Chapter 15 Special Functions

15-1 Gamma & Beta Functions

Gamma function: $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$

Theorem $\Gamma(x+1)=x\Gamma(x)$, $\Gamma'(x)=\int_0^\infty \ell n(t)\cdot t^{x-1}e^{-t}dt$

(Proof) $\Gamma(x+1)=\int_0^\infty t^x e^{-t}dt=-\int_0^\infty t^x de^{-t}$

$$=-t^x e^{-t}\Big|_{t=0}^{t=\infty}+\int_0^\infty e^{-t}dt^x=x\int_0^\infty t^{x-1}e^{-t}dt=x\Gamma(x)$$

Theorem $\Gamma(n)=(n-1)!$, $\Gamma(1)=0!=1$.

Theorem $\Gamma(x)\Gamma(1-x)=\dfrac{\pi}{\sin(\pi x)}=\int_0^\infty \dfrac{t^{x-1}}{1+t}dt$

Theorem $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $\Gamma'(1)=\int_0^\infty e^{-t}\ln t\,dt=-\gamma=-0.5772156....$

($\gamma=\lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}-\ln n\right]$)

(Proof) $\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)=\left[\Gamma\left(\frac{1}{2}\right)\right]^2=\dfrac{\pi}{\sin\left(\frac{\pi}{2}\right)}=\pi\Rightarrow\Gamma(\frac{1}{2})=\sqrt{\pi}$

Stirling's formula: $\Gamma(n+1)=n!\approx\sqrt{2\pi n}\,n^n e^{-n}$

Beta function: $\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Eg. Evaluate $\int_0^1\dfrac{dz}{\sqrt{1-z^4}}$. 【清大材研】

(Sol.) Let $t=z^4$, $dt=4z^3\,dz$, $\int_0^1\dfrac{dz}{\sqrt{1-z^4}}=\int_0^1\dfrac{\frac{1}{4}t^{-\frac{3}{4}}dt}{\sqrt{1-t}}=\frac{1}{4}\int_0^1 t^{\frac{1}{4}-1}\cdot(1-t)^{\frac{1}{2}-1}dt=\frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$

Theorem $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)d\theta$

Theorem $\beta(x,y)=\beta(y,x)$

15-2 Bessel Functions

Bessel differential equation of order ν: $x^2y''+xy'+(x^2-\nu^2)y=0$, its general solution is $y(x)=cJ_\nu(x)+dY_\nu(x)$.

Bessel differential equation of order ν with parameter λ: $x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0$, its solution is $y(x)=cJ_\nu(\lambda x)+dY_\nu(\lambda x)$.

Bessel function of the first kind: $J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{x}{2})^{\nu+2k}}{k!\Gamma(\nu+k+1)}$

Note: $J_{-n}(x)=(-1)^nJ_n(x)$, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, But $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent for $\nu\notin N$.

Orthogonality of $J_n(x)$:

If $\lambda\neq\mu$ and $J_n(\lambda)=J_n(\mu)=0$, then $\int_0^1 xJ_n(\lambda x)J_n(\mu x)dx=0$

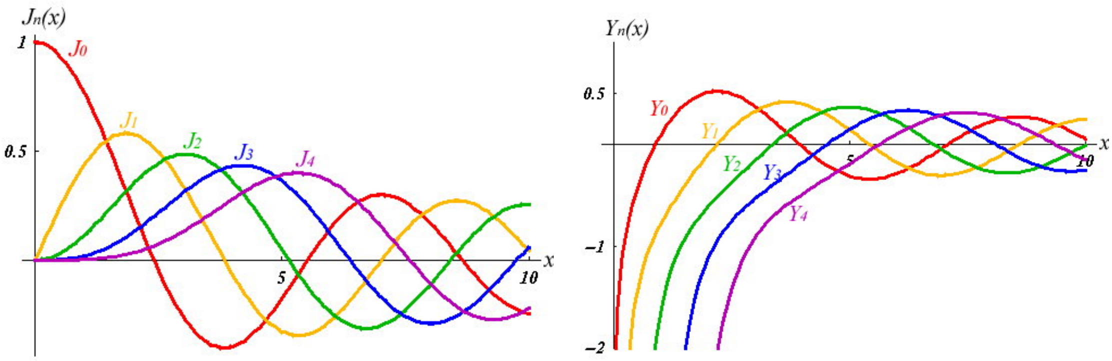
Else if $\lambda\neq\mu$ but $J_n(\lambda)\neq 0\neq J_n(\mu)$, then

$$\int_0^1 xJ_n(\lambda x)J_n(\mu x)dx=\frac{\mu J_n(\lambda)J_n'(\mu)-\lambda J_n(\mu)J_n'(\lambda)}{\lambda^2-\mu^2}$$

Else if $\lambda=\mu$, $J_n(\lambda)\neq 0$, then $\int_0^1 xJ_n^{\,2}(\lambda x)dx=\frac{1}{2}\left[J_n'^{\,2}(\lambda)+\left(1-\frac{n^2}{\lambda^2}\right)J_n^{\,2}(\lambda)\right]$

Bessel function of the second kind:

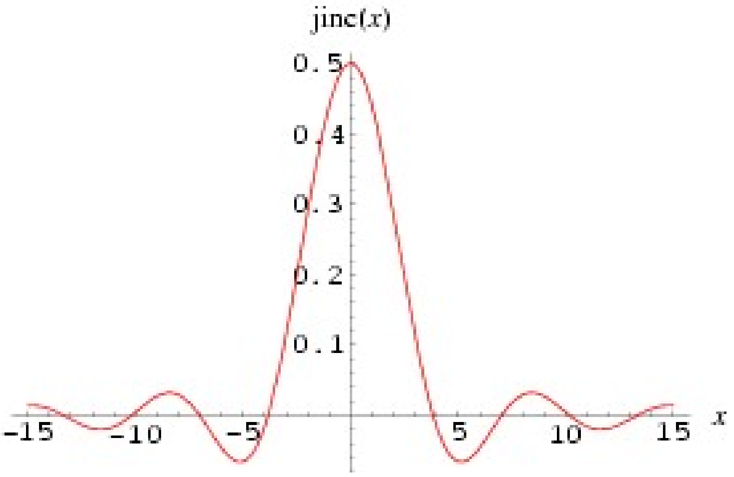
$$Y_\nu(x)=\begin{cases}\dfrac{J_\nu(x)\cos(\nu\pi)-J_{-\nu}(x)}{\sin(\nu\pi)}, & \nu \text{ is not an integer}\\ \lim_{\nu\to n}\dfrac{J_\nu(x)\cos(\nu\pi)-J_{-\nu}(x)}{\sin(\nu\pi)}, & \nu \text{ is an integer } n.\end{cases}$$



Hankel functions: $\begin{cases}H_\nu^{(1)}(x)=J_\nu(x)+iY_\nu(x): \text{the 1st kind}\\ H_\nu^{(2)}(x)=J_\nu(x)-iY_\nu(x): \text{the 2nd kind}\end{cases}$

Note: $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are linearly independent.

Jinc function: $jinc(x)=\frac{J_1(x)}{x}$ and $jinc'(x)=-\frac{J_2(x)}{x}$



Properties of Bessel functions:

1. $B_\nu'(x) = \frac{1}{2}[B_{\nu-1}(x) - B_{\nu+1}(x)]$

2. $[x^\nu B_\nu(x)]' = -x^\nu B_{\nu-1}(x)$

3. $[x^{-\nu} B_\nu(x)]' = -x^\nu B_{\nu+1}(x)$

4. $B_{\nu+1}(x) = \frac{2\nu}{x} B_\nu(x) - B_{\nu-1}(x)$

5. $xB_\nu'(x) = -\nu B_\nu(x) + xB_{\nu-1}(x) = \nu B_\nu(x) - xB_{\nu+1}(x)$

6. $J_n(x+y) = \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y)$

7. Generating function $g(x,t)$ for $J_n(x)$: $g(x,t)$=exp$\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n$

$$g(x,t) = g\left(x, \frac{-1}{t}\right) \Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

8. $\cos(x\sin\theta) = J_0(x) + 2\sum_{m=1}^{\infty} J_{2m}(x)\cdot\cos(2m\theta)$

$$\sin(x\sin\theta) = 2\sum_{m=1}^{\infty} J_{2m-1}(x)\cdot\sin((2m-1)\theta)$$

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) = J_0{}^2(x) + 2\sum_{n=1}^{\infty} J_n{}^2(x)$$

$$\cos(x\cos\theta) = J_0(x) + 2\sum_{n=1}^{\infty} (-1)^n J_{2n}(x)\cos(2n\theta)$$

$$\sin(x\cos\theta) = 2\sum_{n=1}^{\infty} (-1)^{n+1} J_{2n-1}(x)\cos((2n-1)\theta)$$

9. $J_{2n}(x) = \frac{1}{\pi}\int_0^{\pi}\cos(2n\theta)\cdot\cos(x\sin\theta)d\theta = \frac{(-1)^n}{\pi}\int_0^{\pi}\cos(2n\theta)\cdot\cos(x\cos\theta)d\theta$

$$J_{2n+1}(x) = \frac{1}{\pi}\int_0^{\pi}\sin((2n+1)\theta)\cdot\sin(x\sin\theta)d\theta = \frac{(-1)^n}{\pi}\int_0^{\pi}\cos((2n+1)\theta)\cdot\sin(x\cos\theta)d\theta$$

10.
$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$
$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$
as $x \to \infty$

Spherical Bessel functions:

$$J_{n+\frac{1}{2}}(x) = (-1)^n\sqrt{\frac{2}{\pi}}x^{n+\frac{1}{2}}\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right),\quad J_{\frac{1}{2}-n}(x) = \sqrt{\frac{2}{\pi}}x^{n-\frac{1}{2}}\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$

$$Y_{n+\frac{1}{2}}(x) = (-1)^{n+1}J_{-n-\frac{1}{2}}(x),\quad Y_{\frac{1}{2}-n}(x) = (-1)^n J_{n+\frac{1}{2}}(x)$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x,\quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Modified Bessel functions:

$I_\nu(x) = (i)^{-\nu}J_\nu(ix)$: the 1st kind

$K_\nu(x) = \frac{\pi}{2\sin(\nu\pi)}[I_{-\nu}(x) - I_\nu(x)]$: the 2nd kind, where ν is non-integer

15-3 Legendre Differential Equations and Legendre Polynomials

Legendre equation: $(1-x^2)y''-2xy'+n(n+1)y=0$, its general solution is $y(x)=cP_n(x)+dQ_n(x)$.

Legendre polynomial of degree n of the 1st kind:

$$P_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!\,x^{n-2k}}{2^n k!(n-2k)!(n-k)!}$$

$$=\frac{(2n-1)(2n-3)\cdots 1}{n!}\left\{x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4}+\cdots\right\}$$

$$=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)$$

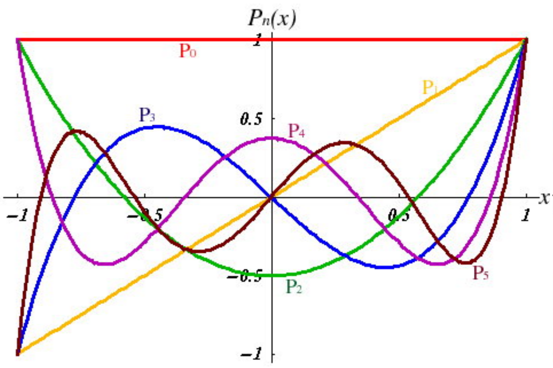
Orthogonality of $P_n(x)$: $\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{mn}$

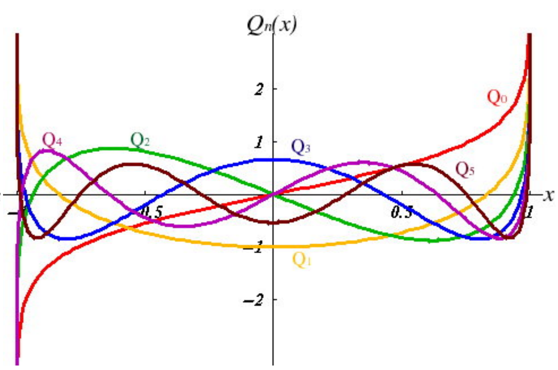
Legendre polynomial of degree n of the 2nd kind:

For $|x|<1$,

$$Q_n(x)=\begin{cases}\dfrac{(-1)^{n/2}2^n\left[\left(\frac{n}{2}\right)!\right]^2}{n!}\cdot\left[x+\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k(n-1)\cdots(n-2k+1)(n+2)\cdots(n+2k)x^{2k+1}}{(2k+1)!}\right], & n:even\\[2ex] \dfrac{(-1)^{\frac{n+1}{2}}\cdot 2^{n+1}\left[\left(\frac{n-1}{2}\right)!\right]^2}{n!}\cdot\left[1+\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k n(n-2)\cdots(n-2k+2)(n+1)\cdots(n+2k-1)x^{2k}}{(2k)!}\right], & n:odd\end{cases}$$

$$Q_n(x)=\sum_{k=0}^{\infty}\frac{2^n(n+k)!(n+2k)!}{k!(2n+2k+1)!}x^{-n-2k-1}$$ if $|x|>1$, but $Q_n(x)$ is divergent at $x=\pm 1$.





Properties of Legendre polynomials:

1. $(n+1)L_{n+1}(x)-(2n+1)xL_n(x)+nL_{n-1}(x)=0$, $n>0$
2. $L'_{n+1}(x)-(2n+1)L_n(x)-L'_{n-1}(x)=0$
3. $(x^2-1)L'_n(x)-nxL_n(x)+nL_{n-1}(x)=0$
4. **Generating functions for $P_n(x)$ and $Q_n(x)$:**

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=-\infty}^{\infty}P_n(x)t^n\ ,\quad \frac{1}{\sqrt{1-2xt+t^2}}\cdot\cosh^{-1}\left(\frac{t-x}{\sqrt{x^2-1}}\right)=\sum_{n=-\infty}^{\infty}Q_n(x)t^n$$

Associated Legendre equations and functions: $(1-x^2)y''-2xy'+\left[n(n+1)-\frac{m^2}{1-x^2}\right]y=0$,

its general solution is $y(x)=cP_n{}^m(x)+dQ_n{}^m(x)$,

where $L_n{}^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}L_n(x)$.

Properties of the associated Legendre polynomials:

1. $L_{n+1}{}^m(x)-(2n+1)\sqrt{1-x^2}L_n{}^{m-1}(x)-L_{n-1}{}^m(x)=0$
2. $xL_n{}^m(x)-L_{n-1}{}^m(x)+(m-n-1)\sqrt{1-x^2}L_n{}^{m-1}(x)=0$
3. $\int_{-1}^{1}P_m{}^k(x)P_n{}^k(x)dx=\frac{2}{2n+1}\cdot\frac{(n+k)!}{(n-k)!}\delta_{mn}$

15-4 Applications of Bessel and Legendre Functions

Electrostatic potentials in the cylindrical coordinate:

In source-free region: $\nabla^2V=\frac{\partial^2V}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial\rho}+\frac{\partial^2V}{\rho^2\partial^2\phi}+\frac{\partial^2V}{\partial z^2}=0$

Set $V(\rho,\phi,z)=R(\rho)\Phi(\phi)Z(z)\Rightarrow\frac{1}{R}\left(\frac{\partial^2R}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial R}{\partial\rho}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\rho^2\partial\phi^2}+\frac{1}{Z}\frac{\partial^2Z}{\partial z^2}=0$

Set $\frac{\partial^2Z}{\partial z^2}=k^2Z$, $\frac{\partial^2\Phi}{\partial\phi^2}=-n^2\Phi$, $\rho=\frac{x}{k}\Rightarrow\frac{d^2\Phi}{d\phi^2}+n^2\Phi=0$

$\Rightarrow\Phi(\phi)=A\cos(n\phi)+B\sin(n\phi)$

$\frac{d^2Z}{dz^2}-k^2Z=0\Rightarrow Z(z)=ce^{kz}+de^{-kz}$

$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+\left(1-\frac{n^2}{x^2}\right)R=0\Rightarrow x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+\left(x^2-n^2\right)R=0$ (Bessel differential equation) $\Rightarrow R(x)=R(k\rho)=eJ_n(k\rho)+fY_n(k\rho)$

(If V is finite at $\rho=0\Rightarrow f=0$)

Special case 1: V is independent of z, $\dfrac{\partial^2}{\partial z^2} = 0$

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0 \Rightarrow \Phi(\phi) = A\cos(n\phi) + B\sin(n\phi)$$

$$\rho^2 \frac{d^2R(\rho)}{d\rho^2} + \rho\frac{dR(\rho)}{d\rho} - n^2R(\rho) = 0 \quad \text{(Euler's equation)} \Rightarrow R(\rho) = c\rho^n + d\rho^{-n}$$

Special case 2: V is independent of φ and z

$$\frac{d}{d\rho}\left[\rho\frac{dR(\rho)}{d\rho}\right] = 0 \Rightarrow V(\rho) = R(\rho) = c\ln\rho + d$$

Electrostatic potentials in the spherical coordinate:

In source-free region: $\nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial V}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$

Set $V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$\Rightarrow \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cdot\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

Set $\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = \ell(\ell+1)$

$\Rightarrow r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \ell(\ell+1)R = 0$ (Euler's equation) $\Rightarrow R(r)=Ar^l+Br^{-(l+1)}$

$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \Rightarrow \Phi(\phi) = C\cos(m\phi) + D\sin(m\phi)$

Let $x=\cos\theta \Rightarrow (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$ (Associated Legendre equation)

$\Rightarrow \Theta = \Theta(x) = EP_\ell^m(x) + FQ_\ell^m(x) = EP_\ell^m(\cos\theta) + FQ_\ell^m(\cos\theta)$

Special case: $m=0 \Rightarrow (1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \ell(\ell+1)\Theta = 0$ (Legendre equation)

$$\Rightarrow \Theta = EP_l(\cos\theta) + FQ_l(\sin\theta)$$

15-5 Elliptic Integral Functions

If we set $v=\sin\theta$, $x=\sin\varphi$, and $0<k<1$,

the 1st-kind elliptic integral: $F(k,\phi)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^{x}\frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$.

1. $\phi=\frac{\pi}{2}$, complete integral, $F\left(k,\frac{\pi}{2}\right)\equiv F(k)$. 2. $0<\phi<\frac{\pi}{2}$, incomplete integral.

the 2nd-kind elliptic integral of: $E(k,\phi)=\int_0^{\phi}\sqrt{1-k^2\sin^2\theta}\,d\theta=\int_0^{x}\sqrt{\frac{1-k^2v^2}{1-v^2}}\,d\upsilon$.

1. $\phi=\frac{\pi}{2}$, complete integral, $E\left(k,\frac{\pi}{2}\right)=E(k)$. 2. $0<\phi<\frac{\pi}{2}$, incomplete integral.

the 3rd-kind elliptic integral of:

$$\Pi(k,n,\theta)=\int_0^{\phi}\frac{d\theta}{(1+n\sin^2\theta)\cdot\sqrt{1-k^2\sin^2\theta}}=\int_0^{x}\frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}.$$

1. $\phi=\frac{\pi}{2}$, complete integral, $\Pi\left(k,\frac{\pi}{2}\right)\equiv\Pi(k)$. 2. $0<\phi<\frac{\pi}{2}$, incomplete integral.

Eg. Evaluate $\int_0^{\pi/2}\sqrt{1+4\sin^2 x}\,dx$.

(Sol.) $\int_0^{\pi/2}\sqrt{1+4\sin^2 x}\,ds=\int_0^{\pi/2}\sqrt{5-4\cos^2 x}\,dx$ (set $x=\frac{\pi}{2}-\theta$)

$$=\sqrt{5}\int_0^{\pi/2}\sqrt{1-\frac{4}{5}\sin^2\theta}\,d\theta=\sqrt{5}E\left(\sqrt{\frac{4}{5}}\right)$$