**Sums of Reciprocals of Polygonal Numbers and a Theorem of Gauss**

In *Problem* 07-003, Hongwei Chen requests a general formula in terms of the digamma function for the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{p_r(n)}, \qquad r \geq 5,$$

where $p_r(n) = n[(r-2)n - (r-4)]/2$ is the $n$th $r$-sided polygonal number.

*Solution by* G. C. Greubel[1], Newport News, VA. Substitution yields

$$S_r = b \sum_{n=1}^{\infty} \frac{1}{n(n-a)},$$

where $b = 2/(r-2)$ and $a = (r-4)/(r-2)$. This sum can be evaluated using the following relation satisfied by the digamma function $\psi(z) := \Gamma'(z)/\Gamma(z)$:

$$\psi(z+1) + \gamma = \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \qquad z \neq -1, -2, -3, \ldots .$$

Note that $r \geq 5$ implies that $a = (r-4)/(r-2)$ is not an integer. Hence setting $z = -a$ leads to

$$\psi(1-a) + \gamma = -a \sum_{n=1}^{\infty} \frac{1}{n(n-a)} = (-a)\frac{S_r}{b};$$

thus

$$\sum_{n=1}^{\infty} \frac{1}{p_r(n)} = S_r = -\frac{2}{r-4}\left[\gamma + \psi\left(\frac{2}{r-2}\right)\right]$$

is the desired formula. Exact values of $S_r$ for $3 \leq r \leq 8$ are given in the table below.

**Some Exact Values of $S_r$**

| $r$ | $S_r$ |
|---|---|
| 3 | $2$ |
| 4 | $\pi^2/6$ |
| 5 | $3\log 3 - \pi\sqrt{3}/3$ |
| 6 | $2\log 2$ |
| 7 | $\pi\sqrt{5}\sqrt{5-2\sqrt{5}}/15 + (2/3)\log 5$ |
| | $+(\sqrt{5}+1)\log(\sqrt{2}\sqrt{5-\sqrt{5}}/2)/3$ |
| | $-(\sqrt{5}-1)\log(\sqrt{2}\sqrt{5+\sqrt{5}}/2)/3$ |
| 8 | $\pi\sqrt{3}/12 + (3/4)\log 3$ |

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*Editorial note.* Gauss proved that $\psi(x)$ can be expressed in terms of elementary functions when $x$ is rational [1, p. 13]. Specifically, for $0 < p < q$,

$$\psi(p/q) = -\gamma - \frac{\pi}{2}\cot\frac{\pi p}{q} - \log q + 2\sum_{n=1}^{\lfloor q/2\rfloor}{}' \cos\left(\frac{2\pi n p}{q}\right)\log\left(2\sin\frac{\pi n}{q}\right),$$

where $\sum'$ signifies that when $q$ is even, the term $q/2$ is divided by 2. This gives rise to the following explicit formula for $S_r$ in terms of $q = r - 2 \geq 3$:

$$S_r = \frac{1}{q-2}\left(\pi\cot\left(\frac{2\pi}{q}\right) + 2\log q - 4\sum_{n=1}^{\lfloor q/2\rfloor}{}' \cos\left(\frac{4\pi n}{q}\right)\log\left(2\sin\left(\frac{\pi n}{q}\right)\right)\right).$$

## REFERENCE


[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.


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