Chapter 8

The Solution of Equations of the Fifth Degree

We seek the solution of the equation $x^5 = 2625x + 61500$.

8.1 This chapter closely follows a talk given in 1977 by the author at the Philips Contest for Young Scientists and Inventors, "Special equations of the fifth degree that are solvable in radicals." The equation presented above is again a classical example. Already in 1762, Leonhard Euler recognized from his studies of solvability of equations that this equation belongs to a class of fifth-degree equations that can be solved in radicals. Like other mathematicians of his time, Euler had attempted to extend the methods for equations of degree less than five to those of fifth degree. Even the mountain of formulas that resulted could not dampen Euler's optimism, for he wrote,

> One may conjecture with apparent certainty that with the correct approach to this elimination procedure, one would finally arrive at an equation of fourth degree. If the result were an equation of higher degree, then ... [the previously used intermediate value for representing the solutions] would itself contain roots of this degree, and that would seem to be unreasonable.

However, in his actual calculations, Euler had to trim his sails somewhat:

> However, since the large number of expressions makes this task so difficult that one cannot achieve any measure of success, it seems appropriate to develop some special cases that do not lead to such complex formulas.[1]

Euler refers to the intermediate results he used as "such values as shorten the calculations." In reality, Euler has avoided not merely calculational difficulties, but the basic impossibility of a general solution. Nonetheless, in this way he arrives at a large class of fifth-degree equations that can be solved in radicals. Since this class does not contain all solvable fifth-degree equations, we will look here at the work of another mathematician. In 1771, thus at almost the same time as the work of Lagrange and Vandermonde, the Italian mathematician Giovanni Francesco Malfatti (1731–1807) was searching for a general formula for equations of the fifth degree. Malfatti, who later, in 1804, commented critically on Ruffini's first attempts at an unsolvability proof based on his own work and thereby motivated Ruffini to refine his work, succeeded in carrying out extremely complicated calculations of a resolvent of the sixth degree. This did not lead to the original goal of a general solution. However, Malfatti noticed that in the special case in which the sixth-degree resolvent possesses a rational solution, the given fifth-degree equation can be solved. Later, using Galois theory, it could be shown that Malfatti had characterized all equations of the fifth degree that are solvable in radicals (in relation to all irreducible fifth-degree polynomials over the rational numbers).

Malfatti's computations are very complicated, and it is very much worth noting that he continued successfully from the point at which Euler had not been able to progress.[2] To get some idea of Malfatti's method of attack, we will consider his calculation, beginning with the

[1] Von der Auflösung der Gleichungen aller Grade, reprinted in: Leonhard Euler, *Drei Abhandlungen über die Auflösung der Gleichungen*, Ostwalds Klassiker Nr. 226, Leipzig, 1928. This quotation and the one following appear on page 45; the equation in the epigraph appears on page 50.

[2] See J. Pierpont, Zur Geschichte der Gleichung V. Grades (bis 1858), *Monatshefte für Mathematik und Physik*, 6 (1895), pp. 15–68. Malfatti's attempts at a solution are described on pages 33 through 36.

equation

$$x^5 + 5ax^3 + 5bx^2 + 5cx + d = 0,$$

only for the case $a = b = 0$, that is, for equations of the type

$$x^5 + 5cx + d = 0.$$

Furthermore, we will assume $cd \neq 0$. We should note further that this does not restrict the generality as much as it seems at first glance. In fact, every equation of degree five can be transformed into an equation of this type using a substitution that eliminates the degree-four term. See the section on the transformations of Tschirnhaus and of Bring and Jerrard.[3]

Malfatti's calculations begin with the assumption, without loss of generality, that the solutions are represented in the form

$$x_{j+1} = -\left(\epsilon^j m + \epsilon^{2j} p + \epsilon^{3j} q + \epsilon^{4j} n\right),$$

for $j = 0, 1, 2, 3, 4$ and with $\epsilon = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right)$. This corresponds precisely to the method employed already by Bézout, Euler, Lagrange,[4] and Vandermonde. If one multiplies the five associated linear factors together, then one obtains, along with Euler, the equation

$$\begin{aligned} &x^5 - 5(mn + pq)x^3 + 5\left(m^2p + n^2q + mp^2 + nq^2\right)x^2 \\ &\quad - 5\left(m^3p + n^3q + mq^3 + np^3 - m^2n^2 + mnpq - p^2q^2\right)x \\ &\quad + m^5 + n^5 + p^5 + q^5 + (mn - pq)\left(mp^2 + nq^2 - m^2q - n^2p\right) = 0. \end{aligned}$$

Finally, one must try to determine the unknowns m, n, p, q by comparing the coefficients with the original equation. We will employ the following shorthand:

$$\begin{aligned} y &= pq = -mn, \\ r &= m^2q + n^2p = -\left(mp^2 + nq^2\right), \\ v &= m^3p + n^3q, \\ w &= mq^3 + np^3. \end{aligned}$$

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[3]For specific applications, however, it is unfortunate that equations with rational coefficients are not transformed into equations of the same type.

[4]Since $m = -\left(x_1 + \epsilon^4 x_2 + \epsilon^3 x_3 + \epsilon^2 x_4 + \epsilon x_5\right)/5$, etc., at issue here are Langrange resolvents for the values m^5, p^5, q^5, n^5.

The two identities mentioned together with the definition of the quantities y and r already contain the result of comparing coefficients for the powers x^3 and x^2. For the other two powers, comparing coefficients gives the pair of equations

$$\begin{aligned} c &= -v - w + 3y^2, \\ d &= m^5 + n^5 + p^5 + q^5 + 20ry. \end{aligned}$$

To be able to formulate as well the last-introduced identity completely in terms of r, v, w, y, we use the relations

$$\begin{aligned} rv &= \left(m^2q + n^2p\right)\left(m^3p + n^3q\right) \\ &= pq\left(m^5 + n^5\right) + (mn)^2\left(mp^2 + nq^2\right) \\ &= \left(m^5 + n^5\right)y - ry^2, \\ rw &= -\left(mp^2 + nq^2\right)\left(mq^3 + np^3\right) \\ &= -mn\left(p^5 + q^5\right) - (pq)^2\left(m^2q + n^2p\right) \\ &= \left(p^5 + q^5\right)y - ry^2, \end{aligned}$$

thereby obtaining for the pair of equations the new form

$$\begin{aligned} c &= -(v + w) + 3y^2, \\ dy &= r(v + w) + 22ry^2. \end{aligned}$$

A calculation of the four unknown quantities r, v, w, y will be possible only if two additional identities are taken into account:

$$\begin{aligned} vw &= \left(m^3p + n^3q\right)\left(mq^3 + np^3\right) \\ &= pq\left(m^4q^2 + n^4p^2\right) + mn\left(m^2p^4 + n^2q^4\right) \\ &= pq\left(m^2q + n^2p\right)^2 + mn\left(mp^2 + nq^2\right)^2 - 4m^2n^2p^2q^2 \\ &= yr^2 + (-y)(-r)^2 - 4y^4 = -4y^4 \end{aligned}$$

and

$$\begin{aligned} -r^2 &= \left(m^2q + n^2p\right)\left(mp^2 + nq^2\right) \\ &= pq\left(m^3p + n^3q\right) + mn\left(mq^3 + np^3\right) = (v - w)y. \end{aligned}$$

Putting these two identities together, we obtain

$$r^4 = (v - w)^2y^2 = (v + w)^2y^2 - 4vwy^2 = (v + w)^2y^2 + 16y^6.$$

Now, using this equation and the pair of equations previously obtained from comparing coefficients, we may determine the values r, v, w, y. First, we eliminate $v + w$ via

$$v + w = 3y^2 - c,$$

so that the following equations remain:

$$dy = \left(25y^2 - c\right) r,$$
$$r^4 = 25y^6 - 6cy^4 + c^2y^2.$$

To eliminate the variable r as well, we take the fourth power of the first of these two equations and then substitute the second equation into the result to obtain

$$d^4y^4 = \left(25y^2 - c\right)^4 \left(25y^4 - 6cy^2 + c^2\right) y^2.$$

Our exclusion of the special case $cd = 0$ helps us in what follows to avoid some complications: First, we have $y \neq 0$, since otherwise, at least three of the values m, n, p, q would be equal to zero, resulting in $c = 0$. Furthermore, we would also have $25y^2 - c \neq 0$, since otherwise we must have $y = 0$.

From $y \neq 0$, we can now multiply the last equation by $25y^{-2}$. We then substitute $z = 25y^2$, so that a *bicubic resolvent* results, that is, an equation of the sixth degree:

$$(z - c)^4 \left(z^2 - 6cz + 25c^2\right) = d^4 z.$$

As we shall see, it is sometimes useful to use the bicubic resolvent in the equivalent form

$$\left(z^3 - 5cz^2 + 15c^2z + 5c^3\right)^2 = \left(d^4 + 256c^5\right) z.$$

Of course, in its general form, the bicubic resolvent cannot be solved in radicals. If it were, then beginning with the variable z, the values y, r, v, w, m, n, p, q could then be calculated in turn:

$$y = \frac{1}{5}\sqrt{z},$$
$$r = \frac{dy}{25y^2 - c},$$
$$v = \frac{3y^3 - cy - r^2}{2y},$$
$$w = \frac{3y^3 - cy + r^2}{2y},$$
$$m, n = \sqrt[5]{\frac{v + y^2}{2y}r \pm \sqrt{\left(\frac{v + y^2}{2y}r\right)^2 + y^5}},$$
$$p, q = \sqrt[5]{\frac{w + y^2}{2y}r \pm \sqrt{\left(\frac{w + y^2}{2y}r\right)^2 - y^5}}.$$

Each equation comes almost directly from the previously derived identities, in the case of the last two equations with the help of Viète's root theorem. Note that the sign of the unknown y can be chosen arbitrarily, since changing the sign merely exchanges the pairs (p, q) and (m, n). Furthermore, note that the ordering of the variables p, q, m, n is always taken such that the equation $v = m^3p + n^3q$ is satisfied.

8.2 Malfatti himself recognized that the bicubic resolvent that he obtained can be used to solve special equations of the fifth degree in radicals. In particular, this is possible when a rational solution to the bicubic resolvent can be found. Here we shall take as an example the equation in the epigraph to this chapter with the coefficients $c = -525$ and $d = -61500$.

Since the bicubic resolvent is a monic polynomial with integer coefficients, all rational solutions, as demonstrated in Chapter 6, must be integers dividing the number $25c^6$. One obtains additional information from the second representation of the bicubic resolvent: Since $d^4 + 256c^5 = 3780900000^2$ is a square, every rational solution must be the square of an integer. And finally, division by 5^6 shows that $\frac{z}{5}$ is also a solution of an equation with integer coefficients, that is, that z is divisible by 5. Having limited the number of possible integer solutions to 112, one obtains the solution $z = 5625$. It then turns out that $y = 15$, $r = -150$, $v = -150$, $w = 1350$, and finally, for $j = 0, 1, 2, 3, 4$,

$$x_{j+1} = \epsilon^j \sqrt[5]{75\left(5+4\sqrt{10}\right)} + \epsilon^{2j} \sqrt[5]{225\left(35-11\sqrt{10}\right)} + \epsilon^{3j} \sqrt[5]{225\left(35+11\sqrt{10}\right)} + \epsilon^{4j} \sqrt[5]{75\left(5-4\sqrt{10}\right)}.$$

8.3 Malfatti's attempt at a solution shows a methodology in the finest classical tradition, namely, to solve equations using suitable substitutions and transformations. In hindsight, we see that the success of Malfatti's approach, to the extent that success was possible, is clarified if one expresses the relevant intermediate values as polynomials in the solutions $x_1, \ldots, x_5$. Thus from the two identities

$$p = -\frac{1}{5}\left(x_1 + \epsilon^2 x_2 + \epsilon^4 x_3 + \epsilon x_4 + \epsilon^5 x_5\right)$$

and

$$q = -\frac{1}{5}\left(x_1 + \epsilon^3 x_2 + \epsilon x_3 + \epsilon^4 x_4 + \epsilon^2 x_5\right)$$

one obtains

$$25y = 25pq = \sum_{j=1}^{5} x_j^2 + \left(\epsilon^2+\epsilon^3\right)\left(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1\right) + \left(\epsilon+\epsilon^4\right)\left(x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2\right).$$

In the special case considered here, $a = b = 0$, since we have

$$\sum_{j=1}^{5} x_j = \sum_{1\le j<k\le 5} x_j x_k = 0$$

and $-\epsilon + \epsilon^2 + \epsilon^3 - \epsilon^4 = -\sqrt{5}$, we obtain for the resolvent solution z the particularly simple representation[5]

$$z = 25y^2 = \frac{1}{5}(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)^2.$$

Furthermore, with this representation it is clear that in the sense of Vandermonde, the existence of a rational solution of the bicubic resolvent can be interpreted as a relation between the solutions.

[5] A derivation of the bicubic resolvent based on Lagrange's universal approach (see Chapter 5) can be found in C. Runge, Über die auflösbaren Gleichungen der Form $x^5 + ux + v = 0$, *Acta Mathematica*, 7 (1885), pp. 173–186; see also Heinrich Weber, *Lehrbuch der Algebra*, volume I, Braunschweig, 1898, pp. 670–676: One first investigates the behavior of the slightly altered polynomial representation

$$y = \frac{\sqrt{5}}{50}(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 - x_1x_3 - x_2x_4 - x_3x_5 - x_4x_1 - x_5x_2)$$

under the 120 possible permutations of the five solutions $x_1, \ldots, x_5$. Ten of these permutations leave the polynomial unchanged. All of these are *even* permutations; that is, they belong to the collection of sixty permutations that leave unchanged the square root of the discriminant:

$$\sqrt{D} = \prod_{i<j}(x_i - x_j).$$

Furthermore, there are ten *odd* permutations whose effect on the polynomial y is to change its sign. Thus the sixty even permutations transform the polynomial y into six different polynomials $y_1 = y, y_2, \ldots, y_6$, and the sixty odd permutations transform y into an additional six polynomials, namely $y_7 = -y_1, \ldots, y_{12} = -y_6$. The first six polynomials are thus solutions of the sixth-degree equation

$$y^6 + \lambda_5 y^5 + \cdots + \lambda_1 y + \lambda_0 = 0,$$

whose coefficients $\lambda_0, \ldots, \lambda_5$ arise from the elementary symmetric polynomials in the polynomials $y_1, \ldots, y_6$. To obtain these coefficients in terms of c and d of the original equation $x^5 + 5cx + d = 0$, the polynomials $y_1, \ldots, y_6$ are expressed in terms of the solutions $x_1, \ldots, x_5$. However, the resulting polynomials are only "almost" symmetric; namely, the polynomials of even degree (in the variables $y_1, \ldots, y_6$) are symmetric, while those of odd degree are altered by a sign change for odd permutations and are unchanged by even permutations. Using the fundamental theorem on symmetric functions and considering the degrees of $c, d, \sqrt{D}, \lambda_0, \ldots, \lambda_5$ as polynomials in the variables $x_1, \ldots, x_5$ (namely 4, 5, 10, and $12 - 2j$ for λ_j), there must exist rational numbers $\mu_0, \mu_1, \mu_2, \mu_4$ satisfying

$$y^6 + \mu_4 c y^4 + \mu_2 c^2 y^2 + \mu_0 c^3 = \mu_1 \sqrt{D} y.$$

After determining the constants, one finally obtains, after squaring the equation obtained, the form of the bicubic resolvent derived by a different route in the main text; here one determines $\sqrt{D}$ by observing that the discriminant D must be representable as a symmetric polynomial of degree 20 of the form $\alpha c^5 + \beta d^4$ with two constants α and β, where the constants can be found using particular equations. One finally obtains $D = 5^5\left(256c^5 + d^4\right)$.

The Transformations of Tschirnhaus and of Bring and Jerrard

The first systematic attempt at a general solution method for equations of degree five was undertaken in 1683 by Ehrenfried Walther, Count of Tschirnhaus (1651–1708). Tschirnhaus's idea is based on the hope that one could generalize the well-known substitutions that cause the second-highest coefficient to disappear so that additional coefficients would disappear as well.

Instead of transforming a given equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = 0$$

using the substitution

$$x = y - \frac{a_{n-1}}{n}$$

into an equation of reduced form

$$y^n + b_{n-2}y^{n-2} + \cdots + b_1 y + b_0 = 0,$$

Tschirnhaus began his investigations with a substitution of the form

$$y = x^2 + px + q$$

with parameters p and q to be determined. The n solutions $x_1, \ldots, x_n$ of the original equation are transformed into the n solutions $y_1, \ldots, y_n$ with $y_j = x_j^2 + px_j + q$, where the coefficients of the powers of y^{n-1} and y^{n-2} are both zero precisely when the two conditions

$$\sum y_j = \sum y_j^2 = 0$$

are satisfied. If one starts with a reduced equation in which the coefficient of the second-highest power is already 0, then one obtains for the parameters p and q the following conditions that must be satisfied:

$$\begin{aligned} 0 &= \sum y_j = \sum \left(x_j^2 + px_j + q\right) = \sum x_j^2 + p\sum x_j + nq \\ &= \sum x_j^2 + nq, \\ 0 &= \sum y_j^2 = \sum \left(x_j^2 + px_j + q\right)^2 \\ &= \sum x_j^4 + 2p\sum x_j^3 + \left(p^2 + 2q\right)\sum x_j^2 + nq^2. \end{aligned}$$

The first of the two conditions immediately permits a unique determination of the parameter q. If one then substitutes the obtained value for q into the second condition, then one obtains for the parameter p a quadratic equation (except in the special case in which the coefficient of the third-highest power is already zero). Thus the so-called *Tschirnhaus* *transformation* of a given nth-degree equation can always be parameterized such that the resulting equation has coefficients equal to zero for the powers y^{n-1} and y^{n-2}.

Tschirnhaus now believed that using transformations of higher degree, which of course contain more parameters to be chosen, would allow further simplification of the equations, so that every equation could be solvable in radicals. Although Tschirnhaus did not succeed in supporting his idea with concrete calculations, it is nevertheless possible to use a transformation of the form

$$y = x^4 + px^3 + qx^2 + rx + s$$

for his special case of a fifth-degree equation

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$

resulting in an equation of the form

$$y^5 + b_1y + b_0 = 0.$$

The parameters can be determined by solving a cubic and a quardratic equation. This fact was first discovered in 1786 by the Swedish mathematician Erland Samuel Bring (1736–1798), though without the mathematical world taking proper note of his achievement. Only much later, in 1864, after George Birch Jerrard (1804–1863) had rediscovered the transformation, were Bring's investigations recalled. The transformation is today generally called the *Bring–Jerrard transformation*. However, its details are so complicated that the actual calculations are difficult to carry out.[6]

Literature on Equations of the Fifth Degree

R. Bruce King, *Behind the Quartic Equation*, Boston, 1996.

Samson Breuer,[7] *Über die irreduktiblen auflösbaren trinomischen Gleichungen fünften Grades*, Borna-Leipzig, 1918.

Sigeru Kabayashi, Hiroshi Nakagawa, Resolution of equation, *Math. Japonica*, 5 (1992), pp. 882–886.

[6] A description of the Bring–Jerrard transformation can be found in J. Pierpont, Zur Geschichte der Gleichung V. Grades (bis 1858), *Monatshefte für Mathematik und Physik*, 6 (1895), pp. 18–19.

[7] The sad fate of the victims of racial and political persecution demands that we recall here the 1933 expulsion of Samson Breuer (1891–1978). See Reinhard Siegmund Schultze, *Mathematiker auf der Flucht vor Hitler*, Braunschweig, 1998, pp. 109, 292.

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Exercises

(1) Solve the equation

$$x^5 + 15x + 12 = 0.$$

(2) Solve the equation

$$x^5 + 330x - 4170 = 0.$$