A

Vector and Dyadic Analysis

*I am greatly astonished when I consider the weakness of my mind and
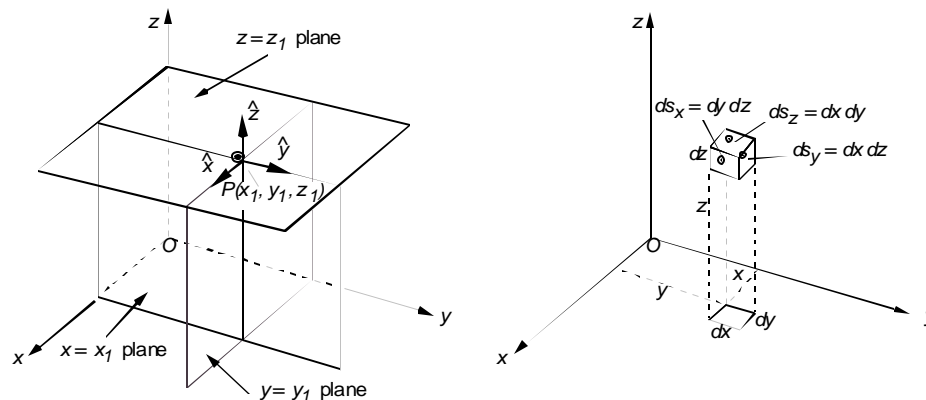its proness to error.*

—Descartes

This appendix summarizes a number of useful relationships and transformations from
vector calculus and dyadic analysis that are especially relevant to electromagnetic theory.
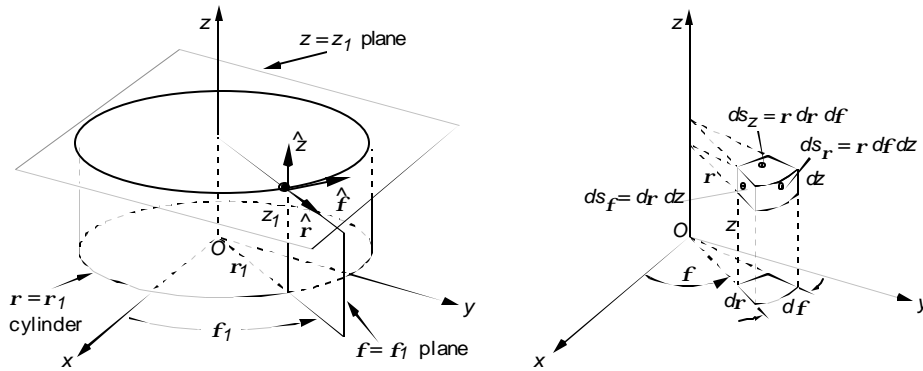
A.1 COORDINATE SYSTEMS

A.1.1 Rectangular Coordinates: $(x; y; z)$

$$
\begin{aligned}
\overline{A} &= A_x \hat{x} + A_y \hat{y} + A_z \hat{z} & d\overline{S} &= dy\,dz\,\hat{x} + dx\,dz\,\hat{y} + dx\,dy\,\hat{z} \\
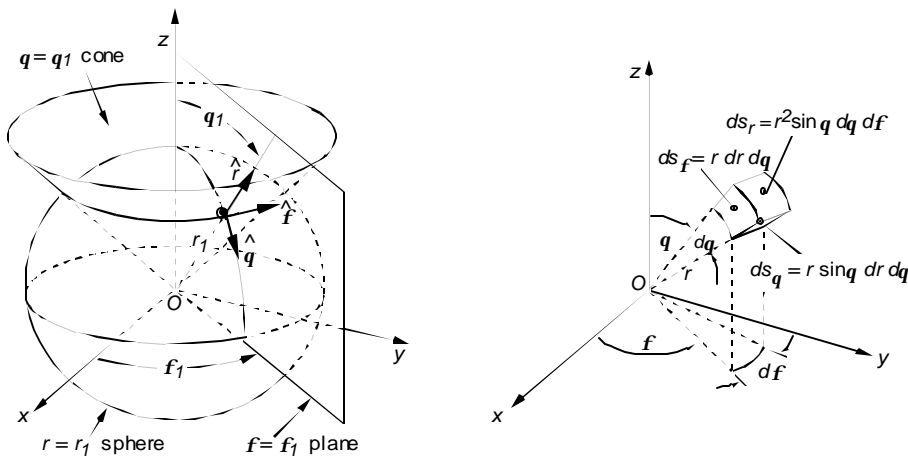d\overline{\ell} &= \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz & dV &= dx\,dy\,dz
\end{aligned}
$$

A.1.2   Cylindrical Coordinates: $(\rho; \phi; z)$

$$
\begin{aligned}
\overline{A} &= A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z} & d\overline{S} &= \rho\, d\phi\, dz\, \hat{\rho} + d\rho\, dz\, \hat{\phi} + \rho\, d\rho\, d\phi\, \hat{z} \\
d\overline{\ell} &= \hat{\rho}\, d\rho + \hat{\phi}\, \rho\, d\phi + \hat{z}\, dz & dV &= \rho\, d\rho\, d\phi\, dz
\end{aligned}
$$

A.1.3   Spherical Coordinates: $(r; \theta; \phi)$

$$
\begin{aligned}
\overline{A} &= A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi} & d\overline{S} &= r^2 \sin\theta\, d\theta\, d\phi\, \hat{r} + r \sin\theta\, dr\, d\phi\, \hat{\theta} + r\, dr\, d\theta\, \hat{\phi} \\
d\overline{\ell} &= \hat{r}\, dr + \hat{\theta}\, r d\theta + \hat{\phi}\, r \sin\theta\, d\phi & dV &= r^2 \sin\theta\, dr\, d\theta\, d\phi
\end{aligned}
$$

A.2 COORDINATE TRANSFORMATIONS

Note that relations between the unit vectors in the different coordinate systems are obtained from the following by replacing the components of $\overline{A}$ with the corresponding unit vector (for example, $A_r \Rightarrow \hat{r}$).

A.2.1 Rectangular $\longleftrightarrow$ Spherical transformation:

These coordinates are related by

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
\tag{A.1}
$$

and conversion between vectors is given by

$$
\text{Spherical} \rightarrow \text{Rectangular} \quad
\begin{aligned}
A_x &= A_r \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi \\
A_y &= A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi \\
A_z &= A_r \cos\theta - A_\theta \sin\theta
\end{aligned}
\tag{A.2}
$$

$$
\text{Rectangular} \rightarrow \text{Spherical} \quad
\begin{aligned}
A_r &= A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta \\
A_\theta &= A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta \\
A_\phi &= -A_x \sin\phi + A_y \cos\phi
\end{aligned}
\tag{A.3}
$$

A.2.2 Rectangular $\longleftrightarrow$ Cylindrical transformation:

These coordinates are related by

$$
\begin{aligned}
x &= \rho \cos\phi \\
y &= \rho \sin\phi \\
z &= z
\end{aligned}
\tag{A.4}
$$

and conversion between vectors is given by

$$
\text{Cylindrical} \rightarrow \text{Rectangular} \quad
\begin{aligned}
A_x &= A_\rho \cos\phi - A_\phi \sin\phi \\
A_y &= A_\rho \sin\phi + A_\phi \cos\phi \\
A_z &= A_z
\end{aligned}
\tag{A.5}
$$

$$
\text{Rectangular} \rightarrow \text{Cylindrical} \quad
\begin{aligned}
A_\rho &= A_x \cos\phi + A_y \sin\phi \\
A_\phi &= -A_x \sin\phi + A_y \cos\phi \\
A_z &= A_z
\end{aligned}
\tag{A.6}
$$

A.2.3 Cylindrical $\longleftrightarrow$ Spherical transformation:

These coordinates are related by

$$
\rho = r \sin\theta
$$

$$z = r \cos \theta \qquad \qquad (A.7)$$

(the azimuthal angle ϕ is common to both coordinate systems). Conversion between vectors is given by

$$\text{Cylindrical} \rightarrow \text{Spherical} \quad \begin{aligned} A_r &= A_\rho \sin \theta + A_z \cos \theta \\ A_\theta &= A_\rho \cos \theta - A_z \sin \theta \\ A_\phi &= A_\phi \end{aligned} \qquad (A.8)$$

$$\text{Spherical} \rightarrow \text{Cylindrical} \quad \begin{aligned} A_\rho &= A_r \sin \theta + A_\phi \cos \theta \\ A_\phi &= A_\phi \\ A_z &= A_r \cos \theta - A_\theta \sin \theta \end{aligned} \qquad (A.9)$$

A.3   ELEMENTS OF VECTOR CALCULUS

A.3.1   Flux and Circulation

Maxwell's equations are expressed in terms of two important vector field concepts: flux and circulation. The flux ψ of a vector field $\overline{A}$ through some surface S is defined as

$$\text{flux of } \overline{A} \text{ through } S \qquad \psi \equiv \iint_S \overline{A} \cdot d\overline{S} \qquad (A.10)$$

and the circulation of $\overline{A}$ around some path C is defined as

$$\text{circulation of } \overline{A} \text{ around } C = \oint_C \overline{A} \cdot d\overline{\ell} \qquad (A.11)$$

These concepts are expressed in differential form as the *divergence* and *curl*

$$\text{Divergence:} \quad \nabla \cdot \overline{A} \equiv \lim_{V \to 0} \frac{\oiint_S \overline{A} \cdot d\overline{S}}{\iiint_V dV} \qquad \qquad \text{Curl:} \quad \nabla \times \overline{A} \equiv \lim_{S \to 0} \frac{\oint_C \overline{A} \cdot d\overline{\ell}}{\iint_S d\overline{S}} \quad (A.12)$$

where we have defined the 'del' operator, which in rectangular coordinates is

$$\nabla \equiv \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \qquad (A.13)$$

The ∇ operator takes on different forms in other coordinate systems. Section A.4 lists explicit divergence and curl operations in the three most common coordinate systems. Note that the concepts of flux/divergence are also related through the Divergence theorem (A.54), and the concepts of circulation/curl are also related through the Stokes theorem (A.59).

A.3.2 The Gradient

Another important operation is the *gradient* of a vector field, which is the vector equivalent of a derivative operation. The gradient only operates on scalar fields, ϕ, and is written as $\nabla \phi$. Explicit forms for the gradient operation in the three common coordinate systems is given in section A.4.

The gradient produces a vector which points in the direction of greatest change of the scalar field. This property is useful in a geometric sense for determining tangent planes and normal directions to an arbitrary surface [2]. In three dimensions an arbitrary surface can be described by the functional relation

$$f(\overline{r}) = C \tag{A.14}$$

where C is a constant, and $f(\overline{r})$ is shorthand for a function of the three coordinate variables; for example, in rectangular coordinates, $f(\overline{r}) = f(x, y, z)$. A plane tangent to this surface at the point $\overline{r}'$ is described by

$$(\overline{r} - \overline{r}') \cdot \nabla f(\overline{r}') = 0 \qquad \text{(tangent plane)} \tag{A.15}$$

The gradient points in the direction normal to the surface, so a unit normal to the surface described by (A.14) at the point $\overline{r}'$ can be found from

$$\hat{n} = \frac{\nabla f(\overline{r}')}{|\nabla f(\overline{r}')|} \qquad \text{(unit normal)} \tag{A.16}$$

A.3.3 Vector Taylor Expansion

The multi-dimensional Taylor series expansion of a function $f(\overline{r} + \overline{a})$ around the point $\overline{r}$ can be represented in vector form as

$$f(\overline{r} + \overline{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\overline{a} \cdot \nabla)^n f(\overline{r}) \tag{A.17}$$

A.3.4 Change of Variables

In three dimensions, a change of variables from the coordinates (x, y, z) to new coordinates (u, v, w) is given by [2]

$$\iiint f(x, y, z) \, dx \, dy \, dz = \iiint g(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du \, dv \, dw \tag{A.18}$$

where

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

is called the *Jacobian* of the transformation, and

$$g(u, v, w) = f\left(x(u, v, w), y(u, v, w), z(u, v, w)\right)$$

and it has been assumed that (x, y, z) can be expressed functionally in terms of (u, v, w) (or vica-versa). A similar result applies to transformations in two-dimensions.

A.4 EXPLICIT DIFFERENTIAL OPERATIONS

A.4.1 Rectangular Coordinates $(x; y; z)$:

$$\nabla \Phi = \hat{x}\frac{\partial \Phi}{\partial x} + \hat{y}\frac{\partial \Phi}{\partial y} + \hat{z}\frac{\partial \Phi}{\partial z} \tag{A.19}$$

$$\nabla \cdot \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{A.20}$$

$$\nabla \times \overline{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{A.21}$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{A.22}$$

$$\nabla^2 \overline{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z \tag{A.23}$$

A.4.2 Cylindrical Coordinates $(½; Á; z)$:

$$\nabla \Phi = \hat{\rho}\frac{\partial \Phi}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} + \hat{z}\frac{\partial \Phi}{\partial z} \tag{A.24}$$

$$\nabla \cdot \overline{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{A.25}$$

$$\nabla \times \overline{A} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]$$

$$+ \hat{\phi}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \hat{z}\frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right] \tag{A.26}$$

$$\nabla^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{A.27}$$

$$\nabla^2 \overline{A} = \hat{\rho}\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_\rho}{\rho^2}\right)$$

$$+ \hat{\phi}\left(\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi} - \frac{A_\phi}{\rho^2}\right) + \hat{z}(\nabla^2 A_z) \tag{A.28}$$

A.4.3 Spherical Coordinates $(r; \mu; \acute{A})$:

$$\nabla\Phi = \hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} \tag{A.29}$$

$$\nabla \cdot \overline{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{A.30}$$

$$\nabla \times \overline{A} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]$$
$$+\frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] \tag{A.31}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \tag{A.32}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} \tag{A.33}$$

$$\nabla^2\overline{A} = \hat{r}\left[\nabla^2 A_r - \frac{2}{r^2}\left(A_r + A_\theta\cot\theta + \csc\theta\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_\theta}{\partial\theta}\right)\right]$$
$$+\hat{\theta}\left[\nabla^2 A_\theta - \frac{1}{r^2}\left(A_\theta\csc^2\theta - 2\frac{\partial A_r}{\partial\theta} + 2\cot\theta\csc\theta\frac{\partial A_\phi}{\partial\phi}\right)\right]$$
$$+\hat{\phi}\left[\nabla^2 A_\phi - \frac{1}{r^2}\left(A_\phi\csc^2\theta - 2\csc\theta\frac{\partial A_r}{\partial\phi} - 2\cot\theta\csc\theta\frac{\partial A_\theta}{\partial\phi}\right)\right] \tag{A.34}$$

A.5 VECTOR RELATIONS

A.5.1 Dot and Cross Product Identities

$$\overline{A} \cdot \overline{A}^* = |\overline{A}|^2 \tag{A.35}$$

$$\overline{A} \cdot \overline{B} = \overline{B} \cdot \overline{A} \tag{A.36}$$

$$\overline{A} \times \overline{B} = -\overline{B} \times \overline{A} \tag{A.37}$$

$$\overline{A} \cdot (\overline{B} \times \overline{C}) = \overline{B} \cdot (\overline{C} \times \overline{A}) = \overline{C} \cdot (\overline{A} \times \overline{B}) \tag{A.38}$$

$$\overline{A} \times (\overline{B} \times \overline{C}) = (\overline{A} \cdot \overline{C})\overline{B} - (\overline{A} \cdot \overline{B})\overline{C} \tag{A.39}$$

$$\overline{A} \times (\overline{B} \times \overline{C}) + \overline{B} \times (\overline{C} \times \overline{A}) + \overline{C} \times (\overline{A} \times \overline{B}) = 0 \tag{A.40}$$

$$(\overline{A} \times \overline{B}) \cdot (\overline{C} \times \overline{D}) = \overline{A} \cdot \left[\overline{B} \times (\overline{C} \times \overline{D})\right]$$
$$= (\overline{A} \cdot \overline{C})(\overline{B} \cdot \overline{D}) - (\overline{A} \cdot \overline{D})(\overline{B} \cdot \overline{C}) \tag{A.41}$$

$$(\overline{A} \times \overline{B}) \times (\overline{C} \times \overline{D}) = (\overline{A} \times \overline{B} \cdot \overline{D})\overline{C} - (\overline{A} \times \overline{B} \cdot \overline{C})\overline{D} \tag{A.42}$$

A.5.2 Vector Differential operations

$$\nabla \cdot \nabla \phi = \nabla^2 \phi \tag{A.43}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{A.44}$$

$$\nabla \cdot (\phi\overline{A}) = \overline{A} \cdot \nabla\phi + \phi\nabla \cdot \overline{A} \tag{A.45}$$

$$\nabla \times (\phi\overline{A}) = \phi\nabla \times \overline{A} - \overline{A} \times \nabla\phi \tag{A.46}$$

$$\nabla \cdot (\overline{A} \times \overline{B}) = \overline{B} \cdot (\nabla \times \overline{A}) - \overline{A} \cdot (\nabla \times \overline{B}) \tag{A.47}$$

$$\nabla \times (\overline{A} \times \overline{B}) = \overline{A}(\nabla \cdot \overline{B}) - \overline{B}(\nabla \cdot \overline{A}) + (\overline{B} \cdot \nabla)\overline{A} - (\overline{A} \cdot \nabla)\overline{B} \tag{A.48}$$

$$\nabla(\overline{A} \cdot \overline{B}) = \overline{A} \times (\nabla \times \overline{B}) + \overline{B} \times (\nabla \times \overline{A}) + (\overline{B} \cdot \nabla)\overline{A} + (\overline{A} \cdot \nabla)\overline{B} \tag{A.49}$$

$$\nabla \times \nabla\phi = 0 \tag{A.50}$$

$$\nabla \cdot (\nabla \times \overline{A}) = 0 \tag{A.51}$$

$$\nabla \times \nabla \times \overline{A} = \nabla(\nabla \cdot \overline{A}) - \nabla^2\overline{A} \tag{A.52}$$

The last identity essentially *defines* the vector Laplacian $\nabla^2\overline{A}$, which reduces to three scalar Laplacians in rectangular coordinates *only*.

A.5.3 Integral relations

From the *Fundamental Theorem of Calculus*,

$$\int_a^b \nabla\phi \cdot d\overline{\ell} = \int_a^b \frac{\partial\phi}{\partial\ell}d\ell = \phi(b) - \phi(a) \tag{A.53}$$

In the following, V is a volume bounded by a closed surface S, with the direction of $d\overline{S}$ taken as pointing outward from the enclosed volume, by convention:

$$\text{(Divergence theorem)} \qquad \iiint_V (\nabla \cdot \overline{A})\, dV = \oiint_S \overline{A} \cdot d\overline{S} \tag{A.54}$$

$$\iiint_V (\nabla\phi)\, dV = \oiint_S \phi\, d\overline{S} \tag{A.55}$$

$$\iiint_V (\nabla \times \overline{A})\, dV = \oiint_S (d\overline{S} \times \overline{A}) \tag{A.56}$$

Note that (A.54) combined with (A.51) gives

$$\oiint_S (\nabla \times \overline{A}) \cdot d\overline{S} = 0 \tag{A.57}$$

and that (A.50) and (A.56) give

$$\oiint_S d\overline{S} \times \nabla\phi = 0 \qquad (A.58)$$

In the following, S is an open surface bounded by a contour C described by line element $d\overline{\ell}$. The direction of $d\overline{\ell}$ is tangent to C. The direction of $d\overline{S}$ is normal to the surface following the right-hand rule with the fingers curled in the direction of C:

(Stokes theorem) $\qquad \iint_S (\nabla \times \overline{A}) \cdot d\overline{S} = \oint_C \overline{A} \cdot d\overline{\ell} \qquad (A.59)$

$$\iint_S (d\overline{S} \times \nabla\phi) = \oint_C \phi \, d\overline{\ell} \qquad (A.60)$$

Note that (A.50) and (A.59) give

$$\oint_C \nabla\phi \cdot d\overline{\ell} = 0 \qquad (A.61)$$

Green's identities and theorems provide additional relations between surface and volume integrals. These are often useful in proving orthogonality of eigenfunctions of the scalar and vector wave equations, and also for boundary-value problems using Green's functions. For two scalar functions ϕ and ψ, which are continuous through the second derivatives in the volume V, we have

(Green's first identity) $\qquad \iiint_V (\nabla\phi \cdot \nabla\psi + \phi\nabla^2\psi) \, dV = \oiint_S \phi\nabla\psi \cdot d\overline{S} \qquad (A.62)$

Interchanging ϕ and ψ and subtracting gives

(Green's theorem) $\qquad \iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi) \, dV = \oiint_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\overline{S}$

$$(A.63)$$

The vector forms of Green's identity and Green's theorem are

$$\iiint_V \left(\nabla \times \overline{A} \cdot \nabla \times \overline{B} - \overline{A} \cdot \nabla \times \nabla \times \overline{B} \right) dV = \oiint_S (\overline{A} \times \nabla \times \overline{B}) \cdot d\overline{S} \qquad (A.64)$$

$$\iiint_V \left(\overline{B} \cdot \nabla \times \nabla \times \overline{A} - \overline{A} \cdot \nabla \times \nabla \times \overline{B} \right) dV = \oiint_S (\overline{A} \times \nabla \times \overline{B} - \overline{B} \times \nabla \times \overline{A}) \cdot d\overline{S} \quad (A.65)$$

which also require that $\overline{A}$ and $\overline{B}$ are continuous through the second derivatives.

A.5.4 Distance Vector Identities

Let $\overline{R}$ be the position vector defined by the two points $\overline{r}$ and $\overline{r}'$ as shown below. Also define $\overline{R} = R\hat{R}$, where R is the distance between the points and $\hat{R}$ is the unit vector in the direction of $\overline{R}$.
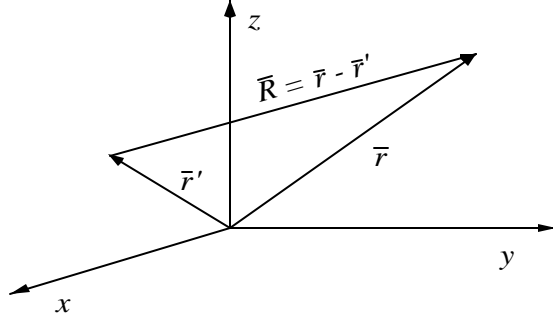
Then the following relations hold, where ∇ operates on *unprimed* coordinates:

$$\nabla R = \hat{R} \tag{A.66}$$

$$\nabla(1/R) = -\hat{R}/R^2 = -\overline{R}/R^3 \tag{A.67}$$

$$\nabla \cdot \overline{R} = 3 \tag{A.68}$$

$$\nabla \cdot \hat{R} = 2/R \quad \text{(from A.67 and A.68)} \tag{A.69}$$

$$\nabla \times \hat{R} = 0 \quad \text{(from A.66)} \tag{A.70}$$

$$\nabla \times \overline{R} = 0 \quad \text{(from A.66 and A.70)} \tag{A.71}$$

$$\nabla^2(1/R) = -4\pi\delta(\overline{R}) \tag{A.72}$$

$$\nabla \cdot (\overline{R}/R^3) = 4\pi\delta(\overline{R}) \quad \text{(from A.67 and A.72)} \tag{A.73}$$

In the following, $\overline{a}$ is any constant vector:

$$\nabla \cdot (\overline{a}/R) = \overline{a} \cdot \nabla(1/R) = -\overline{a} \cdot \hat{n}/R^2 \tag{A.74}$$

$$\nabla^2(\overline{a}/R) = \overline{a}\nabla^2(1/R) = -4\pi\overline{a}\delta(\overline{R}) \tag{A.75}$$

$$\nabla \times [\overline{a} \times (\hat{n}/R^2)] = 4\pi\overline{a}\delta(\overline{R}) - \nabla[(\overline{a} \cdot \hat{n})/R^2] \tag{A.76}$$

$$(\overline{a} \cdot \nabla)\hat{R} = \frac{1}{R}[\overline{a} - \hat{R}(\overline{a} \cdot \hat{R})] \tag{A.77}$$

$$(\overline{a} \cdot \nabla)\overline{R} = \overline{a} \quad \text{(from A.66 and A.77)} \tag{A.78}$$

A.5.5 The Helmholtz theorem

The Helmholtz theorem [3] states that a vector function $\overline{A}(\overline{r})$ can be expressed as the sum of two vector functions, one which has zero divergence (the solenoidal or rotational part) and one with zero curl (the lamellar, or irrotational part); that is,

$$\overline{A}(\overline{r}) = \nabla \times \overline{\xi} + \nabla \varphi \tag{A.79}$$

To show that such a decomposition is possible, take the divergence and curl of (A.79), which gives

$$\nabla^2 \varphi = \nabla \cdot \overline{A} \tag{A.80a}$$

$$\nabla \times \nabla \times \overline{\xi} = \nabla \times \overline{A} \tag{A.80b}$$

Since $\overline{A}$ is assumed known, these two differential equations are uncoupled and can (in principle) be solved independently for the pair of functions $(\varphi, \overline{\xi})$. This essentially proves the theorem. Note that in order to uniquely determine $\overline{A}$, both it's divergence and curl must be specified; this is an alternative statement of the Helmholtz theorem.

There are an infinite number of possible functions $\overline{\xi}$ which can be used to uniquely determine $\overline{A}$, since the gradient of an arbitrary scalar function, $\nabla \phi$, can always be added to $\overline{\xi}$ without changing (A.80); that is, if $\overline{\xi}$ is a solution of (A.80), so is $\overline{\xi} + \nabla \phi$. We can pick any function ϕ that is convenient; if ϕ is chosen such that $\nabla \cdot \overline{\xi} = 0$, then

$$\nabla \times \nabla \times \overline{\xi} = \nabla(\nabla \cdot \overline{\xi}) - \nabla^2 \overline{\xi} = -\nabla^2 \overline{\xi}$$

and (A.80) become

$$\nabla^2 \varphi = \nabla \cdot \overline{A} \tag{A.81a}$$

$$\nabla^2 \overline{\xi} = -\nabla \times \overline{A} \tag{A.81b}$$

From electrostatics, we know these have the solution (for unbounded regions)

$$\varphi(\overline{r}) = -\iiint \frac{\nabla' \cdot \overline{A}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} \, dV' \qquad \overline{\xi}(\overline{r}) = \iiint \frac{\nabla' \times \overline{A}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} \, dV'$$

and so (A.79) can be written as

$$\overline{A}(\overline{r}) = -\nabla \iiint \frac{\nabla' \cdot \overline{A}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} \, dV' + \nabla \times \iiint \frac{\nabla' \times \overline{A}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} \, dV' \tag{A.82}$$

If the field is to be represented in a bounded region, then the solutions to (A.81) must be modified accordingly, and it can be shown that the representation is, more generally,

$$\overline{A}(\overline{r}) = -\nabla \left(\iiint_V \frac{\nabla' \cdot \overline{A}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} \, dV' - \oiint_S \frac{\overline{A}(\overline{r}') \cdot d\overline{S}'}{4\pi|\overline{r} - \overline{r}'|} \right)$$

$$+ \nabla \times \left(\iiint_V \frac{\nabla' \times \overline{A}(\overline{r}')}{4\pi|\overline{r} - \overline{r}'|} \, dV' + \oiint_S \frac{\overline{A}(\overline{r}') \times d\overline{S}'}{4\pi|\overline{r} - \overline{r}'|} \right) \tag{A.83}$$

where S is the surface enclosing the volume V. This is the formal statement of the Helmholtz theorem.

A.5.6   Useful Vector Relations in Two-Dimensions

Situations arise where one dimension (usually taken as $\hat{z}$) can be factored out of the analysis. Let the subscript t represent vector components that are transverse to $\hat{z}$, so that:

$$\nabla = \nabla_t + \hat{z}\frac{\partial}{\partial z} \qquad \overline{A} = \overline{A}_t + \hat{z}A_z$$

Transverse and longitudinal components of other common operations can then be similarly decomposed

$$\overline{A} \times \overline{B} \quad = \quad \underbrace{A_z(\hat{z} \times \overline{B}_t) - B_z(\hat{z} \times \overline{A}_t)}_{\mathrm{transverse}} \quad + \quad \underbrace{\overline{A}_t \times \overline{B}_t}_{\mathrm{longitudinal}} \tag{A.84}$$

$$\nabla \times \overline{A} \quad = \quad \underbrace{-\hat{z} \times (\nabla_t A_z) + \frac{\partial}{\partial z}(\hat{z} \times \overline{A}_t)}_{\mathrm{transverse}} \quad + \quad \underbrace{\nabla_t \times \overline{A}_t}_{\mathrm{longitudinal}} \tag{A.85}$$

We use the earlier vector relations in three dimensions to prove the following identities:

$$\nabla_t \cdot \nabla_t \phi \;=\; \nabla_t^2 \phi \tag{A.86}$$

$$\nabla_t \cdot (\hat{z} \times \nabla_t \phi) \;=\; 0 \tag{A.87}$$

$$\nabla_t \times \nabla_t \phi \;=\; 0 \tag{A.88}$$

$$\hat{z} \times (\hat{z} \times \nabla_t \phi) \;=\; -\nabla_t \phi \tag{A.89}$$

$$\nabla_t \times (\hat{z} \times \nabla_t \phi) \;=\; \hat{z}\nabla_t^2 \phi \tag{A.90}$$

$$\hat{z} \times (\hat{z} \times \overline{A}_t) \;=\; -\overline{A}_t \tag{A.91}$$

$$(\hat{z} \times \overline{A}_t) \cdot (\hat{z} \times \overline{B}_t) \;=\; \overline{A}_t \cdot \overline{B}_t \tag{A.92}$$

$$\overline{A}_t \times (\hat{z} \times \overline{B}_t) \;=\; \hat{z}(\overline{A}_t \cdot \overline{B}_t) \tag{A.93}$$

$$\nabla_t \times (\hat{z} \times \overline{A}_t) \;=\; \hat{z}(\nabla_t \cdot \overline{A}_t) \tag{A.94}$$

$$(\hat{z} \times \overline{A}_t) \times (\hat{z} \times \overline{B}_t) \;=\; \overline{A}_t \times \overline{B}_t \tag{A.95}$$

$$\nabla_t \cdot (\hat{z} \times \overline{A}_t) \;=\; -\hat{z} \cdot (\nabla_t \times \overline{B}_t) \tag{A.96}$$

$$\overline{A}_t \cdot (\hat{z} \times \overline{B}_t) \;=\; -\hat{z} \cdot (\overline{A}_t \times \overline{B}_t) \tag{A.97}$$

In the following, S is an open surface bounded by a contour C described by line element $d\overline{\ell}$. The direction of $d\overline{\ell}$ is tangent to C, while the normal to C is described by $\hat{n}$.

$$\text{(2D Divergence theorem)} \qquad \iint_S (\nabla_t \cdot \overline{A})\, dS = \oint_C \overline{A} \cdot \hat{n}\, d\ell \tag{A.98}$$

Green's identity (A.60) and Green's theorem (A.61) generalize to two dimensions as follows:

$$\text{(2D Green's identity)} \qquad \iint_S (\nabla_t \phi \cdot \nabla_t \psi + \phi \nabla_t^2 \psi)\, dS = \oint_C \phi \frac{\partial \psi}{\partial n} d\ell \qquad \text{(A.99)}$$

$$\text{(2D Green's theorem)} \qquad \iint_S (\phi \nabla_t^2 \psi - \psi \nabla_t^2 \phi)\, dS = \oint_C \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] d\ell \text{ (A.100)}$$

A.5.7 Solid Angle

An element of surface area for a sphere of radius a, centered at the origin of a spherical coordinate system, is given by $dA = a^2 \sin\theta\, d\theta\, d\phi$. It is sometimes convenient to view this element of surface area as subtending a "solid angle", d , so that the angular integration in θ and ϕ is replaced by an integration over the range of "solid angles" subtended by the surface. That is, we write $dA = a^2\, d$, and integrating over the surface of the sphere gives

$$\oiint dA = a^2 \oiint d = 4\pi a^2$$

which is interpreted as meaning that the entire closed surface of the sphere subtends a total solid angle of 4π. The solid-angle is a unitless concept, but it is conventionally given the dimensionless units of *steradians*.
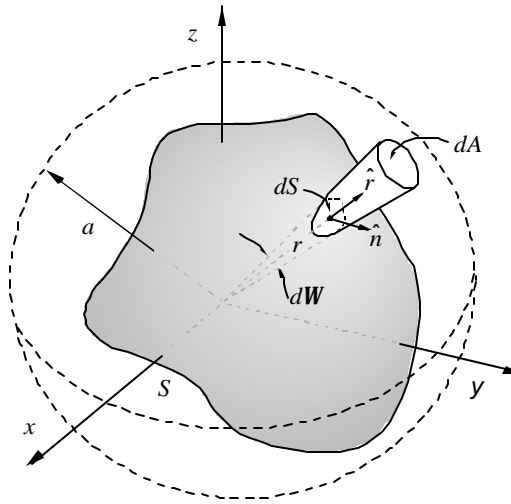
This concept can be extended to any arbitrary surface S by forming the projection of each surface element dS onto a sphere. In the figure above, dA is the projection of the surface element dS along the radial direction onto a sphere of radius a, centered at the origin. In doing so, both dA and dS subtend the same solid angle d , which from the

discussion above is defined as $d\Omega = dA/a^2$. The projection dA is found by taking the dot product of $d\overline{S} = \hat{n}\,dS$ with the radial unit vector, and scaling the result by a factor of a^2/r^2, where r is the distance to the surface element,

$$dA = \hat{r} \cdot d\overline{S}\frac{a^2}{r^2}$$

and therefore

$$d\Omega = \frac{\hat{r} \cdot d\overline{S}}{r^2} \qquad \text{and} \qquad \oiint d\Omega = \oiint \frac{\hat{r} \cdot d\overline{S}}{r^2} = 4\pi \qquad \text{(A.101)}$$

It is important to note that the result $\oint d\Omega = 4\pi$ is critically dependent on having chosen the surface enclose the origin of the coordinate system. Clearly if the origin were outside of the surface S, then the surface no longer subtends a total solid angle of 4π. Mathematically this can be seen as follows. From (A.67) note that

$$d\Omega = -\nabla(1/r) \cdot d\overline{S}$$

From the divergence theorem,

$$\oiint d\Omega = -\iiint \nabla^2(1/r)\,dV = 4\pi \iiint \delta(\overline{r})\,dV$$

where the last equality follows from (A.72). The last integral is zero unless the volume bounded by S contains the point $r = 0$. Shifting the coordinate system by $\overline{r}_0$, this result takes the more general form

$$\oiint_S \frac{\hat{R} \cdot d\overline{S}}{R^2} = \begin{cases} 4\pi & \overline{r}_0 \text{ inside } S \\ 0 & \overline{r}_0 \text{ outside } S \end{cases} \qquad \text{(A.102)}$$

where $\overline{R} = \overline{r} - \overline{r}_0$. This is essentially Gauss' law.

A.6   DIRAC DELTA FUNCTIONS

Dirac delta functions are a convenient mathematical shorthand that are used to help us out of difficult situations. In the context of Maxwell's equations, such difficulties can arise from our description of charge and current distributions as density functions, ρ and $\overline{J}$, respectively. For example, consider the charge density of a single electron—how do we represent such a thing? From a macroscopic point of view, the actual size of the electron is neglible, and acounting for it would unnecessarily complicate the mathematics. For an electron located at $\overline{r} = 0$ with charge q, a mathematical description of the charge density must have the properties

$$\rho(\overline{r}) = 0 \quad \text{for } \overline{r} \neq 0 \qquad \text{and} \qquad \iiint \rho(\overline{r})\,dV = q$$

where the integral is taken over the region contining the charge. A delta function in one dimension, written as $\delta(x)$, is defined to have similar properties, ie.

$$\delta(x - x_0) = 0 \quad \text{for } x \neq x_0 \qquad \text{and} \qquad \int_a^b \delta(x - x_0)\,dx = \begin{cases} 1 & a \leq x_0 \leq b \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.103)}$$

The most important property of the delta function follows from the above definition and involves its appearance in an integrand with another ordinary function, $f(x)$. As long as f is continuous at the location of the delta function singularity, then the only contribution to the integral will come from this point, and we get (in one dimension)

$$\int_a^b f(x)\delta(x - x_0)dx = \begin{cases} f(x_0) & a \leq x_0 \leq b \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.104)}$$

where the range of integration is taken over all values of x. This is called the "sifting" property of the delta function.

The extension to three dimensions is straightforward, at least in rectangular coordinates. We define $\delta(\overline{r} - \overline{r}')$ by the properties

$$\delta(\overline{r} - \overline{r}') = 0 \quad \text{for } \overline{r} \neq \overline{r}' \qquad \text{and} \qquad \iiint_V \delta(\overline{r} - \overline{r}')\,dV = \begin{cases} 1 & \text{if } \overline{r}' \text{ in V} \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.105)}$$

which in turn lead to the sifting property

$$\iiint_V f(\overline{r})\delta(\overline{r} - \overline{r}')\,dV = \begin{cases} f(\overline{r}') & \text{if } \overline{r}' \text{ in V} \\ 0 & \text{otherwise} \end{cases} \quad \text{(A.106)}$$

In rectangular coordinates, $dV = dx\,dy\,dz$, and therefore $\delta(\overline{r} - \overline{r}')$ can be represented as a product of three one dimensional delta functions

$$\text{rectangular:} \quad \delta(\overline{r} - \overline{r}') = \delta(x - x')\delta(y - y')\delta(z - z') \quad \text{(A.107)}$$

Returning to our original example, we find that the charge density function associated with a point charge at $\overline{r}'$ can now be represented concisely as

$$\rho(\overline{r}) = q\delta(\overline{r} - \overline{r}')$$

As another example, consider a current I_0 flowing along a thin wire colinear with the z-axis. Using the delta function, we can represent the corresponding current density as

$$\overline{J}(x, y, z) = I_0\delta(x)\delta(y)\hat{z}$$

Although the current density so defined is singular at $x = y = 0$, the integral over the cross section of the wire will remain finite and provide the correct answer

$$I = \iint \overline{J} \cdot d\overline{S} = \iint I_0\delta(x)\delta(y)\,dx\,dy = I_0$$

These examples also illustrate that the delta function must have units. If x represents a physical length dimension, then $\delta(x)$ has the units of inverse length. Examining the

expressions for the charge and current density above, we see that the correct units of $\left[\text{C/m}^3\right]$ and $\left[\text{A/m}^2\right]$ are obtained, respectively, with this association of units. From the sifting property of the three dimensional delta function, we see that it has the units of inverse volume, $[1/dV]$.

In three dimensions, the differential element of volume takes different forms in different coordinate systems, and so the delta function must be represented somewhat differently in each case. To transform from the representation in rectangular coordinates (A.107) to some other set of coordinates (u, v, w), we use the change of variable theorem of the previous section and note that the volume element in the new coordinate system is given by $|J|\,du\,dv\,dw$, where $|J|$ is the Jacobian of the transformation. Therefore a representation for the delta function is

$$\delta(\overline{r} - \overline{r}') = \frac{1}{|J|}\delta(u - u')\delta(v - v')\delta(w - w') \tag{A.108}$$

Using this we find, for cylindrical coordinates

$$\text{cylindrical:} \quad \delta(\overline{r} - \overline{r}') = \frac{1}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z') \tag{A.109}$$

and for spherical coordinates

$$\text{spherical:} \quad \delta(\overline{r} - \overline{r}') = \frac{\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi')}{r'^2 \sin\theta'} \tag{A.110}$$

There are situations where this approach breaks down, however, corresponding to the singularities of the Jacobian. This occurs when the delta function peak is located such that one of the variables (u, v, w) is irrelevant in the transformation. For example, in cylindrical coordinates if the delta function is located on the z-axis, the azimuthal angle ϕ does not appear in the transformation, and the representation is instead [4]

$$\delta(\overline{r} - \overline{r}') = \frac{1}{2\pi\rho}\delta(\rho)\delta(z - z') \tag{A.111}$$

One can always check the validity of a delta function representation using the integral properties defined above. Similarly in spherical coordinates, points on the z-axis (corresponding to $\theta' = 0$ or $\theta' = \pi$) are represented by

$$\delta(\overline{r} - \overline{r}') = \frac{1}{2\pi r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta') \tag{A.112}$$

For points at the origin, both θ and ϕ are irrelevant, and

$$\delta(\overline{r} - \overline{r}') = \frac{\delta(r)}{4\pi r^2} \tag{A.113}$$

Having shown how the three-dimensional delta function can be represented by products of one-dimensional delta functions, we now list some additional properties of the latter that are useful in electromagnetic analysis:

$$\delta(ax - b) = \frac{1}{|a|}\delta(x - b/a) \tag{A.114}$$

$$\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right] \tag{A.115}$$

$$\int \delta(x-a)\delta(x-b)\,dx = \delta(a-b) \tag{A.116}$$

$$\int f(x)\delta'(x-a)\,dx = -f'(a) \tag{A.117}$$

where in the last relation the prime denotes a derivative with respect to the argument. Another useful transformation is given by

$$\delta\left(f(x)\right) = \sum_i \frac{\delta(x-x_i)}{|df(x_i)/dx|} \tag{A.118}$$

where x_i are the zeroes of $f(x)$, ie. $f(x_i) = 0$, and the summation is over all the possible zeroes. A more exhaustive collection of delta function properties relevant to electromagnetic theory is found in [4].

A.7   DYADIC ANALYSIS

In elementary vector analysis we frequently encounter scalar relationships between two vectors, such as in Ohm's law, $\overline{J} = \sigma\overline{E}$, where σ is a scalar quantity. In matrix form,

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \sigma \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

This is a very simple relationship which takes the vector quantity $\overline{E}$ and scales each component by the number σ to give a new vector, $\overline{J}$, which consequently retains the original direction of $\overline{E}$. A more general linear transformation would allow each component of $\overline{E}$ to influence each component of $\overline{J}$, so that the transformation changes the *direction* as well as the magnitude (ie. involves a rotation in addition to a scaling). We could write this in matrix form as

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

The matrix $[\sigma]$ is referred to as a *second-rank tensor*. Each component of the tensor describes the influence of one field quantity on another; for example, σ_{xz} describes the $\hat{x}$-component of current flow due to the $\hat{z}$-component of the electric field. Such tensor relationships arise in many physical contexts, such as current flow in an anisotropic crystal, or wave propagation in a plasma. Naturally the mathematics becomes more complicated, which is why tensor relationships are rarely covered in elementary electromagnetics courses!

The tensor relationship can be written in a different way using vector notation as

$$\overline{J} = \overline{\overline{\sigma}} \cdot \overline{E} \tag{A.119}$$

where $\overline{\overline{\sigma}}$ is defined as

$$
\begin{aligned}
\overline{\overline{\sigma}} = \; & \sigma_{xx}\hat{x}\hat{x} + \sigma_{xy}\hat{x}\hat{y} + \sigma_{xz}\hat{x}\hat{z} + \\
& \sigma_{yx}\hat{y}\hat{x} + \sigma_{yy}\hat{y}\hat{y} + \sigma_{yz}\hat{y}\hat{z} + \\
& \sigma_{zx}\hat{z}\hat{x} + \sigma_{zy}\hat{z}\hat{y} + \sigma_{zz}\hat{z}\hat{z}
\end{aligned}
$$

The only new feature is the appearance of products of unit vectors. This definition gives the correct result using the normal rules for the vector dot product, provided we *strictly obey* the order of the unit vectors and the dot product. For example, $\hat{x}\hat{y} \cdot \overline{E} = \hat{x}E_y$, but interchanging the order of the unit vectors clearly gives a different result, $\hat{y}\hat{x} \cdot \overline{E} = \hat{y}E_x$. Similarly, we can see that interchanging the order of the dot product in (A.119) also markedly affects the result, ie. $\overline{\overline{\sigma}} \cdot \overline{E} \neq \overline{E} \cdot \overline{\overline{\sigma}}$. This is not surprising given the obvious similarity between this new quantity $\overline{\overline{\sigma}}$ and an ordinary matrix.

Since the components of $\overline{\overline{\sigma}}$ are characterized by pairs of unit vectors, it is called a *dyad*, or a *dyadic* quantity (the word "dyad" means pair). Clearly there is a close relationship between dyads and second-rank tensors.

A dyad or dyadic operator is expressable as the algebraic product of two vectors or vector operators, much like a matrix can be formed from the product of two vectors,

$$\overline{\overline{P}} = \overline{X}\,\overline{Y} \tag{A.120}$$

To the extent that the *vector* fields represent (or can be related to) physically meaningful quantities, a dyad only has meaning when it acts upon another vector. However, we can often ascribe an independent physical significance to dyads such as $\overline{\overline{\sigma}}$, in this case the "conductivity" dyadic. As noted above, dyad-vector multiplications do not obey the familiar vector commutation rules (A.36)-(A.37), but obey instead the matrix-like commutative laws

$$
\begin{aligned}
\overline{\overline{P}} \cdot \overline{A} &= \overline{A} \cdot \overline{\overline{P}}^T \\
\overline{\overline{P}} \times \overline{A} &= -\left(\overline{A} \times \overline{\overline{P}}^T\right)^T
\end{aligned}
$$

where the superscript T suggests a matrix-like transpose operation. For example,

$$
\begin{aligned}
\overline{\overline{\sigma}}^T = \; & \sigma_{xx}\hat{x}\hat{x} + \sigma_{yx}\hat{y}\hat{x} + \sigma_{zx}\hat{z}\hat{x} + \\
& \sigma_{xy}\hat{x}\hat{y} + \sigma_{yy}\hat{y}\hat{y} + \sigma_{zy}\hat{z}\hat{y} + \\
& \sigma_{xz}\hat{x}\hat{z} + \sigma_{yz}\hat{y}\hat{z} + \sigma_{zz}\hat{z}\hat{z}
\end{aligned}
$$

Consequently, one must resist the temptation to use dyads in place of vectors in the vector identities of section A.5, which are derived assuming the simpler vector commutation laws (A.36)-(A.37) where ordering of the vectors is not as significant.

It is frequently useful to employ a unit dyad, $\overline{\overline{I}}$, defined such that

$$\overline{A} \cdot \overline{\overline{I}} = \overline{\overline{I}} \cdot \overline{A} = \overline{A} \tag{A.121}$$

In rectangular coordinates,

$$\overline{\overline{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{A.122}$$

This is analogous to the identity matrix in linear algebra.

In electromagnetic theory, dyadic notation is frequently used for brevity. Once the reader becomes familiar with the notation, we find it can be employed in many situations formerly handled by vector manipulations. A simple example given in the text is the function $\nabla\nabla \cdot \overline{A}$. Ordinarily this expression is understood to mean $\nabla(\nabla \cdot \overline{A})$, but it can also be represented as $(\nabla\nabla) \cdot \overline{A}$, where $\nabla\nabla$ is a dyadic operator. Similarly, the function $\overline{J} - (\hat{r} \cdot \overline{J})\hat{r}$ which appears in the radiation integrals can be represented by $(\overline{\overline{I}} - \hat{r}\hat{r}) \cdot \overline{J}$

A.7.1 Dyadic Dot and Cross Product Identities

$$\overline{A} \cdot \overline{\overline{P}} = \overline{\overline{P}}^T \cdot \overline{A} \tag{A.123}$$

$$\left(\overline{A} \times \overline{\overline{P}}\right)^T = -\overline{\overline{P}}^T \times \overline{A} \tag{A.124}$$

$$\overline{A} \cdot \overline{\overline{P}} \cdot \overline{B} = \left(\overline{A} \cdot \overline{\overline{P}}\right) \cdot \overline{B} = \overline{A} \cdot \left(\overline{\overline{P}} \cdot \overline{B}\right) \tag{A.125}$$

$$\overline{A} \cdot \overline{\overline{P}} \cdot \overline{B} = \overline{B} \cdot \overline{\overline{P}}^T \cdot \overline{A} \tag{A.126}$$

$$\left(\overline{A} \times \overline{B}\right) \cdot \overline{\overline{P}} = \overline{A} \cdot \left(\overline{B} \times \overline{\overline{P}}\right) = -\overline{B} \cdot \left(\overline{A} \times \overline{\overline{P}}\right) \tag{A.127}$$

$$\overline{\overline{P}} \cdot \left(\overline{A} \times \overline{B}\right) = -\left(\overline{\overline{P}} \times \overline{B}\right) \cdot \overline{A} = \left(\overline{\overline{P}} \times \overline{A}\right) \cdot \overline{B} \tag{A.128}$$

$$\overline{A} \times \left(\overline{B} \times \overline{\overline{P}}\right) = \overline{B}\left(\overline{A} \cdot \overline{\overline{P}}\right) - \left(\overline{A} \cdot \overline{B}\right)\overline{\overline{P}} \tag{A.129}$$

$$\left(\overline{A} \times \overline{\overline{P}}\right) \cdot \overline{B} = \overline{A} \times \left(\overline{\overline{P}} \cdot \overline{B}\right) = \overline{A} \times \overline{\overline{P}} \cdot \overline{B} \tag{A.130}$$

$$\left(\overline{A} \cdot \overline{\overline{P}}\right) \times \overline{B} = \overline{A} \cdot \left(\overline{\overline{P}} \times \overline{B}\right) = \overline{A} \cdot \overline{\overline{P}} \times \overline{B} \tag{A.131}$$

$$\left(\overline{A} \times \overline{\overline{P}}\right) \times \overline{B} = \overline{A} \times \left(\overline{\overline{P}} \times \overline{B}\right) = \overline{A} \times \overline{\overline{P}} \times \overline{B} \tag{A.132}$$

$$\left(\overline{\overline{P}} \cdot \overline{\overline{Q}}\right)^T = \overline{\overline{Q}}^T \cdot \overline{\overline{P}}^T \tag{A.133}$$

$$\left(\overline{A} \cdot \overline{\overline{P}}\right) \cdot \overline{\overline{Q}} = \overline{A} \cdot \left(\overline{\overline{P}} \cdot \overline{\overline{Q}}\right) = \overline{A} \cdot \overline{\overline{P}} \cdot \overline{\overline{Q}} \tag{A.134}$$

$$\left(\overline{\overline{P}} \cdot \overline{\overline{Q}}\right) \cdot \overline{A} = \overline{\overline{P}} \cdot \left(\overline{\overline{Q}} \cdot \overline{A}\right) = \overline{\overline{P}} \cdot \overline{\overline{Q}} \cdot \overline{A} \tag{A.135}$$

$$\overline{\overline{P}} \cdot \left(\overline{A} \times \overline{\overline{Q}}\right) = \left(\overline{\overline{P}} \times \overline{A}\right) \cdot \overline{\overline{Q}} \tag{A.136}$$

$$\left(\overline{A} \times \overline{\overline{P}}\right) \cdot \overline{\overline{Q}} = \overline{A} \times \left(\overline{\overline{P}} \cdot \overline{\overline{Q}}\right) = \overline{A} \times \overline{\overline{P}} \cdot \overline{\overline{Q}} \qquad (A.137)$$

$$\left(\overline{\overline{P}} \cdot \overline{\overline{Q}}\right) \times \overline{A} = \overline{\overline{P}} \cdot \left(\overline{\overline{Q}} \times \overline{A}\right) = \overline{\overline{P}} \cdot \overline{\overline{Q}} \times \overline{A} \qquad (A.138)$$

A.7.2   Differential Operations Involving Dyads

$$\nabla \left(\phi \overline{A}\right) = (\nabla \phi)\, \overline{A} + \phi \nabla \overline{A} \qquad (A.139)$$

$$\nabla \cdot \left(\phi \overline{\overline{P}}\right) = (\nabla \phi) \cdot \overline{\overline{P}} + \phi \nabla \cdot \overline{\overline{P}} \qquad (A.140)$$

$$\nabla \times \left(\phi \overline{\overline{P}}\right) = (\nabla \phi) \times \overline{\overline{P}} + \phi \nabla \times \overline{\overline{P}} \qquad (A.141)$$

$$\nabla \cdot \left(\overline{A}\,\overline{B}\right) = (\nabla \cdot \overline{A})\, \overline{B} + \overline{A} \cdot (\nabla \overline{B}) \qquad (A.142)$$

$$\nabla \cdot \left(\overline{A}\,\overline{B} - \overline{B}\,\overline{A}\right) = \nabla \times \left(\overline{B} \times \overline{A}\right) \qquad (A.143)$$

$$\nabla \times \left(\overline{A}\,\overline{B}\right) = (\nabla \times \overline{A})\, \overline{B} - \overline{A} \times \nabla \overline{B} \qquad (A.144)$$

$$\nabla \left(\overline{A} \times \overline{B}\right) = (\nabla \overline{A}) \times \overline{B} - (\nabla \overline{B}) \times \overline{A} \qquad (A.145)$$

$$\nabla \cdot \left(\overline{A} \times \overline{\overline{P}}\right) = (\nabla \times \overline{A}) \cdot \overline{\overline{P}} - \overline{A} \cdot \nabla \times \overline{\overline{P}} \qquad (A.146)$$

$$\nabla \times \left(\nabla \overline{A}\right) = 0 \qquad (A.147)$$

$$\nabla \cdot \left(\nabla \times \overline{\overline{P}}\right) = 0 \qquad (A.148)$$

$$\nabla \times \left(\nabla \times \overline{\overline{P}}\right) = \nabla \left(\nabla \cdot \overline{\overline{P}}\right) - \nabla^2 \overline{\overline{P}} \qquad (A.149)$$

A.7.3   Properties of the Unit Dyad

$$\overline{\overline{I}} = \overline{\overline{I}}^T \qquad (A.150)$$

$$\overline{A} \cdot \overline{\overline{I}} = \overline{\overline{I}} \cdot \overline{A} = \overline{A} \qquad (A.151)$$

$$\overline{\overline{I}} \times \overline{A} = \overline{A} \times \overline{\overline{I}} \qquad (A.152)$$

$$\left(\overline{A} \times \overline{\overline{I}}\right) \cdot \overline{B} = \overline{A} \cdot \left(\overline{\overline{I}} \times \overline{B}\right) = \overline{A} \times \overline{B} \qquad (A.153)$$

$$\overline{\overline{I}} \times \left(\overline{A} \times \overline{B}\right) = \overline{B}\,\overline{A} - \overline{A}\,\overline{B} \qquad (A.154)$$

$$\left(\overline{A} \times \overline{\overline{I}}\right) \cdot \overline{\overline{P}} = \overline{A} \times \overline{\overline{P}} \qquad (A.155)$$

$$\nabla \cdot \left(\phi \overline{\overline{I}}\right) = \nabla \phi \qquad (A.156)$$

$$\nabla \cdot \left(\overline{\overline{I}} \times \overline{A}\right) = \nabla \times \overline{A} \qquad (A.157)$$

$$\nabla \times \left(\phi \overline{\overline{I}}\right) = \nabla \phi \times \overline{\overline{I}} \qquad (A.158)$$

$$(A.159)$$

A.7.4   Integral relations

We can generalize the earlier vector integral theorems in a straightforward manner to accomodate dyadic functions. In the following, V is a volume bounded by a closed surface S, with the direction of $d\overline{S}$ taken as pointing outward from the enclosed volume, by convention:

$$\iiint_V \nabla \cdot \overline{\overline{P}} \, dV \;=\; \oiint_S d\overline{S} \cdot \overline{\overline{P}} \tag{A.160}$$

$$\iiint_V \nabla \overline{A} \, dV \;=\; \oiint_S d\overline{S}\, \overline{A} \tag{A.161}$$

$$\iiint_V \nabla \times \overline{\overline{P}} \, dV \;=\; \oiint_S d\overline{S} \times \overline{\overline{P}} \tag{A.162}$$

In the following, S is an open surface bounded by a contour C described by line element $d\overline{\ell}$. The direction of $d\overline{\ell}$ is tangent to C, while the normal to C is described by $\hat{n}$. The direction of $d\overline{S}$ is normal to the surface following the right-hand rule with the fingers curled in the direction of C:

$$\iint_S d\overline{S} \cdot \left(\nabla \times \overline{\overline{P}} \right) \;=\; \oint_C d\overline{\ell} \cdot \overline{\overline{P}} \tag{A.163}$$

$$\iint_S (d\overline{S} \times \nabla \overline{A}) \;=\; \oint_C d\overline{\ell}\, \overline{A} \tag{A.164}$$

The vector-dyadic form of Green's identity (A.64) and Green's theorem (A.65) are (see [5] for a derivation)

$$\iiint_V \left[(\nabla \times \overline{A}) \cdot \nabla \times \overline{\overline{P}} - \overline{A} \cdot \nabla \times \nabla \times \overline{\overline{P}} \right] dV = \oiint_S \hat{n} \cdot (\overline{A} \times \nabla \times \overline{\overline{P}}) \, dS \tag{A.165}$$

$$\iiint_V \left[(\nabla \times \nabla \times \overline{A}) \cdot \overline{\overline{P}} - \overline{A} \cdot \nabla \times \nabla \times \overline{\overline{P}} \right] dV$$

$$= \oiint_S \hat{n} \cdot \left[\overline{A} \times \nabla \times \overline{\overline{P}} + (\nabla \times \overline{A}) \times \overline{\overline{P}} \right] dS \tag{A.166}$$

where $d\overline{S} = \hat{n}\, dS$. The dyadic-dyadic forms of the above are:

$$\iiint_V \left[\left(\nabla \times \overline{\overline{Q}} \right)^T \cdot \nabla \times \overline{\overline{P}} - \left(\nabla \times \nabla \times \overline{\overline{Q}} \right)^T \cdot \overline{\overline{P}} \right] dV$$

$$= \oiint_S \left(\nabla \times \overline{\overline{Q}} \right)^T \cdot \left(\hat{n} \times \overline{\overline{P}} \right) dS \tag{A.167}$$

$$\iiint \left[\overline{\overline{Q}}^T \cdot \nabla \times \nabla \times \overline{\overline{P}} - \left(\nabla \times \nabla \times \overline{\overline{Q}} \right)^T \cdot \overline{\overline{P}} \right] dV$$

$$= \oiint \left[\left(\nabla \times \overline{\overline{Q}} \right)^T \cdot \left(\hat{n} \times \overline{\overline{P}} \right) + \overline{\overline{Q}}^T \cdot \left(\hat{n} \times \nabla \times \overline{\overline{P}} \right) \right] dS \qquad \text{(A.168)}$$

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