Polynomials of the form $\mathrm{P}(4k, e) = 1 + 4ke + 4ke^2$

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Abstract

We show that if k is not a perfect square, $\mathrm{P}(4k, e)$ is a perfect square for an infinite number of positive integers e. If k is a perfect square then there are two possibilities: If the square root of k is not an element of the matrix A on page 2, then $\mathrm{P}(4k, e)$ is never a perfect square for any positive integer e. Otherwise, if the square root of k belongs to the e-th row of matrix A then $\mathrm{P}(4k, e)$ is the one and only perfect square produced by the polynomial.

Our interest in these polynomials is the fact that

$$\mathrm{P}(4k, e) = (2n+1)^2 \text{ if and only if } \mathrm{t}(n) = k \cdot \mathrm{t}(e)$$

where $\mathrm{t}(n)$ denotes the n-th triangular number $\frac{n(n+1)}{2}$. For example, the condition that $\mathrm{P}(8, e)$ be a perfect square defines the sequence of positive integers e such that $2\,\mathrm{t}(e)$ is a triangular number. This sequence is item A053141 in EIS [1]

Let $\mathcal{P}(k)$ be the set of all positive integers e such that $\mathrm{P}(4k, e)$ is a perfect square. We begin our study to find out exactly those k for which there exists at least one e so that $\mathrm{P}(4k, e)$ is a perfect square. In other words, such that $\mathcal{P}(k) \neq \emptyset$. We first believed that $\mathcal{P}(k)$ was infinite provided k was not a perfect square and $\mathcal{P}(k^2)$ was always empty. The latter assumption came from the fact that the polynomial value $\mathrm{P}(16, e)$, for every positive integer e, is never the square of a positive integer. Eckert's proof: $(4 \cdot e + 1)^2 < \mathrm{P}(16, e) < (4 \cdot e + 3)^2$. Eckert's proof works also for $\mathrm{P}(36, e)$ as well and a slight modification of his proof works for $\mathrm{P}(64, e)$ and $\mathrm{P}(100, e)$. But for

[1] Encyclopedia of Integer Sequences; http://www.research.att.com/~njas/sequences/

P(144, e) the modification fails in this way:

$$(12e+1)^2 = 144e^2 + 24e + 1 < \mathrm{P}(144, e)$$
$$(12e+3)^2 = 144e^2 + 72e + 9 < \mathrm{P}(144, e)$$
$$(12e+5)^2 = 144e^2 + 120e + 25 = \mathrm{P}(144, e),\ \text{when } e = 1.$$

From this last example, we soon discovered, using Mathematica, that $\mathcal{P}(k^2)$ is a singlton set for $k = 6, 10, 14, \cdots$; that is for $k = 4e + 2$ and in fact $\mathcal{P}(k^2) = \{e\}$ when $k = 4e + 2$.

Further complications arise because, again by Mathematica experiments, we find that $\mathcal{P}(k^2) = \{1\}$ when $k = 6, 35, 204, \cdots$ and $\mathcal{P}(k^2) = \{2\}$ when $k = 10, 99, 980, \cdots$ and so forth.

We note that $\mathrm{P}(4k^2, e) = 1 + 8k^2\,\mathrm{t}(e)$ and thus $\mathrm{P}(4k^2, e) = (2m+1)^2$ is an instance of the Pell equation $x^2 - Dy^2 = 1$ where D is not a perfect square. In fact, because $1 + 8\,\mathrm{t}(e)$ is always a perfect square, $8\,\mathrm{t}(e)$ is never one. Therefore, $\mathrm{P}(4k^2, e) = (2m+1)^2$ can be rewritten as

$$(2m+1)^2 - 8\,\mathrm{t}(e)k^2 = 1.$$

The Chebyshev polynomials satisfy a similar Pell equation:

Definition 1. $\mathrm{C}_1(n, x)$ *denotes the* n*-th Chebyshev polynomial of the first kind and* $\mathrm{C}_2(n, x)$ *denotes the* n*-th Chebyshev polynomial of the second kind. We note the following Pell equation for these two types of polynomials:*

$$\mathrm{C}_1^2(n+1, x) - (x^2 - 1)\,\mathrm{C}_2^2(n, x) = 1$$

and if we set $x = 1 + 2e$ *this equation becomes*

$$\mathrm{C}_1^2(n+1, 1+2e) - 8\,\mathrm{t}(e)\,\mathrm{C}_2^2(n, 1+2e) = 1.$$

Thus we have the following sequential solution to $\mathrm{P}(4k_n^2, e) = (2m_n+1)^2$:

$$k_n = C_2(n, 1+2e) \text{ and } 2m_n + 1 = \mathrm{C}_1(n+1, 1+2e)$$

This state of affairs is represented by the following two infinite matrices A and B:

$$\mathrm{A} = \begin{pmatrix} 6 & 35 & 204 & 1189 & 6930 & 40391 & \dots \\ 10 & 99 & 980 & 9701 & 96030 & 950599 & \dots \\ 14 & 195 & 2716 & 37829 & 562890 & 7338631 & \dots \\ 18 & 323 & 5796 & 104005 & 1866294 & 33489287 & \dots \\ 22 & 483 & 10604 & 232805 & 5111106 & 112211527 & \dots \\ 26 & 675 & 17524 & 454949 & 11811150 & 306634951 & \dots \\ \vdots & & & & & & \end{pmatrix}$$

$$B = \begin{pmatrix} 17 & 99 & 577 & 3363 & 19601 & 114243 & \dots \\ 49 & 485 & 4801 & 47525 & 470449 & 4656965 & \dots \\ 97 & 1351 & 18817 & 262087 & 3650401 & 50843527 & \dots \\ 161 & 2889 & 51841 & 930249 & 16692641 & 299537289 & \dots \\ 241 & 5291 & 116161 & 2550251 & 55989361 & 1229215691 & \dots \\ 337 & 8749 & 227137 & 5896813 & 153090001 & 3974443213 & \dots \\ \vdots & & & & & & \end{pmatrix}$$

The first row of A is the sequence of Chebyshev polynomials of the second kind evaluated at 3, the second by 5 and so on. In general the function $C_2(n, 1+2e)$ will produce the e-th row of the matrix A. Rows one and two are respectively A001109 and A004189 in EIS. In particular, row one is the sequence of positive integers whose square is a triangular number and row two is the sequence of positive integers whose square is a triangular number divided by t(2). In general the e-th row is the sequence of positive integers whose square is a triangular number divided by $\mathrm{t}(e)$.

Simiarly, the e-th row of the matrix B is the sequence $C_1(n+1, 1+2e)$. An element of the e-th row and m-th column of B; i.e B_{em}, is the square root of $\mathrm{P}(4\,\mathrm{A}_{em}^2, e)$. The first two rows of B are, respectively, the items A001541 and A001079 in EIS.

These facts are summed up in the following theorem:

Theorem 1. *$\mathcal{P}(4k^2) = \{e\}$ if and only if $k = \mathrm{A}_{en}$ for some positive integer n.*

We note here that the matrices A and B are generated by Chebyshev polynomials evaluated for only odd positive integers. This raises the question of what happens at the even positive integers? It turns out that the Pell equation for Chebychev polynomials

$$\mathrm{C}_1^2(n+1, x) - (x^2 - 1)\,\mathrm{C}_2^2(n, x) = 1$$

applies here as well leading to the equation

$$\mathrm{P}(4C_2^2(n, 2e), \frac{2e-1}{2}) = C_1^2(n+1, 2e).$$

Thus, for example, we have that $\mathrm{P}(4C_2^2(n, 2), \frac{1}{2})$ and $\mathrm{P}(4C_2^2(n, 4), \frac{3}{2})$ are perfect squares.

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We now turn our attention to the equation $(2m+1)^2 = \mathrm{P}(4k, e)$ when k is not a perfect square. We rewrite the equation as:

$$(2m+1)^2 = \mathrm{P}(4k, e) = 1 + 4k(e + e^2) = 1 + k(2e+1)^2 - k$$

and setting $x = 2m + 1$ and $y = 2e + 1$ the equation becomes

$$x^2 - ky^2 = 1 - k$$

which is a classic Pell equation. If (p, q) is one of the infinite number of solutions to

$$x^2 - ky^2 = 1$$

then the equation

$$1 - k = (1-k)(p^2 - kq^2) = (p + kq)^2 - k(p+q)^2$$

shows that $(p + kq, p + q)$ is one of the infinite number of solutions to: [2]

$$x^2 - ky^2 = 1 - k.$$

We have proven

Theorem 2. *If k is not a perfect square then $\mathcal{P}(k)$ is infinite.*

[2]Weisstein, Eric W. "Pell Equation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/PellEquation.html