BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W.   WASHINGTON, D. C. 20024

B70 02042

SUBJECT: Some Characteristics of Coelliptic Orbits - Case 610

DATE: February 17, 1970

FROM: P. H. Whipple

ABSTRACT

Coelliptic orbits can be defined as two orbits that are coplanar and confocal. A property of coelliptic orbits is that the difference in magnitude between aligned radius vectors is nearly the same, regardless of where within the orbits they are positioned. For this and other reasons, coelliptic orbits are useful in rendezvous trajectories for linking together the earlier part of the chase vehicle trajectory where most of the altitude adjustment and required phasing are achieved, with the terminal portion of the rendezvous trajectory.

For conic coelliptic orbits, the differences in perigee radii, apogee radii, and semi-major axes all have the same value. When the aligned radius vectors are not directed along the line of apsides, deviations from this value do occur but they are surprisingly small for conic coelliptic orbits of low eccentricity, typically being about one foot maximum. For non-conic coelliptic earth orbits, these deviations remain small but can increase to 100 to 200 feet due to differential perturbations caused by the earth's asphericity.

(NASA-CR-109655) SOME CHARACTERISTICS OF COELLIPTIC ORBITS (Bellcomm, Inc.) 9 p

N79-71970

Unclas
00/13 12753

BELLCOMM, INC.
955 L'ENFANT PLAZA NORTH, S.W. WASHINGTON, D.C. 20024

B70 02042

SUBJECT: Some Characteristics of Coelliptic Orbits - Case 610

DATE: February 17, 1970

FROM: P. H. Whipple

MEMORANDUM FOR FILE

I. Introduction

In the concentric orbit rendezvous scheme used in the Apollo and Apollo Applications Programs, the trajectory of a chase vehicle from insertion into orbit to rendezvous with an orbiting target vehicle is comprised of an initial phase, a coelliptic orbit phase, and a terminal phase. The completion of the initial phase occurs near one of the apogee or perigee locations where the propulsive maneuvers that accomplish most of the altitude adjustment and phasing are usually confined to occur. The initiation point of the terminal phase is generally considerably removed from this point in order to provide for docking of the spacecraft in daylight and a standardized design of the terminal phase. The coelliptic orbit serves as a linking and storage orbit for the chase vehicle, allowing it to have a small amount of phasing capability while preserving the altitude adjustment accomplished in the initial phase.

In the coelliptic orbit phase, the chase vehicle travels in an orbit that is coelliptic with the target vehicle orbit, i.e., the difference in magnitude between the chase vehicle radius vector and an aligned radius vector in the target orbit is nearly constant and independent of the chase vehicle position within its orbit. While this relationship is obviously satisfied for circular orbits, it is not clear that such a condition does exist for elliptic orbits. This memorandum investigates this feature and presents methods of establishing coelliptic orbits.

II. Discussion

A. Geometric Characteristics

Coelliptic orbits can be defined as having the following properties:

1. The orbits have a common occupied focus.

2. The orbits are coplanar.

3. The perigees of the two orbits lie along the same line from the focus.

4. The difference in perigee and apogee radii are equal.

These conditions establish the meaning of "coelliptic" with mathematical precision even for highly eccentric orbits. While other definitions might be invented, this is the commonly accepted one. In particular, Apollo and AAP guidance programs are based on this definition. Such a pair of orbits are shown in Figure 1. It follows that

$$R_{p_2} - R_{p_1} = R_{a_2} - R_{a_1} = \Delta h$$

$$2a_2 = R_{a_2} + R_{p_2} = R_{a_1} + \Delta h + R_{p_1} + \Delta h$$

$$= 2a_1 + 2\Delta h$$

and

$$a_2 = a_1 + \Delta h,$$

a not unexpected relation between the semi-major axes. From the equation for a conic evaluated at perigee,

$$R = \frac{a(1-e^2)}{1+e\cos f}, f = 0,$$

it follows that

$$R_{p_2} - R_{p_1} = \frac{a_2(1-e_2{}^2)}{1+e_2} - \frac{a_1(1-e_1{}^2)}{1+e_1}$$

$$= \Delta h - a_2e_2 + a_1e_1$$

and

$$a_2 e_2 = a_1 e_1$$

where R = radius
a = semi-major axis
e = eccentricity
f = true anomaly.

Since the distance between foci for any ellipse is equal to 2ae, the two ellipses have the same vacant focus also. If the definition of coelliptic orbits had been taken initially that they simply be coplanar and confocal, all of the above characteristics could easily have been derived.

To see how well the differential radius relationship is preserved around the orbit, the following equation is necessary.

$$\Delta R = R_2 - R_1 = \frac{a_2(1-e_2^2)}{1 + e_2 \cos f_1} - \frac{a_1(1-e_1^2)}{1 + e_1 \cos f_1} \tag{1}$$

The exact error in ΔR, defined as $E\Delta R = \Delta R - \Delta h$, can be evaluated by rearranging equation (1) into the following form.

$$E\Delta R = \Delta R - \Delta h = a_1 e_1 \cos f_1 \left\{ \frac{1-e_1^2}{1+e_1 \cos f_1} - \frac{1-e_2^2}{1+e_2 \cos f_1} \right\} + e_2^2 (\Delta h)\left(1+\frac{\Delta h}{a_1}\right) \tag{2}$$

It is clear that for non-circular coelliptic orbits, ΔR is not equal to Δh exactly. It can be confirmed that $E\Delta R=0$ at f_1=zero and 180 degrees. Furthermore, differentiating (2) with respect to f_1, it is found that the maximum value of this error occurs when

$$\cos f_1 = \cos f_{1_M} = \frac{\sqrt{\dfrac{1 - e_2^2}{1 - e_1^2}} - 1}{e_2 - e_1 \sqrt{\dfrac{1 - e_2^2}{1 - e_1^2}}}$$

where the radical must assume a positive sign. The corresponding exact expression for the maximum error in ΔR is given by

$$E\Delta R_{MAX} = 2a_1 \left(1 + \frac{a_1}{\Delta h}\right) \left(1 - \frac{1}{2}\left(e_1^2 + e_2^2\right) - \sqrt{1-e_1^2}\sqrt{1-e_2^2}\right)$$

$$+ e_2^2 (\Delta h) \left(1 + \frac{\Delta h}{a_1}\right) .$$

After expanding $\sqrt{1 - e_1^2}\ \sqrt{1 - e_2^2}$ in an infinite series, it is seen that

$$1 - \frac{1}{2}\left(e_1^2 + e_2^2\right) - \sqrt{1 - e_1^2}\sqrt{1 - e_2^2} = \frac{1}{8}\left(e_1^2 - e_2^2\right)^2$$

+ higher order terms in e_1 and e_2.

After neglecting these higher order terms,

$$E\Delta R_{MAX} \cong e_2^2 (\Delta h)\left(1 + \frac{\Delta h}{a_1}\right)\left(1 + e_2^2\left(1 + \frac{\Delta h}{2a_1}\right)^2\right).$$

For AAP missions, Δh is typically about ten nautical miles and a_1 is typically about 3650 nautical miles. Therefore $\frac{\Delta h}{a_1} \ll 1$ and

$$E\Delta R_{MAX} \cong e_2^2 (\Delta h) .$$

For an outer orbit with a 185 nautical mile perigee altitude and 210 nautical mile apogee altitude, $e_2 \cong .0034$ and $E\Delta R_{MAX} \cong .71$ feet! For an unlikely outer orbit with an 80 nautical mile perigee altitude and 300 nautical mile apogee altitude, $E\Delta R_{MAX}$ is only about 55 feet. It is clear that for all practical conic coelliptic orbits, ΔR at any value of f_1 can be taken equal to Δh with surprising accuracy.

An alternate exact expression for ΔR can be written in terms of the eccentric anomalies of the two vehicles.

$$\Delta R = R_2 - R_1 = a_2(1 - e_2 \cos E_2) - a_1(1 - e_1 \cos E_1)$$

$$= \Delta h + a_1 e_1 (\cos E_1 - \cos E_2).$$

It can be shown that the maximum value of $E\Delta R$ occurs when $\sin E_1 = \sin E_2$ and $\cos E_1 = -\cos E_2$. An alternate exact expression for the maximum error is then

$$E\Delta R_{MAX} = 2a_1 e_1 \cos E_{1_M} \quad .$$

where E_{1_M} is given by

$$\tan \frac{E_{1_M}}{2} = \sqrt{\frac{1 - e_1}{1 + e_1}} \tan \frac{f_{1_M}}{2} \quad .$$

B. Initiation of the Chase Vehicle Coelliptic Orbit

At the conclusion of the initial phase, a propulsive maneuver is required to establish the chase vehicle coelliptic orbit. The required direction and magnitude of the chase vehicle velocity after the burn must be computed to correctly orient the spacecraft for the burn. This can easily be done by computing the required radial and horizontal velocities of the chase vehicle as functions of the radial and horizontal velocities of the target vehicle and other known orbital characteristics.

Assuming that the chase vehicle is traveling the inner orbit of Figure 1, the magnitude of the total velocity in the coelliptic orbit is easily computed from

$$V_1{}^2 = \mu\left(\frac{2}{R_1} - \frac{1}{a_1}\right) \tag{3}$$

where $a_1 = a_2 - \Delta h$ and

$$a_2 = \frac{1}{\frac{2}{R_2} - \frac{V_2^2}{\mu}} \quad .$$

The radial velocity of the chase vehicle is given by

$$\dot{R}_1 = \frac{dR_1}{dt} = \frac{\mu}{h_1} e_1 \sin f_1$$

where h is the magnitude of the angular momentum. After some algebraic manipulation, the exact expression for the ratio of the radial velocities of the two vehicles is seen to be

$$\frac{\dot{R}_1}{\dot{R}_2} = \left(\frac{a_2}{a_1}\right)^{1.5}\left[\frac{1 - e_2{}^2}{1 - e_1{}^2}\right]^{.5} \tag{4}$$

where R_2 is aligned with R_1. For $e_1{}^2 \ll 1$ and $e_2^2 \ll 1$

$$\dot{R}_1 \cong \left(\frac{a_2}{a_1}\right)^{1.5} \dot{R}_2 \quad . \tag{5}$$

The required horizontal velocity of the chase vehicle can then be computed from

$$V_{1H}^2 = V_1^{\ 2} - (\dot{R}_1)^2 \quad . \tag{6}$$

Alternatively, the horizontal velocity can be computed direct from

$$V_{1H} = R_1 \dot{f}_1 = \frac{\mu}{h_1} (1 + e_1 \cos f_1),$$

$$\frac{V_{1H}}{V_{2H}} = \left(\frac{R_2}{R_1}\right) \left(\frac{a_1(1-e_1^2)}{a_2\ (1-e_2^{\ 2})}\right)^{.5} \tag{7}$$

and

$$V_{1H} \cong \left(\frac{R_2}{R_1}\right) \left(\frac{a_1}{a_2}\right)^{.5} V_{2H} \tag{8}$$

where R_2 is again aligned with R_1 and (7) and (8) are exact and approximate expressions respectively.

These expressions have been used to establish coelliptic orbits in trajectory simulations made with the Bellcomm Apollo Simulation Program. Simulations were made for an outer orbit with a 185 nautical mile perigee altitude, 210 nautical mile apogee altitude, a Δh of ten nautical miles, and with spherical and aspherical gravitational models of the earth. For each of these conditions, various combinations of the exact and approximate expressions for $\dot{R}_1$ and V_H as given by equations (4) through (8) were used with almost identical results. When the spherical gravitational model of the earth was used, the maximum value of $E\Delta R$ encountered through a complete revolution in the coelliptic orbit ranged from two to five feet, depending upon which combination of radial and horizontal velocity expressions were used. When the aspherical gravitational model of the earth was used, the maximum $E\Delta R$ encountered throughout a complete revolution in the coelliptic orbit varied from 44 to 139 feet, depending primarily upon where within the orbit the chase vehicle orbit was established and to a much small extent, upon the expressions used for the radial and horizontal velocities at the establishment of the orbit.

Simulations were made with similar results for a circular outer orbit of 235 nautical miles altitude and an outer orbit with a 110 nautical mile perigee altitude and a 210 nautical mile apogee altitude. The maximum value of $E\Delta R$ throughout a complete revolution varied from 42 to 134 feet in the first case and from 7 to 166 feet in the latter case.

The logic for the coelliptic burn initiation in the Command Module and Lunar Module computers uses the approximate equations (5) and (6) to determine the required velocity vector of the chase vehicle.

III. Summary

It has been shown that coelliptic orbits are very effective in preserving a desired difference in magnitude between aligned radius vectors in the chase and target vehicle orbits. By coasting in its orbit for an appropriate interval of time, the chase vehicle can achieve the required phase relationship with the target vehicle for the terminal phase without disturbing the altitude adjustment accomplished in the initial phase of the rendezvous.

The deviation from the desired difference in magnitude of the radii can be attributed primarily to the perturbations due to the earth's asphericity and to a much smaller degree, to the geometric characteristics of the coelliptic orbits. For practical earth orbits, this deviation due to the earth's asphericity can be as much as 100 to 200 feet while the deviation due to the geometric characteristics is typically about one foot.

In establishing the chase vehicle orbit to be coelliptic with the target vehicle orbit, simplified approximate expressions for the required radial and horizontal velocities can be used with essentially no penalty in accuracy.

P H Whipple

1025-PHW-li P. H. Whipple

Attachment

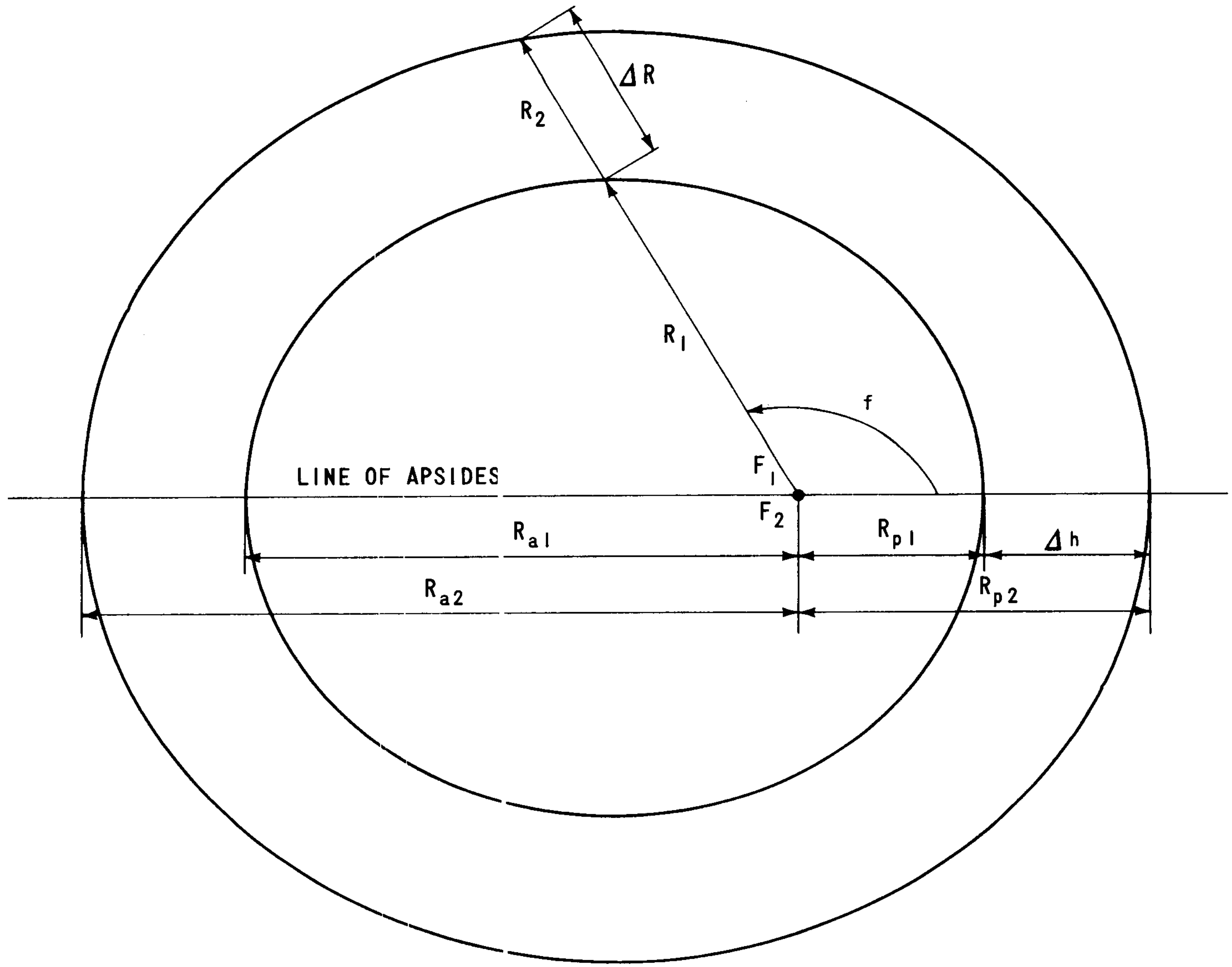

FIGURE 1 - COELLIPTIC ORBITS