \cdots a_n^{m_n} \, U_m^s(\mathfrak{x}).$$

In the last two expansions, summation is over all non-negative $m_1, \ldots, m_n$.

The case $n = 2$ has been investigated in greater detail [see Appell and Kampé de Fériet (1926, Part II, Chapter VII), and the papers quoted in sections 12.5 - 12.7 of the present chapter]. An alternative approach to orthogonal polynomials in spherical regions was suggested by Brinkman and Zernike (1935) and by Gröbner (1948). Polynomials connected with the partial differential equation $\Delta^q F = 0$ in spherical regions were studied by Giulotto (1939) who obtained a biorthogonal system for this case. Devisme (1932) introduced polynomials defined by the generating functions

$$(11)\quad (1 - 3ax + 3a^2 y - a^3)^{-\nu}, \qquad [1 - 3ax + 3(a^2 - b)y - a^3]^{-\nu}$$

which arise in the study of the partial differential equation

$$(12)\quad \Delta_3 u = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3 \frac{\partial^3 u}{\partial x \, \partial y \, \partial z} = 0.$$

A generalization of the polynomials U_m^s, V_m^s may be defined by means of a fixed quadratic form, $\phi(\mathfrak{x})$, its reciprocal form $\psi(\mathfrak{x})$, and the bilinear form $\phi(\mathfrak{x}, \mathfrak{y})$ [see 12.8(6) to (8)]. The generating functions are

(13) $\{[\phi(\mathfrak{a}, \mathfrak{x}) - 1]^2 + \phi(\mathfrak{a})[1 - \phi(\mathfrak{x})]\}^{-\frac{1}{2}s} = \sum a_1^{m_1} \cdots a_n^{m_n} \mathfrak{U}_m^s(\mathfrak{x})$

(14) $[1 - 2(\mathfrak{a}, \mathfrak{x}) + \psi(\mathfrak{a})]^{-\frac{1}{2}n - \frac{1}{2}s + \frac{1}{2}} = \sum a_1^{m_1} \cdots a_n^{m_n} \mathfrak{V}_m^s(\mathfrak{x})$.

These polynomials have been introduced by Hermite and were studied by Angelescu (1916). If $\phi(\mathfrak{x}) = (\mathfrak{x}, \mathfrak{x}) = \psi(\mathfrak{x})$, the polynomials defined by (13) and (14) are U_m^s and V_m^s respectively.

HERMITE POLYNOMIALS OF SEVERAL VARIABLES

12.8. Definition of the Hermite polynomials

As in the preceding sections,

(1) $\mathfrak{x} = (x_1, \dots, x_n)$

will be a (real) vector,

(2) $||\mathfrak{x}|| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$

the length of $\mathfrak{x}$, and

(3) $(\mathfrak{a}, \mathfrak{x}) = a_1 x_1 + \cdots + a_n x_n$

the scalar product of two such vectors. C will be a fixed positive definite symmetric square matrix of real elements, i.e.,

(4) $C = [c_{ij}]$ $\qquad i, j = 1, \dots, n$

$$c_{ij} = c_{ji} \text{ real}, \qquad \sum_{i,j=1}^{n} c_{ij} x_i x_j > 0 \qquad \mathfrak{x} \neq 0.$$

The reciprocal matrix will be denoted by C^{-1}: its elements are γ_{ij}/Δ, where

(5) $\Delta = \det c_{ij}$ $\qquad i, j = 1, \dots, n$

is the determinant of C, and γ_{ij} is the cofactor of c_{ji} in Δ. With C we associate the positive definite quadratic form

(6) $\phi(\mathfrak{x}) = (C\mathfrak{x}, \mathfrak{x}) = (\mathfrak{x}, C\mathfrak{x}) = \sum_{i,j=1}^{n} c_{ij} x_i x_j$

and the symmetric bilinear form

(7) $\phi(\mathfrak{x}, \mathfrak{y}) = (C\,\mathfrak{x}, \mathfrak{y}) = (\mathfrak{x}, C\,\mathfrak{y}) = \sum_{i,j=1}^{n} c_{ij}\, x_i\, y_j$.

We also have the reciprocal form

(8) $\psi(\mathfrak{x}) = \phi(C^{-1}\,\mathfrak{x}) = (C^{-1}\,\mathfrak{x}, \mathfrak{x}) = (\mathfrak{x}, C^{-1}\,\mathfrak{x})$,

which is also a positive definite quadratic form, and the reciprocal symmetric bilinear form

(9) $\psi(\mathfrak{x}, \mathfrak{y}) = (C^{-1}\,\mathfrak{x}, \mathfrak{y}) = (\mathfrak{x}, C^{-1}\,\mathfrak{y})$,

These forms are connected by a number of relations.

(10) $\phi(\mathfrak{x}+\mathfrak{y}) = \phi(\mathfrak{x}) + 2\phi(\mathfrak{x}, \mathfrak{y}) + \phi(\mathfrak{y})$

$\psi(\mathfrak{x}+\mathfrak{y}) = \psi(\mathfrak{x}) + 2\,\psi(\mathfrak{x}, \mathfrak{y}) + \psi(\mathfrak{y})$

$\phi(\mathfrak{x}) = \psi(C\,\mathfrak{x}), \quad \psi(\mathfrak{x}) = \phi(C^{-1}\,\mathfrak{x})$

(11) $\phi(\mathfrak{x} + C^{-1}\,\mathfrak{y}) = \phi(\mathfrak{x}) + 2(\mathfrak{x}, \mathfrak{y}) + \psi(\mathfrak{y})$

(12) $\psi(\mathfrak{x} + C\,\mathfrak{y}) = \psi(\mathfrak{x}) + 2(\mathfrak{x}, \mathfrak{y}) + \phi(\mathfrak{y})$.

Lastly we mention the integral formula

(13) $\int \exp[-\tfrac{1}{2}\phi(\mathfrak{x}) + (\mathfrak{a}, \mathfrak{x})]\, dx = (2\pi)^{\frac{1}{2}n}\, \Delta^{-\frac{1}{2}} \exp[\tfrac{1}{2}\psi(\mathfrak{a})]$

where integration is over the whole space, dx stands for $dx_1 \cdots dx_n$, and $\mathfrak{a}$ is a constant vector. The formula may be proved by using (11) and then transforming the quadratic form $\phi(\mathfrak{x} + C^{-1}\,\mathfrak{a})$ into a sum of squares.

The notations introduced above will be used throughout this section and the following sections.

Hermite polynomials of several variables are a biorthogonal system of polynomials associated with the weight function

(14) $w(\mathfrak{x}) = \Delta^{\frac{1}{2}}\, (2\pi)^{-\frac{1}{2}n} \exp[-\tfrac{1}{2}\phi(\mathfrak{x})]$,

the region being the entire n-dimensional space. The relation

(15) $\int w(\mathfrak{x})\, dx = 1$

is a consequence of (13). These polynomials are clearly n-dimensional generalizations of the orthogonal polynomials defined by 10.13 (1). They were introduced by Hermite (1864) and since then studied by many authors. Appell and Kampé de Fériet (1926, Part III) give a detailed presentation of the theory, as of 1926, and a bibliography. Additional references are listed at the end of this chapter under Caccioppoli, Erdélyi, Feldheim, Grad, Koschmieder, Mazza, Picone, Thijssen, and Tortrat. Extensions to infinite-dimensional spaces are due to Cameron and Martin (1947), and Friedrichs (1951, see in particular p. 212 ff).

Two systems of polynomials,

$$(16)\quad G_m(\mathfrak{x}) = G_{m_1, \dots, m_n}(x_1, \dots, x_n)$$

$$H_m(\mathfrak{x}) = H_{m_1, \dots, m_n}(x_1, \dots, x_n)$$

will be defined by means of the *generating functions*

$$(17)\quad \exp[(C\mathfrak{a}, \mathfrak{x}) - \tfrac{1}{2}\phi(\mathfrak{a})] = \exp[\tfrac{1}{2}\phi(\mathfrak{x}) - \tfrac{1}{2}\phi(\mathfrak{x} - \mathfrak{a})]$$

$$= \sum \frac{a_1^{m_1}}{m_1!} \cdots \frac{a_n^{m_n}}{m_n!} H_m(\mathfrak{x}).$$

$$(18)\quad \exp[(\mathfrak{a}, \mathfrak{x}) - \tfrac{1}{2}\psi(\mathfrak{a})] = \exp[\tfrac{1}{2}\phi(\mathfrak{x}) - \tfrac{1}{2}\phi(\mathfrak{x} - C^{-1}\mathfrak{a})]$$

$$= \sum \frac{a_1^{m_1}}{m_1!} \cdots \frac{a_n^{m_n}}{m_n!} G_m(\mathfrak{x}),$$

which are extensions to several dimensions of the generating function 10.13 (19). In all sums $m_1, \dots, m_n$ run through all non-negative integers, unless other regions of summation are explicitly stated. The polynomials defined by (17) and (18) are of degree m_i in x_i, and their (total) *degree* is

$$(19)\quad m = m_1 + \cdots + m_n.$$

In these definitions we followed Appell and Kampé de Fériet (1926, sec. CXVIII). For $n = 1$ we have the Hermite polynomials defined in sec. 10.13 if we take $c_{11} = 2$.

If the coefficients of $a_1^{m_1} \cdots a_n^{m_n}$ in the generating functions (17) and (18) are computed by means of Taylor's theorem, one obtains

$$(20)\quad H_m(\mathfrak{x}) = (-1)^m \exp[\tfrac{1}{2}\phi(\mathfrak{x})] \frac{\partial^m}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \exp[-\tfrac{1}{2}\phi(\mathfrak{x})]$$

$$(21)\quad G_m(C^{-1}\mathfrak{x}) = (-1)^m \exp[\tfrac{1}{2}\psi(\mathfrak{x})] \frac{\partial^m}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \exp[-\tfrac{1}{2}\psi(\mathfrak{x})]$$

corresponding to 10.13(7). Koschmieder (1925) has given an alternative expression in terms of partial derivatives for certain Hermite polynomials of two variables. Either (17) and (18) or (20) and (21) may be regarded as definitions of Hermite polynomials in several variables.

An alternative notation, in case of a special quadratic form $\phi(\mathfrak{x})$, has been proposed by H. Grad (1949).

12.9. Basic properties of Hermite polynomials

The most important feature of Hermite polynomials is the *biorthogonal property*

(1) $\int w(\mathfrak{x})\, G_l(\mathfrak{x})\, H_m(\mathfrak{x})\, dx = \delta_{l_1 m_1} \cdots \delta_{l_n m_n}\, m_1! \cdots m_n!,$

where $w(\mathfrak{x})$ is the weight function defined by 12.8(14), δ_{pq} is defined in sec. 12.2, and

$$l = l_1 + \cdots + l_n.$$

To prove the biorthogonal property, we remark that by 12.8(14), (17) (18), the integral on the left-hand side of (1) is the coefficient of

(2) $$\frac{a_1^{l_1}}{l_1!} \cdots \frac{a_n^{l_n}}{l_n!}\ \frac{b_1^{m_1}}{m_1!} \cdots \frac{b_n^{m_n}}{m_n!}$$

in

(3) $$(2\pi)^{-\frac{1}{2}n}\,\Delta^{-\frac{1}{2}} \int \exp[-\tfrac{1}{2}\phi(\mathfrak{x}) + (\mathfrak{a}, \mathfrak{x}) - \tfrac{1}{2}\psi(\mathfrak{a}) + (C\mathfrak{b}, \mathfrak{x}) - \tfrac{1}{2}\phi(\mathfrak{b})]\,dx.$$

By 12.8(13), the expression (3) is equal to

(4) $$\exp[\tfrac{1}{2}\psi(\mathfrak{a} + C\mathfrak{b}) - \tfrac{1}{2}\psi(\mathfrak{a}) - \tfrac{1}{2}\phi(\mathfrak{b})],$$

and by 12.8(12) this is

(5) $$\exp[(\mathfrak{a}, \mathfrak{b})] = \sum \frac{(a_1 b_1)^{m_1}}{m_1!} \cdots \frac{(a_n b_n)^{m_n}}{m_n!}$$

The coefficient of (2) in the series (5) gives the right-hand side of (1).

A *bilinear generating function*, corresponding to Mehler's formula 10.13(22), may be obtained in a similar manner, computing the integral

$$(6)\quad (\pi^n t_1 \cdots t_n)^{-1} \int\int \exp[-\sum_{j=1}^{n} (u_j^2 + v_j^2)/t_j + \tfrac{1}{2}\phi(\mathfrak{x})$$
$$- \tfrac{1}{2}\phi(\mathfrak{x} - \mathfrak{u} - i\mathfrak{v}) + \tfrac{1}{2}\phi(\mathfrak{y}) - \tfrac{1}{2}\phi(\mathfrak{y} - \mathfrak{u} + i\mathfrak{v})]\, du\, dv$$

for sufficiently small positive $t_1, \dots, t_n$ in two different ways, once by using 12.8(13), and another time by first using 12.8(17) and (18), and then integrating directly. Putting

$$(7)\quad \phi_1(\mathfrak{x}) = \sum_{j=1}^{n} x_j^2/t_j + \phi(\mathfrak{x})$$
$$\phi_2(\mathfrak{x}) = \sum_{j=1}^{n} x_j^2/t_j - \phi(\mathfrak{x}),$$

noting that for sufficiently small positive $t_1, \dots, t_n$ the quadratic forms $\phi_k(\mathfrak{x})$, $k = 1, 2$, are positive definite, denoting the determinant of ϕ_k by Δ_k, and the reciprocal quadratic form by $\psi_k(\mathfrak{x})$, the result is

$$(8)\quad \sum \frac{t_1^{m_1}}{m_1!} \cdots \frac{t_n^{m_n}}{m_n!} H_m(\mathfrak{x})\, H_m(\mathfrak{y})$$
$$= (t_1 \cdots t_n)^{-1} (\Delta_1 \Delta_2)^{-\frac{1}{2}} \exp[\tfrac{1}{4}\psi_1(C\mathfrak{x}+C\mathfrak{y}) - \tfrac{1}{4}\psi_2(C\mathfrak{x}-C\mathfrak{y})].$$

In this form the result was obtained by Erdélyi (1938a) together with the corresponding result for the generating function of $H_m(\mathfrak{x})\, G_m(\mathfrak{y})$, thus extending results by Koschmieder (1938, 1938a) who gave explicit formulas for $n = 2$. The bilinear generating function was also discussed by Tortrat (1948, 1948a).

The system of partial differential equations satisfied by $H_m(\mathfrak{x})$ may also be derived from the generating function. The function on the left-hand side of 12.8(17) satisfies the system of partial differential equations

$$\sum_{j=1}^{n} \gamma_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j} - \sum_{k=1}^{n} c_{ik} x_k \sum_{j=1}^{n} \gamma_{ij} \frac{\partial F}{\partial x_j} - \Delta a_i \frac{\partial F}{\partial a_i} = 0$$
$$i = 1, 2, \dots, n$$

where Δ is the determinant of the c_{ij} and γ_{ij} is the cofactor of c_{ji} in Δ. Expanding in powers of a_i, we obtain the following system of partial differential equations for $H_m(\mathfrak{x})$

$$(9)\quad \sum_{j=1}^{n} \gamma_{ij}\left[\frac{\partial^2 H}{\partial x_i \partial x_j} - \sum_{k=1}^{n} c_{ik} x_k \frac{\partial H}{\partial x_j}\right] - m_i \Delta H = 0$$

$$i = 1, \ldots, n.$$

The partial differential equation

$$(10)\quad \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \frac{\partial^2 H}{\partial x_i \partial x_j} - \Delta \sum_{k=1}^{n} x_k \frac{\partial H}{\partial x_k} - m\Delta H = 0$$

is obtained by adding the n equations (9) and is common to all polynomials of the same degree m.

The proof of the system of partial differential equations

$$(11)\quad \sum_{j=1}^{n} \gamma_{ij} \frac{\partial^2 G}{\partial x_i \partial x_j} - \Delta x_i \frac{\partial G}{\partial x_i} + m_i \Delta G = 0 \qquad i = 1, \ldots, n$$

$$(12)\quad \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \frac{\partial^2 G}{\partial x_i \partial x_j} - \Delta \sum_{k=1}^{n} x_k \frac{\partial G}{\partial x_k} + m\Delta G = 0$$

satisfied by $G_m(\mathfrak{x})$ is similar.

Recurrence and differentiation formulas may also be obtained from the generating functions. For $n = 2$ they are recorded in Appell and Kampé de Fériet (1926, sec. CXXII).

There are many connections between Hermite polynomials in one and those in several variables. Replacing $\mathfrak{a}$ by $t\mathfrak{a}$ in 12.8(17) and (18), and expanding in powers of t by 10.13(19) we obtain

$$(13)\quad \sum_{m_1+\cdots+m_n=m} \frac{a_1^{m_1}}{m_1!} \cdots \frac{a_n^{m_n}}{m_n!} H_{m_1,\ldots,m_n}(x_1, \ldots, x_n)$$

$$= \frac{[\frac{1}{2}\phi(\mathfrak{a})]^{\frac{1}{2}m}}{m!} H_m\left(\frac{\phi(\mathfrak{a}, \mathfrak{x})}{[2\phi(\mathfrak{a})]^{\frac{1}{2}}}\right)$$

(14) $$\sum_{m_1+\cdots+m_n=m} \frac{a_1^{m_1}}{m_1!} \cdots \frac{a_n^{m_n}}{m_n!} G_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$$

$$= \frac{[\tfrac{1}{2}\psi(\mathfrak{a})]^{\frac{1}{2}m}}{m!} H_m\left(\frac{(\mathfrak{a},\mathfrak{x})}{[2\psi(\mathfrak{a})]^{\frac{1}{2}}}\right).$$

For other connections between Hermite polynomials of one and those of several variables see the book by Appell and Kampé de Fériet, and the papers by Feldheim, listed at the end of this chapter. Note that Feldheim's notation differs from our notation.

An addition theorem for Hermite polynomials in two variables was obtained by Koschmieder (1930a).

12.10. Further investigations

By a comparison of the generating functions it is easy to see that Hermite polynomials of several variables are limiting cases of the polynomials defined by 12.7(13) and (14).

(1) $$\lim_{s\to\infty} s^{-\frac{1}{2}m}\, \mathfrak{U}_m^s\left(\frac{\mathfrak{x}}{s^{\frac{1}{2}}}\right) = \frac{1}{m_1!\cdots m_n!} H_m(\mathfrak{x})$$

(2) $$\lim_{s\to\infty} s^{-\frac{1}{2}m}\, \mathfrak{V}_m^s\left(\frac{\mathfrak{x}}{s^{\frac{1}{2}}}\right) = \frac{1}{m_1!\cdots m_n!} G_m(\mathfrak{x}).$$

For the further investigation of Hermite polynomials one may use the *multi-dimensional Gauss transform*

(3) $$\mathfrak{G}_{\mathfrak{x}}^u[F(\mathfrak{y})] = \Delta^{\frac{1}{2}}(2\pi u)^{-\frac{1}{2}n} \int F(\mathfrak{y}) \exp\left[-\frac{1}{2u}\phi(\mathfrak{x}-\mathfrak{y})\right] dy$$

[see equations 10.13(30), (31)]. The first of the formulas

(4) $$\mathfrak{G}_{\mathfrak{x}}^u[H_m(\lambda\mathfrak{y})] = (1-\lambda^2 u)^{\frac{1}{2}m} H_m\left[\frac{\lambda\mathfrak{x}}{(1-\lambda^2 u)^{\frac{1}{2}}}\right]$$

(5) $$\mathfrak{G}_{\mathfrak{x}}^1[H_m(\mathfrak{y})] = \prod_{i=1}^{n}\left(\sum_{j=1}^{n} c_{ij}x_j\right)$$

(6) $$\mathfrak{G}_{\mathfrak{x}}^1\left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} c_{kj}y_j\right)\right] = i^{-m} H_m(i\mathfrak{x})$$

may be proved from the generating function 12.8(17), and is an integral equation satisfied by Hermite polynomials; the second is a limiting case of the first; and the third, which is also a limiting case ($\lambda \to \infty$) of the first, is an integral representation of Hermite polynomials. The corresponding formulas for G_m are

(7) $$\mathfrak{G}^u_{\mathfrak{x}}[G_m(\lambda\mathfrak{y})] = (1-\lambda^2 u)^{\frac{1}{2}m}\, G_m\left[\frac{\lambda\mathfrak{x}}{(1-\lambda^2 u)^{\frac{1}{2}}}\right]$$

(8) $$\mathfrak{G}^1_{\mathfrak{x}}[G_m(\mathfrak{y})] = \prod_{j=1}^{n} x_j$$

(9) $$\mathfrak{G}^1_{\mathfrak{x}}\left[\prod_{j=1}^{n} y_j\right] = i^{-m}\, G_m(i\mathfrak{x}).$$

Feldheim (1942) used a more general definition

(10) $$\mathfrak{G}^u_{\mathfrak{x}}[F(\mathfrak{y})] = \frac{\Delta^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}n}}{(u_1 \cdots u_n)^{\frac{1}{2}}} \times \int F(\mathfrak{y}) \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} c_{ij}\, \frac{x_i - y_i}{u_i^{\frac{1}{2}}}\, \frac{x_j - y_j}{u_j^{\frac{1}{2}}}\right) dy$$

and investigated the behavior of Hermite polynomials under the functional transformation defined by (10).

The biorthogonal property 12.9 (1) shows that an "arbitrary" function $f(\mathfrak{x})$ may be expanded in series of Hermite polynomials in either of the two forms

(11) $\Sigma\, a_m\, G_m(\mathfrak{x})$

(12) $\Sigma\, b_m\, H_m(\mathfrak{x})$

where

(13) $m_1! \cdots m_n!\, a_m = \int w(\mathfrak{x})\, f(\mathfrak{x})\, H_m(\mathfrak{x})\, dx.$

(14) $m_1! \cdots m_n!\, b_m = \int w(\mathfrak{x})\, f(\mathfrak{x})\, G_m(\mathfrak{x})\, dx.$

The convergence of such expansions was discussed by Thijssen (1926, 1927) for the case $n = 2$ and functions $f(\mathfrak{x})$ which vanish identically outside a bounded region, and satisfy certain continuity requirements in that region. The problem of approximations in mean square (see sec. 10.2) was discussed by Caccioppoli (1932a) for functions of the class L^2_w, that is to say for functions for which the integral

$$\int |f(\underset{\sim}{x})|^2 \exp[-\tfrac{1}{2}\phi(\underset{\sim}{x})]\, dx$$

is convergent. The approximation to arbitrary functions in unbounded regions has also been discussed by Picone (1935).

Mazza (1940) has also discussed Hermite polynomials and constructed an orthogonal system. Devisme (1932) defined systems of polynomials which are, in some measure, analogous to Hermite polynomials. The generating functions are

$$(15) \quad \exp(ax - a^2 y + a^3/3), \qquad \exp[ax - (a^2 - b)\, y + a^3/3].$$

The polynomials generated by (15) are related to certain partial differential equations involving the differential operator 12.7 (12).

REFERENCES

Angelescu, Aurel, 1915: *C.R. Acad. Sci. Paris* 161, 490-492.

Angelescu, Aurel, 1915-16: *Bull. Math. Soc. Roumaine Sci.* 4, 30-35.

Angelescu, Aurel, 1916: *Sur des polynomes généralisant les polynomes de Legendre et d'Hermite et sur le calcul approché des integrales multiples*, Thesis, Paris, 140 pp.

Appell, Paul, 1881: *Arch. Math. Physik* (1) 66, 238-245.

Appell, Paul, 1882: *J. Math. Pures Appl.* (3) 8, 173-216.

Appell, Paul, 1901: *Arch. Math. Physik* (3) 1, 69-71.

Appell, Paul, 1903: *Arch. Math. Physik* (3) 4, 20-21.

Appell, Paul and Joseph Kampé de Fériet, 1926: *Fonctions hypergéométriques et hypersphériques, Polynomes d' Hermite*, Gauthier-Villars, Paris.

Brinkman, H.C. and Frits Zernike, 1935: *Nederl. Akad. Wetensch, Proc.* 38, 161-170.

Caccioppoli, Renato, 1932: *Rend. Sem. Mat. Univ. Padova* 3, 163-182.

Caccioppoli, Renato, 1932a: *Giorn. Ist. Ital. Attuari* 3, 364-375.

Cameron, R.H. and W.T. Martin, 1947: *Ann. of Math.* 48, 385-389.

Chen, K.K., 1928: *Sci. Rep. Tohoku Imp.Univ.* Ser. I, 17, 1073-1086.

Devisme, Jacques, 1932: *C.R. Acad. Sci. Paris* 195, 437-439, 936-938.

Didon, François, 1868: *Ann. Sci. École Norm. Sup.* (1) 5, 229-310.

Dinghas, Alexander, 1950: *Math. Z.* 53, 76-83.

Erdélyi, Arthur, 1938: *Math. Ann.* 11, 456-465.

Erdélyi, Arthur, 1938a: *Math. Z.* 44, 301-311.

Feldheim, Ervin, 1940: *Ann. Scuola Norm. Super. Pisa* (2) 9, 225-252.

Feldheim, Ervin, 1941: *C.R. (Doklady) Acad. Sci. URSS* (*N.S.*) 31, 534-537.

Feldheim, Ervin, 1942: *Pont. Acad. Sci. Comment.* 6, 1-25.

Friedrichs, K.O., 1951: *Comm. Pure Appl. Math.* 4, 161-224.

Giulotto, Virgilio, 1939: *Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat.* 72, 37-57.

Grad, Harold, 1949: *Comm. Pure Appl. Math.* 2, 325-330.

Gröbner, Wolfgang, 1948: *Monatsh. Math.* 52, 38-54.

Hermite, Charles, 1864: *C.R. Acad. Sci. Paris*, 58, 93-100, 266-273.

Hermite, Charles, 1865: *J. Reine Angew. Math.* 64, 294-296.

Hermite, Charles, 1865a: *C.R. Acad. Sci. Paris* 60, 370-377, 432, 440, 461-466, 512-518.

REFERENCES

Jackson, Dunham, 1937: *Duke Math. J.* 2, 423-434.

Jackson, Dunham, 1938: *Duke Math. J.* 4, 441-454.

Jackson, Dunham, 1938 a: *Ann. of Math.* (2) 39, 262-268.

Kampé de Fériet, Joseph, 1915: *Sur les fonctions hyperspheriques*, Thesis, Paris, 111 pp.

Koschmieder, Lothar, 1924: *Math. Ann.* 91, 62-81.

Koschmieder, Lothar, 1925: *Jber. Deutsch. Math. Verein* 34, 57-64.

Koschmieder, Lothar, 1926: *Revista Mat. Hisp.-Amer.* (2) 1, 97-107.

Koschmieder, Lothar, 1929: *Math. Ann.* 101, 120-125.

Koschmieder, Lothar, 1930: *Revista Soc. Mat. Espanola* (2) 5, 1-14.

Koschmieder, Lothar, 1930 a: *Revista Soc. Mat. Espanola* (2) 5, 274-280.

Koschmieder, Lothar, 1931: *Math. Ann.* 104, 387-402.

Koschmieder, Lothar, 1933: *Monatsh. Math. Phys.* 40, 223-232.

Koschmieder, Lothar, 1934: *Math. Ann.* 110, 734-738.

Koschmieder, Lothar, 1934 a: *Monatsh. Math. Phys.* 41, 58-63.

Koschmieder, Lothar, 1938: *Math. Z.* 43, 248-254.

Koschmieder, Lothar, 1839 a: *Math. Z.* 43, 783-792.

Koschmieder, Lothar, 1940: *Anz. Akad. Wiss. Wien. Math.-Nat. Kl.* 41-43.

Mazza, S.C., 1940: *An. Soc. Ci. Argentina* 130, 137-148.

Orloff, G.A., 1881: *On some polynomials in one or several variables*, Thesis, St. Petersburg, 124 pp.

Orlow, G.A., 1881 a: *Nouv. Ann.* (2) 40, 481-489.

Picone, Mauro, 1935: *Giorn. Ist. Ital. Attuari* 5, 155-195.

Schmeidler, Werner, 1941: *J. Reine Angew. Math.* 183, 175-182.

Thijssen, W.P., 1926: *Verslagen Amsterdam* 35 (2) 1100-1111.

Thijssen, W.P., 1927: *Nederl. Akad. Wetensch, Proc.* 30 (1), 69-80.

Tortrat, Albert, 1948: *C.R. Acad. Sci. Paris* 226, 298-300.

Tortrat, Albert, 1948 a: *C.R. Acad. Sci. Paris* 226, 543-545, errata 758-759.

CHAPTER XIII

ELLIPTIC FUNCTIONS AND INTEGRALS

13.1. Introduction

Elliptic integrals were encountered by John Wallis in 1655-59. They were known to Euler who, in 1753, obtained their addition theorem. Legendre, whose work on elliptic integrals stretches over several decades, introduced the normal forms which are still in use. Jacobi, in 1828, introduced elliptic functions obtained as inversions of (indefinite) elliptic integrals; he also studied systematically theta functions. Abel obtained some of Jacobi's results independently, and he also studied what is now called hyperelliptic and abelian integrals. Weierstrass showed how the theory of elliptic functions fits in with the theory of functions of a complex variable, and developed the general theory of doubly periodic functions.

The history of elliptic functions is given in the article by R. Fricke in the *Encyklopädie* (1913). This article also contains a list of references up to 1913. The more important books on elliptic functions which appeared since 1913 are listed at the end of this chapter; for the older literature the reader may be referred to Fricke's article.

The present chapter consists of two parts, one on elliptic integrals, and the other on elliptic functions. In the second part, both Jacobian and Weierstrassian functions are treated, the former on account of their usefulness in connection with numerical work, the latter on account of the symmetry and simplicity of the basic relations. It may be mentioned here that Neville (1944) developed a systematic notation for Jacobian elliptic functions which simplifies the formulas to a considerable extent: in the present chapter we shall adhere to the traditional notation for the sole reason that it is still generally used. Theta functions are also included in the second part, and there is also a brief section on elliptic modular functions. For further information on modular functions see Chapter XIV.

PART ONE: ELLIPTIC INTEGRALS

13.2. Elliptic integrals

The simplest (indefinite) integrals are integrals of a rational function. The next simplest type consists of integrals of the form

(1) $I = \int R(x, y)\, dx,$

in which R is a rational function of its two variables, and y is an *algebraic function* of x, that is to say, is given by an equation of the form

(2) $P(x, y) = 0$

where P is a polynomial of degree n, say, in its two variables. Such integrals are called *abelian* integrals.

One of the striking features of the theory of abelian integrals is the fact that the behavior of the integral (1) depends not so much on the nature of R as on the nature of P, or rather on the nature of the *algebraic curve* C_n in the x,y-plane represented by equation (2). For the theory of abelian integrals, algebraic curves of degree n are classified according to their *genus* (or *deficiency*),

(3) $p = \binom{n-1}{2} - d,$

that is the difference between the largest possible number $\binom{n-1}{2}$ of double points of a non-degenerate curve of degree n, and the actual number of double points, d, of the curve in question. The genus is a *birational invariant*, that is, it remains unchanged if the curve is subjected to a *birational transformation*

(4) $x = R_1(\xi, \eta), \quad y = R_2(\xi, \eta),$

where the rational functions R_1 and R_2 are such that two further rational functions R_3, R_4 exist so that

(5) $\xi = R_3(x, y), \quad \eta = R_4(x, y).$

Curves of genus zero are *unicursal* (or *rational*) curves. It is known that for such curves x and y can be expressed as *rational functions* of a parameter. Since rational functions are single-valued, this parameter is a *uniformizing variable* for the curve. On introduction of this parameter as a new variable of integration in (1), the integrand is a rational function of the parameter, and the integral may be evaluated in terms of elementary functions (of the parameter). The parameter itself is an algebraic function of x, and hence *abelian integrals of genus zero may be expressed in terms of elementary and algebraic functions.*

For algebraic curves of genus unity, Clebsch (1865) proved that x and y can be expressed as rational functions of *two* parameters ξ and η where η^2 is a polynomial in ξ of degree three or four. Introducing ξ as a new variable of integration in (1), it is seen that every integral of genus unity can be reduced to a form in which the equation (2) becomes

$$(6) \quad y^2 = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4$$

where either $a_0 \neq 0$ or $a_0 = 0$ and $a_1 \neq 0$. Integrals defined by (1), (6) are called *elliptic integrals*, and we have proved that *abelian integrals of genus unity may be reduced to elliptic integrals* by a rational change of the variable of integration. We shall see later, in sec. 13.14, that in equation (6), and hence for any algebraic curve of genus unity, x and y may be expressed rationally in terms of single-valued elliptic functions of a variable z which is a uniformizing variable for the curve in question.

For $p \geq 2$ the situation is much more involved. We have here *hyper-elliptic integrals* for which equation (2) takes the form

$$(7) \quad y^2 = a_0 x^n + na_1 x^{n-1} + \cdots + a_n,$$

but it is no longer true that every curve can be transformed, by a birational transformation, to the form (7). Accordingly, hyperelliptic functions do not suffice for the uniformization of algebraic curves of genus $p \geq 2$, and *automorphic functions* must be used: see also sec. 14.9.

In this chapter we restrict ourselves to elliptic integrals defined by (1), (6), and to the elliptic functions associated with such integrals. The polynomial on the right-hand side of (6) will be denoted by $G_4(x)$ when $a_0 \neq 0$, and by $G_3(x)$ when $a_0 = 0$, $a_1 \neq 0$. If the polynomial on the right-hand side of (6) has a double zero, the integral I may be evaluated in terms of elementary functions. Thus we may assume that G_4 (or G_3, as the case may be) has no double zero.

13.3. Reduction of elliptic integrals

It has been stated in the preceding section that for the behavior of the elliptic integral

$$(1) \quad I = \int R(x, y)\, dx, \qquad y^2 = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4$$

the polynomial $a_0 x^4 + \cdots + a_4$ is more important than the rational function R. This statement is justified, and is given a precise meaning, by the following theorem due to Legendre. *The elliptic integral* (1) *may be expressed as a linear combination (with constant coefficients) of an integral of a rational function of x and of integrals of the following types:*

$$(2)\quad I_1 = \int \frac{dx}{y}, \quad I_2 = \int \frac{\frac{1}{2}a_0 x^2 + a_1 x}{y}\,dx, \quad I_3 = \int \frac{dx}{(x-c)\,y}$$

where c is a constant parameter, and

$$(3)\quad I_3^* = \int \frac{x\,dx}{y}$$

is interpreted as the integral I_3 corresponding to the case $c = \infty$. The reduction will be effected in several steps.

Since even powers of y may be expressed as polynomials in x, we may write R in the form

$$(4)\quad R(x, y) = \frac{M_1(x) + M_2(x)\,y}{N_1(x) + N_2(x)\,y} = \frac{[M_1(x) + M_2(x)y][N_1(x) - N_2(x)y]\,y}{\{[N_1(x)]^2 - [N_2(x)\,y]^2\}\,y}$$

where M_1, M_2, N_1, N_2 are polynomials in x, and this may be written as

$$(5)\quad R(x, y) = R_1(x) + \frac{R_2(x)}{y}$$

with two rational functions, R_1 and R_2, of x, thus completing the first step in the reduction process.

As a second step we remark that $R_2(x)$, being a rational function of x, may be decomposed into a polynomial in x and a sum of partial fractions. Thus,

$$(6)\quad I = \int R(x, y)\,dx = \int R_1(x)\,dx + \sum_n a_n \int \frac{x^n}{y}dx + \sum_{m,r} A_{m,r} \int \frac{dx}{(x-c_m)^r y},$$

and it is sufficient to consider further the integrals

$$(7)\quad J_n = \int \frac{x^n}{y}dx \qquad n = 0, 1, 2, \ldots$$

$$H_r = \int \frac{dx}{(x-c)^r y} \qquad r = 1, 2, \ldots .$$

The third step is based on certain recurrence relations for J_n and H_r. Let us define $b_0, \dots, b_4$ by means of the identity

$$(8)\quad a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$$
$$= b_0(x-c)^4+4b_1(x-c)^3+6b_2(x-c)^2+4b_3(x-c)+b_4$$

in x. We then have the following identities

$$(9)\quad \frac{d}{dx}(x^m y) = mx^{m-1}y + x^m y' = \frac{1}{y}\left[mx^{m-1}y^2 + \frac{1}{2}x^m \frac{d(y^2)}{dx}\right]$$
$$= \frac{1}{y}\Big[mx^{m-1}(a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4)$$
$$+\frac{1}{2}x^m(4a_0x^3+12a_1x^2+12a_2x+4a_3)\Big]$$
$$= (m+2)a_0\frac{x^{m+3}}{y} + 2(2m+3)a_1\frac{x^{m+2}}{y} + 6(m+1)a_2\frac{x^{m+1}}{y}$$
$$+ 2(2m+1)a_3\frac{x^m}{y} + ma_4\frac{x^{m-1}}{y}$$

$$(10)\quad \frac{d}{dx}[(x-c)^m y] = (m+2)b_0\frac{(x-c)^{m+3}}{y} + 2(2m+3)b_1\frac{(x-c)^{m+2}}{y}$$
$$+ 6(m+1)b_2\frac{(x-c)^{m+1}}{y} + 2(2m+1)b_3\frac{(x-c)^m}{y} + mb_4\frac{(x-c)^{m+1}}{y}.$$

Putting $m = 0, 1, 2, \dots$ in (9), $m = -1, -2, -3, \dots$ in (10) and integrating, we obtain successively

$$(11)\quad 2a_0J_3 + 2\cdot 3a_1J_2 + 6\cdot 1a_2J_1 + 2\cdot 1a_3J_0 = y$$
$$3a_0J_4 + 2\cdot 5a_1J_3 + 6\cdot 2a_2J_2 + 2\cdot 3a_3J_1 + a_4J_0 = xy$$
$$4a_0J_5 + 2\cdot 7a_1J_4 + 6\cdot 3a_2J_3 + 2\cdot 5a_3J_2 + 2a_4J_1 = x^2y$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(12)\quad b_0\int\frac{(x-c)^2}{y}dx+2\cdot 1\cdot b_1\int\frac{x-c}{y}dx-2\cdot 1\cdot b_3H_1-1\cdot b_4H_2=\frac{y}{x-c}$$

$$-2\cdot 1\cdot b_1J_0-6\cdot 1\cdot b_3H_1-2\cdot 3\cdot b_3H_2-2\cdot b_4H_3=\frac{y}{(x-c)^2}$$

$$-b_0J_0-2\cdot 3\cdot b_1H_1-6\cdot 2\cdot b_2H_2-2\cdot 5\cdot b_3H_3-3\cdot b_4H_4=\frac{y}{(x-c)^3}$$

. . . .

Now,

$$(13)\quad \int\frac{x-c}{y}dx=J_1-cJ_0,\qquad \int\frac{(x-c)^2}{y}dx=J_2-2cJ_1+J_0$$

and hence equations (11) and (12) serve to express all J_n and H_r in terms of J_0, J_1, J_2, H_1, and certain rational functions of x and y. Moreover, a comparison of (7) with (2) and (3) shows that

$$(14)\quad J_0=I_1,\quad J_1=I_3^*,\quad a_0J_2=2I_2-2a_1I_3^*,\quad H_1=I_3,$$

and thus proves Legendre's theorem.

If $a_0=0$, and hence also $b_0=0$, there is a slight simplification. In this case

$$(15)\quad I_2=a_1I_3^* \qquad a_0=0$$

and hence all integrals reduce to a linear combination of I_1, I_3, I_3^*. Also from (11) and (12) it is seen that in this case all J_n and H_r may be expressed in terms of only J_0, J_1, H_1, and rational functions of x and y.

The integrals I_1, I_2, I_3 may be called *elliptic integrals of the first, second, and third kinds*, respectively.

A linear fractional transformation of the variable of integration in (1) changes the polynomial y^2, and an appropriate transformation of this kind may be used to reduce the polynomial to a standard form (see sec. 13.5). There are two such standard forms in use, and we shall give the more important results of the present section for each of these two standard forms, adding a brief note on a third form.

Weierstrass' form. Here

$$(16)\quad y^2=4x^3-g_2x-g_3.$$

The integrals of the first, second and third kinds are, respectively,

(17) $$I_1 = J_0 = \int \frac{dx}{(4x^3 - g_2 x - g_3)^{1/2}}$$

$$I_2 = I_3^* = J_1 = \int \frac{x\,dx}{(4x^3 - g_2 x - g_3)^{1/2}}$$

$$I_3 = H_1 = \int \frac{dx}{(x-c)(4x^3 - g_2 x - g_3)^{1/2}}.$$

The first few recurrence relations are

(18) $$J_2 = \int \frac{x^2\,dx}{(4x^3 - g_2 x - g_3)^{1/2}} = \frac{1}{6}\,(4x^3 - g_2 x - g_3)^{1/2} + \frac{1}{12} g_2 J_0$$

$$J_3 = \int \frac{x^3\,dx}{(4x^3 - g_2 x - g_3)^{1/2}} = \frac{1}{10} x (4x^3 - g_2 x - g_3)^{1/2}$$

$$+ \frac{3}{20} g_2 J_1 + \frac{1}{10} g_3 J_0$$

$$H_2 = \int \frac{dx}{(x-c)^2 (4x^3 - g_2 x - g_3)^{1/2}}$$

$$= \frac{2(J_1 - cJ_0) - (6c^2 - \tfrac{1}{2} g_2) H_1 - (4x^3 - g_2 x - g_3)^{1/2} (x-c)^{-1}}{4c^3 - g_2 c - g_3}.$$

Legendre's form. Here

(19) $y^2 = (1 - x^2)(1 - k^2 x^2).$

It is customary to define the corresponding elliptic integrals of the first, second and third kinds respectively, as

(20) $$F = \int \frac{dx}{[(1-x^2)(1-k^2x^2)]^{1/2}} = \int \frac{d\phi}{(1 - k^2 \sin^2 \phi)^{1/2}}$$

$$E = \int \left(\frac{1 - k^2 x^2}{1 - x^2}\right)^{1/2} dx = \int (1 - k^2 \sin^2 \phi)^{1/2}\, d\phi$$

$$x = \sin\phi$$

$$(20)\quad \Pi = \int \frac{dx}{(1 - x^2/c^2)[(1 - x^2)(1 - k^2x^2)]^{1/2}}$$

$$= \int \frac{d\phi}{(1 - c^{-2}\sin^2\phi)(1 - k^2\sin^2\phi)^{1/2}} \qquad x = \sin\phi .$$

The basic integrals of the general theory are

$$(21)\quad I_1 = J_0 = F$$

$$(22)\quad I_2 = ½(F - E)$$

$$(23)\quad I_3 = H_1 = \int \frac{(x + c)\,dx}{(x^2 - c^2)[(1 - x^2)(1 - k^2x^2)]^{1/2}}$$

$$= \int \frac{½\,d(x^2)}{(x^2 - c^2)[(1 - x^2)(1 - k^2x^2)]^{1/2}} - \frac{1}{c}\Pi$$

$$(24)\quad I_3^* = J_1 = \int \frac{x\,dx}{[(1 - x^2)(1 - k^2x^2)]^{1/2}} .$$

The first integral on the second line of (23), and the integral in (24), may be evaluated in terms of elementary functions so that everything may be expressed in terms of E, F, Π. The recurrence relation for the J_n are

$$(25)\quad 2k^2J_3 - (1 + k^2)J_1 = [(1 - x^2)(1 - k^2x^2)]^{1/2}$$

$$3k^2J_4 - 2(1 + k^2)J_2 + J_0 = x[(1 - x^2)(1 - k^2x^2)]^{1/2}$$

$$4k^2J_5 - 3(1 + k^2)J_3 + 2J_1 = x^2[(1 - x^2)(1 - k^2x^2)]^{1/2}$$

. . . .

and the recurrence relations for the H_r may be obtained from equation (12).

A third canonical form,

$$(26)\quad y^2 = x(x - m)(x - 1)$$

has been suggested by A.R. Low (1950). In a sense it is between Weierstrass' and Legendre's form and has some of the advantages of both. It may be obtained from Weierstrass' form by a translation and normalization, or from Legendre's form by the substitution

$$x^2 = 1/\xi, \quad y^2 = \eta^2/\xi^3 .$$

The latter derivation shows that the parameter m corresponds to k^2 in Legendre's form.

13.4. Periods and singularities of elliptic integrals

We shall now consider

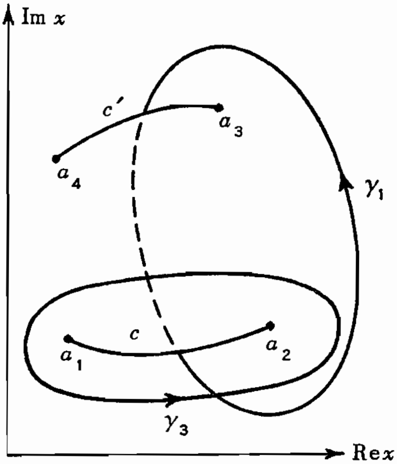
(1) $I(x) = \int_a^x R(\xi, \eta)\, d\xi,$

where

(2) $\eta^2 = G(\xi) = a_0 \xi^4 + 4a_1 \xi^3 + 6a_2 \xi^2 + 4a_3 \xi + a_4,$

and regard $I(x)$ as a function of x, the lower limit, a, being fixed (and the integrand regular at $\xi = a$).

The integrand is a two-valued function of ξ whose branch-points coincide with those of η; and we shall study the behavior of $I(x)$ on the Riemann surface of $[G(x)]^{1/2}$ rather than in the x-plane. If $a_0 \neq 0$, let a_1, a_2, a_3, a_4 be the four (distinct) zeros of $G_4(x)$; if $a_0 = 0$ (and $a_1 \neq 0$), let a_1, a_2, a_3 be the three (distinct) zeros of $G_3(x)$, and let $a_4 = \infty$. In either of these cases, join a_1 and a_2 by an arc c, and a_3 and a_4 by an arc c' which has no point in common with c. Cut two copies of the complex x-plane along the arcs c and c', and join them crosswise along the cuts, thus obtaining a model of the Riemann surface, $\mathfrak{R}$, of $[G(x)]^{1/2}$. The integrand, $R(x, y)$, is a meromorphic function on $\mathfrak{R}$, that is to say $R(x, y)$ is a single-valued function of x on $\mathfrak{R}$, and is analytic, except possibly at a finite number of points where it has poles. On the other hand, $I(x)$ is a many-valued function on $\mathfrak{R}$, since there are closed curves, Γ, on $\mathfrak{R}$ which cannot be deformed into a point, and for which $\int_\Gamma R\, d\xi \neq 0$. The closed curves γ_1 and γ_3 of the figure are such curves. (The curve γ_1 crosses the branch-cuts and its dotted portion lies in the "second sheet" of the Riemann surface.) In addition there is a closed curve encircling each pole at which the residue is $\neq 0$. Let b_i be one of the poles of R, and let r_i be the residue at b_i of R. Given any closed curve, C, on $\mathfrak{R}$, it follows by deformation of contours that there are integers m, n, p_i (positive, negative, or zero) such that



$$\int_C R(\xi, \eta)\, d\xi = m \int_{\gamma_1} R\, d\xi + n \int_{\gamma_3} R\, d\xi + \sum_i p_i 2\pi i r_i .$$

This means that $I_0(x)$ being one of the possible values of $I(x)$, any other of the possible values of this function is of the form

(3) $I(x) = I_0(x) + m_1 \Omega_1 + m_2 \Omega_2 + \cdots + m_k \Omega_k$

where $m_1, \ldots, m_k$ are arbitrary integers and $\Omega_1, \ldots, \Omega_k$ are certain complex numbers independent of x. They are known as the *periods* or *moduli of periodicity* of $I(x)$.

Every elliptic integral has at least two periods (for instance, the periods corresponding to γ_1 and γ_3). The integrands of I_1 and I_2 in 13.3 (2) have no non-zero residues in the cut plane, and hence elliptic integrals of the first and second kinds have *exactly two* (independent) periods. On the other hand, $x = c$ is a simple pole with residue $[G(c)]^{-1/2}$ of the integrand of I_3, and accordingly elliptic integrals of the third kind have *three* (independent) periods.

We are now in a position to describe the singularities of elliptic integrals of the first, second, and third kinds. They all have *branch-points* at $x = a_1, a_2, a_3, a_4$, and their values at these branch-points are *finite*, with the single exception of the point $a_4 = \infty$ for I_2 in the case $a_0 = 0$. In addition we have the following behavior of these integrals.

Elliptic integrals of the first kind are analytic on $\mathfrak{R}$, *except at* $x = a_1$, a_2, a_3, a_4. *They are finite at every point of* $\mathfrak{R}$. This is clear from the behavior of their integrand.

Elliptic integrals of the second kind are analytic on $\mathfrak{R}$ *except at* $x = a_1, a_2, a_3, a_4$, *and* ∞. *At* ∞ *they have poles if* $a_0 \neq 0$. (If $a_0 = 0$, then $a_4 = \infty$, and I_2 has a branch-point, and becomes infinite there.) There are two poles at infinity if $a_0 \neq 0$, one in each of the sheets of $\mathfrak{R}$, and the residues at these poles are zero.

Elliptic integrals of the third kind are analytic on $\mathfrak{R}$ *except at* $x = a_1$, a_2, a_3, a_4, *and* c. *They have logarithmic singularities at* $x = c$. There are two points $x = c$, one in each sheet of $\mathfrak{R}$, and the behavior of I_3 in the neighborhood of these points is like that of

$$\pm [G(c)]^{-1/2} \log (x - c).$$

The different behavior of these elliptic integrals shows clearly that in general (i.e., apart from special values of c or x), an elliptic integral of the third kind cannot be reduced to integrals of the first and second kinds.

Another interesting feature of elliptic integrals of the third kind is expressed by the *interchange theorem*. Let

$$I_3(x, c) = \int_{\infty}^{x} \frac{d\xi}{(\xi - c)\,\eta}.$$

Then

$$I_3(x, c) - I_3(c, x) = I_1(c)\, I_2(x) - I_1(x)\, I_2(c) + (2m+1)\,\pi.$$

For the proofs of the statements presented in this section and for further details, see Tricomi (1937).

13.5. Reduction of $G(x)$ to normal form

In considering elliptic integrals it is convenient to reduce the polynomial

(1) $$G(x) = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = y^2$$

to one of the two standard forms given in sec. 13.3. The reduction is achieved by means of a linear fractional transformation of x. For Weierstrass' form, one of the zeros of $G(x)$ is mapped on ∞, and then the centroid of the remaining three zeros is taken as the origin. For the Legendre form, a pair of points is chosen which is apolar with respect to (forms cross-ratio -1 with) each of two pairs of roots of $G(x)$, and these points are mapped on 0 and ∞. The four roots of $G(x)$ can be grouped in two pairs in three distinct ways, and accordingly there are three distinct ways of the reduction to Legendre's form of any given $G(x)$. Weierstrass' form is more symmetric, and hence more suitable for theoretical investigations; Legendre's form is more highly standardized and hence more suitable for numerical computations. Most of the existing numerical tables have been computed for Legendre's form. We shall describe briefly the reduction to each of the two standard forms.

Reduction to Weierstrass' normal form. If $a_0 \neq 0$, we reduce $G(x)$ to a cubic by the transformation

(2) $$x = a_4 - \frac{1}{X}, \qquad y = \frac{Y}{X^2}$$

where a_4 is one of the zeros of $G(x)$. This transformation changes (1) into

(3) $$4A_1 X^3 + 6A_2 X^2 + 4A_3 X + A_4 = Y^2$$

where

(4) $$A_1 = \tfrac{1}{4} G'(a_4) = a_0\, a_4^3 + 3a_1 a_4^2 + 3a_2 a_4 + a_3,$$

$$A_2 = \frac{1}{12} G''(a_4) = a_0 a_4^2 + 2a_1\, a_4 + a_2,$$

$$A_3 = \frac{1}{24} G'''(a_4) = a_0 a_4 + a_1, \qquad A_4 = \frac{1}{24} G''''(a_4) = a_0.$$

If $a_0 = 0$, then (1) is already of the form (3) and no preliminary transformation is needed.

Next, we eliminate the quadratic term by the transformation

$$(5) \quad X = \frac{\xi - \frac{1}{2}A_2}{A_1}, \qquad Y = \frac{\eta}{A_1}$$

which changes (3) into Weierstrass' form

$$(6) \quad 4\xi^3 - g_2\xi - g_3 = \eta^2$$

where

$$(7) \quad g_2 = 3A_2^2 - 4A_1A_3, \qquad g_3 = 2A_1A_2A_3 - A_2^3 - A_1^2A_4.$$

From (4) and (7) it is seen that

$$(8) \quad g_2 = a_0a_4 + 3a_2^2 - 4a_1a_3$$

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

are *invariants* of the quartic $G(x)$; see, for instance, Burnside and Panton (1892, sec. 160) where the expression of these invariants as symmetric functions of the roots is given. It should be noted that the final form (6) is independent of the zero, a_4, of $G(x)$ which was selected for the transformation and that the coefficients in (6) are rational functions (actually polynomials) of the coefficients of (1). In particular, if $a_0, \dots, a_4$ are real, then also g_2 and g_3 are real.

Reduction to Legendre's normal form. We first show that $G(x)$ may be factorized in the form

$$(9) \quad G(x) = [B_1(x-\beta)^2 + C_1(x-\gamma)^2][B_2(x-\beta)^2 + C_2(x-\gamma)^2].$$

In fact, $G(x)$ may certainly be factorized as

$$(10) \quad G(x) = Q_1(x)\,Q_2(x)$$

$$Q_1(x) = p_1x^2 + 2q_1x + r_1, \qquad Q_2(x) = p_2x^2 + 2q_2x + r_2.$$

With a constant multiplier λ, $Q_1 - \lambda Q_2$ will be a perfect square if

$$(11) \quad (p_1 - \lambda p_2)(r_1 - \lambda r_2) - (q_1 - \lambda q_2)^2 = 0.$$

Let λ_1, λ_2 be the two roots of this equation. Then

(12) $Q_1 - \lambda_1 Q_2 = (p_1 - \lambda_1 p_2)(x-\beta)^2$

$Q_1 - \lambda_2 Q_2 = (p_1 - \lambda_2 p_2)(x-\gamma)^2$

and hence

(13) $Q_1 = B_1(x-\beta)^2 + C_1(x-\gamma)^2$

$Q_2 = B_2(x-\beta)^2 + C_2(x-\gamma)^2$

with certain constants $B_1, \dots, \gamma$; and this proves (9). Moreover, if $a_0, \dots, a_4$ are real and $G(x)$ has at least one pair of complex roots, let $Q_1(x)$ have complex roots. Then the left-hand side of (11) is > 0 when $\lambda = 0$, and ≤ 0 when $\lambda = p_1/p_2$, so that λ_1 and λ_2 are real, β and γ in (12) are real and so are $B_1, \dots, C_2$ in (13). If $a_0, \dots, a_4$ are real and all zeros of $G(x)$ are real, the factorization (10) may be arranged so that the zeros of $Q_1(x)$ do not interlace those of $Q_2(x)$, and in this case it is easy to see that $B_1, \dots, \gamma$ are real. Thus for a real $G(x)$ there is always a real factorization of the form (9). Furthermore, this factorization is valid both for G_4 and G_3; in the latter case either $B_1 + C_1 = 0$ or $B_2 + C_2 = 0$.

In (9) we put

(14) $$\frac{x-\gamma}{x-\beta} = \left(-\frac{B_1}{C_1}\right)^{1/2} \xi, \quad \frac{y}{(x-\beta)^2} = (B_1 B_2)^{1/2} \eta$$

and obtain Legendre's normal form

(15) $$(1-\xi^2)(1-k^2\xi^2) = \eta^2$$

where

(16) $$k^2 = \frac{B_1 C_2}{B_2 C_1}.$$

The quantity k is the *modulus*. Clearly we may take $|k^2| \le 1$, and if $|k^2| = 1$, a different grouping of the zeros may be used to make $|k^2| < 1$, except in the so-called *equianharmonic* case when $-k^2$ is a complex cube root of unity. This exceptional case arises when the zeros of $(1-\xi^2)(1-k^2\xi^2)$ lie at the end-points of two diameters of the unit circle and the angle between the two diameters is $\pi/6$.

We shall now give more specific reduction formulas for the case that the coefficients in (1) are real and that $G(x) \ge 0$ in the interval of integration. It will be seen that in this case the reduction may be effected by a *real transformation* in such a manner that $0 < k^2 < 1$. We shall give the reduction to trigonometric form [the variable ϕ in equations 13.3(20)],

(17) $y^2 = \cos^2\phi(1 - k^2 \sin^2\phi)$.

By division by a positive number we may make the leading coefficient (a_0 in G_4, or a_1 in G_3) ± 1, and we shall assume that this has been done so that

(18) $G(x) = \pm \prod_i (x - a_i)$

where $i = 1, 2, 3, 4$ or $i = 1, 2, 3$, according as G is G_4 or G_3. We shall use the abbreviations

(19) $a_{rs} = a_s - a_r$

(20) $(\alpha, \beta, \gamma, \delta) = \dfrac{\alpha - \gamma}{\alpha - \delta} \dfrac{\beta - \delta}{\beta - \gamma}$

(21) $\mu = \left(\dfrac{1 - k^2 \sin^2\phi}{G(x)}\right)^{\frac{1}{2}} \dfrac{dx}{d\phi}$

where μ is a constant and

(22) $\dfrac{dx}{[G(x)]^{\frac{1}{2}}} = \mu \dfrac{d\phi}{(1 - k^2 \sin^2\phi)}$

so that μ occurs in the conversion of the elliptic integrals of the first kind.

Table 1 gives the transformation formulas for the case that all roots of $G(x)$ are real, it being assumed that

(23) $a_1 > a_2 > a_3 > a_4$

(in the case of G_3, omit a_4). For each of the two possible leading coefficients, 1 and -1, of $G(x)$, the table lists the intervals in which $G(x) \geq 0$, the transformation formulas, some corresponding values of x and ϕ, the values of k^2 and μ.

Table 2 gives the corresponding transformations in the case that there are complex roots. In the case of G_3, the real root is a_1, and the complex roots are

(24) $b \pm ic$ $\qquad c > 0$.

TABLE 1. TRANSFORMATION TO

All zeros

$G(x)$ zeros	Leading coefficient	Interval	Transformation $x =$
$G_4(x)$ four real zeros	$+1$	$a_1 \leq x$ or $x \leq a_4$	$\dfrac{a_1 a_{42} - a_2 a_{41} \sin^2\phi}{a_{42} - a_{41} \sin^2\phi}$
		$a_3 \leq x \leq a_2$	$\dfrac{a_3 a_{42} - a_4 a_{32} \sin^2\phi}{a_{42} - a_{32} \sin^2\phi}$
	-1	$a_4 \leq x \leq a_3$	$\dfrac{a_4 a_{31} + a_1 a_{43} \sin^2\phi}{a_{31} + a_{43} \sin^2\phi}$
		$a_2 \leq x \leq a_1$	$\dfrac{a_2 a_{31} - a_3 a_{21} \sin^2\phi}{a_{31} - a_{21} \sin^2\phi}$
$G_3(x)$ three real zeros	$+1$	$a_3 \leq x \leq a_2$	$a_3 + a_{32} \sin^2\phi$
		$a_1 \leq x$	$\dfrac{a_1 - a_2 \sin^2\phi}{1 - \sin^2\phi}$
	-1	$x \leq a_3$	$a_1 - \dfrac{a_{31}}{\sin^2\phi}$
		$a_2 \leq x \leq a_1$	$\dfrac{a_2 a_{31} - a_3 a_{21} \sin^2\phi}{a_{31} - a_{21} \sin^2\phi}$

LEGENDRE'S NORMAL FORM

of $G(x)$ real

<table>
<tr><th rowspan="2">$\sin^2 \phi =$</th><th colspan="2">Corresponding values</th><th rowspan="2">k^2</th><th rowspan="2">μ</th></tr>
<tr><th>x</th><th>ϕ</th></tr>
<tr><td rowspan="2">$\frac{\alpha_{42}}{\alpha_{41}} \frac{x-\alpha_1}{x-\alpha_2}$</td><td>$\alpha_1$</td><td>0</td><td rowspan="4">$(\alpha_1\, \alpha_2\, \alpha_4\, \alpha_3)$</td><td rowspan="8">$\frac{2}{(\alpha_{31}\, \alpha_{42})^{1/2}}$</td></tr>
<tr><td>α_4</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{\alpha_{42}}{\alpha_{32}} \frac{x-\alpha_3}{x-\alpha_4}$</td><td>$\alpha_3$</td><td>0</td></tr>
<tr><td>α_2</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{\alpha_{31}}{\alpha_{43}} \frac{x-\alpha_4}{\alpha_1-x}$</td><td>$\alpha_4$</td><td>0</td><td rowspan="4">$(\alpha_3\, \alpha_2\, \alpha_4\, \alpha_1)$</td></tr>
<tr><td>α_3</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{\alpha_{31}}{\alpha_{21}} \frac{x-\alpha_2}{x-\alpha_3}$</td><td>$\alpha_2$</td><td>0</td></tr>
<tr><td>α_1</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{x-\alpha_3}{\alpha_{32}}$</td><td>$\alpha_3$</td><td>0</td><td rowspan="4">$\frac{\alpha_{32}}{\alpha_{31}}$</td><td rowspan="8">$\frac{2}{(\alpha_{31})^{1/2}}$</td></tr>
<tr><td>α_2</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{x-\alpha_1}{x-\alpha_2}$</td><td>$\alpha_1$</td><td>0</td></tr>
<tr><td>∞</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{\alpha_{31}}{\alpha_1-x}$</td><td>$-\infty$</td><td>0</td><td rowspan="4">$\frac{\alpha_{21}}{\alpha_{31}}$</td></tr>
<tr><td>α_3</td><td>$\frac{1}{2}\pi$</td></tr>
<tr><td rowspan="2">$\frac{\alpha_{31}}{\alpha_{21}} \frac{x-\alpha_2}{x-\alpha_3}$</td><td>$\alpha_2$</td><td>0</td></tr>
<tr><td>α_1</td><td>$\frac{1}{2}\pi$</td></tr>
</table>

TABLE 2. TRANSFORMATION TO

$G(x)$ has

$G(x)$ zeros	Leading coefficient	Interval	Transformation
$G_4(x)$ two real and two complex zeros	1	$a_1 \leq x$ or $x \leq a_2$	$x = \frac{a_1 + a_2}{2} - \frac{a_1 - a_2}{2}\,\frac{\nu - \cos\phi}{1 - \nu\cos\phi}$
	-1	$a_2 \leq x \leq a_1$	$(\tan \frac{1}{2}\phi)^2 = \frac{\cos\theta_1}{\cos\theta_2}\,\frac{a_1 - x}{x - a_2}$
$G_3(x)$ two complex zeros	1	$a_1 \leq x$	$x = a_1 - \frac{c}{\cos\theta_1}\,\frac{1 - \cos\phi}{1 + \cos\phi}$
	-1	$x \leq a_1$	$(\tan \frac{1}{2}\phi)^2 = \frac{\cos\theta_1}{c}\,(a_1 - x)$
$G_4(x)$ four complex zeros $b_1 > b_2$	1	$-\infty < x < \infty$	$x = b_1 + c_1 \tan(\phi + \frac{1}{2}\theta_3 + \frac{1}{2}\theta_4)$ $\tan(\phi + \frac{1}{2}\theta_3 + \frac{1}{2}\theta_4) = (x - b_1)/c_1$
$G_4(x)$ four complex zeros $b_1 = b_2$, $c_1 > c_2$			$x = b_1 - c_1 \operatorname{ctn}\phi$ $\tan\phi = \frac{c_1}{b_1 - c}$

LEGENDRE'S NORMAL FORM

complex zeros

Auxiliary Quantities	Corresponding values x	Corresponding values ϕ	k^2	μ
θ_1 acute, θ_2 obtuse	a_1	0	$[\sin \frac{1}{2}(\theta_1 - \theta_2)]^2$	$\frac{(-\cos\theta_1 \cos\theta_2)^{1/2}}{c}$
θ_1, θ_2 acute	a_2	π		$-\frac{(\cos\theta_1 \cos\theta_2)^{1/2}}{c}$
θ_1 obtuse	a_1	0	$[\sin(\frac{1}{2}\theta_1 + \frac{1}{4}\pi)]^2$	$\left(\frac{-\cos\theta_1}{c}\right)^{1/2}$
θ_1 acute	∞	π		$-\left(\frac{\cos\theta_1}{c}\right)^{1/2}$
$\theta_3, \theta_4, \frac{1}{2}\theta_5$ acute	$-\infty$	$-\frac{1}{2}\pi - \frac{1}{2}\theta_3 - \frac{1}{2}\theta_4$	$\sin^2\theta_5$	$\left(\frac{\cos\theta_5}{c_1 c_2}\right)^{1/2}$
	b_1	$-\frac{1}{2}\theta_3 - \frac{1}{2}\theta_4$		
$\theta_3 = \theta_4 = \frac{1}{2}\pi$			$1 - \left(\frac{c_2}{c_1}\right)^2$	$\frac{1}{c_1}$
	∞	$\frac{1}{2}\pi - \frac{1}{2}\theta_3 - \frac{1}{2}\theta_4$		

In the case of G_4 with two real and a pair of complex roots, $a_1 > a_2$ are the real roots and equation (24) represents the complex roots. In the case of G_4 with two pairs of complex roots, the roots are

$$(25)\quad b_1 \pm ic_1, \quad b_2 \pm ic_2 \qquad\qquad b_1 \geq b_2, \quad c_1 > 0, \quad c_2 > 0,$$

In this table, the transformation formulas, k^2, and μ are expressed in terms of certain auxiliary quantities defined as follows

$$(26)\quad \tan\theta_1 = \frac{a_1 - b}{c}, \quad \tan\theta_2 = \frac{a_2 - b}{c}$$

$$\nu = \tan(\tfrac{1}{2}\theta_2 - \tfrac{1}{2}\theta_1)\tan(\tfrac{1}{2}\theta_2 + \tfrac{1}{2}\theta_1)$$

$$(27)\quad \tan\theta_3 = \frac{c_1 + c_2}{b_1 - b_2}, \quad \tan\theta_4 = \frac{c_1 - c_2}{b_1 - b_2}$$

$$(\tan\tfrac{1}{2}\theta_5)^2 = \cos\theta_3 / \cos\theta_4 \, .$$

The transformation formulas given in these tables remain valid when the zeros of $G(x)$ do not satisfy the conditions given in the first column of the tables and equations (23) to (25); however, in this case the transformations, and k^2, will in general be complex.

There are several integral tables, textbooks, and works of reference which give tables of reduction formulas for elliptic integrals to normal form. We mention Gröbner and Hofreiter (1949 sections 241 to 246, 1950 sections 221 to 223); Jahnke-Emde (1938, p. 58, 59); Magnus and Oberhettinger (1949, Chapter VII); Meyer zur Capellen (1950, sec. 2.3); Oberhettinger and Magnus (1949, sec. 2), and Tricomi (1937, p. 76, 77). The tables given here are adapted from Tricomi's book. See also a forthcoming book by Byrd and Friedman. See errata!

For the evaluation of elliptic integrals by means of elliptic functions see sec. 13.14; for the evaluation in terms of theta functions see sec. 13.20.

13.6. Evaluation of Legendre's elliptic integrals

In sections 13.3 and 13.5 the reduction of any elliptic integral to elliptic integrals of the first, second, and third kinds in normal form has been described. The evaluation of integrals in Weierstrass' normal form by means of Weierstrassian elliptic functions will be given in sec. 13.14; in the present section we discuss the evaluation of Legendre's elliptic integrals.

First we make the definitions 13.3 (20) more specific by setting

(1) $F(\phi, k) = \int_0^{\phi} (1 - k^2 \sin^2 t)^{-1/2}\, dt$

(2) $E(\phi, k) = \int_0^{\phi} (1 - k^2 \sin^2 t)^{1/2}\, dt$

(3) $\Pi(\phi, \nu, k) = \int_0^{\phi} (1 + \nu \sin^2 t)^{-1} (1 - k^2 \sin^2 t)^{-1/2}\, dt.$

We also recall that, with the exception of the equianharmonic case, the reduction may be performed in such a manner that

(4) $|k| < 1.$

The integrals of the first and second kinds may be evaluated by binomial expansion of the integrand.

(5) $$F(\phi, k) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} k^{2n} S_{2n}(\phi) \qquad |k| < 1, \quad |\sin \phi| \le 1$$

(6) $$E(\phi, k) = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n}{n!} k^{2n} S_{2n}(\phi) \qquad |k| < 1, \quad |\sin \phi| \le 1$$

where

(7) $$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

(8) $$S_{2n}(\phi) = \int_0^{\phi} (\sin t)^{2n}\, dt = 2^{-2n} \left[\binom{2n}{n} \phi + \sum_{m=1}^{n} (-1)^m \binom{2n}{n-m} \frac{\sin(2m\phi)}{m} \right]$$

Thus in the real case there is always a convenient convergent series for computing F and E. When the modulus k is near unity, the convergence of the series is slow, and alternative, less simple, expansions must be used. Some such expansions were given by Radon (1950), who also gave the expansions of F and E as trigonometric series. There are extensive numerical tables of elliptic integrals of the first and second kinds; see Jahnke-Emde (1938, p. 52-89); Fletcher, Miller, and Rosenhead (1946, sec. 21).

Elliptic integrals of the third kind present a far more formidable computational problem on account of their dependence on three parameters. The analogue of equations (5) and (6) is

(9) $$\Pi(\phi, \nu, k) = \sum_{n=0}^{\infty} (-\nu)^n B_n^{(-1/2)}(k^2/\nu)\, S_{2n}(\phi)$$

$$|k| < 1, \quad |\nu| < 1, \quad |\sin \phi| \le 1$$

where

(10) $$B_n^{(\alpha)}(z) = \sum_{m=0}^{n} \binom{\alpha}{m} z^m$$

is the truncated binomial expansion. The condition $|\nu| < 1$ in (9) limits the usefulness of this expansion. For alternative expansions see Radon (1950).

For the computation of $\Pi(\phi, \nu, k)$ by means of theta functions and Jacobian elliptic functions see sec. 13.20.

We note here that

(11) $$\Pi(\phi, 0, k) = F(\phi, k)$$

(12) $$(1-k^2)\,\Pi(\phi, -k^2, k) = E(\phi, k) - (1-k^2\sin^2\phi)^{-\frac{1}{2}} k^2 \sin\phi\cos\phi$$

(13) $$(1-k^2)\,\Pi(\phi, -1, k) = (1-k^2)\,F(\phi, k) - E(\phi, k) + \tan\phi\,(1-k^2\sin^2\phi)^{\frac{1}{2}}$$

13.7. Some further properties of Legendre's elliptic normal integrals

The integrals

(1) $$\mathbf{K} = \mathbf{K}(k) = F(\tfrac{1}{2}\pi, k), \qquad \mathbf{E} = \mathbf{E}(k) = E(\tfrac{1}{2}\pi, k)$$

are the *complete elliptic integrals* of the first and the second kinds, respectively. With the *complementary modulus*

(2) $$k' = (1-k^2)^{\frac{1}{2}}$$

we also have

(3) $$\mathbf{K}' = \mathbf{K}'(k) = F(\tfrac{1}{2}\pi, k'), \qquad \mathbf{E}' = \mathbf{E}'(k) = E(\tfrac{1}{2}\pi, k').$$

The incomplete elliptic integrals $F(\phi, k)$ and $E(\phi, k)$ are many-valued functions on the Riemann surface $\mathfrak{R}$ of the function y defined by equation 13.3(19). The branch-points are $x = \sin\phi = \pm 1, \pm k^{-1}$. The periods may be evaluated in terms of complete elliptic integrals.

Integrals	Periods	
$F(\phi, k)$	$4\mathbf{K}$,	$2i\mathbf{K}'$
$E(\phi, k)$	$4\mathbf{E}$,	$2i(\mathbf{K}' - \mathbf{E}')$.

In each case the first of these periods is called the *real*, the second the *imaginary*, *period* (because they are respectively real and imaginary when $0 < k < 1$).

Although $F(\phi, k)$ and $E(\phi, k)$ are many-valued functions of $x = \sin\phi$ on $\mathfrak{R}$, E considered as a function of F is single-valued on $\mathfrak{R}$ provided that corresponding values of E and F are obtained by integration over the same path. This gives rise to Jacobi's function $E(u)$, see sec. 13.16.

Elliptic integrals, like elliptic functions, possess *addition theorems*. Given ϕ and ψ, determine χ from the equations

$$(4)\quad (1-k^2\sin^2\phi\sin^2\psi)\sin\chi = \sin\phi\cos\psi(1-k^2\sin^2\psi)^{1/2} + \sin\psi\cos\phi(1-k^2\sin^2\phi)^{1/2}$$

$$(1-k^2\sin^2\phi\sin^2\psi)\cos\chi = \cos\phi\cos\psi - \sin\phi\sin\psi(1-k^2\sin^2\phi)^{1/2}(1-k^2\sin^2\psi)^{1/2}$$

and denote by $\equiv$ the relation (*congruence*) between two functions which differ by a (constant) linear combination of their periods. Then

$$(5)\quad F(\chi)\equiv F(\phi)+F(\psi)$$

$$(6)\quad E(\chi)\equiv E(\phi)+E(\psi)-k^2\sin\phi\sin\psi\sin\chi$$

are the addition theorems of $E(\phi, k)$, $F(\phi, k)$.

The *interchange theorem* mentioned in sec. 13.4 is most conveniently expressed in terms of the elliptic integral of the third kind

$$(7)\quad \Pi^*(\phi, \psi, k) = \int_0^\phi \frac{k^2\cos\psi\sin\psi(1-k^2\sin^2\psi)^{1/2}\sin^2 t}{(1-k^2\sin^2\psi\sin^2 t)(1-k^2\sin^2 t)^{1/2}}\,dt$$

$$= \operatorname{ctn}\psi\,(1-k^2\sin^2\psi)^{1/2}\,[\Pi(\phi, -k^2\sin^2\psi, k)-F(\phi, k)]$$

when it reads

$$(8)\quad \Pi^*(\phi, \psi)-\Pi^*(\psi, \phi) = F(\phi)E(\psi)-F(\psi)E(\phi)+n\pi i.$$

Here k has been omitted from all symbols of elliptic integrals and n is an integer.

Both the addition theorems and the interchange theorem depend on the connection between elliptic integrals and elliptic functions.

In sec. 13.5 it has been mentioned that a regrouping of the zeros of $G(x)$ results in changing the modulus. If k was the original modulus, such a regrouping will lead to one of the following values

$$(9)\quad k,\quad \frac{ik}{k'},\quad k',\quad \frac{1}{k},\quad \frac{1}{k'},\quad \frac{k'}{ik}.$$

Elliptic integrals belonging to any two of these moduli are connected by rational relations (*linear transformations*). To the expressions enumerated in (9) we add

$$(10)\quad \frac{1-k'}{1+k'}.$$

TABLE 3. TRANSFORMATIONS OF ELLIPTIC INTEGRALS

$\dot{k}$	$\sin\dot{\phi}$	$\cos\dot{\phi}$	$F(\dot{\phi},\dot{k})$	$E(\dot{\phi},\dot{k})$
$\frac{1}{k}$	$k\sin\phi$	$\Delta(\phi,k)$	$kF(\phi,k)$	$\frac{1}{k}[E(\phi,k)-k'^2F(\phi,k)]$
k'	$-i\tan\phi$	$\sec\phi$	$-iF(\phi,k)$	$i[E(\phi,k)-F(\phi,k)-\tan\phi\,\Delta(\phi,k)]$
$\frac{1}{k'}$	$-ik'\tan\phi$	$\frac{\Delta(\phi,k)}{\cos\phi}$	$-ik'F(\phi,k)$	$\frac{i}{k'}[E(\phi,k)-k'^2F(\phi,k)-\tan\phi\,\Delta(\phi,k)]$
$\frac{ik}{k'}$	$\frac{k'\sin\phi}{\Delta(\phi,k)}$	$\frac{\cos\phi}{\Delta(\phi,k)}$	$k'F(\phi,k)$	$\frac{1}{k'}\left[E(\phi,k)-k^2\frac{\sin\phi\cos\phi}{\Delta(\phi,k)}\right]$
$\frac{k'}{ik}$	$-\frac{ik\sin\phi}{\Delta(\phi,k)}$	$\frac{1}{\Delta(\phi,k)}$	$-ikF(\phi,k)$	$\frac{i}{k}\left[E(\phi,k)-F(\phi,k)-k^2\frac{\sin\phi\cos\phi}{\Delta(\phi,k)}\right]$
$\frac{1-k'}{1+k'}$	$\frac{(1+k')\sin\phi\cos\phi}{\Delta(\phi,k)}$	$\frac{\cos^2\phi-k'\sin^2\phi}{\Delta(\phi,k)}$	$(1+k')F(\phi,k)$	$\frac{2}{1+k'}[E(\phi,k)+k'F(\phi,k)]-(1-k')\frac{\sin\phi\cos\phi}{\Delta(\phi,k)}$

Elliptic integrals of moduli k and (10) are also connected by rational relations (*Landen's transformation*).

Table 3 (p. 316) gives for any of the moduli (9) or (10), denoted by $\dot{k}$, the transformed values $\dot{\phi}$ in terms of ϕ and k, $F(\dot{\phi}, \dot{k})$ and $E(\dot{\phi}, \dot{k})$ in terms of $F(\phi, k)$, $E(\phi, k)$, ϕ, and k. We continue to use the notation (2) and introduce the abbreviation

(11) $$\Delta(\phi, k) = (1 - k^2 \sin^2 \phi)^{1/2}, \qquad \Delta(\tfrac{1}{2}\pi, k) = k'.$$

The quantity $\dot{\phi}$ in the table is determined up to multiples of 2π by giving both $\sin\dot{\phi}$ and $\cos\dot{\phi}$.

We also note the differentiation formulas

(12) $$\frac{\partial F}{\partial k} = \frac{1}{k'^2}\left[\frac{E - k'^2 F}{k} - \frac{\sin\phi\cos\phi}{\Delta(\phi, k)}\right]$$

$$\frac{\partial E}{\partial k} = \frac{E - F}{k}.$$

13.8. Complete elliptic integrals

We use the following notations for the complete elliptic integrals of the first, second, and third kind.

(1) $$\mathbf{K} = \mathbf{K}(k) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\Delta(\phi, k)} = \int_0^1 \frac{dx}{[(1 - x^2)(1 - k^2 x^2)]^{1/2}}$$

(2) $$\mathbf{E} = \mathbf{E}(k) = \int_0^{\frac{1}{2}\pi} \Delta(\phi, k)\, d\phi = \int_0^1 \left(\frac{1 - k^2 x^2}{1 - x^2}\right)^{1/2} dx$$

(3) $$\Pi_1 = \Pi_1(\nu, k) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 + \nu\sin^2\phi)\,\Delta(\phi, k)}$$

$$= \int_0^1 \frac{dx}{(1 + \nu x^2)\,[(1 - x^2)(1 - k^2 x^2)]^{1/2}}.$$

From 13.6(8),

(4) $$S_{2n}(\tfrac{1}{2}\pi) = \int_0^{\frac{1}{2}\pi} \sin^{2n}\phi\, d\phi = \frac{(\frac{1}{2})_n}{n!}\,\frac{\pi}{2}$$

and using this in 13.6(5), (6), and (9),

(5) $\mathbf{K}(k) = \frac{1}{2}\pi\ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ $\qquad |k| < 1$

(6) $\mathbf{E}(k) = \frac{1}{2}\pi\ {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; k^2)$ $\qquad |k| < 1$

(7) $\Pi_1(\nu, k) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} (-\nu)^n B_n^{(-\frac{1}{2})}\left(\frac{k^2}{\nu}\right)$ $\qquad |k| < 1, \quad |\nu| < 1.$

In (5) and (6)

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} z^n$$

is Gauss' hypergeometric series, see chapter 2.

Tricomi (1935, 1936) also gave the expansion

(8) $\mathbf{K}(\sin\alpha) = \pi \sum_{n=0}^{\infty} \left[\frac{(\frac{1}{2})_n}{n!}\right]^2 \sin[(4n+1)\alpha]$ $\qquad 0 < \alpha < \frac{1}{2}\pi$

and the inequality

(9) $\log 4 \leq \mathbf{K} + \log k' \leq \frac{1}{2}\pi.$

From (5) it is seen that $\mathbf{K}(k)$ is a monotonic increasing function of k for $0 < k < 1$. $\mathbf{K}(0) = \frac{1}{2}\pi$, and from (9) it is seen that $\mathbf{K}$ becomes logarithmically infinite as $k \to 1$. More precisely,

(10) $\mathbf{K} = \log(4/k') + O(k'^2 \log k')$ $\qquad k' \to 0.$

On the other hand, (6) shows that $\mathbf{E}$ is decreasing for $0 < k < 1$, and from (2)

(11) $1 \leq \mathbf{E} \leq \frac{1}{2}\pi$ $\qquad 0 \leq k \leq 1.$

Expansions valid near $k = 1$ have been given by several authors; see, for instance Radon (1950). We also mention a formula for the integrals of the third kind developed by Hamel (1932).

For the computation of complete elliptic integrals of the first and second kinds by means of theta functions see sec. 13.20.

Corresponding to the transformations of Table 3, there are transformations of complete elliptic integrals. These are listed in Table 4 (p. 319).

The transformation

(12) $\dot{k} = \frac{1-k'}{1+k'}, \quad \mathbf{K}\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2}\mathbf{K}(k)$

TABLE 4. TRANSFORMATIONS OF COMPLETE ELLIPTIC INTEGRALS

$\dot{k}$	$\mathbf{K}(\dot{k})$	$\mathbf{K}'(\dot{k})$	$\mathbf{E}(\dot{k})$	$\mathbf{E}'(\dot{k})$
$\frac{1}{k}$	$k(\mathbf{K}+i\mathbf{K}')$	$k\mathbf{K}'$	$\frac{1}{k}(\mathbf{E}+i\mathbf{E}'-k'^2\mathbf{K}-ik^2\mathbf{K}')$	$\frac{1}{k}\mathbf{E}'$
k'	$\mathbf{K}'$	$\mathbf{K}$	$\mathbf{E}'$	$\mathbf{E}$
$\frac{1}{k'}$	$k'(\mathbf{K}'+i\mathbf{K})$	$k'\mathbf{K}$	$\frac{1}{k'}(\mathbf{E}'+i\mathbf{E}-k^2\mathbf{K}'-ik'^2\mathbf{K})$	$\frac{1}{k'}\mathbf{E}$
$\frac{ik}{k'}$	$k'\mathbf{K}$	$k'(\mathbf{K}'-i\mathbf{K})$	$\frac{1}{k'}\mathbf{E}$	$\frac{1}{k'}(\mathbf{E}'+i\mathbf{E}-k^2\mathbf{K}'-ik'^2\mathbf{K})$
$\frac{k'}{ik}$	$k\mathbf{K}'$	$k(\mathbf{K}+i\mathbf{K}')$	$\frac{1}{k}\mathbf{E}'$	$\frac{1}{k}(\mathbf{E}-i\mathbf{E}'-k'^2\mathbf{K}+ik^2\mathbf{K}')$
$\frac{1-k'}{1+k'}$	$\frac{1+k'}{2}\mathbf{K}$	$(1+k')\mathbf{K}'$	$\frac{\mathbf{E}+k'\mathbf{K}}{1+k'}$	$\frac{2\mathbf{E}'-k^2\mathbf{K}'}{1+k'}$
$\frac{2k^{1/2}}{1+k}$	$(1+k)\mathbf{K}$	$\frac{1+k}{2}\mathbf{K}'$	$\frac{2\mathbf{E}-k'^2\mathbf{K}}{1+k}$	$\frac{\mathbf{E}'+k\mathbf{K}'}{1+k}$

is especially important since it may be used to compute **K** numerically. The first equation in (12) may also be written

$$\dot{k}'=\frac{2k'^{1/2}}{1+k'} .$$

Here $k' < \dot{k}' < 1$ if $0 < k' < 1$, and if the transformation is repeated, k' tends rapidly to unity. The corresponding $\mathbf{K}(0)$ is $\frac{1}{2}\pi$. Now define

(13) $$k_0' = k', \quad k_{n+1}' = \frac{2k_n'^{1/2}}{1+k_n'} \qquad n = 0, 1, 2, \ldots .$$

Then by repeated application of (12),

(14) $$\mathbf{K}(k) = \frac{\pi}{2} \prod_{n=0}^{\infty} \frac{2}{1+k_n'} .$$

For the four complete elliptic integrals belonging to complementary moduli we have *Legendre's relation*

(15) $$\mathbf{KE}' + \mathbf{K}'\mathbf{E} - \mathbf{KK}' = \tfrac{1}{2}\pi.$$

For particular values of k we list the following relations.

(16) $$\mathbf{K}(2^{-1/2}) = \mathbf{K}'(2^{-1/2}) = \frac{[\Gamma(\frac{1}{4})]^2}{4\pi^{1/2}}$$

(17) $$\mathbf{K}'\left(\sin\frac{\pi}{18}\right) = 3^{1/2}\,\mathbf{K}\left(\sin\frac{\pi}{18}\right)$$

(18) $$\mathbf{K}'(2^{1/2}-1) = 2^{1/2}\,\mathbf{K}(2^{1/2}-1)$$

(19) $$\mathbf{K}'\left(\frac{2^{1/2}-1}{2^{1/2}+1}\right) = 2\mathbf{K}\left(\frac{2^{1/2}-1}{2^{1/2}+1}\right)$$

(20) $$\mathbf{K}'(e^{i\pi/3}) = e^{i\pi/6}\,\mathbf{K}(e^{i\pi/3}) = \frac{\pi^{1/2}\,\Gamma(1/6)}{2\cdot 3^{3/4}\,\Gamma(2/3)}\; e^{-i\pi/6}.$$

The first of these relations corresponds to the *lemniscate functions* which arise from the inversion of the integral

$$\int (1-x^4)^{-1/2}\,dx,$$

and the last relation corresponds to the *equianharmonic* case of elliptic integrals.

The complete elliptic integrals of the third kind $\Pi_1(\nu, k)$ may be expressed in terms of incomplete elliptic integrals of the first and second kind. For $\nu > -1$ this was observed by Legendre, for $\nu < -1$ (when the Cauchy principal value of the integral must be taken) it was proved by Tricomi. The parameter ν is expressed in terms of an auxiliary quantity θ, different expressions being valid in the intervals $(-\infty, -1)$, $(-1, -k^2)$ $(-k^2, 0)$ and $(0, \infty)$. The results are

(21) $\operatorname{ctn}\theta\, \Delta(\theta, k)\, \Pi_1(-\csc^2\theta, k) = \mathbf{E}(k)\, F(\theta, k) - \mathbf{K}(k)\, E(\theta, k)$

(22) $$k' \frac{\sin\theta\cos\theta}{\Delta(\theta, k')}\, [\Pi_1(-\Delta^2(\theta, k'), k) - \mathbf{K}(k)]$$

$$= \tfrac{1}{2}\pi - [\mathbf{E}(k) - \mathbf{K}(k)]\, F(\theta, k') - \mathbf{K}(k)\, E(\theta, k)$$

(23) $$\operatorname{ctn}\theta\, \Delta(\theta, k)\, [\Pi_1(-k^2\sin^2\theta, k) - \mathbf{K}(k)]$$

$$= -\mathbf{E}F(\theta, k) + \mathbf{K}E(\theta, k)$$

(24) $$\frac{\sin\theta\cos\theta}{\Delta(\theta, k')}\, [\Pi_1(k^2\tan^2\theta, k) - \mathbf{K}(k)\cos^2\theta)]$$

$$= [\mathbf{E}(k) - \mathbf{K}(k)]\, F(\theta, k') + \mathbf{K}(k)\, E(\theta, k').$$

Beside $\mathbf{K}$, $\mathbf{E}$, Π_1, it is sometimes convenient to introduce

(25) $$\mathbf{D}(k) = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\phi}{\Delta(\phi, k)} d\phi, \qquad \mathbf{B}(k) = \int_0^{\frac{1}{2}\pi} \frac{\cos^2\phi}{\Delta(\phi, k)}\, d\phi$$

$$\mathbf{C}(k) = \int_0^{\frac{1}{2}\pi} \frac{(\sin\phi\cos\phi)^2}{[\Delta(\phi, k)]^3}\, d\phi.$$

With $\kappa = k^2$ we have the differentiation and integration formulas, and connections between various integrals

(26) $$\mathbf{D} = \frac{\mathbf{K} - \mathbf{E}}{k^2}, \qquad \mathbf{B} = \mathbf{K} - \mathbf{D} = \frac{\mathbf{E} - k'^2\mathbf{K}}{k^2}$$

$$\mathbf{C} = \frac{\mathbf{D} - \mathbf{B}}{k^2} = \frac{1}{k^4}\, [(2 - k^2)\, \mathbf{K} - 2\mathbf{E}].$$

(27) $2\frac{d\mathbf{K}}{d\kappa}=\frac{\mathbf{B}}{1-\kappa}, \quad 2\frac{d\mathbf{E}}{d\kappa}=-\mathbf{D}, \quad 2\frac{d\mathbf{D}}{d\kappa}=\frac{\mathbf{D}-\mathbf{C}}{1-\kappa}$

$$2\frac{d\mathbf{B}}{d\kappa}=\mathbf{C}, \quad 2\kappa\frac{d\mathbf{C}}{d\kappa}=\frac{\mathbf{B}}{1-\kappa}-4\mathbf{C}$$

(28) $\int \mathbf{K}\,d\kappa = 2\kappa\mathbf{B}, \quad \int \mathbf{E}\,d\kappa=\frac{2}{3}\kappa(\mathbf{E}+\mathbf{B})$

$$\int \mathbf{D}\,d\kappa=-2\mathbf{E}, \quad \int \mathbf{B}\,d\kappa=2(\mathbf{E}+\kappa\mathbf{B}), \quad \int \mathbf{C}\,d\kappa=2\mathbf{B}.$$

For series expansions and other formulas for these integrals and for short numerical tables see Jahnke-Emde (1938, p. 73-84).

PART TWO: ELLIPTIC FUNCTIONS

13.9. Inversion of elliptic integrals

Historically, *elliptic functions* were introduced by inverting elliptic integrals. To obtain *Jacobian elliptic functions* consider the relation

(1) $u=\int_0^{\phi}(1-k^2\sin^2 t)^{-1/2}\,dt=F(\phi, k)$

between the complex variables u and ϕ. We already know that u is a many-valued function of $x=\sin\phi$; conversely, equation (1) also defines ϕ, or $\sin\phi$, as a (possibly many-valued) function of u. Jacobi puts

(2) $\phi=\operatorname{am} u=\operatorname{am}(u, k)$

and adopts as basic functions

(3) $\operatorname{sn} u=\operatorname{sn}(u, k)=\sin(\operatorname{am} u)$

$\operatorname{cn} u=\operatorname{cn}(u, k)=\cos(\operatorname{am} u)$

$\operatorname{dn} u=\operatorname{dn}(u, k)=\Delta(\operatorname{am} u, k)=[1-k^2\sin^2(\operatorname{am} u)]^{1/2}.$

Beside these, the following nine functions are often used

(4) $\operatorname{ns} u=1/\operatorname{sn} u, \quad \operatorname{nc} u=1/\operatorname{cn} u, \quad \operatorname{nd} u=1/\operatorname{dn} u,$

$\operatorname{cs} u=\operatorname{cn} u/\operatorname{sn} u, \quad \operatorname{sc} u=\operatorname{sn} u/\operatorname{cn} u, \quad \operatorname{sd} u=\operatorname{sn} u/\operatorname{dn} u,$

$\operatorname{ds} u=\operatorname{dn} u/\operatorname{sn} u, \quad \operatorname{dc} u=\operatorname{dn} u/\operatorname{cn} u, \quad \operatorname{cd} u=\operatorname{cn} u/\operatorname{dn} u,$

the notation being due to Glaisher.

At $u=0$, we may put

(5) $\operatorname{sn} 0=0, \quad \operatorname{cn} 0=\operatorname{dn} 0=1,$

and this clearly defines the three basic functions, and hence also the nine functions (4), as single-valued analytic functions in some neighborhood of the origin (except for ns u, cs u, ds u which have simple poles at $u = 0$ and are analytic in a punctured neighborhood of this point). The crucial fact of the theory of elliptic functions is the circumstance that the functions obtained by analytic continuation of the twelve functions thus defined in a neighborhood of $u = 0$ are all *single-valued* functions of u, analytic except for an infinity of (simple) poles. This result may be established by a discussion of the *inversion problem* for the integral (1), see Hancock (1917), Neville (1944).

Weierstrass' elliptic functions present a similar problem. The relation

$$(6) \quad z = \int_{\infty}^{w} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}}\, dt$$

between the two complex variables z and w may be inverted to yield *Weierstrass'* $\wp$-function

$$(7) \quad w = \wp(z) = \wp(z;\, g_2, g_3),$$

and $\wp(z)$ turns out to be single-valued, and analytic except for an infinity of poles (of the second order).

In either case the inversion problem is a formidable one (except in the case of the integral (1) in the real field and for $0 < k < 1$), and it is of interest to note that an alternative approach exists and has many advantages. Weierstrass has shown that a study of doubly periodic analytic functions leads quite naturally to elliptic functions. Since then it has become customary to approach elliptic functions from the general theory of analytic functions. We shall do so in this chapter and establish the connection with elliptic integrals later, see sec. 13.14.

13.10. Doubly-periodic functions

Let $f(z)$ be a single-valued function which is analytic save for isolated singularities. A period of this function is a complex number, p, such that

$$(1) \quad f(z) = f(z + p)$$

for all z for which f is analytic. A function which has one (non-zero) period has an infinity of periods (for instance np for all integers n). Let Ω be the set of all points in the complex plane which correspond to periods of a fixed function $f(z)$. If $f(z)$ happens to be a constant, then Ω is the whole plane. This case excepted it may be proved [see for instance Tricomi (1937 Chap. I, sec. 2)] that Ω is *either* a system of equidistant points on a straight line through the origin, or else a *point-lattice* formed by the intersections of two families of equidistant parallel lines (*line-lattice*). In the former case $f(z)$ is *simply-periodic*, in the latter case *doubly-periodic*.

We now consider a doubly-periodic function $f(z)$ and the corresponding point-lattice Ω. The point-lattice may be generated (in many ways) as the points of intersection of two families of equidistant parallel lines, that is to say by the repetition of congruent parallelograms. Take one such parallelogram with one of its vertices at 0, and let the other three vertices be 2ω, $2\omega'$, $2\omega + 2\omega'$. Then 2ω and $2\omega'$ are called a pair of *primitive periods* of $f(z)$, and all periods are of the form

(2) $$2\omega_{m,n} = 2m\omega + 2n\omega' \qquad m, n \text{ integers.}$$

Clearly, ω'/ω is not real, and we may choose the primitive periods in such a manner that

(3) $$\operatorname{Im}(\omega'/\omega) > 0.$$

This convention will be adhered to throughout this chapter.

A point-lattice may be generated in infinitely many ways from a line-lattice, that is to say it possesses infinitely many pairs of primitive periods. Let ω, ω' be primitive half-periods of Ω, and let α, β, γ, δ be any integers. Then

(4) $$\dot\omega = \alpha\omega + \beta\omega', \qquad \dot\omega' = \gamma\omega + \delta\omega'$$

is certainly a pair of half-periods. If

(5) $$\alpha\delta - \beta\gamma = 1,$$

then we have, from (4)

(6) $$\omega = \delta\dot\omega - \beta\dot\omega, \qquad \omega' = -\gamma\dot\omega + \alpha\dot\omega'$$

so that ω, ω', and hence any half-period of $f(z)$, is a linear combination, with integer coefficients, of $\dot\omega$, $\dot\omega'$ and (4) gives another pair of primitive half-periods. *Equivalent pairs of primitive half-periods are connected by unimodular transformations*

(7) $$\begin{bmatrix} \dot\omega \\ \dot\omega' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \omega \\ \omega' \end{bmatrix}, \qquad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1.$$

It can be shown [see, for instance, Tricomi (1937, Chap. I, sec. 2)] that a pair of primitive periods may be chosen in such a manner that

(8) $$|\omega| \le |\omega'| \quad \text{and} \quad \operatorname{Im}(\omega'/\omega) \ge \tfrac{1}{2} \cdot 3^{1/2}$$

but such a choice will not be assumed in this chapter.

Two points of the z-plane are said to be *congruent* if they differ by a period. A connected set of points is called a *fundamental region* if every point of the plane is congruent to exactly one point of the set. We shall always choose the fundamental region as a parallelogram, with two sides and the vertex at which they intersect being counted as part of the parallelogram, the other two sides and three vertices not forming part of it. Fixing a z_0, the points

$$(9) \quad z = z_0 + 2\xi\omega + 2\eta\omega' \qquad 0 \leq \xi < 1, \; 0 \leq \eta < 1$$

form the *fundamental period-parallelogram*. Any parallelogram obtained from this by a translation by a period, that is every set of points

$$(10) \quad z = z_0 + 2(m + \xi)\,\omega + 2(n + \eta)\,\omega' \qquad 0 \leq \xi < 1, \quad 0 \leq \eta < 1$$

with a fixed pair of integers (m, n) is a *period-parallelogram*, or, shortly, a *mesh*.

Since a doubly-periodic function assumes the same value at congruent points, it is sufficient to describe the behavior of such a function in any one mesh. Since $f(z)$ has only isolated singularities and isolated zeros, it is possible to choose the fundamental period parallelogram (i.e., z_0) so that no singularities or zero of $f(z)$ lies on the boundary of a mesh. This will be assumed in the general theorems of sec. 13.11, and such a mesh will be called a *cell*.

13.11. General properties of elliptic functions

A doubly-periodic meromorphic function is called an *elliptic function*, that is to say, an elliptic function is defined to be a single-valued doubly-periodic analytic function whose only possible singularities in the finite part of the plane are poles. In this section $f(z)$ will be such a function, ω, ω' a pair of primitive half-periods of $f(z)$, and Ω the point-lattice associated with $f(z)$.

It may be mentioned here that often Weierstrass' sigma- and zeta-functions, theta functions, and other functions associated with elliptic functions are also referred to as elliptic functions (in the wider sense), but in the present chapter the term "elliptic function" will be used in the sense of the definition given above.

Every non-constant elliptic function has poles. For if $f(z)$ has no poles in a mesh then it is bounded there, and hence in the entire plane. By Liouville's theorem it is then a constant.

An elliptic function has only a finite number of poles in any mesh, and, if it does not vanish identically, only a finite number of zeros there. For an infinity of poles in a mesh implies the existence of a limiting point of these poles, and hence an essential singularity. Similarly, an infinity of zeros of an elliptic function which does not vanish identically implies the existence of an essential singularity.

The number of poles in a cell, each pole counted according to its multiplicity, is called the *order* of the elliptic function. The set of poles or zeros in a given cell is called an irreducible set.

The sum of the residues of an elliptic function at its poles in any cell is zero. Let C be the boundary of the cell. The sum of the residues is

$$\frac{1}{2\pi i}\int_C f(z)\,dz$$

and this is zero, since the integrals along opposite sides cancel.

There is no elliptic function of order one. For such a function has exactly one simple pole in each cell, and the residue is zero by the preceding theorem.

An elliptic function of order r assumes, in any mesh, every value exactly r times (counting multiplicity). To show that $f(z)-c$ has exactly r zeros, take the mesh so that $f'(z)/[f(z)-c]$ is regular on its boundary C. The difference between the number of poles and the number of zeros of $f(z)-c$ is

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-c}\,dz$$

and in this integral the contributions of opposite sides cancel.

The sum of an irreducible set of zeros is congruent to the sum of an irreducible set of poles (each zero and pole being repeated according to its multiplicity). Let C be the boundary of a cell, and let $a_1, \dots, a_r$ be the zeros and $\beta_1, \dots, \beta_r$ the poles of $f(z)$ within C. The function $f'(z)/f(z)$ has a simple pole with residue k at a zero of order k, and a simple pole with residue $-k$ at a pole of order k.

$$(1)\quad \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}\,dz = \sum_{h=1}^{r}(a_r-\beta_r).$$

If the vertices of the cell are z_0, $z_0 + 2\omega$, $z_0 + 2\omega + 2\omega'$, $z_0 + 2\omega'$, the integral in (1) is

$$\frac{1}{2\pi i}\int_0^{2\omega}\left[\frac{(z_0+t)\,f'(z_0+t)}{f(z_0+t)}-\frac{(z_0+2\omega'+t)\,f'(z_0+2\omega'+t)}{f(z_0+2\omega'+t)}\right]dt$$

$$-\frac{1}{2\pi i}\int_0^{2\omega'}\left[\frac{(z_0+t)f'(z_0+t)}{f(z_0+t)}-\frac{(z_0+2\omega+t)f'(z_0+2\omega+t)}{f(z_0+2\omega'+t)}\right]dt$$

$$=\frac{1}{2\pi i}\,\{2\omega[\log f(z_0+t)]_0^{2\omega'}-2\omega'[\log f(z_0+t)]_0^{2\omega}\}.$$

Since $f(z)$ has periods 2ω, $2\omega'$, we see that $\log f(z_0)$, $\log f(z_0+2\omega')$, and $\log f(z_0+2\omega)$ differ from each other by integer multiples of $2\pi i$, and hence the integral in (1) has the value $2m\omega + 2n\omega'$.

From these fundamental theorems some corollaries follow immediately. We mention only two of these.

Two elliptic functions which have the same periods, the same poles, and the same principal parts at each pole differ by a constant.

The quotient of two elliptic functions whose periods, poles, and zeros (and multiplicities of poles and zeros) *are the same, is a constant.*

All elliptic functions with the same periods $(2\omega, 2\omega')$ form a *field*, $\mathfrak{K}$, that is the sum, difference, product, or quotient of two such functions has the same periods. Clearly, any rational function (with constant coefficients) of such functions belongs to $\mathfrak{K}$. Moreover, the derivative of any function of $\mathfrak{K}$ belongs also to $\mathfrak{K}$, so that $\mathfrak{K}$ is a *differential field*. An integral of a function of $\mathfrak{K}$ does not necessarily belong to $\mathfrak{K}$. Although $(2\omega, 2\omega')$ is a pair of primitive periods for some functions in $\mathfrak{K}$, and a pair of periods for all functions of $\mathfrak{K}$, it is not necessarily a pair of primitive periods for all functions of $\mathfrak{K}$.

From the representation of elliptic functions in terms of certain standard functions (see sec. 13.14) some additional results easily follow.

Any two functions, f and g, of $\mathfrak{K}$ are connected by an algebraic equation $P(f, g) = 0$, *where* $P(x, y)$ *is a polynomial with constant coefficients, and the algebraic curve* $P(x, y) = 0$ *is unicursal.*

Any elliptic function satisfies an algebraic differential equation of the first order, $P(f, f') = 0$. Here again $P(x, y)$ is a polynomial with constant coefficients and of genus zero.

Any elliptic function, $f(z)$, satisfies an algebraic addition theorem

(2) $A[f(u), f(v), f(u+v)] = 0$

where $A(x, y, z)$ is a polynomial whose coefficients are independent of u, v, and (2) is satisfied identically in u, v.

Conversely, it may be shown that *a single-valued analytic function of z which satisfies an algebraic addition theorem* of the form (2) *is either a rational function of z, or a rational function of $e^{\lambda z}$* for some λ, *or else an elliptic function.*

The simplest (non-trivial) elliptic functions are functions of order two. Among these one may select as standard functions either a function which has one double pole (with residue zero) in each cell, or else a function which has two simple poles (with residues equal in magnitude but opposite in sign) in each cell. The former possibility is chosen in Weierstrass' theory, the latter in Jacobi's.

13.12. Weierstrass' functions

Let 2ω, $2\omega'$ be a fixed pair of primitive periods,

(1) $\tau = \omega'/\omega, \quad \operatorname{Im} \tau > 0$

(2) $w = w_{mn} = 2m\omega + 2n\omega'$.

Σ and Π will indicate infinite sums and products taken over all integers m, n, and Σ' and Π' sums and products taken over all integers m, n with the exception of $m = n = 0$.

Weierstrass' function $\wp(z) = \wp(z|\omega, \omega')$ in an elliptic function of periods 2ω, $2\omega'$ which is of order two, has a double pole at $z = 0$, the principal part of the function at this pole being z^{-2}, and for which $\wp(z) - z^{-2}$ is analytic in a neighborhood of, and vanishes at, $z = 0$. These conditions define $\wp(z)$ uniquely. To obtain an analytic expression we first construct a meromorphic function which has double poles, with principal parts, $(z - w)^{-2}$, at all points $w = w_{mn}$. The partial fraction expansion of such a function is

(3) $f(z) = z^{-2} + \Sigma'[(z-w)^{-2} - w^{-2}]$.

Moreover, $f(z) - z^{-2}$ vanishes at $z = 0$. We prove that $f(z + 2\omega) = f(z) = f(z + 2\omega')$ by rearranging the series and then conclude that $f(z) = \wp(z)$ or

(4) $$\wp(z) = \wp(z|\omega, \omega') = \frac{1}{z^2} + \sum{}' \left[\frac{1}{(z - 2m\omega - 2n\omega')^2} - \frac{1}{(2m\omega + 2n\omega')^2}\right]$$

The function $\wp(z)$ is an even function of z. Also

(5) $\wp'(z) = -2z^{-3} - 2\Sigma'(z-w)^{-3} = -2\Sigma(z-w)^{-3}$.

Integrating term by term we obtain *Weierstrass' zeta function* which is a meromorphic function with simple poles.

(6) $\zeta(z) = \zeta(z|\omega, \omega') = z^{-1} + \Sigma'[(z-w)^{-1} + w^{-1} + zw^{-2}]$

(7) $\wp(z) = -\zeta'(z)$.

The function $\zeta(z)$ is an odd function of z. It is not doubly- periodic and hence *not* an elliptic function. It is usual to put

(8) $\zeta(z+2\omega) = \zeta(z) + 2\eta, \quad \zeta(z+2\omega') = \zeta(z) + 2\eta'$.

Since $\zeta(z)$ is an odd function of z,

(9) $\eta = \zeta(\omega), \quad \eta' = \zeta(\omega')$.

By integrating $\zeta(z)$ around a cell one obtains *Legendre's relation*

(10) $\eta\omega' - \eta'\omega = \frac{1}{2}\pi i$.

Weierstrass' sigma function is an entire function whose logarithmic derivative is the zeta function

(11) $\sigma(z) = \sigma(z|\omega, \omega') = z\,\Pi'\left\{\left(1-\frac{z}{w}\right)\exp\left[\frac{z}{w}+\frac{1}{2}\left(\frac{z}{w}\right)^2\right]\right\}$

(12) $\zeta(z) = \dfrac{\sigma'(z)}{\sigma(z)}, \quad \wp(z) = \dfrac{\sigma'^2(z) - \sigma(z)\sigma''(z)}{\sigma^2(z)}$.

With the abbreviations

(13) $g_2 = 60\,\Sigma' w^{-4}, \quad g_3 = 140\,\Sigma' w^{-6}$,

the power series expansion of $\sigma(z)$, and the Laurent series expansions of $\zeta(z)$, $\wp(z)$, $\wp'(z)$, in a neighborhood of the origin are

(14) $$\sigma(z) = z - \frac{g_2 z^5}{2^4\cdot 3\cdot 5} - \frac{g_3 z^7}{2^3\cdot 3\cdot 5\cdot 7} - \frac{g_2^2 z^9}{2^9\cdot 3^2\cdot 5\cdot 7} - \frac{g_2 g_3 z^{11}}{2^7\cdot 3^2\cdot 5^2\cdot 7\cdot 11} - \cdots$$

(15) $$\zeta(z) = \frac{1}{z} - \frac{g_2 z^3}{2^2\cdot 3\cdot 5} - \frac{g_3 z^5}{2^2\cdot 5\cdot 7} - \frac{g_2^2 z^7}{2^4\cdot 3\cdot 5^2\cdot 7} + \cdots$$

(16) $$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{2^2 \cdot 5} + \frac{g_3 z^4}{2^2 \cdot 7} + \frac{g_2^2 z^6}{2^4 \cdot 3 \cdot 5^2} + \cdots$$

(17) $$\wp'(z) = -\frac{2}{z^3} + \frac{g_2 z}{2 \cdot 5} + \frac{g_3 z^3}{7} + \frac{g_2^2 z^5}{2^3 \cdot 5^2} + \cdots .$$

The radius of convergence of these series is equal to the smallest distance of two points of the point-lattice Ω, i.e., the smallest of the four numbers $|2\omega|$, $|2\omega'|$, $|2\omega \pm 2\omega'|$.

Formulas with Weierstrass' functions may be expressed more symmetrically when the notation

(18) $\omega_1 = \omega, \quad \omega_2 = -\omega - \omega', \quad \omega_3 = \omega'$

(19) $\eta_\alpha = \zeta(\omega_\alpha) \qquad \alpha = 1, 2, 3$

is used. We then have

(20) $\zeta(z + 2\omega_\alpha) = \zeta(z) + 2\eta_\alpha \qquad \alpha = 1, 2, 3$

(21) $\sigma(z + 2\omega_\alpha) = -\sigma(z) \exp[2\eta_\alpha(z + \omega_\alpha)] \qquad \alpha = 1, 2, 3.$

It is convenient to introduce the three functions

(22) $$\sigma_\alpha(z) = \frac{\sigma(z + \omega_\alpha)}{\sigma(\omega_\alpha)} \exp(-z\eta_\alpha) \qquad \alpha = 1, 2, 3.$$

For these we have

(23) $\sigma_\alpha(z + 2\omega_\alpha) = -\sigma_\alpha(z) \exp[2\eta_\alpha(z + \omega_\alpha)] \qquad \alpha = 1, 2, 3$

$\sigma_\alpha(z + 2\omega_\beta) = \sigma_\alpha(z) \exp[2\eta_\beta(z + \omega_\beta)] \qquad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta.$

The function $\wp'(z)$ is an odd elliptic function of order three with periods $2\omega_\alpha$, $\alpha = 1, 2, 3$: it has three zeros in every cell. Now, $\wp'(-\omega_\alpha) = \wp'(\omega_\alpha)$ since $\wp'$ has period $2\omega_\alpha$, and $\wp'(-\omega_\alpha) = -\wp'(\omega_\alpha)$ since $\wp'(z)$ is an odd function of z. Thus we see that $z = \omega_\alpha$, $\alpha = 1, 2, 3$ is an irreducible set of zeros for $\wp'(z)$. It is customary to put

(24) $e_\alpha = \wp(\omega_\alpha) \qquad \alpha = 1, 2, 3.$

The function $\wp(z) - \wp(\omega_\alpha)$ is an elliptic function of order two. It has double poles at points congruent to 0, and double zeros at points congruent to ω_α. Since it is of order two, these are the only poles and zeros, and hence the function $[\wp(z) - e_\alpha]^{1/2}$ may be defined as a single-valued function (but it need not have periods 2ω, $2\omega'$, see sec. 13.13, 13.16).

13.13. Further properties of Weierstrass' functions

The dependence of $\wp(z)$ on the half-periods ω, ω', is indicated by writing $\wp(z|\omega, \omega')$, the dependence on the invariants g_2, g_3 by $\wp(z; g_2, g_3)$; and similarly for the other functions of Weierstrass.

From the definitions we have the homogeneity relations for arbitrary $t \neq 0$,

$$(1)\quad \begin{aligned} &\wp'(tz|t\omega, t\omega') = t^{-3}\,\wp'(z|\omega, \omega')\\ &\wp(tz|t\omega, t\omega') = t^{-2}\,\wp(z|\omega, \omega')\\ &\zeta(tz|t\omega, t\omega') = t^{-1}\,\zeta(z|\omega, \omega')\\ &\sigma(tz|t\omega, t\omega') = t\,\sigma(z|\omega, \omega') \end{aligned}$$

$$(2)\quad \begin{aligned} &\wp'(tz; t^{-4}g_2, t^{-6}g_3) = t^{-3}\,\wp'(z; g_2, g_3)\\ &\wp(tz; t^{-4}g_2, t^{-6}g_3) = t^{-2}\,\wp(z; g_2, g_3)\\ &\zeta(tz; t^{-4}g_2, t^{-6}g_3) = t^{-1}\,\zeta(z; g_2, g_3)\\ &\sigma(tz; t^{-4}g_2, t^{-6}g_3) = t\,\sigma(z; g_2, g_3). \end{aligned}$$

Thus it is seen that Weierstrass' functions depend essentially on two parameters which may be chosen, for instance, as the ratios of z, ω, ω'. The expressions of the invariants in terms of the periods are given in 13.12(13). Conversely, from 13.9(6) and 13.12(24),

$$\omega_\alpha = \int_\infty^{e_\alpha} (4t^3 - g_2 t - g_3)^{-1/2}\, dt.$$

The functions

$$\wp'^2(z) \quad \text{and} \quad [\wp(z) - e_1][\wp(z) - e_2][\wp(z) - e_3]$$

are both elliptic functions of order six with periods $2\omega_\alpha$, $\alpha = 1, 2, 3$. They both have an irreducible set of double zeros at ω_α, $\alpha = 1, 2, 3$, and a pole of order six at 0. By the general theorems of sec. 13.11, their quotient is constant. The value of this constant may be computed from the expansions 13.12(4) and (5). Thus we obtain the *algebraic differential equation* of Weierstrass' $\wp$-function,

$$(3)\quad \wp'^2(z) = 4[\wp(z) - e_1][\wp(z) - e_2][\wp(z) - e_3].$$

An alternative form of this differential equation may be obtained from the remark that

$$\wp'^2(z) - 4\wp^3(z) + g_2\wp(z)$$

is an elliptic function of order six at most, and that all possible poles of this function are congruent to 0. From the expansions 13.12(16) and (17) it follows that this function is regular at $z = 0$, and hence a constant by sec. 13.11. The value of this constant, obtained by means of 13.12(16) and (17), is $-g_3$, and hence the alternative differential equation

(4) $\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3.$

A comparison of the right-hand sides of (3) and (4) shows that e_α, $\alpha = 1, 2, 3$ are the roots of the algebraic equation $4t^3 - g_2 t - g_3 = 0$, and the formulas for symmetric functions of the roots of algebraic equations lead to the following formulas

(5) $e_1 + e_2 + e_3 = 0,$

$$-4(e_2e_3 + e_3e_1 + e_1e_2) = g_2, \quad 4e_1e_2e_3 = g_3$$

(6) $e_1^2 + e_2^2 + e_3^2 = \dfrac{1}{2}g_2, \quad e_1^3 + e_2^3 + e_3^3 = \dfrac{3}{4}g_3, \quad e_1^4 + e_2^4 + e_3^4 = \dfrac{1}{8}g_2^2$

(7) $16(e_2 - e_3)^2(e_3 - e_1)^2(e_1 - e_2)^2 = g_2^3 - 27g_3^2 = \Delta.$

The last of these expressions is the *discriminant* of the cubic equation.

The differential equation (4), together with the remark that $\wp(z)$ has a pole, and hence becomes infinite, at $z = 0$, establishes the relations 13.9(6) and (7), and the connection between Weierstrass' canonical form of elliptic integrals of the first kind, and Weierstrass' $\wp$- function. From (4) we also have

(8) $\wp''(z) = 6\wp^2(z) - \tfrac{1}{2}g_2, \quad \wp'''(z) = 12\wp(z)\wp'(z)$

and, by induction,

$$\wp^{(2n-2)}(z) \quad \text{and} \quad \wp^{(2n+1)}(z)/\wp'(z)$$

are polynomials of degree n in $\wp(z)$.

The *addition theorem* of the $\wp$-function may be written in several forms.

(9) $\wp(u + v) = \dfrac{1}{4}\left[\dfrac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2 - \wp(u) - \wp(v)$

(10) $\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(u+v) & -\wp'(u+v) \end{vmatrix} = 0$

$$(11)\quad \wp(u+v)=\wp(u)-\frac{1}{2}\frac{\partial}{\partial u}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]$$

$$=\wp(v)-\frac{1}{2}\frac{\partial}{\partial v}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]$$

$$(12)\quad \wp(u+v)+\wp(u-v)=2\wp(u)-\frac{\partial^2}{\partial u^2}\{\log[\wp(u)-\wp(v)]\}.$$

These addition theorems may be obtained in several ways. They may be proved by observing that the functions on the two sides of the equation are elliptic functions with the same periods, poles, and principal parts, and have the same value at some specified point.

From the addition theorems many formulas for Weierstrass' function follow. We note

$$(13)\quad \wp(z+\omega_\alpha)=e_\alpha+\frac{(e_\alpha-e_\beta)(e_\alpha-e_\gamma)}{\wp(z)-e_\alpha} \qquad \alpha=1,\,2,\,3$$

$$(14)\quad \wp(2z)=-2\wp(z)+\left[\frac{\wp''(z)}{2\wp'(z)}\right]^2$$

$$(15)\quad \wp(\tfrac{1}{2}z)=\wp(z)+[\wp(z)-e_2]^{1/2}[\wp(z)-e_3]^{1/2}$$

$$+[\wp(z)-e_3]^{1/2}[\wp(z)-e_1]^{1/2}+[\wp(z)-e_1]^{1/2}[\wp(z)-e_2]^{1/2}.$$

In the first of these, α, β, γ is any permutation of 1, 2, 3. Equation (14) is the *duplication formula*. The square roots in (15) are to be taken in accordance with (22).

There are also corresponding formulas for Weierstrass' zeta and sigma functions.

$$(16)\quad \zeta(u+v)=\zeta(u)+\zeta(v)+\frac{1}{2}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}$$

$$(17)\quad \sigma(u+v)\,\sigma(u-v)=-\sigma^2(u)\,\sigma^2(v)[\wp(u)-\wp(v)].$$

These formulas are sometimes called the addition theorems of the zeta and the sigma function, although they are not addition theorems as defined in 13.11(2). Since $\zeta(u)$ and $\sigma(u)$ are not elliptic functions, they cannot possess addition theorems. The following formulas may be deduced from (16) and (17).

(18) $\zeta(z \pm \omega_\alpha) = \zeta(z) \pm \eta_\alpha + \frac{1}{2}\,\frac{\wp'(z)}{\wp(z) - e_\alpha}$ $\qquad \alpha = 1, 2, 3$

(19) $\zeta(z + 2m\omega + 2n\omega') = \zeta(z) + 2m\eta + 2n\eta'$ $\qquad m, n$ integers

(20) $\sigma(z + 2m\omega + 2n\omega') = (-1)^{m+n+mn}\sigma(z)$
$\times \exp[(z + m\omega + n\omega')(2m\eta + 2n\eta')]$ $\qquad m, n$ integers.

Equations (16) to (18) may be proved by expressing the elliptic functions, $[\wp'(u) - \wp'(v)]/[\wp(u) - \wp(v)]$ in terms of zeta functions, $\wp(u) - \wp(v)$ in terms of sigma functions, and $\wp'(z)/[\wp(z) - e_\alpha]$ in terms of zeta functions (see the following sections).

It has been mentioned in sec. 13.12, in the lines following 13.12(24), that $[\wp(z) - e_\alpha]^{1/2}$ may be defined as a single-valued function of z. This may be done by taking that square root which will make $z = 0$ a simple pole with residue unity for this function. Since the principal part near the origin of $\wp'(z)$ is $-2z^{-3}$, this definition implies that

(21) $\wp'(z) = -2[\wp(z) - e_1]^{1/2}\,[\wp(z) - e_2]^{1/2}\,[\wp(z) - e_3]^{1/2}$.

To obtain an explicit formula for $[\wp(z) - e_\alpha]^{1/2}$, put $u = z$, $v = \omega_\alpha$, in (17) and use (20) and 13.12(21).

$$\wp(z) - e_\alpha = -\frac{\sigma(z + \omega_\alpha)\,\sigma(z - \omega_\alpha)}{\sigma^2(z)\,\sigma^2(\omega_\alpha)}$$

$$= \frac{\sigma^2(z + \omega_\alpha)}{\sigma^2(z)\,\sigma^2(\omega_\alpha)}\ \exp[-2\eta_\alpha(z + \omega_\alpha)] = \left[\frac{\sigma_\alpha(z)}{\sigma(z)}\right]^2 .$$

Extracting the square root according to the definition made above,

(22) $[\wp(z) - e_\alpha]^{1/2} = \sigma_\alpha(z)/\sigma(z)$.

In particular, putting $z = \omega_\beta$,

(23) $(e_\beta - e_\alpha)^{1/2} = \sigma_\alpha(\omega_\beta)/\sigma(\omega_\beta)$.

In relations involving square roots, such as (15), we shall always assume that the square roots are determined as in (22) and (23). From (23) and 13.12(22) we have

(24) $(e_\beta - e_\alpha)^{1/2} = \frac{\sigma(\omega_\alpha + \omega_\beta)}{\sigma(\omega_\alpha)\,\sigma(\omega_\beta)}\ \exp(-\eta_\alpha\omega_\beta)$,

and this equation in combination with Legendre's relation 13.12(10) shows that

$$(25)\quad (e_1 - e_3)^{1/2} = i(e_3 - e_1)^{1/2}, \quad (e_1 - e_2)^{1/2} = i(e_2 - e_1)^{1/2}$$
$$(e_2 - e_3)^{1/2} = i(e_3 - e_2)^{1/2}.$$

13.14. The expression of elliptic functions and elliptic integrals in terms of Weierstrass' functions

We shall now consider the problem of expressing any elliptic function in terms of standard functions, either as a rational combination of $\wp$ and $\wp'$ (linear in $\wp'$), or as a linear combination of zeta functions and their derivatives, or else as a quotient of two products of sigma functions. Let $f(z)$ be an elliptic function with periods 2ω, $2\omega'$, and let $\wp(z)$, $\zeta(z)$, $\sigma(z)$ be Weierstrass' functions constructed with primitive periods 2ω, $2\omega'$.

Expression in terms of $\wp(z)$ *and* $\wp'(z)$. First, let $f(z)$ be an even function of z. If $f(z)$ has a zero or pole at $z = 0$, this zero or pole must be of even order, and hence $f(z)\,[\wp(z)]^s$ will be analytic and $\neq 0$ at $z = 0$, for some integer s. The zeros and poles of the even function $f(z)[\wp(z)]^s$ are situated symmetrically to the origin. Let $a_1, \dots, a_h, -a_1, \dots, -a_h$ be an irreducible set of zeros, and $\beta_1, \dots, \beta_h, -\beta_1, \dots, -\beta_h$ an irreducible set of poles, each zero and pole repeated according to its multiplicity. Then

$$f(z)\,[\wp(z)]^s \prod_{r=1}^{h} \frac{\wp(z) - \wp(\beta_r)}{\wp(z) - \wp(a_r)}$$

is an elliptic function without zeros or poles and hence a constant. *An even elliptic function may be expressed as a rational function of* $\wp(z)$.

Let $f(z)$ be any elliptic function

$$f(z) = \frac{1}{2}\,[f(z) + f(-z)] + \wp'(z)\,\frac{f(z) - f(-z)}{2\wp'(z)}\,.$$

Here $f(z) + f(-z)$ and $[f(z) - f(-z)]/\wp'(z)$ are even elliptic functions and hence rational functions of $\wp(z)$. Thus, *any elliptic function may be expressed in the form*

$$(1)\quad f(z) = R_1[\wp(z)] + R_2[\wp(z)]\,\wp'(z)$$

where $R_1(w)$ and $R_2(w)$ are rational functions of w.

From this, in conjunction with the differential equation and addition theorem of the $\wp$-function it follows easily that any elliptic function has an algebraic differential equation and an algebraic addition theorem, and that any two elliptic functions with the same periods are algebraically connected (see sec. 13.11).

Expression in terms of zeta functions. The function $\zeta(z)$ is not an elliptic function, but it is easy to see by means of 13.13(19) that

$$\sum_{r=1}^{h} c_r\, \zeta(z-\gamma_r)$$

is an elliptic function if and only if

$$\sum_{r=1}^{h} c_r = 0.$$

Moreover, $\zeta'(z) = -\wp(z)$ so that all derivatives of $\zeta(z)$ are elliptic functions.

Let $\beta_1, \dots, \beta_h$ be an irreducible set of distinct poles of $f(z)$, and let

$$\sum_{s=1}^{m_r} b_{r,s}(z-\beta_r)^{-s}$$

be the principal part (the sum of the negative powers in the Laurent expansion) of $f(z)$ for the neighborhood of $z=\beta_r$ which is a pole of order m_k. Consider

$$\Phi(z) = f(z) - \sum_{r=1}^{h} \sum_{s=1}^{m_r} \frac{(-1)^{s-1}}{(s-1)!} b_{r,s}\, \zeta^{(s-1)}(z-\beta_r).$$

Now,

$$\sum_{r=1}^{h} b_{r,1}\, \zeta(z-\beta_r)$$

is an elliptic function since $\Sigma\, b_{r,1}$, being the sum of residues at an irreducible set of poles, is zero (see sec. 13.11). Also $\zeta^{(s-1)}(z-\beta_r)$ is an elliptic function for $s = 2, 3, \dots$, and hence $\Phi(z)$ is an elliptic function. Since the principal part of $\zeta(z-\beta_r)$ at $z=\beta_r$ is $(z-\beta_r)^{-1}$, it follows that $\Phi(z)$ has no poles at $z=\beta_1, \dots, \beta_h$, hence no poles at all, and thus is constant. *Any elliptic function may be expressed as*

$$(2) \quad f(z) = b_0 + \sum_{r=1}^{h} \sum_{s=1}^{m_r} \frac{(-1)^{s-1}}{(s-1)!}\, b_{r,s}\, \zeta^{(s-1)}(z-\beta_r).$$

Such an expression is especially useful when integrating elliptic functions. From (2), 13.12(7), and 13.12(12),

$$(3)\quad \int f(u)\,du = b_0 u + c + \sum_{r=1}^{h} \Big\{ b_{r,1} \log[\sigma(u-\beta_r)] - b_{r,2}\zeta(u-\beta_r) + \sum_{s=3}^{m_r} \frac{(-1)^s}{(s-1)!} b_{r,s}\, \wp^{(s-3)}(u-\beta_r) \Big\}.$$

The expansion (2) may be used to establish 13.13(16) and (18).

Expression in terms of sigma functions. Although $\sigma(z)$ itself is not an elliptic function, it is easy to see by means of 13.13(20) that

$$(4)\quad \Psi(z) = \prod_{r=1}^{h} \frac{\sigma(z-\alpha_r)}{\sigma(z-\beta_r)}$$

is an elliptic function if and only if $\sum_{r=1}^{h} (\alpha_r - \beta_r) = 0$. Now let $\alpha_1, \dots, \alpha_h$; $\beta_1, \dots, \beta_h$ be an irreducible set of zeros and poles of $f(z)$, each repeated according to its multiplicity. We know (sec. 13.11) that $\sum_{r=1}^{h} (\alpha_r - \beta_r)$ is a period, and replacing some of the zeros and poles by congruent ones, we may assume that $\sum_{r=1}^{h} (\alpha_r - \beta_r) = 0$. We then form $\Psi(z)$ according to (4), and see that $f(z)/\Psi(z)$ is an elliptic function without zeros and poles and hence a constant. *Any elliptic function may be expressed as*

$$(5)\quad f(z) = c \prod_{r=1}^{h} \frac{\sigma(z-\alpha_r)}{\sigma(z-\beta_r)}$$

where $\alpha_1, \dots, \alpha_h$ is an irreducible set of zeros, and $\beta_1, \dots, \beta_h$ an irreducible set of poles, of $f(z)$, each zero and pole repeated according to its multiplicity, and the sets are so chosen that

$$(6)\quad \sum_{r=1}^{h} \alpha_r = \sum_{r=1}^{h} \beta_r.$$

The representation (5) may be used to prove 13.13(17).

Elliptic integrals. Given an elliptic integral in Weierstrass' canonical form

$$(7)\quad I = \int R(x, y)\,dx, \qquad y^2 = 4x^3 - g_2 x - g_3,$$

we may put

(8) $x = \wp(z; g_2, g_3), \quad y = \wp'(z; g_2, g_3)$

to reduce (7) to

(9) $I = \int R[\wp(z), \wp'(z)]\,\wp'(z)\,dz.$

The integrand is a rational function of $\wp(z)$ and $\wp'(z)$ and hence an elliptic function, say $f(z)$: it has an expansion (2), and the integral itself may be evaluated in the form (3).

The substitution (8) represents points on the algebraic curve

(10) $y^2 = 4x^3 - g_2 x - g_3.$

The coordinates, x and y, appear as single-valued functions of a parameter z, which is a *uniformizing variable* for (10) (see also sec. 13.2).

Any elliptic integral

(11) $I = \int R(x, y)\,dx$

(12) $y^2 = G(x) = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4$

may be reduced to Weierstrass' functions. We first reduce (12) to Weierstrass' canonical form as in sec. 13.5, and then proceed as above. To some extent the computations indicated in sec. 13.5 may be avoided by using the expressions 13.5(8) for the invariants with which to form Weierstrass' functions. See, for instance, Bianchi (1916, 371-374) where the computation of an elliptic integral of the first kind involving (12) is carried out.

13.15. Descriptive properties and degenerate cases of Weierstrass' functions

In many applications the coefficients of $G(x)$ are real. In this case 13.5(8) shows that also the invariants g_2, g_3 are real. We shall describe briefly the behavior of $\wp(z)$ for real g_2 and g_3, distinguishing two cases according as the discriminant $\Delta = g_2^3 - 27g_3^2$ is positive or negative.

First let $\Delta > 0$. In this case there exists a pair of primitive periods 2ω, $2\omega'$ so that ω is real and ω' is imaginary. The point-lattice of all periods may be generated by a rectangular line lattice. The function $\wp(z)$ is real on the lines of the lattice,

$\operatorname{Re} z = 2m\,\omega$ m integer

$i \operatorname{Im} z = 2n\,\omega'$ n integer

and also on the half-way lines

$$\operatorname{Re} z = (2m+1)\,\omega \qquad m \text{ integer}$$

$$i \operatorname{Im} z = (2n+1)\,\omega' \qquad n \text{ integer.}$$

We have the following symmetry relations in which z_1 and z_2 are real.

$$\wp(z_1 + iz_2) = \overline{\wp(z_1 - iz_2)} = \wp(-z_1 - iz_2) = \overline{\wp(-z_1 + iz_2)},$$

the bars denoting conjugate complex quantities. In this case e_1, e_2, e_3 are real, $e_1 > e_2 > e_3$, $e_1 > 0$, $e_3 < 0$. As z describes the boundary of the rectangle 0, ω, $\omega + \omega'$, ω', 0, the function $\wp(z)$ decreases from ∞ to $e_1 = \wp(\omega)$, to $e_2 = \wp(\omega + \omega')$, to $e_3 = \wp(\omega')$, to $-\infty$.

Now let $\Delta < 0$. This case is very different from the first one. There is again a pair of periods, the first of which is real and the second imaginary, but they are not primitive periods. There exists, however, a pair of conjugate complex primitive periods giving a rhombic fundamental parallelogram. If 2ω, $2\omega'$ are a pair of conjugate complex primitive periods, the diagonals of the period parallelograms are the lines

$$\operatorname{Re} z = m(\omega + \omega') \qquad m \text{ integer}$$

$$i \operatorname{Im} z = n(\omega - \omega') \qquad n \text{ integer}$$

and these are the only lines on which $\wp(z)$ is real. Only e_2 is real in this case: e_1 and e_3 are conjugate complex. As z varies along diagonals of period parallelograms from 0 to $\omega + \omega'$ to 2ω (or $2\omega'$) $\wp(z)$ decreases from $+\infty$ to e_2 to $-\infty$.

Degenerate cases of Weierstrass' functions occur when one or both of the periods become infinite, or, what is the same, two or all three of e_1, e_2, e_3 coincide. We list the following three cases.

(i) Real period infinite.

(1) $e_1 = e_2 = a, \quad e_3 = -2a$

(2) $g_2 = 12a^2, \quad g_3 = -8a^3, \quad \omega = \infty, \quad \omega' = (12a)^{-1/2}\pi i$

(3) $\wp(z;\, 12a^2, -8a^3) = a + 3a\,\{\sinh[(3a)^{1/2} z]\}^{-2}$

(4) $\zeta(z;\, 12a^2, -8a^3) = -au + (3a)^{1/2} \operatorname{ctnh}[(3a)^{1/2} z]$

(5) $\sigma(z;\, 12a^2, -8a^3) = (3a)^{-1/2} \sinh[(3a)^{1/2} z] \exp(-\tfrac{1}{2}az^2)$.

(ii) Imaginary period infinite.

(6) $e_1 = 2a, \quad e_2 = e_3 = -a$

(7) $g_2 = 12a^2, \quad g_3 = 8a^3, \quad \omega = (12a)^{-1/2}\pi, \quad \omega' = i\infty.$

(8) $\wp(z;\, 12a^2,\, 8a^3) = -a + 3a\{\sin[(3a)^{1/2} z]\}^{-2}$

(9) $\zeta(z;\, 12a^2,\, 8a^3) = az + (3a)^{1/2} \operatorname{ctn}[(3a)^{1/2} z]$

(10) $\sigma(z;\, 12a^2,\, 8a^3) = (3a)^{-1/2} \sin[(3a)^{1/2} z] \exp(\tfrac{1}{2} az^2)$.

(iii) Both periods infinite.

(11) $e_1 = e_2 = e_3 = 0, \quad g_2 = g_3 = 0, \quad \omega = -i\omega' = \infty$

(12) $\wp(z;\, 0,\, 0) = z^{-2}, \quad \zeta(z;\, 0,\, 0) = z^{-1}, \quad \sigma(z;\, 0,\, 0) = z.$

In all three cases $\Delta = 0$.

13.16. Jacobian elliptic functions

Jacobi's function

(1) $w = \operatorname{sn} u = \operatorname{sn}(u, k)$

may be defined as in sec. 13.9 by the integral

(2) $u = \int_0^w [(1 - x^2)(1 - k^2 x^2)]^{-1/2}\, dx$

in which the square root has the value 1 at $x = 0$. Also $\operatorname{sn}(0, k) = 1$. The integral may be evaluated in terms of Weierstrass' functions (see sec. 13.14). It turns out that

(3) $e_1 : e_2 : e_3 = (2 - k^2) : (2k^2 - 1) : -(1 + k^2), \quad z = (e_1 - e_3)^{-1/2} u$

and

(4) $\operatorname{sn}(u, k) = \dfrac{(e_1 - e_3)^{1/2}}{[\wp(z) - e_3]^{1/2}}$.

For the other two basic functions of Jacobi's we have

(5) $\operatorname{cn}(u, k) = \dfrac{[\wp(z) - e_1]^{1/2}}{[\wp(z) - e_3]^{1/2}}$

(6) $\operatorname{dn}(u, k) = \dfrac{[\wp(z) - e_2]^{1/2}}{[\wp(z) - e_3]^{1/2}}$.

In (4), (5), (6),

(7) $u = (e_1 - e_3)^{1/2} z, \quad k^2 = \dfrac{e_2 - e_3}{e_1 - e_3}$

and all square roots occurring here are uniquely defined by 13.13(22) and (23). Using these latter relations we may rewrite (4) to (6) as

$$(8)\quad \operatorname{sn}(u,k) = (e_1 - e_3)^{1/2}\,\frac{\sigma(z)}{\sigma_3(z)}, \qquad \operatorname{cn}(u,k) = \frac{\sigma_1(z)}{\sigma_3(z)}$$

$$\operatorname{dn}(u,k) = \frac{\sigma_2(z)}{\sigma_3(z)}.$$

The nine subsidiary functions 13.9(4) may similarly be expressed in terms of sigma functions. In what follows these nine functions will be omitted in general, since the formulas relating to them may easily be obtained from the formulas for the three basic functions (8).

In sec. 13.9, the Jacobian functions have been established in a neighborhood of the origin by the inversion of an elliptic integral. Equation (8) shows that an analytic continuation of these functions leads to *single-valued analytic functions* with poles at the zeros of $\sigma_3(z)$. Moreover, it is easy to see from (8) and 13.12(23) that *the Jacobian functions are doubly-periodic*. We put

$$(9)\quad u = (e_1 - e_3)^{1/2} z, \qquad \mathbf{K} = (e_1 - e_3)^{1/2}\,\omega, \quad i\mathbf{K}' = (e_1 - e_3)^{1/2}\,\omega',$$

call $\mathbf{K}$ the *real quarter-period* and $\mathbf{K}'$ the *imaginary quarter-period*, and verify that (9) is in accordance with the definition of $\mathbf{K}$ and $\mathbf{K}'$ as complete elliptic integrals in 13.7(1) and (2). The primitive periods of sn, cn, dn may now be found by means of 13.12(23). The zeros of $\sigma(z)$ are all simple and may be read off 13.12(11), those of $\sigma_\alpha(z)$ follow from 13.12(22). This gives the (simple) zeros and poles of Jacobi's functions. Lastly, 13.12(14) in conjunction with 13.13(23) enables us to determine the residues of the three functions (8). The results are shown in Table 5.

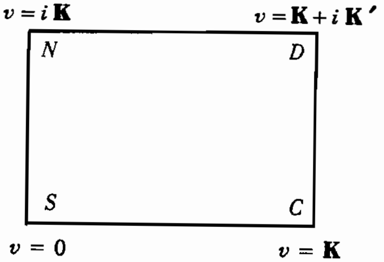
TABLE 5. PERIODS, ZEROS, POLES, AND RESIDUES OF JACOBI'S ELLIPTIC FUNCTIONS

m and *n* are integers

Function	Primitive Periods	Zeros	Poles	Residues
$\operatorname{sn}(u,k)$	$4\mathbf{K}$ $2i\mathbf{K}'$	$2m\mathbf{K} + 2ni\mathbf{K}'$	$2m\mathbf{K} + (2n+1)i\mathbf{K}'$	$\dfrac{(-1)^m}{k}$
$\operatorname{cn}(u,k)$	$4\mathbf{K}$ $2\mathbf{K} + 2i\mathbf{K}'$	$(2m+1)\mathbf{K} + 2ni\mathbf{K}'$		$\dfrac{(-1)^{m+n}}{ik}$
$\operatorname{dn}(u,k)$	$2\mathbf{K}$ $4i\mathbf{K}'$	$(2m+1)\mathbf{K} + (2n+1)i\mathbf{K}'$		$(-1)^{n+1}\,i$

If $0 < k^2 < 1$, then $\mathbf{K}$ and $\mathbf{K}'$ are real, and taking also e_α real, we see from (3) that we may take $e_1 > e_2 > e_3$, when ω becomes real and ω' imaginary. This is the case $\Delta > 0$ of sec. 13.15.

For any k^2 ($\neq 0, 1$) we take the parallelogram which is one eighth of the fundamental parallelogram for sn or dn and denote its vertices by the letters S, C, D, N as in the figure. With this notation, the first letter in the symbol of the twelve Jacobian functions shows the position of a zero, and the second, the position of a pole. Zeros and poles are repeated at half-periods.



From Table 5 it is easy to verify that any cell contains two simple poles (with zero residue-sum) and two simple zeros of any Jacobian elliptic function. Thus *Jacobi's functions* sn u, cn u, dn u *are elliptic functions of order* 2. Given the modulus, k, the quarter-periods $\mathbf{K}$, $\mathbf{K}'$ are determined by 13.7(1) and (2), uniquely if the k-plane is cut from $-\infty$ to -1 and from 1 to ∞. Thereupon the data given in Table 5 determine Jacobi's functions uniquely. We have expressed these in terms of sigma functions, but an independent construction in the manner of the construction of sec. 13.12 is possible. See Neville (1944) where the construction of all twelve Jacobian functions is carried out in a symmetric manner. (The reader should note, however, that Neville's notation differs somewhat from the customary notation adopted in this book.)

Legendre's complete elliptic integrals of the second kind are also expressible in terms of values of Weierstrass' functions

$$(10)\quad \mathbf{E} = \frac{e_1\,\omega + \eta}{(e_1 - e_3)^{1/2}}, \qquad \mathbf{E}' = i\,\frac{e_3\,\omega' + \eta'}{(e_1 - e_3)^{1/2}}.$$

The modulus, k, and the complementary modulus, k', are determined uniquely as

$$(11)\quad k = \frac{(e_2 - e_3)^{1/2}}{(e_1 - e_3)^{1/2}}, \qquad k' = \frac{(e_1 - e_2)^{1/2}}{(e_1 - e_3)^{1/2}}.$$

Given any modulus, k, equation (3) determines the e_α (up to an irrelevant factor), and hence the invariants according to 13.13(5). The Weierstrass functions constructed with these invariants then fully define the Jacobian functions, their periods, the complete elliptic integrals.

Conversely, the Weierstrass functions formed with any invariants determine Jacobian functions whose modulus is given by (11).

In sec. 13.7 it has been pointed out that the (incomplete) elliptic integral of the second kind is a single-valued function of u. This defines Jacobi's function $E(u)$. Putting $\phi = \operatorname{am}(u, k)$, $\sin\phi = \operatorname{sn}(u, k)$, and $\sin t = \operatorname{sn}(x, k)$ in 13.6(2) we find

(12) $E(u) = \int_0^u \operatorname{dn}^2(x, k)\, dx.$

Jacobi's function $E(u)$ is not periodic since

(13) $E(u + 2\mathbf{K}) = E(u) + 2\mathbf{E},$

$$E(u + 2i\mathbf{K}') = E(u) + 2i(\mathbf{K}' - \mathbf{E}').$$

Sometimes it is convenient to use the function

(14) $Z(u) = E(u) - \dfrac{\mathbf{E}}{\mathbf{K}} u$

which is singly-periodic, since

(15) $Z(u + 2\mathbf{K}) = Z(u), \qquad Z(u + 2i\mathbf{K}') = Z(u) - i\pi/\mathbf{K}.$

Although the functions $E(u)$, $Z(u)$ are not elliptic functions, they have many properties similar to those of elliptic functions. See, for instance, Whittaker and Watson (1927, p. 517-520).

13.17. Further properties of Jacobian elliptic functions

We shall often use the abbreviations

(1) $s = \operatorname{sn}(u, k), \quad c = \operatorname{cn}(u, k), \quad d = \operatorname{dn}(u, k).$

The following basic formulas are consequences of the definitions of Jacobian functions and of the properties of Weierstrass' $\wp$-function. Differentiation with respect to u will be indicated by a prime. Thus,

$$(s)' = ds/du \qquad (s)'' = d^2s/du^2, \text{ etc.}$$

(2) $s^2 + c^2 = 1, \quad k^2s^2 + d^2 = 1, \quad d^2 - k^2c^2 = k'^2$

(3) $(s)' = cd, \quad (c)' = -sd, \quad (d)' = -k^2sc$

(4) $(s)'' = -s(d^2 + k^2c^2), \quad (c)'' = -c(d^2 - k^2s^2),$

$(d)'' = -k^2d(c^2 - s^2)$

(5) $(s)'^2 = (1 - s^2)(1 - k^2s^2)$

(6) $(c)'^2 = (1 - c^2)(k^2c^2 + k'^2)$

(7) $(d)'^2 = (1 - d^2)(d^2 - k'^2)$

(8) $\operatorname{sn}(-u) = -\operatorname{sn} u, \quad \operatorname{cn}(-u) = \operatorname{cn} u, \quad \operatorname{dn}(-u) = \operatorname{dn} u$

(9) $\operatorname{sn}(2\mathbf{K} - u) = \operatorname{sn} u, \quad \operatorname{cn}(2\mathbf{K} - u) = -\operatorname{cn} u, \quad \operatorname{dn}(2\mathbf{K} - u) = \operatorname{dn} u$

(10) $\operatorname{sn}(2i\mathbf{K}' - u) = -\operatorname{sn} u, \quad \operatorname{cn}(2i\mathbf{K}' - u) = -\operatorname{cn} u,$
$\operatorname{dn}(2i\mathbf{K}' - u) = -\operatorname{dn} u.$

The power series expansions

(11) $$\operatorname{sn}(u, k) = u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} - \cdots$$

$$\operatorname{cn}(u, k) = 1 - \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - (1 + 44k^2 + 16k^4)\frac{u^6}{6!} + \cdots$$

$$\operatorname{dn}(u, k) = 1 - k^2\frac{u^2}{2!} + k^2(4 + k^2)\frac{u^4}{4!} - k^2(16 + 44k^2 + k^4)\frac{u^6}{6!} + \cdots$$

have a radius of convergence

(12) $\min(|\mathbf{K}'|, \; |2\mathbf{K} + i\mathbf{K}'|, \; |2\mathbf{K} - i\mathbf{K}'|).$

The *addition theorems* may be obtained from the addition theorems of the $\wp$-function in combination with the transformation (see Table 11, sec. 13.22)

(13) $\operatorname{sn}(iu, k) = i \operatorname{sc}(u, k'), \quad \operatorname{cn}(iu, k) = \operatorname{nc}(u, k')$
$\operatorname{dn}(iu, k) = \operatorname{dc}(u, k').$

In the addition theorems we shall use the abbreviations

(14) $s_1 = \operatorname{sn}(u_1, k), \quad s_2 = \operatorname{sn}(u_2, k), \quad s_2' = \operatorname{sn}(u_2, k')$

with similar abbreviations for cn, dn. We then have

(15) $$\operatorname{sn}(u_1 + u_2, k) = (s_1 c_2 d_2 + c_1 d_1 s_2)/(1 - k^2 s_1^2 s_2^2)$$

$$\operatorname{cn}(u_1 + u_2, k) = (c_1 c_2 - s_1 d_1 s_2 d_2)/(1 - k^2 s_1^2 s_2^2)$$

$$\operatorname{dn}(u_1 + u_2, k) = (d_1 d_2 - k^2 s_1 c_1 s_2 c_2)/(1 - k^2 s_1^2 s_2^2)$$

(16) $$\operatorname{sn}(u_1 + iu_2, k) = (s_1 d_2' + ic_1 d_1 s_2' c_2')/(c_2'^2 + k^2 s_1^2 s_2'^2)$$

$$\operatorname{cn}(u_1 + iu_2, k) = (c_1 c_2' - is_1 d_1 s_2' d_2')/(c_2'^2 + k^2 s_1^2 s_2'^2)$$

$$\operatorname{dn}(u_1 + iu_2, k) = (d_1 c_2' d_2' - ik^2 s_1 c_1 s_2')/(c_2'^2 + k^2 s_1^2 s_2'^2)$$

(17) $\operatorname{sn}(2u, k) = 2scd/(1 - k^2 s^4)$

$\operatorname{cn}(2u, k) = (c^2 - s^2 d^2)/(1 - k^2 s^4)$

$\operatorname{dn}(2u, k) = (d^2 - k^2 s^2 c^2)/(1 - k^2 s^4)$

(18) $\operatorname{sn}(\tfrac{1}{2}u, k) = (1 - c)^{\frac{1}{2}} (1 + d)^{-\frac{1}{2}}$

$\operatorname{cn}(\tfrac{1}{2}u, k) = (d + c)^{\frac{1}{2}} (1 + d)^{-\frac{1}{2}}$

$\operatorname{dn}(\tfrac{1}{2}u, k) = (d + k^2 c + k'^2)^{\frac{1}{2}} (1 + d)^{-\frac{1}{2}}.$

In (17) and (18) we reverted to the notation (1). Equations (16) show that the values of Jacobi's elliptic functions for any complex u may be computed if the values of these functions, and also of the functions with the complementary modulus, are known on the real axis.

We also note the following Fourier expansions

(19) $$\operatorname{sn} u = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1 - q^{2n-1}} \sin(2n - 1) \frac{\pi u}{2\mathbf{K}}$$

$$\operatorname{cn} u = \frac{2\pi}{k\mathbf{K}} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1 + q^{2n-1}} \cos(2n - 1) \frac{\pi u}{2\mathbf{K}}$$

$$\operatorname{dn} u = \frac{2\pi}{\mathbf{K}} \left[\frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos n \frac{\pi u}{\mathbf{K}}\right]$$

in which

(20) $q = e^{i\pi\tau} = \exp(-\pi\mathbf{K}'/\mathbf{K}).$

The expansions (19) are valid in the strip of the complex plane bounded by the lines $\pm i\mathbf{K}' + \lambda\mathbf{K}$, $-\infty < \lambda < \infty$.

The values of sn, cn, dn at the points $m\mathbf{K} + ni\mathbf{K}'$ (m, n integers) may be found by means of 13.12(24): from these the values at the points

(21) $\tfrac{1}{2}m\mathbf{K} + \tfrac{1}{2}ni\mathbf{K}'$ $\qquad m, n$ integers

may be found by means of (18). The results for $0 \le m, n \le 3$ are shown in Table 6. The points chosen in Table 6 range in each case over one-half of a cell. The values at the points (21) in the other half of the cell may be found by means of Table 7, at other points (21) by the periodic properties of sn, cn, dn. All square roots in this table are to be taken as positive when $0 < k < 1$, and are defined by analytic continuation otherwise.

TABLE 6. SPECIAL VALUES OF JACOBIAN ELLIPTIC FUNCTIONS

$\mathrm{sn}\,(\tfrac{1}{2}m\mathbf{K} + \tfrac{1}{2}ni\mathbf{K}')$

$\frac{1}{2}ni\mathbf{K}'$ \ $\frac{1}{2}m\mathbf{K}$	0	$\frac{1}{2}\mathbf{K}$	$\mathbf{K}$	$\frac{3}{2}\mathbf{K}$
0	0	$(1+k')^{-1/2}$	1	$(1+k')^{-1/2}$
$\frac{1}{2}i\mathbf{K}'$	$ik^{-1/2}$	$(2k)^{-1/2}[(1+k)^{1/2} + i(1-k)^{1/2}]$	$k^{-1/2}$	$(2k)^{-1/2}[(1+k)^{1/2} - i(1-k)^{1/2}]$
$i\mathbf{K}'$	∞	$(1-k')^{-1/2}$	k^{-1}	$(1-k')^{-1/2}$
$\frac{3}{2}i\mathbf{K}'$	$-ik^{-1/2}$	$(2k)^{-1/2}[(1+k)^{1/2} - i(1-k)^{1/2}]$	$k^{-1/2}$	$(2k)^{-1/2}[(1+k)^{1/2} + i(1-k)^{1/2}]$

TABLE 6 continued

$\mathrm{cn}\,(\frac{1}{2}m\mathbf{K} + \frac{1}{2}ni\mathbf{K}')$

$\frac{1}{2}ni\mathbf{K}'$ \ $\frac{1}{2}m\mathbf{K}$	0	$\frac{1}{2}\mathbf{K}$	$\mathbf{K}$	$\frac{3}{2}\mathbf{K}$
0	1	$k'^{\frac{1}{2}}(1+k')^{-\frac{1}{2}}$	0	$-k'^{\frac{1}{2}}(1+k')^{-\frac{1}{2}}$
$\frac{1}{2}i\mathbf{K}'$	$k^{-\frac{1}{2}}(1+k)^{\frac{1}{2}}$	$k'^{\frac{1}{2}}(2k)^{-\frac{1}{2}}(1-i)$	$-ik^{-\frac{1}{2}}(1-k)^{\frac{1}{2}}$	$-k'^{\frac{1}{2}}(2k)^{-\frac{1}{2}}(1+i)$
$i\mathbf{K}'$	∞	$-ik'^{\frac{1}{2}}(1-k')^{-\frac{1}{2}}$	$-ik^{-1}k'$	$-ik'^{\frac{1}{2}}(1-k')^{-\frac{1}{2}}$
$\frac{3}{2}i\mathbf{K}'$	$-k^{-\frac{1}{2}}(1+k)^{\frac{1}{2}}$	$-k'^{\frac{1}{2}}(2k)^{-\frac{1}{2}}(1+i)$	$-ik^{-\frac{1}{2}}(1-k)^{\frac{1}{2}}$	$k'^{\frac{1}{2}}(2k)^{-\frac{1}{2}}(1-i)$

TABLE 6 continued

$\mathrm{dn}\,(\tfrac{1}{2} m \mathbf{K} + \tfrac{1}{2} n i \mathbf{K}')$

$\frac{1}{2} ni\mathbf{K}'$ \ $\frac{1}{2} m\mathbf{K}$	0	$\frac{1}{2}\mathbf{K}$	$\mathbf{K}$	$\frac{3}{2}\mathbf{K}$
0	1	$k'^{1/2}$	k'	$k'^{1/2}$
$\frac{1}{2} i\mathbf{K}'$	$(1+k)^{1/2}$	$(\tfrac{1}{2}k')^{1/2}[(1+k')^{1/2} - i(1-k')^{1/2}]$	$(1-k)^{1/2}$	$(\tfrac{1}{2}k')^{1/2}[(1+k')^{1/2} + i(1-k')^{1/2}]$
$i\mathbf{K}'$	∞	$-ik'^{1/2}$	0	$ik'^{1/2}$
$\frac{3}{2} i\mathbf{K}'$	$-(1+k)^{1/2}$	$-(\tfrac{1}{2}k')^{1/2}[(1+k')^{1/2} + i(1-k')^{1/2}]$	$-(1-k)^{1/2}$	$-(\tfrac{1}{2}k')^{1/2}[(1+k')^{1/2} - i(1-k')^{1/2}]$

From the addition theorems and Table 6, we may obtain the values of Jacobi's functions at the point $\frac{1}{2}m\mathbf{K} + \frac{1}{2}ni\mathbf{K}' + u$ in terms of their values at u. Table 7 shows the results for the points $m\mathbf{K} + ni\mathbf{K} \pm u$. The table covers more than a cell in order to exhibit the symmetry around the points S, C, D, N of Jacobi's functions. In the table the abbreviations (1) have been used and when double signs appear, the upper signs refer to $m\mathbf{K} + ni\mathbf{K}' + u$, the lower to $m\mathbf{K} + ni\mathbf{K}' - u$.

Jacobian elliptic functions may be used for the computation of Weierstrass' functions when e_1, e_2, e_3 are given. The modulus of the Jacobian functions, and the variable of the Jacobian functions are given by 13.16 (7). The periods of Weierstrass' functions follow from 13.16(9), the quantities η and η' from 13.16(10). Weierstrass' basic function is

$$(22)\quad \wp(z) = e_3 + \frac{e_1 - e_3}{\operatorname{sn}^2(u, k)}.$$

The three e_α may always be numbered in such a fashion that $|k| \leq 1$.

13.18. Descriptive properties and degenerate cases of Jacobi's elliptic functions

In many applications we have $0 < k < 1$. In this case also $0 < k' < 1$, and 13.8(1) shows that $\mathbf{K}$ and $\mathbf{K}'$ are real. The point-lattice $m\mathbf{K} + ni\mathbf{K}'$ may then be generated by a rectangular line-lattice (although the latter need not correspond to primitive periods). We shall indicate the behavior of sn u, cn u, dn u in this case by diagrams (see below). The notations outside the figure indicate the position of the lattice-points $m\mathbf{K} + ni\mathbf{K}'$, the notations inside the figure give the value of the function in question at the lattice points. Along fully drawn lines the function is real and between any two consecutive lattice-points it is monotonic. Along the broken lines the function is imaginary and between any two consecutive lattice-points it is monotonic. Along lines joining a zero and a pole of a function the sign of the imaginary part is not at once obvious from the figure and will be indicated by a $-$ or $+$ symbol.

From these diagrams we see that all three functions are real and periodic on the lines $\operatorname{Im} u = 2n\mathbf{K}'$. The functions sn and cn have periods $4\mathbf{K}$ and oscillate between 1 and -1, the function dn has period $2\mathbf{K}$ and oscillates between 1 and k' on lines corresponding to even n, and between -1 and $-k'$ on lines corresponding to an odd value of n.

TABLE 7. CHANGE OF THE VARIABLE BY QUARTER- AND HALF-PERIODS. SYMMETRY.

$\mathrm{sn}\,(m\mathbf{K}+ni\mathbf{K}'\pm u)$

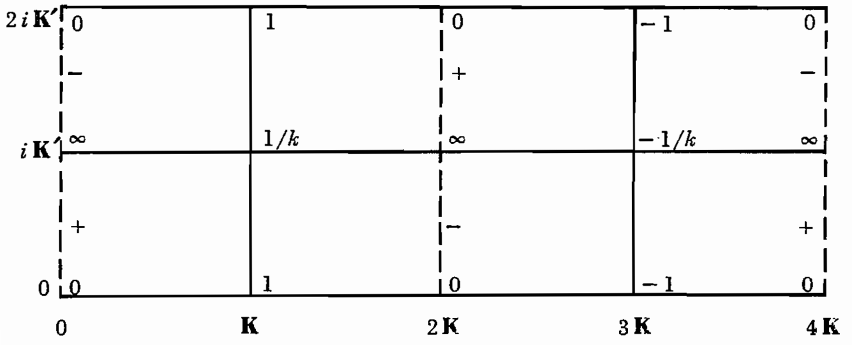
$ni\mathbf{K}'$ \ $m\mathbf{K}$	$-\mathbf{K}$	0	$\mathbf{K}$	$2\mathbf{K}$	$3\mathbf{K}$
$-i\mathbf{K}'$	$-d/(kc)$	$\pm 1/(ks)$	$d/(kc)$	$\mp 1/(ks)$	$-d/(kc)$
0	$-c/d$	$\pm s$	c/d	$\mp s$	$-c/d$
$i\mathbf{K}'$	$-d/(kc)$	$\pm 1/(ks)$	$d/(kc)$	$\mp 1/(ks)$	$-d/(kc)$
$2i\mathbf{K}'$	$-c/d$	$\pm s$	c/d	$\mp s$	$-c/d$

$\mathrm{cn}\,(m\mathbf{K}+ni\mathbf{K}'\pm u)$

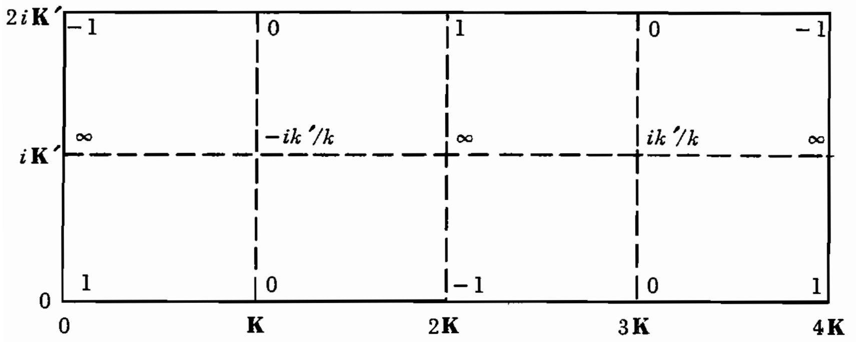
$ni\mathbf{K}'$ \ $m\mathbf{K}$	$-\mathbf{K}$	0	$\mathbf{K}$	$2\mathbf{K}$	$3\mathbf{K}$
$-i\mathbf{K}'$	$-ik'/(kc)$	$\pm id/(ks)$	$ik'/(kc)$	$\mp id/(ks)$	$-ik'/(kc)$
0	$\pm k's/d$	c	$\mp k's/d$	$-c$	$\pm k's/d$
$i\mathbf{K}'$	ik'/kc	$\mp id/(ks)$	$-ik'/(kc)$	$\pm id/(ks)$	$ik'/(kc)$
$2i\mathbf{K}'$	$\mp k's/d$	$-c$	$\pm k's/d$	c	$\mp k's/d$

$\mathrm{dn}\,(m\mathbf{K}+ni\mathbf{K}'\pm u)$

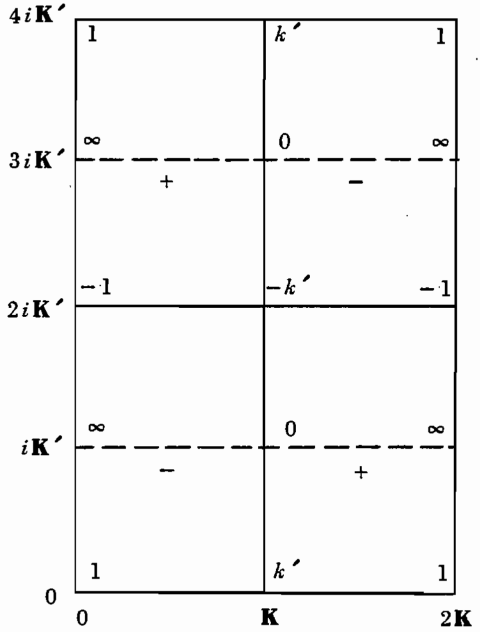
$ni\mathbf{K}'$ \ $m\mathbf{K}$	$-\mathbf{K}$	0	$\mathbf{K}$	$2\mathbf{K}$
$-i\mathbf{K}'$	$\mp ik's/c$	$\pm ic/s$	$\mp ik's/c$	$\pm ic/s$
0	k'/d	d	k'/d	d
$i\mathbf{K}'$	$\pm ik's/c$	$\mp ic/s$	$\pm ik's/c$	$\mp ic/s$
$2i\mathbf{K}'$	$-k'/d$	$-d$	$-k'/d$	$-d$
$3i\mathbf{K}'$	$\mp ik's/c$	$\pm ic/s$	$\mp ik's/c$	$\pm ic/s$



sn u for $0 \leq \text{Re}\, u \leq 4\mathbf{K}$, $\quad 0 \leq \text{Im}\, u \leq 2\mathbf{K}'$



cn u for $0 \leq \text{Re}\, u \leq 4\mathbf{K}$, $\quad 0 \leq \text{Im}\, u \leq 2\mathbf{K}'$



dn u for $0 \leq \mathrm{Re}\, u \leq 2\mathbf{K}$, $0 \leq \mathrm{Im}\, u \leq 4\mathbf{K}'$

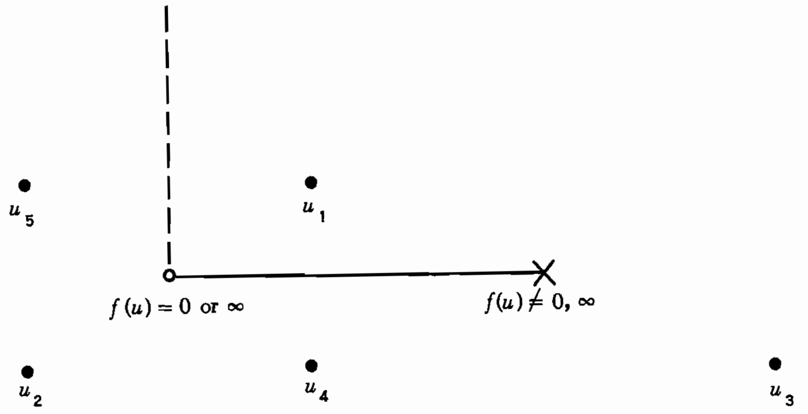
These diagrams may also be used to determine the signs of the real and imaginary parts of Jacobi's functions in any of the rectangles. Take, for instance, the rectangle whose vertices are $\mathbf{K}$, $2\mathbf{K}$, $2\mathbf{K} + i\mathbf{K}'$, $\mathbf{K} + i\mathbf{K}'$. From the diagrams we have on the boundary of this rectangle

$$\mathrm{Re\ sn}\, u \geq 0, \quad \mathrm{Im\ sn}\, u \leq 0;$$

$$\mathrm{Re\ cn}\, u \leq 0, \quad \mathrm{Im\ cn}\, u \leq 0;$$

$$\mathrm{Re\ dn}\, u \geq 0, \quad \mathrm{Im\ dn}\, u \geq 0;$$

and by the theory of conformal mappings these inequalities hold also in the interior.



SYMMETRIES OF JACOBIAN ELLIPTIC FUNCTIONS

The symmetries of Jacobi's functions may also be read off the diagrams. Let u_1 and u_2 lie symmetrically with respect to a zero or pole of one of Jacobi's functions, $f(u)$, say, let u_1 and u_3 be symmetric with respect to a lattice point which is neither a zero nor a pole, u_1 and u_4 symmetric with respect to a line on which $f(u)$ is real, and u_1 and u_5 symmetric with respect to a line on which $f(u)$ is imaginary. Then

$$f(u_1) = -f(u_2) = f(u_3) = \overline{f(u_4)} = -\overline{f(u_5)}.$$

We also note that

(1) $|\text{sn } u| = k^{-1/2}$ $\qquad$ $\text{Im } u = (n + \tfrac{1}{2})\mathbf{K}'$

(2) $|\text{dn } u| = k'^{1/2}$ $\qquad$ $\text{Re } u = (n + \tfrac{1}{2})\mathbf{K}.$

A rotation by a right angle carries the diagram of sn u essentially into the diagram of dn u; a rotation by a right angle does not change the diagram of cn u essentially.

A more complete description of the Jacobian elliptic functions for $0 < k < 1$ is contained in the relief diagrams given in Jahnke-Emde (1938, p. 92, 93).

The Jacobian elliptic functions *degenerate* if one or both of the periods become infinite, that is, if k^2 is 0, 1, or indefinite (the last case being trivial). As in the case of Weierstrass' functions (see. 13.15), we list three cases.

(i) Real period infinite.

(3) $k = 1, \quad k' = 0, \quad \mathbf{K} = \infty, \quad \mathbf{K}' = \frac{1}{2}\pi$

(4) $\operatorname{sn}(u,\ 1) = \tanh u, \quad \operatorname{cn}(u,\ 1) = \operatorname{dn}(u,\ 1) = \operatorname{sech} u.$

(ii) Imaginary period infinite.

(5) $k = 0, \quad k' = 1, \quad \mathbf{K} = \frac{1}{2}\pi, \quad \mathbf{K}' = \infty$

(6) $\operatorname{sn}(u,\ 0) = \sin u, \quad \operatorname{cn}(u,\ 0) = \cos u, \quad \operatorname{dn}(u,\ 0) = 1.$

(iii) Both periods infinite.

(7) $\mathbf{K} = \mathbf{K}' = \infty, \quad \operatorname{sn} u = 0, \quad \operatorname{cn} u = \operatorname{dn} u = 1.$

13.19. Theta functions

Although functions closely related to theta functions were encountered by Euler, Jakob Bernoulli, and Fourier, their systematic study and their exploitation for the theory of elliptic functions is due to Jacobi. Jacobi's theta functions correspond to the sigma functions of Weierstrass' theory. Like the sigma functions, theta functions are entire functions and hence certainly not doubly-periodic, yet such that they show a simple behavior under a translation by a period. Theta functions are more highly standardized than sigma functions. They are simply periodic, can be represented by series whose convergence is extraordinarily rapid, and they are the best means for the numerical computation of elliptic functions.

For Weierstrass' functions we had the variable z, the half-periods ω, ω', we put $\tau = \omega'/\omega$, and assumed $\operatorname{Im} \tau > 0$. Jacobi's functions were represented in terms of u, and the quarter-periods $\mathbf{K}$, $\mathbf{K}'$, where

(1) $u = (e_1 - e_3)^{\frac{1}{2}} z, \quad \mathbf{K} = (e_1 - e_3)^{\frac{1}{2}}\, \omega, \quad i\mathbf{K}' = (e_1 - e_3)^{\frac{1}{2}}\, \omega'.$

Theta functions will be expressed in terms of the variable

(2) $v = \dfrac{z}{2\omega} = \dfrac{u}{2\mathbf{K}},$

the parameter being either

(3) $\tau = \dfrac{\omega'}{\omega} = i\,\dfrac{\mathbf{K}'}{\mathbf{K}} \qquad \operatorname{Im} \tau > 0$

or

(4) $q = e^{i\pi\tau} = e^{i\pi\omega'/\omega} = \exp(-\pi \mathbf{K}'/\mathbf{K})$ $\qquad |q| < 1.$

The half-periods are $1, \tau$. Making use of 13.10(8), we may always achieve

(5) $|q| < \exp(-\frac{1}{2}\pi \cdot 3^{\frac{1}{2}})$,

but such a choice of the primitive periods will not be assumed in what follows.

The definition of the four theta functions is

(6) $$\theta_1(v) = \theta_1(v, q) = \theta_1(v|\tau) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin[(2n+1)\pi v]$$

(7) $$\theta_2(v) = \theta_2(v, q) = \theta_2(v|\tau) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} q^{n(n+1)} \cos[(2n+1)\pi v]$$

(8) $$\theta_3(v) = \theta_3(v, q) = \theta_3(v|\tau) = 1 + 2 \sum_{n=0}^{\infty} q^{n^2} \cos(2n\pi v)$$

(9) $$\theta_4(v) = \theta_4(v, q) = \theta_4(v|\tau) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2} \cos(2n\pi v).$$

The last of these functions is sometimes denoted by $\theta_0(v)$ or $\theta(v)$ simply. These series converge for all (complex) v and all q satisfying (4). On account of the factor q^{n^2} we have excellent convergence. The four series may be rewritten in the form

(10) $$\theta_1(v) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-\frac{1}{2})^2} e^{i\pi(2n-1)v}$$

(11) $$\theta_2(v) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2} e^{i\pi(2n-1)v}$$

(12) $$\theta_3(v) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{i\pi 2nv}$$

(13) $$\theta_4(v) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{i\pi 2nv},$$

when they appear as Laurent expansions in the variable $\exp(i\pi v)$, and are convergent for all finite non-zero values of this variable.

All four theta functions are entire functions of v. All four are periodic, the period of θ_1 and θ_2 being 2, and that of θ_3 and θ_4 being 1. Their behavior under the addition of half- and quarter-periods may be seen from Table 8 in which the abbreviations

TABLE 8. CHANGE OF VARIABLE BY QUARTER- AND HALF-PERIODS. SYMMETRY.

$\theta(v)$	$\theta(-v)$	$\theta(v+1)$	$\theta(v+\tau)$	$\theta(v+1+\tau)$	$\theta(v+\frac{1}{2})$	$\theta(v+\frac{1}{2}\tau)$	$\theta(v+\frac{1}{2}+\frac{1}{2}\tau)$
$\theta_1(v)$	$-\theta_1(v)$	$-\theta_1(v)$	$-A(v)\,\theta_1(v)$	$A(v)\,\theta_1(v)$	$\theta_2(v)$	$iB(v)\,\theta_4(v)$	$B(v)\,\theta_3(v)$
$\theta_2(v)$	$\theta_2(v)$	$-\theta_2(v)$	$A(v)\,\theta_2(v)$	$-A(v)\,\theta_2(v)$	$-\theta_1(v)$	$B(v)\,\theta_3(v)$	$-iB(v)\,\theta_4(v)$
$\theta_3(v)$	$\theta_3(v)$	$\theta_3(v)$	$A(v)\,\theta_3(v)$	$A(v)\,\theta_3(v)$	$\theta_4(v)$	$B(v)\,\theta_2(v)$	$iB(v)\,\theta_1(v)$
$\theta_4(v)$	$\theta_4(v)$	$\theta_4(v)$	$-A(v)\,\theta_4(v)$	$-A(v)\,\theta_4(v)$	$\theta_3(v)$	$iB(v)\,\theta_1(v)$	$B(v)\,\theta_2(v)$

(14) $A(v) = e^{-i\pi(2v+\tau)}, \quad B(v) = e^{-i\pi(v+\frac{1}{4}\tau)}$

have been used. Table 8 also shows the parity of the four theta functions.

Table 8 shows that all four theta functions may be generated by any one of them by the addition of quarter-periods. From the table, θ_1 has a zero at $v = 0$, and hence zeros at $m + n\tau$, where m,n are integers. It can be proved (by integrating θ_1'/θ_1 over the boundary of a parallelogram with vertices $\pm \frac{1}{2} \pm \frac{1}{2}\tau$) that these are the only zeros of θ_1; and Table 8 then may be used to determine the zeros of the other three theta functions. In Table 9, m and n are integers.

TABLE 9. ZEROS OF THETA FUNCTIONS

$\theta(v)$	$\theta_1(v)$	$\theta_2(v)$	$\theta_3(v)$	$\theta_4(v)$
zeros	$m + n\tau$	$m + \frac{1}{2} + n\tau$	$m + \frac{1}{2} + (n + \frac{1}{2})\tau$	$m + (n + \frac{1}{2})\tau$

From the knowledge of the zeros it is possible to obtain infinite products representing the theta functions, and from these products the partial fraction expansions of $\log \theta(v)$ and $\theta'(v)/\theta(v)$ follow. From (17) we also have (19). In the products we use the notation

(15) $$q_0 = \prod_{n=1}^{\infty} (1 - q^{2n})$$

and have

(16) $$\theta_1(v) = 2q_0 q^{\frac{1}{4}} \sin \pi v \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi v + q^{4n})$$

$$\theta_2(v) = 2q_0 q^{\frac{1}{4}} \cos \pi v \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2\pi v + q^{4n})$$

$$\theta_3(v) = q_0 \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2\pi v + q^{4n-2})$$

$$\theta_n(v) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi v + q^{4n-2}).$$

$$(17)\quad \log\left[\pi\frac{\theta_1'(0)}{\theta_1(v)}\right] = \log(\sin\pi v) + 4\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\,\frac{\sin^2 m\pi v}{m}$$

$$\log\left[\frac{\theta_2(v)}{\theta_2(0)}\right] = \log(\cos\pi v) + 4\sum_{m=1}^{\infty}(-1)^m\frac{q^{2m}}{1-q^{2m}}\,\frac{\sin^2 m\pi v}{m}$$

$$\log\left[\frac{\theta_3(v)}{\theta_3(0)}\right] = 4\sum_{m=1}^{\infty}(-1)^m\frac{q^{m}}{1-q^{2m}}\,\frac{\sin^2 m\pi v}{m}$$

$$\log\left[\frac{\theta_4(v)}{\theta_4(0)}\right] = 4\sum_{m=1}^{\infty}\frac{q^{m}}{1-q^{2m}}\,\frac{\sin^2 m\pi v}{m}$$

$$(18)\quad \frac{\theta_1'(v)}{\theta_1(v)} = \pi\,\mathrm{ctn}\,\pi v + 4\pi\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}\sin 2m\pi v$$

$$\frac{\theta_2'(v)}{\theta_2(v)} = -\pi\tan\pi v + 4\pi\sum_{m=1}^{\infty}(-1)^m\frac{q^{2m}}{1-q^{2m}}\sin 2m\pi v$$

$$\frac{\theta_3'(v)}{\theta_3(v)} = 4\pi\sum_{m=1}^{\infty}(-1)^m\frac{q^{m}}{1-q^{2m}}\sin 2m\pi v$$

$$\frac{\theta_4'(v)}{\theta_4(v)} = 4\pi\sum_{m=1}^{\infty}\frac{q^{m}}{1-q^{2m}}\sin 2m\pi v$$

$$(19)\quad \frac{1}{2}\log\left[\frac{\theta_1(v+w)}{\theta_1(v-w)}\right] = \frac{1}{2}\log\left[\frac{\sin\pi(v+w)}{\sin\pi(v-w)}\right] + 2\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{q^{2m}}{1-q^{2m}}\sin 2m\pi v\,\sin 2m\pi w,$$

$$\frac{1}{2}\log\left[\frac{\theta_2(v+w)}{\theta_2(v-w)}\right] = \frac{1}{2}\log\left[\frac{\cos\pi(v+w)}{\cos\pi(v-w)}\right] + 2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\,\frac{q^{2m}}{1-q^{2m}}\sin 2m\pi v\,\sin 2m\pi w,$$

$$(19)\quad \frac{1}{2}\log\left[\frac{\theta_3(v+w)}{\theta_3(v-w)}\right] = 2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\frac{q^m}{1-q^{2m}}\sin 2m\pi v\,\sin 2m\pi w,$$

$$\frac{1}{2}\log\left[\frac{\theta_4(v+w)}{\theta_4(v-w)}\right] = 2\sum_{m=1}^{\infty}\frac{1}{m}\frac{q^m}{1-q^{2m}}\sin 2m\pi v\,\sin 2m\pi w,$$

Equations (16) are valid in the entire v-plane. Of equations (17) and (18) those relating to θ_1 and θ_2 are valid in the strip $|\mathrm{Im}\,v| < \mathrm{Im}\,\tau$, those relating to θ_3 and θ_4 in the strip $|\mathrm{Im}\,v| < \frac{1}{2}\mathrm{Im}\,\tau$. Of equations (19), the first two are valid when $|\mathrm{Im}\,v| + |\mathrm{Im}\,w| < \mathrm{Im}\,\tau$, the last two are valid when $|\mathrm{Im}\,v| + |\mathrm{Im}\,w| < \frac{1}{2}\mathrm{Im}\,\tau$. From (18) we have

$$(20)\quad \frac{\theta_\alpha'(v+m+n\tau)}{\theta_\alpha(v+m+n\tau)} = \frac{\theta_\alpha'(v)}{\theta_\alpha(v)} - 2n\pi i \qquad \alpha = 1, 2, 3, 4; \qquad m, n \text{ integers.}$$

Between the squares of theta functions of the same variable there are the following relations

$$(21)\quad \theta_1^2(v)\,\theta_2^2(0) = \theta_4^2(v)\,\theta_3^2(0) - \theta_3^2(v)\,\theta_4^2(0)$$

$$\theta_1^2(v)\,\theta_3^2(0) = \theta_4^2(v)\,\theta_2^2(0) - \theta_2^2(v)\,\theta_4^2(0)$$

$$\theta_1^2(v)\,\theta_4^2(0) = \theta_3^2(v)\,\theta_2^2(0) - \theta_2^2(v)\,\theta_3^2(0)$$

$$\theta_4^2(v)\,\theta_4^2(0) = \theta_3^2(v)\,\theta_3^2(0) - \theta_2^2(v)\,\theta_2^2(0).$$

Each of these relations may be proved by remarking that the ratio of its two sides is a doubly periodic function (with periods 1 and τ) without zeros or poles and hence a constant, and evaluating this constant by using special values of v (half-periods).

Equations (21) are special cases of the so-called addition formulas of the theta functions which express

$$\theta_\alpha(v+w)\,\theta_\alpha(v-w)\,\theta_4^2(0)$$

in terms of squares of theta functions of v and w (see Whittaker and Watson, 1927, p. 487).

The "theta functions of zero argument"

$$\theta_1'(0), \quad \theta_2(0), \quad \theta_3(0), \quad \theta_4(0)$$

are of especial importance (see sec. 13.20). They satisfy several identities among which the most important are

$$(22)\quad \theta_1'(0) = \pi\,\theta_2(0)\,\theta_3(0)\,\theta_4(0)$$

(23) $\theta_2^4(0)+\theta_4^4(0)=\theta_3^4(0)$.

For graphs illustrating the behavior of the theta functions of argument zero, and for a description and graph of the behavior of $\theta_\alpha(v|0.1)$ for real v see Tricomi's book (1937, p. 137-140).

Theta functions arise, independently of the theory of elliptic functions, in the theory of heat conduction and similar boundary value problems. As is seen from (10)-(13), the functions $\theta_\alpha(\frac{1}{2}x|i\pi t)$, $\alpha = 1, 2, 3, 4$, satisfies the partial differential equation

$$(24)\quad \frac{\partial^2 y}{\partial x^2}=\frac{\partial y}{\partial t}.$$

In this connection it is worth noting that theta functions have remarkably simple Laplace transforms.

There are also non-linear differential equations of the first order (the variable is v) satisfied by quotients of theta functions. These can be derived very easily from the connection between elliptic functions and theta quotients (see sec. 13.20).

Hermite has studied the function

$$(25)\quad \Theta_{\mu,\nu}(v|\tau)=\sum_{n=-\infty}^{\infty}\exp\left[i\pi\tau(n+\tfrac{1}{2}\mu)^2+2i\pi v(n+\tfrac{1}{2}\mu)+i\pi n\nu\right]$$

(see Hurwitz and Courant, 1925, p. 198-201). Jacobi's four theta functions are particular cases of Hermite's function.

13.20. The expression of elliptic functions and elliptic integrals in terms of theta functions. The problem of inversion

Theta functions are very closely related to Weierstrass' sigma functions: hence the expression of Weierstrass' functions in terms of theta functions. Jacobi's functions have already been expressed in terms of Weierstrass' functions and may now be expressed in terms of theta functions. Lastly, theta functions may also be used to write down expressions for complete and incomplete elliptic integrals of the third kind. We shall use the variable z for Weierstrass' functions, u for Jacobian elliptic functions and v for theta functions. These are connected by 13.19(2). The connection between the various notations of periods and other quantities is given by equations 13.19(1) to (4).

Weierstrass' functions

$$(1)\quad \sigma(z)=2\omega\exp\left(\frac{\eta z^2}{2\omega}\right)\frac{\theta_1(v)}{\theta_1'(0)}$$

(2) $\sigma_\alpha(z) = \exp\left(\frac{\eta z^2}{2\omega}\right)\frac{\theta_{\alpha+1}(v)}{\theta_{\alpha+1}(0)}$ $\qquad \alpha = 1, 2, 3$

(3) $\zeta(z) = \frac{\eta}{\omega} z + \frac{1}{2\omega}\;\frac{\theta_1'(v)}{\theta_1(v)}$

(4) $\wp(z) = e_\alpha + \frac{1}{4\omega^2}\left[\frac{\theta_1'(0)}{\theta_{\alpha+1}(0)}\;\frac{\theta_{\alpha+1}(v)}{\theta_1(v)}\right]^2$ $\qquad \alpha = 1, 2, 3$

(5) $\wp'(z) = -\frac{1}{4\omega^3}\;\frac{\theta_2(v)\,\theta_3(v)\,\theta_4(v)\,\theta_1'^3(0)}{\theta_2(0)\,\theta_3(0)\,\theta_4(0)\,\theta_1^3(v)}$

(6) $12\,\omega^2 e_1 = \pi^2[\theta_3^4(0) + \theta_4^4(0)]$

$12\,\omega^2 e_2 = \pi^2[\theta_2^4(0) - \theta_4^4(0)]$

$12\,\omega^2 e_3 = -\pi^2[\theta_2^4(0) + \theta_3^4(0)]$

(7) $(e_2 - e_3)^{1/2} = i(e_3 - e_2)^{1/2} = \frac{\pi}{2\omega}\;\theta_2^2(0)$

$(e_1 - e_3)^{1/2} = i(e_3 - e_1)^{1/2} = \frac{\pi}{2\omega}\;\theta_3^2(0)$

$(e_1 - e_2)^{1/2} = i(e_2 - e_1)^{1/2} = \frac{\pi}{2\omega}\;\theta_4^2(0)$

(8) $g_2 = \frac{2}{3}\left(\frac{\pi}{2\omega}\right)^4[\theta_2^8(0) + \theta_3^8(0) + \theta_4^8(0)]$

$g_3 = \frac{4}{27}\left(\frac{\pi}{2\omega}\right)^6[\theta_2^4(0) + \theta_3^4(0)]\,[\theta_3^4(0) + \theta_4^4(0)][\theta_4^4(0) - \theta_2^4(0)]$

(9) $\Delta^{1/4} = \frac{\pi}{4\omega^3}\;\theta_1'^2(0) = \frac{\pi^3}{4\omega^3}\;[\theta_2(0)\,\theta_3(0)\,\theta_4(0)]^2.$

(10) $\eta = -\frac{1}{12\omega}\,\frac{\theta_1'''(0)}{\theta_1'(0)}, \qquad \eta' = -\frac{\pi i}{2\omega} - \frac{\tau}{12\omega}\,\frac{\theta_1'''(0)}{\theta_1'(0)}$

Equation (1) may be proved by remarking that the quotient of the functions on its two sides is a doubly-periodic function without poles or zeros, and approaches 1 as v and z tend to 0. Equation (2) follows by 13.12(22) and Table 8 of sec. 13.19. Equation (3) follows by logarithmic differentiation of (1), (4) from (2) and 13.13(22), (5) by (4) and 13.13(21), (6) and (7) from 13.13(23), (8) from 12.13(5) and (6), (9) from 13.13(7), (10) from (1) and (3). All of Weierstrass' functions are formed with periods 2ω, $2\omega'$, and variable z. The variables v and q in the theta functions are given by 13.19(2) and (4).

Jacobian elliptic functions. The following relations are obtained from the formulas of sec. 13.16 by means of equations (1) to (10).

(11) $k^{1/2} = \theta_2(0)/\theta_3(0), \quad k'^{1/2} = \theta_4(0)/\theta_3(0)$

(12) $\mathbf{K}^{1/2} = (\tfrac{1}{2}\pi)^{1/2}\, \theta_3(0), \quad \mathbf{K}'^{1/2} = (-\tfrac{1}{2}\tau i)^{1/2}\, \theta_3(0)$

(13) $$\operatorname{sn} u = \frac{\theta_3(0)\,\theta_1(v)}{\theta_2(0)\,\theta_4(v)}, \quad \operatorname{cn} u = \frac{\theta_4(0)\,\theta_2(v)}{\theta_2(0)\,\theta_4(v)}$$

$$\operatorname{dn} u = \frac{\theta_4(0)\,\theta_3(v)}{\theta_3(0)\,\theta_4(v)}, \quad Z(u) = E(u) - \frac{\mathbf{E}}{\mathbf{K}}\,u = \frac{1}{2\mathbf{K}}\,\frac{\theta_4'(v)}{\theta_4(v)}.$$

Given τ, equation (11) determines the modulus of the Jacobian elliptic functions, (12) the quarter-periods, and (13) the functions themselves. In applications of elliptic functions, usually k^2 is given and the question arises whether there always exists a q such that $|q| < 1$ and

(14) $$k^2 = \frac{\theta_2^4(0, q)}{\theta_3^4(0, q)} = 1 - \frac{\theta_4^4(0, q)}{\theta_3^4(0, q)}.$$

This is known as the *problem of inversion*. In many practical applications $0 < k^2 < 1$. In this case consider

$$\frac{\theta_4^4(0, q)}{\theta_3^4(0, q)} = \prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 + q^{2n+1}}$$

by 13.19(16). As q increases from 0 through real values to 1, the infinite product decreases monotonically from 1 to 0 and hence (14) has exactly one solution q for which $0 < q < 1$. For other values of k^2 the discussion is much more difficult (see for instance Whittaker and Watson, 1927, p. 480-483) and involves complex values of q. The proof of a unique system of Jacobian elliptic functions for any given $k^2 \neq 0, 1$ may be based on the theory of elliptic modular functions.

Elliptic integrals. The basic elliptic integrals in Legendre's normal form, 13.6(1)-(3), may be computed by means of theta functions. We form Jacobian elliptic functions with modulus k, determine the quarter-periods $\mathbf{K}$ and $\mathbf{K}'$, and put

$$(15) \quad v = \frac{F(\phi, k)}{2\mathbf{K}}, \qquad q = \exp(-\pi \mathbf{K}'/\mathbf{K})$$

for the parameter and variable of the theta functions. We then have from (13)

$$(16) \quad E(\phi, k) = \frac{1}{2\mathbf{K}} \frac{\theta_4'(v)}{\theta_4(v)} + 2\mathbf{E}v.$$

The computation of elliptic integrals of the third kind is more difficult. We shall give the results for real ϕ, ν, and $0 < k < 1$, shall express ν in terms of an auxiliary real parameter γ, different expressions being valid in the intervals $(-\infty, -1)$, $(-1, -k^2)$, $(-k^2, 0)$, $(0, \infty)$, use (15), and put

$$(17) \quad \beta = \frac{\gamma}{2\mathbf{K}} .$$

We then have (see Tricomi, 1937, p. 153-158)

$$(18) \quad \frac{\mathrm{cn}(\gamma, k)\,\mathrm{dn}(\gamma, k)}{\mathrm{sn}(\gamma, k)} \Pi\left[\phi, -\frac{1}{\mathrm{sn}^2(\gamma, k)}, k\right]$$

$$= \frac{1}{2} \log\left[\frac{\theta_1(v+\beta)}{\theta_1(v-\beta)}\right] - \frac{\theta_4'(\beta)}{\theta_4(\beta)} v \qquad 0 < \gamma < \mathbf{K}, \quad |v| > \beta$$

$$= \frac{1}{2} \log\left[\frac{\theta_1(\beta+v)}{\theta_1(\beta-v)}\right] - \frac{\theta_4'(\beta)}{\theta_4(\beta)} v \qquad 0 < \gamma < \mathbf{K}, \quad |v| < \beta$$

$$(19) \quad k'^2 \frac{\mathrm{sn}(\gamma, k')\,\mathrm{cn}(\gamma, k')}{\mathrm{dn}(\gamma, k')} \Pi[\phi, -\mathrm{dn}^2(\gamma, k'), k]$$

$$= -\frac{1}{2i} \log\left[\frac{\theta_2(v+i\beta)}{\theta_2(v-i\beta)}\right] - i\frac{\theta_3'(i\beta)}{\theta_3(i\beta)} v \qquad 0 < \gamma < \mathbf{K}'$$

(20) $$\frac{\operatorname{cn}(\gamma, k)\operatorname{dn}(\gamma, k)}{\operatorname{sn}(\gamma, k)}\Pi[\phi, -k^2\operatorname{sn}^2(\gamma, k), k]$$

$$= -\frac{1}{2}\log\left[\frac{\theta_4(v+\beta)}{\theta_4(v-\beta)}\right] + \frac{\theta_1'(\beta)}{\theta_1(\beta)}v \qquad 0 < \gamma < \mathbf{K}$$

(21) $$\frac{\operatorname{dn}(\gamma, k')}{\operatorname{sn}(\gamma, k')\operatorname{cn}(\gamma, k')}\Pi\left[\phi, k^2\frac{\operatorname{sn}^2(\gamma, k')}{\operatorname{cn}^2(\gamma, k')}, k\right]$$

$$= \frac{1}{2i}\log\left[\frac{\theta_4(v+i\beta)}{\theta_4(v-i\beta)}\right] + i\frac{\theta_1'(i\beta)}{\theta_1(i\beta)}v \qquad 0 < \gamma < \mathbf{K}'.$$

In all these formulas logarithms have their principal values. In (18) and (20) these are real, in (19) and (21), $-\pi \leq \operatorname{Im}\log[\cdots] \leq \pi$. The right-hand sides of (19) and (21) are real. From 13.19(18) and (19) we have

(22) $$i\frac{\theta_1'(i\beta)}{\theta_1(i\beta)} = \pi\operatorname{ctnh}\pi\beta - 4\pi\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sinh 2n\pi\beta$$

(23) $$i\frac{\theta_3'(i\beta)}{\theta_3(i\beta)} = 4\pi\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1-q^{2n}}\sinh 2n\pi\beta$$

(24) $$\frac{1}{2i}\log\left[\frac{\theta_2(v+i\beta)}{\theta_2(v-i\beta)}\right] = -\tan^{-1}(\tanh\pi\beta\cdot\tan\pi v)$$

$$+ 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^{2n}}{1-q^{2n}}\sin 2n\pi v\cdot\sinh 2n\pi\beta$$

(25) $$\frac{1}{2i}\log\left[\frac{\theta_4(v+i\beta)}{\theta_4(v-i\beta)}\right] = \sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^{2n}}\sin 2n\pi v\cdot\sinh 2n\pi\beta.$$

The convergence of the infinite series in (23) and (25) is not always as rapid as one would wish. When q is not small, the expansions

(26) $$\theta_4(v \pm i\beta) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2n\pi v)\cosh(2n\pi\beta)$$

$$\pm i\sum_{n=1}^{\infty}(-1)^{n-1}q^{n^2}\sin(2n\pi v)\sinh(2n\pi\beta)$$

(27) $\theta_3(i\beta) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cosh(2n\pi\beta)$

(28) $i\theta_3'(i\beta) = 4\pi \sum_{n=1}^{\infty} nq^{n^2} \sinh(2n\pi\beta)$

may be used for the computation of the right-hand sides of (19) and (21). These expansions, and some others which are useful in these computations, follow from 13.19(6) to (9).

Complete elliptic integrals of the first kind have already been expressed in terms of theta functions, see (12). For complete elliptic integrals of the second kind, we have from (6), (7), (10), and 13.16(10),

(29) $\mathbf{E} = \dfrac{\theta_3^4(0) + \theta_4^4(0)}{3\theta_3^4(0)} \mathbf{K} - \dfrac{1}{12\mathbf{K}} \dfrac{\theta_1'''(0)}{\theta_1'(0)}.$

Complete elliptic integrals of the third kind have already been reduced, in 13.8(21)-(24), to elliptic integrals of the first and second kinds and hence may be computed by means of theta functions.

Finally we mention that in applying theta functions to the computation of Jacobian elliptic functions or of elliptic integrals with a given modulus k, $0 < k < 1$, the parameter q of the theta functions may be computed from

(30) $q = \epsilon + 2\epsilon^5 + 15\epsilon^9 + 150\epsilon^{13} + \cdots \qquad 2\epsilon = (1 - k'^{1/2})/(1 + k'^{1/2}).$

13.21. The transformation theory of elliptic functions

The transformation theory of elliptic functions deals with the relations between elliptic functions belonging to different pairs of primitive periods. Since any elliptic functions of periods 2ω, $2\omega'$ may be expressed algebraically in terms of $\wp(z|\omega, \omega')$, it is sufficient to discuss relations between $\wp$-functions. We shall always assume

(1) $\operatorname{Im}(\omega'/\omega) > 0, \quad \operatorname{Im}(\dot\omega'/\dot\omega) > 0,$

and will summarize briefly the results of the general transformation theory, referring for proofs and fuller details to the books listed at the end of this chapter.

We shall say that two functions $f(z)$ and $g(z)$ are *algebraically connected* if there is a polynomial in two variables, $P(x, y)$, such that $P[f(z), g(z)] = 0$ identically in z.

A necessary and sufficient condition for $\wp(u|\omega, \omega')$ *and* $\wp(u|\dot\omega, \dot\omega')$ *to be algebraically connected is the existence of integers* α, β, γ, δ, ρ *such that*

(2) $\rho\dot{\omega} = \alpha\omega + \beta\omega', \quad \rho\dot{\omega}' = \gamma\omega + \delta\omega', \quad D = \alpha\delta - \beta\gamma > 0.$

Given (2), clearly both $\wp(u|\omega, \omega')$ and $\wp(u|\dot{\omega}, \dot{\omega}')$ are even elliptic functions of periods $\rho\omega$, $\rho\omega'$, and hence they are rational functions of $\wp(u|\rho\omega, \rho\omega')$. Thus, it is sufficient to envisage substitutions (2) with $\rho = 1$, and these we shall write in matrix notation as

(3) $$\begin{bmatrix} \dot{\omega} \\ \dot{\omega}' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \omega \\ \omega' \end{bmatrix}, \quad D = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} > 0.$$

Then the relation between

(4) $x = \wp(z|\omega, \omega'), \quad y = \wp(z|\dot{\omega}, \dot{\omega}')$

is of the form

(5) $P(x, y) = 0$

where P is a polynomial in x and y, linear in x, and of degree D in y. (The degree in y is elucidated by counting poles.) We call D the *degree* or *order* of the transformation

(6) $$T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

and shall multiply transformations as matrices,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{bmatrix}.$$

The transformations (3) may also be envisaged as Moebius transformations of the upper half of the complex plane onto itself,

(7) $$\dot{\tau} = \frac{\gamma + \delta\tau}{\alpha + \beta\tau}.$$

All transformations of the first order form a group (the modular group). *A necessary and sufficient condition for* $\wp(u|\omega, \omega') = \wp(u|\dot{\omega}, \dot{\omega}')$ *is that* ω, ω' *and* $\dot{\omega}$, $\dot{\omega}'$ *be connected by a transformation of the first order* (unimodular transformation).

The modular group is generated by the transformations

(8) $$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

that is, any unimodular transformation is a product of powers of A and B. Thus, the study of transformations of the first order may be limited to the study of A and B.

Similarly, the study of *transformations of the second order* may be limited to *Landen's transformation*

$$(9) \quad L = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

since *any transformation of the second order, S, may be decomposed as $S = HLK$, where H and K are unimodular transformations.*

13.22. Transformations of the first order

A transformation of the first order leaves the point-lattice Ω, of all periods (sec. 13.10) unchanged. Since Weierstrass' functions $\sigma(z)$, $\zeta(z)$, $\wp(z)$, and the invariants g_2, g_3, $\Delta = g_2^3 - 27 g_3^2$ depend only on Ω, they do not change. The e_α may undergo a permutation. From 13.12(19) and 13.13(19)

$$\dot{\eta} = \zeta(\dot{\omega}|\dot{\omega}, \dot{\omega}') = \zeta(\dot{\omega}|\omega, \omega') = \zeta(a\omega + \beta\omega'|\omega, \omega') = a\eta + \beta\eta'$$

$$\dot{\eta}' = \gamma\eta + \delta\eta'$$

so that η, η' undergo the same transformation as ω, ω'. The functions $\sigma_\alpha(z)$ may undergo a permutation. A straightforward computation shows that A of equation 13.21(8) interchanges the indices 2 and 3, and B the indices 1 and 3, in e_1, e_2, e_3 and $\sigma_1(z)$, $\sigma_2(z)\,\sigma_3(z)$.

The behavior of Jacobian elliptic functions under unimodular transformations is more involved. If a and δ are odd integers, and β and γ even integers, in 13.21(6), we call T a λ-transformation. It is easy to verify that all λ-transformations form a *subgroup* of the modular group, and this subgroup is called the λ-group. For a λ-transformation,

$$\dot{e}_1 = \wp(\dot{\omega}|\dot{\omega}, \dot{\omega}') = \wp(a\omega + \beta\omega'|\omega, \omega') = \wp(\omega) = e_1,$$

since $\beta\omega'$ is a period for even β, and $a\omega$ differs from ω by a period for odd a. Similarly $\dot{e}_2 = e_2$ and $\dot{e}_3 = e_3$. From 13.16(4)-(6) it is seen that *Jacobi's functions* sn, cn, dn *are invariant under λ-transformations*. Any other unimodular transformation affects Jacobi's elliptic functions.

We shall consider the five transformations

$$(1) \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

The last three may be expressed in terms of A and B.

(2) $C = ABA, \quad D = ABAB, \quad E = BABA.$

In Table 10 the six transformations U (identity), A, ... , E are listed together with the permutations of e_α effected by them. Every permutation of e_1, e_2, e_3 occurs. Since the permutation of the e_α completely determines the transformations of Jacobian elliptic functions, it is sufficient to consider the transformations (1) in order to obtain all possible transformations of the first order of Jacobi's elliptic functions.

TABLE 10. PERMUTATIONS OF THE e_α

Transformation	$\dot{\omega}$	$\dot{\omega}'$	$\dot{e}_1$	$\dot{e}_2$	$\dot{e}_3$
U	ω	ω'	e_1	e_2	e_3
A	ω	$\omega+\omega'$	e_1	e_3	e_2
B	ω'	$-\omega$	e_3	e_2	e_1
C	$\omega+\omega'$	ω'	e_2	e_1	e_3
D	$-\omega+\omega'$	$-\omega$	e_2	e_3	e_1
E	ω'	$-\omega-\omega'$	e_3	e_1	e_2

This table in combination with 13.16(4), (5), (6), (9), and (11) at once leads to the transformation formulas recorded in Table 11.

For the transformation of elliptic integrals see Table 3, sec. 13.7, and Table 4, sec. 13.8.

The transformations of the four theta functions may be derived from the expression

$$(3) \quad \theta_1(v|\tau) = \frac{\omega^{1/2}\,\Delta^{1/8}}{\pi^{1/2}} \exp\left(-\frac{\eta z^2}{2\omega}\right)\sigma(z) \qquad v = \frac{z}{2\omega}, \quad \tau = \frac{\omega'}{\omega}$$

which follows from 13.20(1), (9), and 13.19(2), (3). We know already how the right-hand side behaves under a transformation 13.21(6) of the first order and note in particular that

$$\frac{\eta}{\omega} - \frac{\dot{\eta}}{\dot{\omega}} = \frac{\eta}{\omega} - \frac{\alpha\eta + \beta\eta'}{\alpha\omega + \beta\omega'} = \frac{\beta(\eta\omega' - \eta'\omega)}{\omega\dot{\omega}} = \frac{\beta\pi i}{2\omega\dot{\omega}}$$

TABLE 11. TRANSFORMATIONS OF THE FIRST ORDER OF JACOBI'S ELLIPTIC FUNCTIONS

Transformation	$\dot{\omega}$ $\dot{\omega}'$	$\dot{u}$	$\dot{k}$	$\dot{k}'$	$\dot{\mathbf{K}}$	$\dot{\mathbf{K}}'$	$\mathrm{sn}(\dot{u}, \dot{k})$	$\mathrm{cn}(\dot{u}, \dot{k})$	$\mathrm{dn}(\dot{u}, \dot{k})$
A	ω $\omega + \omega'$	$k'u$	$\frac{ik}{k'}$	$\frac{1}{k'}$	$k'\mathbf{K}$	$k'(\mathbf{K}' - i\mathbf{K})$	$k'\frac{\mathrm{sn}(u, k)}{\mathrm{dn}(u, k)}$	$\frac{\mathrm{cn}(u, k)}{\mathrm{dn}(u, k)}$	$\frac{1}{\mathrm{dn}(u, k)}$
B	ω' $-\omega$	$-iu$	k'	k	$\mathbf{K}'$	$\mathbf{K}$	$-i\frac{\mathrm{sn}(u, k)}{\mathrm{cn}(u, k)}$	$\frac{1}{\mathrm{cn}(u, k)}$	$\frac{\mathrm{dn}(u, k)}{\mathrm{cn}(u, k)}$
C	$\omega + \omega'$ ω'	ku	$\frac{1}{k}$	$\frac{k'}{ik}$	$k(\mathbf{K} + i\mathbf{K}')$	$k\mathbf{K}'$	$k\,\mathrm{sn}(u, k)$	$\mathrm{dn}(u, k)$	$\mathrm{cn}(u, k)$
D	$-\omega + \omega'$ $-\omega$	$-ik'u$	$\frac{1}{k'}$	$\frac{k}{ik'}$	$k'(\mathbf{K}' + i\mathbf{K})$	$k'\mathbf{K}$	$-ik'\frac{\mathrm{sn}(u, k)}{\mathrm{cn}(u, k)}$	$\frac{\mathrm{dn}(u, k)}{\mathrm{cn}(u, k)}$	$\frac{1}{\mathrm{cn}(u, k)}$
E	ω' $-(\omega + \omega')$	$-iku$	$\frac{k'}{ik}$	$\frac{1}{k}$	$k\mathbf{K}'$	$k(\mathbf{K} + i\mathbf{K}')$	$-ik\frac{\mathrm{sn}(u, k)}{\mathrm{dn}(u, k)}$	$\frac{1}{\mathrm{dn}(u, k)}$	$\frac{\mathrm{cn}(u, k)}{\mathrm{dn}(u, k)}$

by 13.12(10), and also that

$$\dot{v} = \frac{z}{2\dot{\omega}} = \frac{z}{2(a\omega + \beta\omega')} = \frac{v}{a+\beta\tau}, \qquad \dot{\tau} = \frac{\gamma + \delta\tau}{a+\beta\tau}.$$

Then we have, from (3), the *general transformation formula of $\theta_1(v|\tau)$ for transformations of the first order,*

(4) $$\theta_1\left(\frac{v}{a+\beta\tau}\,\middle|\,\frac{\gamma+\delta\tau}{a+\beta\tau}\right) = \epsilon(a+\beta\tau)^{\frac{1}{2}} \exp\left(\frac{i\pi\beta v^2}{a+\beta\tau}\right)\theta_1(v|\tau),$$

where $\epsilon^8 = 1$. The factor ϵ accounts for the ambiguity in the fractional powers in (3) and may be determined by dividing (4) by v, making $v \to 0$, and then comparing both sides. The transformations of the other three theta functions then follow from Table 8, sec. 13.19.

The explicit formulas for the transformations A and B of (1), which generate the modular group, are as follows.

Transformation A.

(5) $$\dot{v} = v, \quad \dot{\tau} = 1 + \tau, \quad \dot{q} = -q$$

(6) $$\theta_1(v|\tau+1) = e^{\frac{1}{4}\pi i}\,\theta_1(v|\tau), \qquad \theta_2(v|\tau+1) = e^{\frac{1}{4}\pi i}\,\theta_2(v|\tau)$$

$$\theta_3(v|\tau+1) = \theta_4(v|\tau), \qquad \theta_4(v|\tau+1) = \theta_3(v|\tau).$$

Transformation B.

(7) $$\dot{v} = v/\tau, \quad \dot{\tau} = -1/\tau, \quad \log\dot{q} = \pi^2/\log q$$

(8) $$\theta_1\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = -i(-i\tau)^{\frac{1}{2}}\exp(i\pi v^2/\tau)\,\theta_1(v|\tau)$$

$$\theta_2\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\exp(i\pi v^2/\tau)\,\theta_4(v|\tau)$$

$$\theta_3\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\exp(i\pi v^2/\tau)\,\theta_3(v|\tau)$$

$$\theta_4\left(\frac{v}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\exp(i\pi v^2/\tau)\,\theta_2(v|\tau).$$

In these formulas $(-i\tau)^{\frac{1}{2}}$ has its principal value (lies in the right half-plane). Transformation B is known as *Jacobi's imaginary transformation.*

Transformation B may be used for the numerical computation of theta functions when q is near 1, or τ is very small, when the series for $\theta_1(v|\tau)$ converge somewhat slowly, but those for $\theta_1(v/\tau|-1/\tau)$ converge very rapidly. In particular, the asymptotic behavior as $q \to 1$ may be investigated in this way, and one obtains

(9) $\theta_2(0, q) \sim \theta_3(0, q) \sim (-\pi/\log q)^{1/2} \qquad q \to 1.$

13.23. Transformations of the second order

There is essentially only one transformation of the second order, in the sense that any transformation of the second order is a combination of *Landen's transformation*

(1) $L = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

and two unimodular transformations. In writing down the transformation formulas we shall observe the following convention. All Weierstrassian functions whose periods are not indicated are formed with primitive periods ω, ω', and all e_α, η_α (without dots) are derived from such functions.

Landen's transformation. Weierstrass' functions.

(2) $\dot{\omega} = \frac{1}{2}\omega, \quad \dot{\omega}' = \omega'$

(3) $\dot{e}_1 = e_1 + 2(e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}$

$\dot{e}_2 = e_1 - 2(e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}$

$\dot{e}_3 = -2e_1$

(4) $\dot{\eta}_1 = \eta_1 + \frac{1}{2}e_1\omega_1, \quad \dot{\eta}_2 = \eta_2 - \eta_3 + \frac{1}{2}e_1(\omega_2 - \omega_3)$

$\dot{\eta}_3 = 2\eta_3 + e_1\omega_3.$

(5) $\sigma(z|\frac{1}{2}\omega, \omega') = \exp(\frac{1}{2}e_1 z^2)\,\sigma(z)\,\sigma_1(z)$

$\sigma_1(z|\frac{1}{2}\omega, \omega') = \exp(\frac{1}{2}e_1 z^2)\,[\sigma_1^2(z) - (e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}\sigma^2(z)]$

$\sigma_2(z|\frac{1}{2}\omega, \omega') = \exp(\frac{1}{2}e_1 z^2)\,[\sigma_1^2(z) + (e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}\sigma^2(z)]$

$\sigma_3(z|\frac{1}{2}\omega, \omega') = \exp(\frac{1}{2}e_1 z^2)\,\sigma_2(z)\,\sigma_3(z)$

(6) $\zeta(z|\frac{1}{2}\omega, \omega') = \zeta(z) + \zeta(z + \omega) + e_1 z - \eta_1$

(7) $$\wp(z|\tfrac{1}{2}\omega, \omega') = \wp(z) + \wp(z-\omega_1) - e_1$$
$$= \wp(z) + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp(z) - e_1}.$$

Since Landen's transformation of Weierstrass' functions involves e_α, η_α, which are not invariant under unimodular transformations, we record the basic formulas for two other transformations of the second order.

Gauss' transformation. Weierstrass' $\wp$-function.

(8) $$G = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = -BLB$$

(9) $$\wp(z|\omega, \tfrac{1}{2}\omega') = \wp(z) + \wp(z-\omega_3) - e_3$$
$$= \wp(z) + \frac{(e_1 - e_3)(e_2 - e_3)}{\wp(z) - e_3}$$

The irrational transformation. Weierstrass' $\wp$-function

(10) $$I = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = -ABLABAB$$

(11) $$\wp(z|\omega, \tfrac{1}{2}\omega + \tfrac{1}{2}\omega') = \wp(z) + \wp(z-\omega_2) - e_2$$
$$= \wp(z) - \frac{(e_1 - e_2)(e_2 - e_3)}{\wp(z) - e_2}.$$

Landen's transformation. Jacobian elliptic and theta functions. When the parameter in a theta function is not indicated, it is understood to be τ.

(12) $$\dot{u} = (1+k')u, \quad \dot{k} = (1-k')/(1+k'), \quad \dot{k}' = 2k'^{1/2}/(1+k')$$

(13) $$\operatorname{sn}\left[(1+k')u, \frac{1-k'}{1+k'}\right] = (1+k')\frac{\operatorname{sn}(u,k)\operatorname{cn}(u,k)}{\operatorname{dn}(u,k)}$$

$$\operatorname{cn}\left[(1+k')u, \frac{1-k'}{1+k'}\right] = \frac{1-(1+k')\operatorname{sn}^2(u,k)}{\operatorname{dn}(u,k)}$$

$$\operatorname{dn}\left[(1+k')u, \frac{1-k'}{1+k'}\right] = \frac{1-(1-k')\operatorname{sn}^2(u,k)}{\operatorname{dn}(u,k)}$$

(14) $$\dot{v} = 2v, \quad \dot{\tau} = 2\tau, \quad \dot{q} = q^2$$

(15) $\theta_1'(0|2\tau) = \frac{1}{2}\pi^{\frac{1}{2}}\,[\theta_2^3(0)\,\theta_1'(0)]^{\frac{1}{2}} = \dfrac{1}{2}\,\dfrac{\theta_2(0)\,\theta_1'(0)}{[\theta_3(0)\,\theta_4(0)]^{\frac{1}{2}}}$

$$\theta_2(0|2\tau) = 2^{-\frac{1}{2}}\,[\theta_3^2(0) - \theta_4^2(0)]^{\frac{1}{2}}$$

$$\theta_3(0|2\tau) = 2^{-\frac{1}{2}}\,[\theta_3^2(0) + \theta_4^2(0)]^{\frac{1}{2}}$$

$$\theta_4(0|2\tau) = [\theta_3(0)\,\theta_4(0)]^{\frac{1}{2}}$$

(16) $\theta_1(2v|2\tau) = \dfrac{\theta_1(v)\,\theta_2(v)}{\theta_4(0|2\tau)}$

$$\theta_2(2v|2\tau) = \frac{\theta_2^2(v) - \theta_1^2(v)}{2\,\theta_3(0|2\tau)} = \frac{\theta_3^2(v) - \theta_4^2(v)}{2\theta_2(0|2\tau)}$$

$$\theta_3(2v|2\tau) = \frac{\theta_2^2(v) + \theta_1^2(v)}{2\theta_2(0|2\tau)} = \frac{\theta_3^2(v) + \theta_4^2(v)}{2\theta_3(0|2\tau)}$$

$$\theta_4(2v|2\tau) = \frac{\theta_3(v)\,\theta_4(v)}{\theta_4(0|2\tau)}.$$

Gauss' transformation. Jacobian elliptic functions

(17) $\dot{u} = (1+k)\,u, \quad \dot{k} = 2k^{\frac{1}{2}}/(1+k), \quad \dot{k}' = (1-k)/(1+k)$

(18) $\operatorname{sn}\left[(1+k)\,u, \dfrac{2k^{\frac{1}{2}}}{1+k}\right] = (1+k)\,\dfrac{\operatorname{sn}(u,k)}{1+k\ \operatorname{sn}^2(u,k)}$

$$\operatorname{cn}\left[(1+k)\,u, \frac{2k^{\frac{1}{2}}}{1+k}\right] = \frac{\operatorname{cn}(u,k)\,\operatorname{dn}(u,k)}{1+k\ \operatorname{sn}^2(u,k)}$$

$$\operatorname{dn}\left[(1+k)\,u, \frac{2k^{\frac{1}{2}}}{1+k}\right] = \frac{1-k\ \operatorname{sn}^2(u,k)}{1+k\ \operatorname{sn}^2(u,k)}.$$

Transformations of higher orders are more involved. We mention here only the transformation $(LB)^2$ which is of the fourth order and leads to the following *duplication formulas for the theta functions*. All theta functions have the same parameter τ.

$$(19)\quad \theta_1(2v) = 2\,\frac{\theta_1(v)\,\theta_2(v)\,\theta_3(v)\,\theta_4(v)}{\theta_2(0)\,\theta_3(0)\,\theta_4(0)}$$

$$\theta_2(2v) = \frac{\theta_2^2(v)\,\theta_3^2(v) - \theta_1^2(v)\,\theta_4^2(v)}{\theta_2(0)\,\theta_3^2(0)}$$

$$\theta_3(2v) = \frac{\theta_2^2(v)\,\theta_3^2(v) + \theta_1^2(v)\,\theta_4^2(v)}{\theta_2^2(0)\,\theta_3(0)}$$

$$\theta_4(2v) = \frac{\theta_3^4(v) - \theta_2^4(v)}{\theta_4^3(0)} = \frac{\theta_4^4(v) - \theta_1^4(v)}{\theta_4^3(0)}.$$

13.24. Elliptic modular functions

An *elliptic modular function*, $f(\tau)$, is a function which is regular save for poles, when $\operatorname{Im}\tau > 0$, and has the property that $f(\tau)$ and $f(\dot{\tau})$ are *algebraically connected* whenever τ and $\dot{\tau}$ are connected by a transformation of the *modular group*

$$(1)\quad \dot{\tau} = \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \qquad \alpha,\ \beta,\ \gamma,\ \delta \text{ integers}, \quad \alpha\delta - \beta\gamma = 1.$$

[Note that $\alpha, \ldots, \gamma$ have been renamed as against 13.21 (7).] If $f(\tau) = f(\dot{\tau})$ for any transformation of the modular group, then $f(\tau)$ is called an *automorphic function of the modular group*.

A first example of such a modular function is the square of the modulus of the Jacobian elliptic functions. From 13.16(7) and 13.20(14)

$$(2)\quad k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} = \lambda(\tau),$$

is an analytic function of τ for $\operatorname{Im}\tau > 0$, with the real τ-axis as a natural boundary. From the invariance of e_1, e_2, e_3 under λ-transformations (α, δ odd, β, γ even, see sec. 13.22) it follows that $\lambda(\tau)$ *is an automorphic function of the* λ*-group*. In general, a transformation of the modular group will permute the e_α and hence change $\lambda(\tau)$ into one of the six values

$$(3)\quad \lambda(\tau), \quad 1 - \lambda(\tau), \quad \frac{1}{\lambda(\tau)}, \quad \frac{1}{1 - \lambda(\tau)}, \quad \frac{\lambda(\tau)}{\lambda(\tau) - 1}, \quad 1 - \frac{1}{\lambda(\tau)}.$$

Since all these are algebraically connected with $\lambda(\tau)$, *this function is an elliptic modular function.*

From 13.12(13), g_2, g_3, and $\Delta = g_2^3 - 27 g_3^2$ are homogeneous functions of degree -4, -6, -12 respectively in ω and ω' and the *absolute invariant*

$$(4) \quad \frac{g_2^3}{\Delta} = \frac{g_2^3}{g_2^3 - 27 g_3^2} = J(\tau)$$

is a function of τ alone: it is analytic in the upper half-plane. A transformation of the modular group leaves g_2 and Δ unchanged (see sec. 13.22), showing that *$J(\tau)$ is an automorphic function of the modular group.* From 13.13(6), (7) and 13.16(3), J may be expressed in terms of λ, and from 13.20(8), (9) in terms of theta functions

$$(5) \quad J(\tau) = \frac{4}{27}\,\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2} = \frac{1}{54}\,\frac{[\theta_2^8(0|\tau)+\theta_3^8(0|\tau)+\theta_4^8(0|\tau)]^3}{\theta_2^8(0|\tau)\,\theta_3^8(0|\tau)\,\theta_4^8(0|\tau)}.$$

We call two points τ, τ' in the upper half of the complex τ-plane *equivalent* if they are connected by a transformation (1) of the modular group. The *fundamental region* of the modular group is defined by

$$|\tau| \geq 1, \quad |\tau+1| > |\tau|, \quad |\tau-1| \geq |\tau|.$$

The upper τ half-plane may be subdivided into an infinity of regions, each bounded by three circular arcs (one or two of which may degenerate into segments of straight lines), and each equivalent to the fundamental region. In fact every point in the upper half-plane is equivalent to exactly one point of the fundamental region.

Given an automorphic function of the modular group, it is sufficient to investigate the behavior of this function in the fundamental region. For instance, it may be proved that $J(\tau)$ assumes every finite value exactly once in the fundamental region, and this shows that to every (finite) value of J there is exactly one system of Weierstrassian functions.

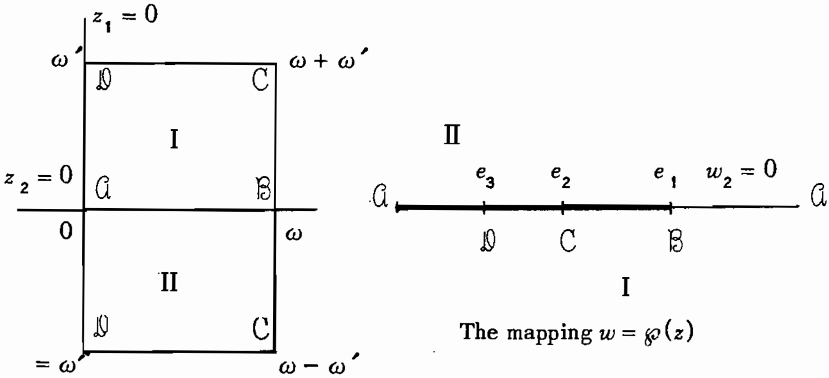
The fundamental region of the λ-group is bounded by the straight lines $\operatorname{Re}\tau = \pm 1$ and the circles $|2\tau \pm 1| = 1$; the boundary points in $\operatorname{Re}\tau \geq 0$ belong to the region, the boundary points for which $\operatorname{Re}\tau < 0$ do not. It may be proved that $\lambda(\tau)$ assumes every finite value different from zero and unity exactly once in the fundamental region of the λ-group, and this is the key to the problem of inversion (sec. 13.20): it may be used to prove that the Jacobian elliptic functions are uniquely determined when the square of the modulus is assigned as any number $\neq 0, 1$.

13.25. Conformal mappings

Elliptic integrals, elliptic functions and related functions occur in many important conformal mappings. Many examples of such conformal mappings, and some further references, are to be found in H. Kober's "Dictionary of conformal representations" (1952, p. 170-200). In this section we shall describe some of the most important mappings briefly. Throughout the section we assume the "real" case,

$$0 < k < 1, \quad 0 < q < 1, \quad \omega \text{ real}, \quad \omega' \text{imaginary}, \quad \mathbf{K}, \mathbf{K}'\text{real},$$

and put $e_1 > e_2 > e_3$. We put $\operatorname{Re} z = z_1$, $\operatorname{Im} z = z_2$, and similarly for other complex variables. In diagrams illustrating conformal mappings from the plane of one complex variable to the plane of another such variable, corresponding points will be indicated by the same letter.

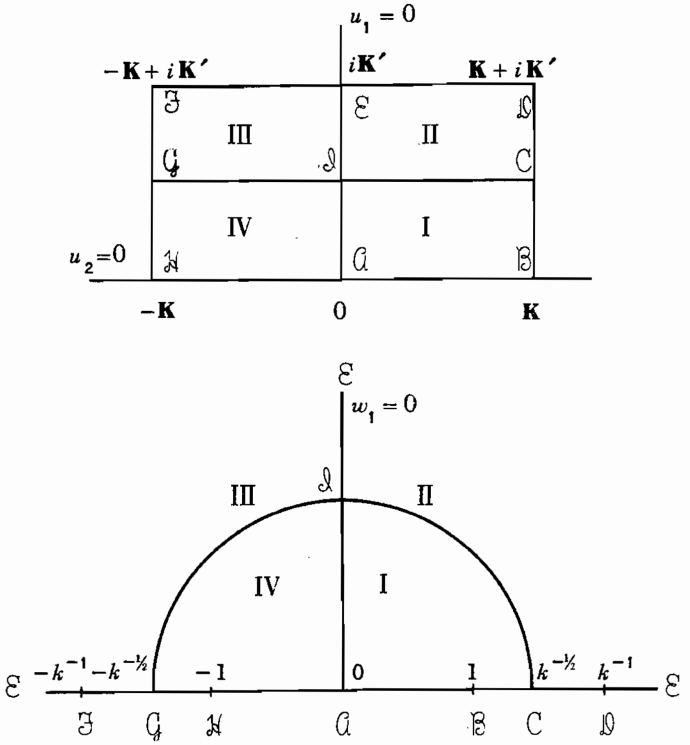


The mapping $w = \wp(z)$

The function $w = \wp(z)$. As z describes the boundary of the rectangle with vertices 0, ω, $\omega + \omega'$, ω', the variable w is real and decreases from ∞ to e_1, e_2, e_3, $-\infty$ (see sec. 13.15). The function maps the *interior* of the rectangle on the lower w half-plane. By Schwarz's reflection principle, the rectangle with vertices $-\omega'$, $\omega - \omega'$, $\omega + \omega'$, ω' in the z-plane is mapped on the whole w plane cut from $-\infty$ to e_1.

In the *lemniscatic* case, $g_3 = 0$, $g_2 > 0$, we have $e_2 = 0$, $e_3 = -e_1$. The rectangle in the z-plane becomes a *square*, the diagonal 𝔄ℭ joining 0 and $\omega + \omega'$ is mapped on the negative imaginary axis in the w-plane, and the diagonal 𝔅𝔇 joining ω and ω' is mapped on the lower half of the circle with center at $e_2 = 0$ and radius e_1, in the w-plane. The interior of the rectangular isosceles triangle with vertices $\frac{1}{2}\omega + \frac{1}{2}\omega'$, ω, $\omega' + \omega$ in the z-plane is mapped on the fourth quadrant of the circle with radius e_1 in the w-plane.

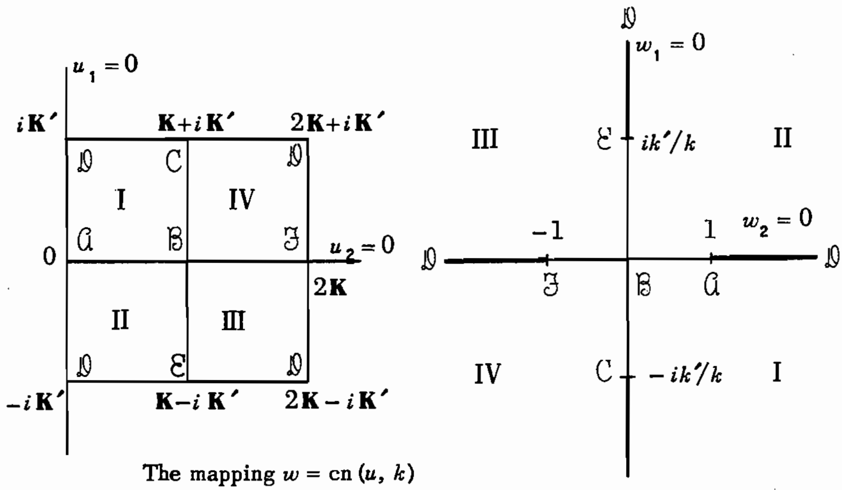
The function $w = \mathrm{sn}\,(u, k)$. From sec. 13.18 it is seen that the interior of the rectangle with vertices 0, $\mathbf{K}$, $\mathbf{K} + i\mathbf{K}'$, $i\mathbf{K}'$ in the u-plane is mapped on the first quadrant of the w-plane, the rectangle $-\mathbf{K}$, $\mathbf{K}$, $\mathbf{K} + i\mathbf{K}'$,



The mapping $w = \mathrm{sn}\,(u, k)$

$-\mathbf{K} + i\mathbf{K}'$ is mapped on the upper half of the w-plane, and the rectangle with vertices $\pm\mathbf{K} \pm i\mathbf{K}'$ is mapped on the whole w-plane cut from $-\infty$ to -1 and from 1 to ∞. It can be proved (see for instance Dixon, 1894, Appendix A) that the lines $u_1 =$ const., $u_2 =$ const., are mapped on the doubly orthogonal system of confocal bicircular quartics in the w-plane whose foci are ± 1, $\pm k^{-1}$. These quartics are symmetric with respect to both the w_1 and w_2 axes. The quartics corresponding to $u_1 =$ const. have two branches, one, encircling 𝔅𝔇, corresponding to $u_1 > 0$, the other, encircling 𝔉ℌ, to $u_1 < 0$. The quartics corresponding to $u_2 = 0$ are ovals encircling ℌ𝔅. In particular, for $u_2 = (n + \tfrac{1}{2})\,\mathbf{K}'$, we have a circle, see 13.18 (1). See the figure for further details.

The function $w = \text{cn}(u, k)$. The interior of the rectangle with vertices 0, $\mathbf{K}$, $\mathbf{K} + i\mathbf{K}'$, $i\mathbf{K}'$ in the u-plane is mapped on the fourth quadrant of the w-plane, the rectangle $-\mathbf{K}$, $\mathbf{K}$, $\mathbf{K} + i\mathbf{K}'$, $-\mathbf{K} + i\mathbf{K}'$ is mapped on the right w half-plane cut from 0 to 1, the rectangle $-i\mathbf{K}'$, $\mathbf{K} - i\mathbf{K}'$, $\mathbf{K} + i\mathbf{K}'$, $i\mathbf{K}'$ is mapped on the right half-plane cut from 1 to ∞, and the rectangle with vertices $\pm i\mathbf{K}'$, $2\mathbf{K} \pm i\mathbf{K}'$ is mapped on the whole w-plane cut from $-\infty$ to -1, from 1 to ∞, from $-i\infty$ to $-ik'/k$, and from ik'/k to $i\infty$. The lines u_1 = const, u_2 = const are mapped on the doubly orthogonal system of confocal bicircular quartics in the w-plane whose foci are ± 1, $\pm ik'/k$. Both families are ovals, those corresponding to u_1 = const around $\mathfrak{CE}$, those corresponding to u_2 = const around $\mathfrak{AF}$. Both families are symmetric with respect to the axes $w_1 = 0$, $w_2 = 0$.

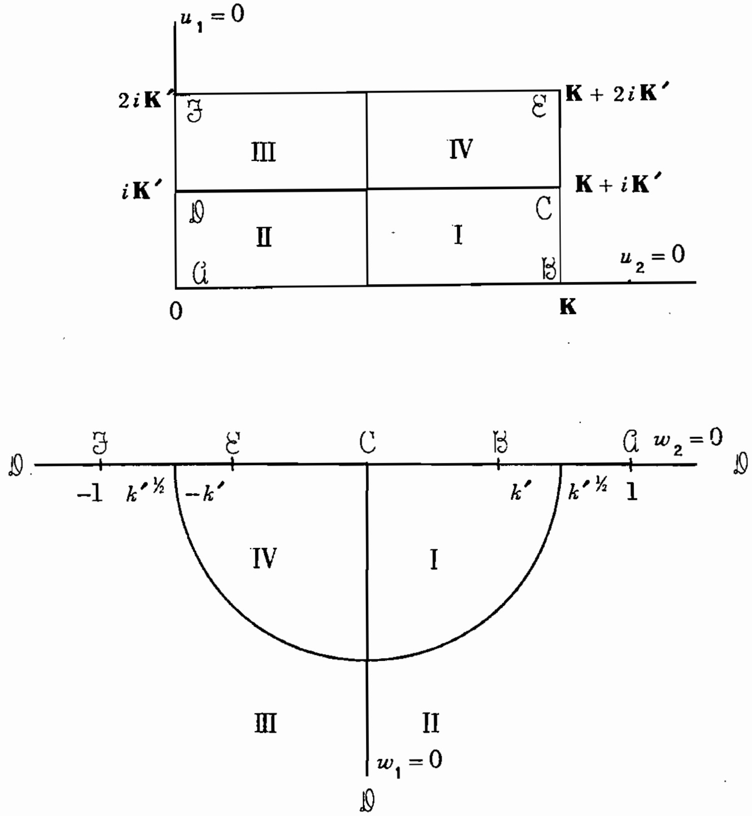


The mapping $w = \text{cn}(u, k)$

The function $w = \text{dn}(u, k)$. Since it follows from tables 7 (sec. 13.17) and 11 (sec. 13.22) that

$$\text{dn}(u, k) = k'\,\text{sn}(\mathbf{K}' - i\mathbf{K} + iu, k'),$$

the mapping $w = \text{dn}\, u$ may be derived from $w = \text{sn}\, u$.



The mapping $w = \mathrm{dn}\,(u,\, k)$.

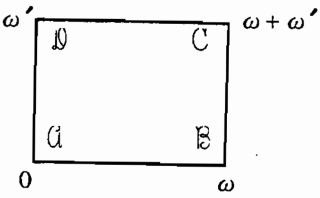
In particular, the rectangle with vertices 0, $\mathbf{K}$, $\mathbf{K} + 2i\mathbf{K}'$, $2i\mathbf{K}'$ is mapped on the lower w half-plane in the manner indicated in the figure, and the rectangle with vertices 0, $2\mathbf{K}$, $2\mathbf{K} + 2i\mathbf{K}'$, $2i\mathbf{K}'$ is mapped on the whole w-plane cut from $-\infty$ to -1 and from 1 to ∞. The lines u_1 = const., u_2= const. are mapped on the doubly orthogonal system of confocal bicircular quartics with foci ± 1, $\pm k'$, and the lines $u_1 = (m + \tfrac{1}{2})\mathbf{K}$ in particular are mapped on the circle with center at $w = 0$ and radius $k'^{\frac{1}{2}}$.

The *functions* $w = \zeta(z) + e_\alpha z$. Clearly $\zeta(z_1)$ is real, $\zeta(iz_2)$ is imaginary, and since we have from 13.13 (18) that

$$\zeta(\omega + iz_2) - \zeta(\omega) = \zeta(iz_2) + \frac{1}{2}\,\frac{\wp'(iz_2)}{\wp(iz_2) - e_1}$$

$$\zeta(\omega' + z_1) - \zeta(\omega') = \zeta(z_1) + \frac{1}{2}\,\frac{\wp'(z_1)}{\wp'(z_1) - e_3}\ ,$$

the first of these two expressions is imaginary, the second real. Investigating the mapping of the rectangle with vertices 0, ω, $\omega+\omega'$, ω' in the z-plane, we find that $\mathfrak{A}\mathfrak{B}$ and $\mathfrak{C}\mathfrak{D}$ are mapped on horizontal lines, and $\mathfrak{B}\mathfrak{C}$ and $\mathfrak{A}\mathfrak{D}$ on vertical lines in the w-plane ($a = 1, 2, 3$). Moreover,



z-plane

$$w(\mathfrak{A}) = \infty, \quad w(\mathfrak{B}) = \eta + e_\alpha \omega,$$
$$w(\mathfrak{C}) = \eta + \eta' + e_\alpha(\omega + \omega'), \quad w(\mathfrak{D}) = \eta' + e_\alpha \omega'.$$

We have to discuss the signs of $\eta + e_\alpha \omega$ and of $(\eta' + e_\alpha \omega')/i$. From 13.16(9), (10), (11) and 13.8(25), (26), we have

$$(e_1 - e_3)^{-\frac{1}{2}} (\eta + e_1 \omega) = \mathbf{E} > 0$$

$$(e_1 - e_3)^{-\frac{1}{2}} (\eta + e_2 \omega) = \mathbf{E} - (e_1 - e_3)^{-\frac{1}{2}} (e_1 - e_2)\omega$$
$$= \mathbf{E} - k'^2 \mathbf{K} = k^2 \mathbf{B} > 0$$

$$(e_1 - e_3)^{-\frac{1}{2}} (\eta + e_3 \omega) = \mathbf{E} - (e_1 - e_3)^{\frac{1}{2}} \omega$$
$$= \mathbf{E} - \mathbf{K} = -k^2 \mathbf{D} < 0$$

$$-i(e_1 - e_3)^{-\frac{1}{2}} (\eta' + e_3 \omega') = -\mathbf{E}' < 0$$
$$-i(e_1 - e_3)^{-\frac{1}{2}} (\eta' + e_2 \omega') = -\mathbf{E}' - (e_1 - e_3)^{-\frac{1}{2}} (e_2 - e_3) i \omega'$$
$$= -\mathbf{E}' + k^2 \mathbf{K}' = -k'^2 \mathbf{B}' < 0$$
$$-i(e_1 - e_3)^{-\frac{1}{2}} (\eta' + e_1 \omega') = -\mathbf{E}' - (e_1 - e_3)^{\frac{1}{2}} i \omega'$$
$$= \mathbf{K}' - \mathbf{E}' = k'^2 \mathbf{D}' > 0.$$

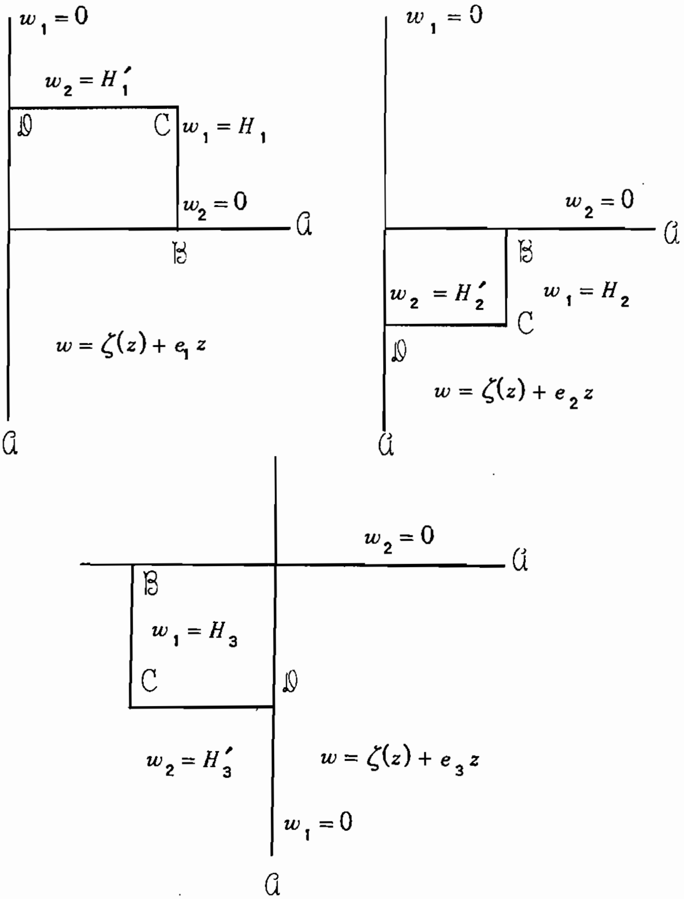
In the figures illustrating the mapping $w = \zeta(z) + e_\alpha z$ of the rectangle $\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}$, the abbreviations

$$\eta + e_\alpha \omega = H_\alpha, \quad \eta' + e_\alpha \omega' = H'_\alpha i$$

were used. From our discussion,

$$H_1 > H_2 > 0 > H_3, \quad H'_1 > 0 > H'_2 > H'_3.$$

In each case that portion of the plane which is to the left of $\mathfrak{ABCDA}$ (in this order) is the map of the rectangle. By reflection on the sides of the rectangle we find the following results. The function $w = \zeta(z) + e_1 z$ maps



the interior of the rectangle with vertices $\pm\,\omega$, $\pm\,\omega + 2\omega'$ in the z-plane on the region exterior to two semi-infinite horizontal strips with corners $\pm H_1$, $\pm H_1 + 2iH_1'$ in the w-plane. The function $w = \zeta(z) + e_2 z$ maps the interior of the rectangle with vertices $\pm\,\omega \pm \omega'$ in the z-plane on the exterior of the rectangle with vertices $\pm H_2 \pm iH_2'$ in the w-plane. The function $w = \zeta(z) + e_3 z$ maps the interior of the rectangle with corners $\pm\,\omega'$, $2\omega \pm \omega'$ in the z-plane on the region exterior to two semi-infinite vertical strips with corners $\pm\,iH_3'$, $2H_3 \pm iH_3'$ in the w-plane.

The mapping $w = \zeta(z) + e_2 z$ may be combined with one of the preceding mappings to map the exterior of a rectangle on a half-plane.

REFERENCES

Appell, Paul and Émile Lacour, 1922: *Fonctions elliptiques et applications*, Gauthier-Villars, Paris.

Bianchi, Luigi, 1916: *Lezioni sulla teoria delle funzioni di variabile complessa e delle funzioni ellittiche*, 2nd edition, Spoerri, Pisa.

Burkhardt, Heinrich and Georg Faber, 1920: *Elliptische Funktionen* 3rd edition, Gruyter, Berlin.

Burnside, W.S. and A.W. Panton, 1892: *Theory of equations*, 3rd edition, Longmans, Roberts and Green, London.

Clebsch, Alfred, 1865: *J. f. Math.* 64, 210-270.

Dixon, A.C., 1894: *Elliptic functions with examples*, Macmillan, and Co. Ltd., London.

Enriques, Federigo and Oscar Chisini, 1934: *Funzioni ellittiche ed abeliane Vol. IV Della Teoria geometrica delle equazioni, ecc.* Zanichelli, Bologna.

Fletcher, Allan, J.C.P. Miller, and Louis Rosenhead, 1946: *An index of mathematical tables*, Scientific Computing Service, London.

Fricke, Robert, 1913: *Elliptische Funktionen, Encyklopädie, der mathematischen Wissenschaften*, v. 2, pt. 2, p. 181-348, B.G. Teubner, Leipzig.

Fricke, Robert, 1916-1922: *Die elliptischen Funktionen und ihre Anwendungen* I, II, B.G. Teubner, Leipzig.

Gröbner, Wolfgang and Nikolaus Hofreiter, 1949: *Integraltafel, Erster Teil, Unbestimmte Integrale*, Springer-Verlag Wien.

Gröbner, Wolfgang and Nikolaus Hofreiter, 1950: *Integraltafel, Zweiter Teil, Bestimmte Integrale*, Springer-Verlag Wien.

Hamel, Georg, 1932: *S.-B. Berlin Math. Ges.* 31, 17-22.

Hancock, Harris, 1917: *Elliptic integrals*, Wiley.

Humbert, Pierre, 1922: *Introduction a l'étude des fonctions elliptiques*, Hermann and Cie., Paris.

Hurwitz, Adolph and Richard Courant, 1925: *Funktionentheorie*, 2nd edition, Springer, Berlin.

Jahnke, Eugen and Fritz Emde, 1938: *Tables of functions with formulae and curves*, B.G. Teubner, Leipzig and Berlin.

Kober, Hermann, 1952: *Dictionary of conformal representations*, Dover.

König, Robert and Maximilian Krafft, 1928: *Elliptische Funktionen*, Gruyter, Berlin.

Low, A.R., 1950: *Normal elliptic functions*, University of Toronto press.

REFERENCES

Magnus, Wilhelm and Fritz Oberhettinger, 1949: *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik*, Springer-Verlag, Berlin, Göttingen, Heidelberg.

Meyer zur Capellen, Walther, 1950: *Integraltafeln*, Springer-Verlag, Berlin, Göttingen, Heidelberg.

Milne-Thomson, L.M., 1950: *Jacobian elliptic function tables*, Dover, New York.

Neville, E.H., 1944: *Jacobian Elliptic Functions*, Oxford, Clarendon Press.

Oberhettinger, Fritz and Wilhelm Magnus, 1949: *Anwendung der elliptischen Funktionen in Physik und Technik*, Springer, Berlin.

Prasad, Ganesh, 1948: *An introduction to the theory of elliptic functions and higher transcendentals*, University of Calcutta.

Radon, Brigitte, 1950: *Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat.* (8) 2, 69-109.

Roberts, W.R. Westropp, 1938: *Elliptic and hyperelliptic integrals and allied theory*, Cambridge University Press.

Spenceley, G.W. and R.M. Spenceley, 1947: *Smithsonian Elliptic Functions Tables*, The Smithsonian Institute Washington.

Tricomi, F.G., 1935: *Boll. Un. Mat. Ital.* 14, 213-218 and 277-282.

Tricomi, F.G., 1936: *Boll. Un. Mat. Ital.* 15, 102-195.

Tricomi, F.G., 1937: *Funzioni ellittiche*, Bologna, Zanichelli, 1951 (German edition, Leipzig, Akad. Verlagsgesellschaft, 1948: second Italian edition 1951).

Whittaker, E.T. and G.N. Watson, 1927: *Modern Analysis*, 4th edition, Cambridge University press.

SUBJECT INDEX

All numbers refer to pages. Numbers in italics refer to the definitions.

F

G

R

S

T

U

W

INDEX OF NOTATIONS

MISCELLANEOUS NOTATIONS