

Chapter 5

Alicia Boole Stott and four-dimensional polytopes

5.1 Introduction

In this chapter we present the life and work of Alicia Boole Stott (1860-1940), an Irish woman who considerably contributed to four-dimensional geometry. Although she never studied mathematics, she learned herself to ‘see’ the fourth dimension. Using the special capacity of her mind, she developed a new method to visualize four-dimensional polytopes. In particular, she constructed the three-dimensional sections of these four-dimensional objects. The result is a series of three-dimensional polyhedra, which she illustrated making drawings and three-dimensional models. The presence of an extensive collection in the University of Groningen (The Netherlands) reveals a collaboration between Boole Stott and the Groningen professor of geometry P. H. Schoute. This collaboration lasted more than 20 years and combined Schoute’s analytical methods with Boole Stott unusual ability to visualize the fourth dimension. After Schoute’s death (1913), the University of Groningen in 1914 awarded an honorary doctorate to Boole Stott. She remained isolated from the mathematical community until about 1930, when she was introduced to the geometer H. S. M. Coxeter with whom she collaborated until her death in 1940.

5.2 Boole Stott's life

5.2.1 Gems from the basement

In the spring of 2001 an old paper roll containing drawings of polyhedra was found in the basement of the Mathematics, Astronomy and Physics building at the Zernike Campus of the University of Groningen (see Figure 5.14 in section 5.3.1). The drawings, carefully made and beautifully coloured, looked like a series of related Archimedean solids, first increasing and then decreasing in size. The roll was unsigned but the drawings were quickly recognised to be representations of three-dimensional models held at the Groningen University Museum and known to be the work of Alicia Boole Stott (1860-1940), the daughter of the logician George Boole (1815-1864). Further investigation revealed that Boole Stott had enjoyed a fruitful collaboration with the Groningen Professor of Geometry, Pieter Hendrik Schoute (1846-1913) for over twenty years¹, and had been awarded an honorary doctorate by the University of Groningen in 1914. After Schoute's death in 1913, Boole Stott's drawings and models remained in the Groningen University Mathematics Department. The drawings appear to display three-dimensional sections of regular four-dimensional polytopes, obtained by intersecting the four-dimensional polytopes with a three-dimensional space. Looking at the complete set of drawings it is possible to see that one section develops into another by a further shift of the three-dimensional space.

In this chapter we trace the history of Boole Stott's drawings and models, beginning with a biography of Boole Stott, and finishing with a detailed description of two of her publications. As will be described, Boole Stott, had a rather special education under the tutelage of her mother (her father George Boole having died when she was only four). We set Boole Stott's work into its historical context with an account of the early history of four-dimensional geometry, followed by a discussion of the work of Boole Stott's predecessors, notably Ludwig Schlaefli (1814-1895) and Washington Irving Stringham (1847-1900).

It is clear that Boole Stott developed a mental capacity to understand the fourth dimension in way that differed considerably from the analytic approach of other geometers of the time, in particular that of Schoute. But how did she come to develop such an understanding of four-dimensional

¹There is an error in the Dictionary of Scientific Biography (and it is repeated on the St Andrew's website) where Schoute's date of death is given as 1923.

geometry? Did her isolation from the mathematical community and her special education play a role in her discoveries? In Section 5.2.4 we consider these questions and discuss the origins of Boole Stott's interest in polytopes.

Boole Stott's collaboration with Schoute, which is described in detail in Section 5.2.6, raises several questions concerning their actual working practice - How often, when and where did they meet? Why did the models and drawings end up in Groningen? - which we attempt to answer. With Schoute's death, Boole Stott's mathematical activity seems to have drawn to a halt and it was only several years later that Boole Stott's interest in polytopes was revived. In 1930 Boole Stott's nephew, Geoffrey Ingham Taylor (1886-1975), introduced her to the young HSM Coxeter (1907-2003), the two became friends, and Coxeter later made several references to her in his works.

Boole Stott's published her main results on polytopes in two papers of 1900 [B-S 2] and 1910 [B-S 4]. As the discussion in Section 5.3 shows, the first paper [B-S 2] relates to the drawings and the models. This publication studies the three-dimensional sections of the regular polytopes, which are series of three-dimensional polyhedra. In order to illustrate these sections, Boole Stott made drawings and cardboard models of the sections of the two most complicated polytopes. Boole Stott's work was receipted by some of her contemporaries, but was almost forgotten later on.

5.2.2 The beginnings of four-dimensional geometry

Geometry as studied to the middle of the 19th century dealt with objects of dimension no greater than three. The interest among mathematicians in the fourth dimension seems to have arisen in the middle of the 19th century after the *Habilitation* lecture of Riemann (1826-1866), given on June 10th, 1854. In this lecture [Rie], published by Richard Dedekind after Riemann's death, Riemann introduced the notion of an n -dimensional manifold. The lecture had few mathematical details but was presented with many ideas about what geometry should be. With the increasing use of analytical and algebraic methods, the step to a higher number of dimensions became necessary. Various mathematicians generalized their theories to n dimensions. From that moment on, interest in higher dimensional spaces was booming. By 1885 several articles on the topic had appeared, written by mathematicians such as William Clifford (1845-1879) [Cli] or Arthur Cayley (1821-1895) [Cay]. Another important figure who popularized the topic was Howard Hin-

ton (1853-1907), an English high school teacher of mathematics. In his book *The fourth dimension* [Hin2] Hinton introduced the term *tesseract* for an unfolded hypercube.

5.2.3 Polytopes and modelling

Four-dimensional polytopes are the four-dimensional analogue of polyhedra. They were discovered by the Swiss mathematician L. Schläefli. Between 1850 and 1852, Schläefli had developed a theory of geometry in n -dimensions. His work, *Theorie der vielfachen Kontinuität* [Schl1], contained the definition of the n -dimensional sphere and the introduction of the concept of four-dimensional polytopes, which he called *polyschemes*. He proved that there are exactly six regular polytopes in four dimensions and only three in dimensions higher than four. Unfortunately, because of its size, his work was not accepted for publication. Some fragments of it were sent by Schläefli to Cayley, who acted as an intermediary and published them in the *Quarterly Journal Of Pure And Applied Mathematics* [Schl2]. The manuscript was not published in full until after his death [Schl1]. Thus, mathematicians writing about the subject during the second half of the century were partly unaware of Schläefli's discoveries.

The first person to rediscover Schläefli's polytopes was W. I. Stringham. His paper [Stri], much referred to, became important since it provides an intuitive proof of the existence of the six regular polytopes, and gives explicit constructions for each of them. It also includes one of the earliest known illustrations of four-dimensional figures, displayed in Figure 5.1.

5.2.4 A special education

Alicia Boole Stott was born in Castle Road, near Cork (Ireland) on June 8th, 1860 [McH]. She was the third daughter to the today famous logician George Boole (1815-1864) and Mary Everest (1832-1916). George Boole died from fever at the age of 49. George's widow Mary and five daughters were left with very little money, so Mrs Boole was forced to move to London, taking Alicia's four sisters with her. Alicia had to stay at Cork with her grandmother Everest and an uncle of her mother [McH]. At the age of eleven, she moved to London to live with her mother and sisters for seven years. Her stay in London was only interrupted by one visit to Cork in 1876, where she worked in a children's hospital for a short period [Cox5].

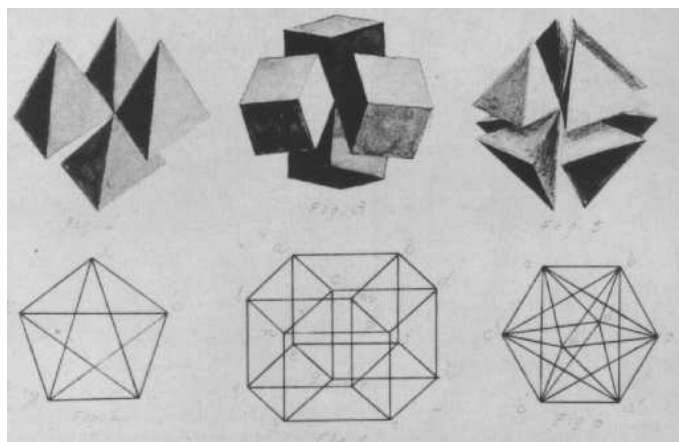


Figure 5.1: Projections of four-dimensional polytopes by Stringham [Stri].

Since Alicia was a woman born around the middle of the nineteenth century, she had hardly any educational opportunity. In England colleges did not offer degrees to women, and women could only aspire to study some classical literature and other arts, and hardly any science [Mich]. Alicia's knowledge of science consisted only of the first two books of Euclid [Cox1].

Having so little knowledge of science, how is it possible that Alicia developed such an understanding of four-dimensional geometry? Did her special environment stimulate her? The family situation certainly provided her with a very particular education. She was only four years old when her father died, so she could not have received much mathematical influence from him. However, she certainly received a good tuition from her mother. Mary Everest Boole had studied with her husband, George Boole. When Boole died, Everest Boole moved to England and was offered a job at Queen's College in London as a librarian. Her passion however was teaching, and she liked giving advice to the students [Mich]. She had innovating ideas about education, believing for example that children should manipulate things in order to make the unconscious understanding of mathematical ideas grow [Mich]. Her belief that models should be used in order to visualize and understand geometrical objects is reflected in the following words:

There is another set of models, the use of which is to provide people who have left school with a means of learning the relation between three dimensions and four. [Eve1]

The geometric education may begin as soon as the child's hands

can grasp objects. Let him have, among his toys, the five regular solids and a cut cone. [Eve2]

Most of Everest Boole's books were published years after they were written. Michalowicz [Mich] proffers some explanation as to why Everest Boole's work remained unknown for so long. It is therefore probable that Everest Boole had these ideas on the use of models by the time she educated her daughters, in particular her daughter Alicia. Apart from the education provided by her mother, Alicia was also strongly influenced by the amateur mathematician Howard Hinton, whom she met during her London period. Hinton was a school teacher, and was very interested in four-dimensional geometry.

Hinton was fascinated by the possibility of life in either two or four dimensions. He used hundreds of small colour cubes and assigned a Latin name to each of them. After having contemplated them for years he claimed that he had learned to visualize the fourth dimension [McH]. He stimulated Alicia and her sisters during his visits to the family by putting together the small cubes and by trying to make them perceive the hypercube. He also made them memorize the arbitrary list of Latin names he had assigned to them [McH]. This seems to have strongly inspired Alicia in her later work, and she soon started surprising Hinton with her ability to visualize the fourth dimension. Little more is known about their contact, apart from Alicia's contribution to Hinton's book: *A new era of thought* [Hin1]. She wrote part of the preface, as well as some chapters and appendices on sections of some three-dimensional solids. Hinton is also remembered for his books *The fourth dimension* [Hin2] and *An episode of flatland* [Hin3].

In 1889 Alicia lived near Liverpool working there as a secretary [Cox5]. She married the actuary Walter Stott in 1890 and had two children, Mary (?-?)² and Leonard (1892-1963) [McH]. Boole Stott returned to do research by the time the children were growing up. On a family picture dated around 1895, Boole Stott is present with her two children in the company of her four sisters, her mother and some of her nephews (see Figure 5.2).

²I have not been able to find the precise years of Mary, but she was born before 1895 (see E. I. Taylor's diary [Tay-E] and Boole Stott's letter to her sister Margaret [B-S 1]) and was still alive in December 1958, when she wrote her will [Sto]



Figure 5.2: *From left to right, from up to down: Margaret Taylor, Ethel L. Voynich, Alicia Boole Stott, Lucy E. Boole, Mary E. Hinton, Julian Taylor, Mary Stott, Mary Everest Boole, George Hinton, Geoffrey Ingram Taylor, Leonard Stott. Boole Stott's three sisters Margaret, Ethel and Mary married E. I. Taylor, W. M. Voynich and H. Hinton respectively.* (Courtesy of the University of Bristol)

5.2.5 Boole Stott's models of polytopes

Inspired by Howard Hinton, Boole Stott undertook a study of four-dimensional geometry between 1880 and 1890. She worked as an amateur, without any scientific education or scientific contacts. Probably unaware of the existence of the six regular polytopes in the fourth dimension, she succeeded in finding them by herself again [McH]. As reminded earlier, these six polytopes, first discovered by Schlaefli in 1840, were independently rediscovered by Stringham [Stri] and other mathematicians [Cox1]. Five of these polytopes are the four-dimensional analogues of the five regular polyhedra, namely the hypercube, hyperoctahedron, hypertetrahedron, 120-cell and 600-cell. The extra one is called the 24-cell and has no three-dimensional analogue.

Boole Stott also calculated series of sections of all six three-dimensional regular polytopes, building them in beautiful cardboard models. These sections consist of a set of increasing semiregular polyhedra, that vary in shape and colour. Figure 5.3 shows a picture of a showcase [B-S 7] kept at the Groningen University Museum containing Boole Stott's models.

Her method to obtain these sections was completely based on geometrical



Figure 5.3: Models of sections of the 600-cell and the 120-cell. [B-S 7]. (Courtesy of the University Museum, Groningen)

visualization. How do the models Boole Stott built help in the understanding of these four-dimensional bodies? To answer this, one must step into a dimension lower: consider the two-dimensional sections of a tetrahedron that consist of a series of increasing triangles. By looking at these triangles, we get information about the shape of the tetrahedron. Since we live in a three-dimensional space, we cannot visualize four-dimensional bodies, in particular the regular polytopes. However, by looking at the increasing polyhedra built by Boole Stott, we can get an idea of the shape of such a polytope.

Boole Stott's method consisted on unfolding the four-dimensional polytopes so that we can visualize them in the three-dimensional space.

5.2.6 Boole Stott and the Netherlands

After the drawings and models made by Boole Stott had been found in Groningen, the present research started. We soon found out about Boole Stott's collaboration with P. H. Schoute, who was a professor of mathematics at the University of Groningen.

For a short biography and a portrait of Schoute, we refer to Section 1.4.3 in Chapter 1. In 1894, Schoute ([Schou1], [Schou2] and [Schou3]) described by analytical methods the three-dimensional central sections of the four-

dimensional polytopes. According to Coxeter [Cox1], Boole Stott got to know about Schoute's publications from her husband (it remains unclear how Walter Stott would have known about Schoute's work). She realized that Schoute's drawings of the sections were identical to her cardboard models and sent photographs of the models to Schoute. These models showed that her central diagonal³ sections agreed with his results. This is displayed in Figure 5.4, where the drawing on the left is from [Schou3] and the photograph on the right is a model by Boole Stott, currently present at Groningen University Museum.

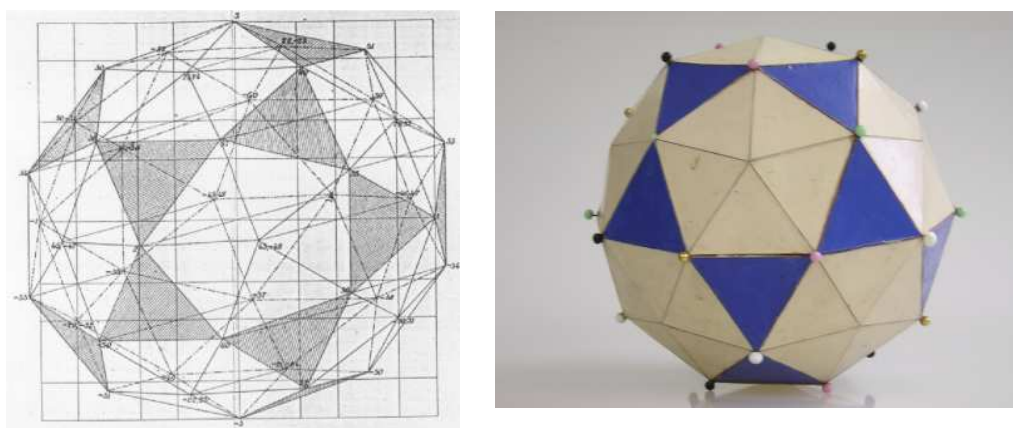


Figure 5.4: *Schoute's drawing [Schou3] and Boole Stott's model of the central diagonal section of the 600-cell from [B-S 7]. (Courtesy of the University Museum, Groningen)*

Schoute was very surprised and immediately answered asking to meet her and proposing a collaboration [McH]. How did this collaboration actually work? Schoute came to England during some of his summer holidays to stay with Boole Stott at her maternal cousin's house in Hever [Cox1]. In the photograph Figure 5.5, Schoute is present in the company of Boole Stott and some of her family during one of his visits.

Boole Stott and Schoute worked together for almost twenty years, combining Boole Stott's ability for visualizing four-dimensional geometry with Schoute's analytical method [McH]. Schoute persuaded her to publish her results. Her main publications are [B-S 2] and [B-S 4] published in the journal of the Dutch Academy of Science *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam*.

³ See Definition 10.



Figure 5.5: *From left to right, from up to down: Mary Stott, G. I. Taylor, Margaret Taylor, Pieter Schoute, Alicia Boole Stott.* [B-S 10]. (Courtesy of the University of Bristol)

Apart from Schoute's visits to England, the collaboration between the two partners also worked via correspondence. Boole Stott, in a letter [B-S 5] to her nephew Geoffrey Ingram Taylor, discusses a manuscript that she received, probably from Schoute, about one of her publications:

... I have not done anything more interesting than staining very shabby floors and such like household thing for some time; but last night I received by post a M.S. of 70 very closely written pages containing an analytical counterpart of my last geometrical paper. Of course I must read it. It is the second attempt and was only written because I did not like the first but I am such a duffer at analytical work anyhow that I don't suppose I shall like this very much better. [B-S 5].

Boole Stott's words show the contrast of every day life with her mathematical work, and reveal much modesty concerning her analytical abilities.

The journal *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam* where Boole Stott published her results was the journal of the Dutch Academy of Sciences; it was read internationally by the mathematicians of the time. Some of them refer to Boole Stott's work in their work:

for example, E. Jouffret refers to Boole Stott publication of 1900 in [Jou, pp. 107]. Other references to her work can be found, for example, in the work of the Dutch mathematicians Willem Abraham Wythoff (1865-1939) [Wyth] and Jacob Cardinal (1848-1922) [Car].

Although Boole Stott published her results in a Dutch journal and a big part of her collection is present in Groningen, it remains unclear whether she ever went to The Netherlands or not. The set of models were proven to be a present from Boole Stott to Schoute [Schou4] what suggests that they could have been sent from England. Concerning her drawings at Groningen, a closer look at the paper she used reveals that some of the sheets are originally English, and others Dutch, which leads us to no conclusion either.

5.2.7 An honorary doctorate

Because of Boole Stott's important contributions to mathematics, the board of the University of Groningen decided to award a doctorate to her in 1914. With that purpose, Johan Antony Barrau (1873-1953) (the successor of Schoute after the death of the latter in 1913) wrote a recommendation letter to the board of the University together with a list of Boole Stott's publications [Bar]. This list included the three joint publications with Schoute [BS-Sch 1], [BS-Sch 2] and [BS-Sch 3], and Boole Stott's papers [B-S 3a] and [B-S 4], but Barrau missed [B-S 2]. The text in Barrau's letter, originally in Dutch, reads:

From these papers, one infers a very special gift for seeing the position and forms in a space of four dimensions. Three of these papers are written jointly with late prof. Dr. P. H. Schoute connected during so many years to the University of Groningen; And this fruitful cooperation with the professor that she lost, is the reason for the Faculty of Mathematics and Physics to propose Mrs A. Boole Stott for the doctorate honoris causa in Mathematics and Physics, to confer on the occasion of the coming festive commemoration of the 300th birthday of the University.

Boole Stott was informed about the doctorate by the University in a printed announcement, 20 April 1914 (Figure 5.6 and [B-S 6]), in which she was invited to attend the festive promotion ceremony on July the 1st. Note in the figure that there was not a female version of such a document, which indicates how rare it was for a woman to receive an honorary doctorate at

that time. Contrary to what Coxeter claims [Cox1], Boole Stott did not come to Groningen to attend the ceremonies ([**]). The plan was that Boole Stott would stay with Schoute's widow [Sch-J]. However, the list of accommodation for the 68 honorus causa candidates contains the remark *does not come* for Alicia Stott [*]. Some small question remains open. Why did Boole Stott not go to the ceremony?

Concerning the honorus causa diploma, note that the message in figure 5.6 is not the original diploma but an announcement of it. What happened to the original document? A private communication with Boole Stott's grand-nephews Geoffrey and James Hinton revealed that the original diploma had been in their possession for some time, but had somehow disappeared afterward.

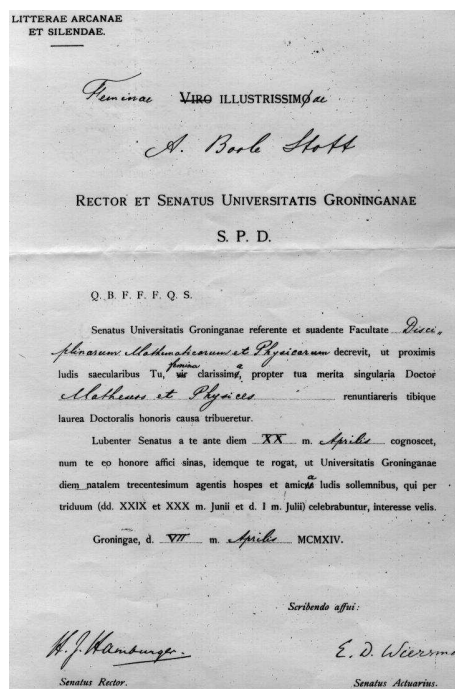


Figure 5.6: Message to Boole Stott that she was awarded an honorary doctorate. [B-S 6]. (Courtesy of the University of Cambridge)

5.2.8 Boole Stott and Coxeter

Boole Stott resumed her mathematical work in 1930, which had stopped since Schoute's death in 1913, when she met H. S. M. Coxeter [Cox1] by her

nephew Geoffrey Ingram Taylor⁴(1886-1975). The correspondence between Taylor and his aunt Boole Stott reveals a closeness between the two relatives. Taylor, aware of her mathematical activities, might have decided to introduce her aunt to Coxeter. When Boole Stott and Coxeter met, she was then a 70 year old woman whilst Coxeter was only 23. Despite this difference of age, they became friends. They used to meet and work at several topics in mathematics. In a particular occasion [Rob], Coxeter invited Boole Stott to a tea party at Cambridge University, where they would deliver a joint lecture. She attended the party bringing along her a set of models which she donated for permanent exhibition to the department of mathematics. These models are currently in the office of Professor Lickorish at the department. Coxeter's own words [Cox1] describe Boole Stott as:

The strength and simplicity of her character combined with the diversity of her interests to make her an inspiring friend.

In his later work, Coxeter often made reference to her and her work, and called her "Aunt Alice", as Boole Stott's nephew Taylor used to do. Coxeter [Cox1] describes her married life saying:

In 1890 she married Walter Stott, an actuary; and for some years she led a life of drudgery, rearing her two children on a very small income.

Boole Stott died at 12 Hornsey Lane, Highgate, Middlesex, on December 17, 1940 [The Times, December 18, 1940].

Since Coxeter and Boole Stott do not have any common publications, it is not always easy to know what precise contributions Boole Stott has made. However, we have some idea about this thanks to several remarks about her work that Coxeter made in his publications.

5.3 Boole Stott's mathematics

Alicia Boole Stott published her main results in two papers. The first one called *On certain series of sections of the regular four-dimensional hypersolids* was published in 1900 and deals with three-dimensional sections of four-dimensional polytopes. In her second publication *Geometrical deduction of*

⁴Taylor became one of the most brilliant and influential mathematical physicist of the 20th century. He was expert on fluid dynamics and wave theory [OxDNB]

semiregular from regular polytopes and space fillings, published in 1910, Boole Stott gives a method to obtain semi-regular polyhedra and polytopes from regular ones.

We next present a mathematical description of Boole Stott's result in her paper [B-S 2].

5.3.1 1900 paper on sections of four-dimensional polytopes

Boole Stott publication *On certain series of sections of the regular four-dimensional hypersolids* [B-S 2] describes a rather intuitive and elegant method to obtain the three-dimensional sections of the regular polytopes. To ease understanding of the paper, we present some definitions.

Definition 8. A *regular polytope* in 4-dimensional space is a subset of \mathbf{R}^4 bounded by isomorphic 3-dimensional regular polyhedra. These polyhedra are called *cells* in the papers of Boole Stott and Schoute. If two cells have a non empty intersection, then the intersection is a 2-dimensional face of both cells.

The number of vertices, edges, faces and cells of these polytopes are given in table Figure 5.7.

| Polytope | v | e | f | c | cell |
|----------------------------|-----|------|------|-----|--------------|
| Hypertetrahedron or 5-cell | 5 | 10 | 10 | 5 | tetrahedron |
| Hypercube or 8-cell | 16 | 32 | 24 | 8 | cube |
| Hyperoctahedron or 16-cell | 8 | 24 | 32 | 16 | tetrahedron |
| 24-cell | 24 | 96 | 96 | 24 | octahedron |
| 120-cell | 600 | 1200 | 720 | 120 | dodecahedron |
| 600-cell | 120 | 720 | 1200 | 600 | tetrahedron |

Figure 5.7: Polytopes in four dimensions.

Boole Stott gave an intuitive proof of the uniqueness of the six regular polytopes in four dimensions. Her proof went as follows. Let P be a regular polytope whose cells are cubes. Let V be one of the vertices of P , and consider the diagonal section of P corresponding to an affine space K , close enough to V so that K intersects all the edges coming from V . The

corresponding section must be a regular polyhedron bounded by equilateral triangles. Furthermore, there are only three regular polyhedra bounded by triangles, namely the tetrahedron (bounded by 4 triangles), the octahedron (bounded by 8 triangles) and the icosahedron (bounded by 20 triangles). Then, the polytope can only have 4, 8, or 20 cubes meeting at each vertex. Considering the possible angles in 4 dimensions, Boole Stott shows that the only possibility for P is to have 4 cubes at a vertex (8 and 20 are too many), which gives the 8-cell (or hypercube).

Proceeding in the same way, she found the remaining 5 polytopes.

After her proof, Boole Stott proceeded by considering three dimensional perpendicular sections, which are defined below.

Definition 9. Let O be the center of a given polytope P , and C be the center of one of its cells. Let H be some affine 3-dimensional subspace, perpendicular to OC . A *perpendicular section* is $H \cap P$.

Although the publication [B-S 2] treats only perpendicular sections, Boole Stott also made models of diagonal sections of polytopes (see for example the model on the right in Figure 5.4). These sections are defined as:

Definition 10. Let O be the center of the polytope P , and V be one of its vertices. Let K be some affine 3-dimensional subspace, perpendicular to OV . A *diagonal section* (or *cross section*) is $K \cap P$.

We now give Boole Stott's description of the perpendicular sections for two of the polytopes in detail, namely, the hypercube and the 24-cell. We also discuss the other polytopes and remark that the drawings reproduced in the figures are Boole Stott's drawings. We note that Boole Stott's method consisted of unfolding the four-dimensional polytope into the third dimension. Once the polytope is represented in a three-dimensional space, the calculations become much simpler and easier to visualize.

The 8-cell or the hypercube

Boole Stott begins by studying the perpendicular sections of the hypercube (see Definition 9). The three-dimensional sections will result from intersecting the polytope with particular three-dimensional spaces.

Let P_1 denote the hypercube. P_1 is the analogue of the cube in four dimensions. It is characterized by having 8 cubes, 4 of them meeting at every vertex. Figure 5.8 is Boole Stott's representation of part of the unfolding

hypercube. Since it is an unfolding of a 4-dimensional object, vertices, edges and faces occur more than once (this, of course, can only be understood in 4 dimensions).

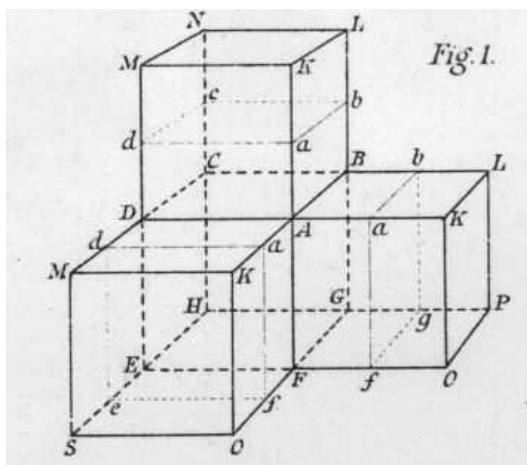


Figure 5.8: Hypercube: four cubes meeting at a vertex A . [B-S 2].

The first three-dimensional section is the result of intersecting the polytope with a three-dimensional space H_i containing the cube $ABCDEFGH$ in Figure 5.8. To obtain the second section, the space H_1 is moved towards the center of the polytope, until it passes through the point a . Call this new three-dimensional space H_2 . The second section is $P_1 \cap H_2$. Note that the faces of the new section must be parallel to the faces of the cube $ABCDEFGH$. In particular, the section $P_1 \cap H_2$ contains the squares $abcd$, $abfg$ and $adef$ (see Figure 5.8). After the necessary identification of the points, edges and faces that occur more than once in the unfolded polytope, and using the symmetry of the polytope, one can conclude that the section $P_1 \cap H_2$ is again a cube isomorphic to the cube $ABCDEFGH$.

Analogously, the third section will again be a cube. This simple example gives the idea of Boole Stott's method. Let us now consider the more interesting case of the 24-cell, which is still not too difficult to visualize.

The 24-cell

Let P_2 denote the 24-cell. We note that this polytope is the only one without an analogues in 3 dimensions. Its 24 cells are octahedra, and 6 of them meet at every vertex. In Boole Stott's representation (see Figure 5.9), only 4

octahedra are drawn. Note that the figure is again an unfolding, and the two A' should be identified and similarly, for AE and AC .

The perpendicular sections that Boole Stott described are parallel to the octahedron $ABCDEF$.

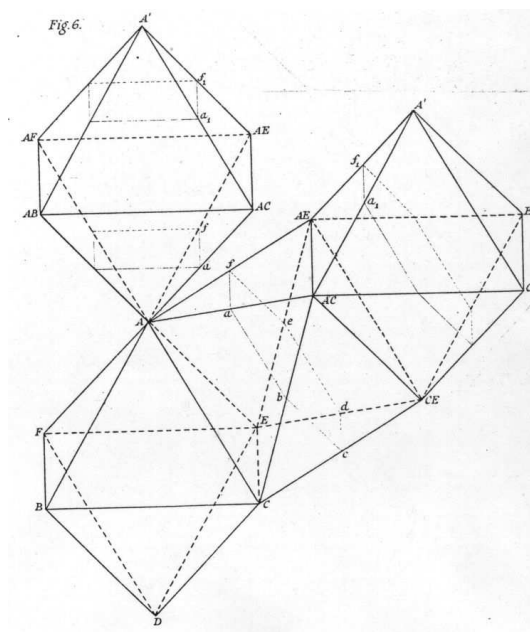


Figure 5.9: Four octahedra of the 24-cell. [B-S 2].

1. Let H_1 be the space containing the octahedron $ABCDEF$. The first section $H_1 \cap P_2$ is clearly the octahedron $ABCDEF$ itself.
2. Let H_2 be the space parallel to H_1 and passing through the point a , the mid-point between A and AC . The second section $H_2 \cap P_2$ is a 3-dimensional solid whose faces are parallel to the faces of the octahedron $ABCDEF$ or the rectangle $BCEF$. In Figure 5.10 two of these faces are shadowed. Since the drawing of the octahedra meeting at A is not complete (3 octahedra are missing), we only see part of the final section. The remaining part can be deduced by symmetry.
3. Let H_3 be the space parallel to H_1 and passing through the vertex AC . The section $H_3 \cap P$ contains the following faces: A rectangle $ABACAEAF$ parallel to the rectangle $BCEF$, and a triangle

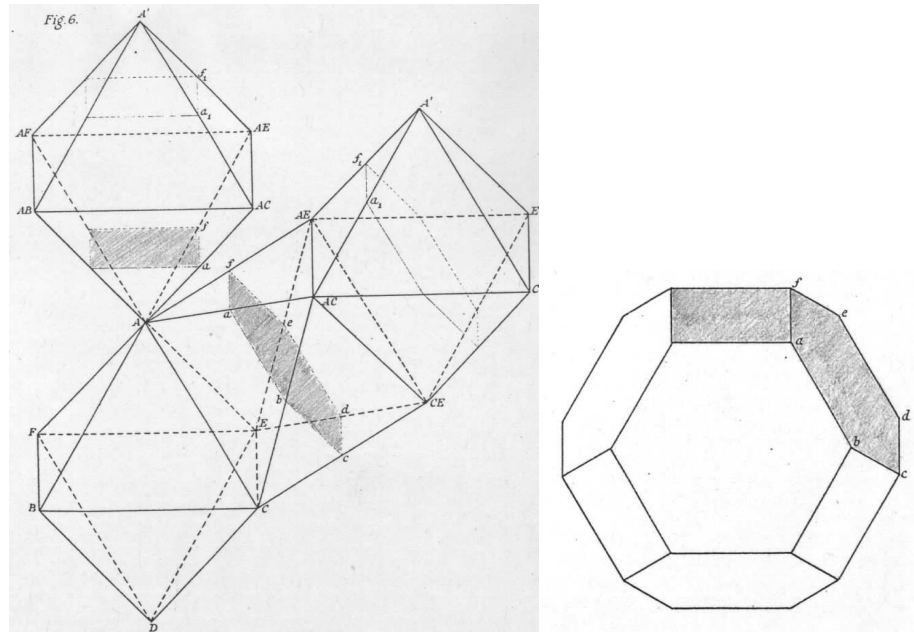


Figure 5.10: Section $H_2 \cap P$ of the 24-cell. [B-S 2].

$AEACCE$ parallel to the face ACE (the shadowed faces of Figure 5.11).

4. By symmetry, the fourth section passing through a_1 , the mid-point between AC and A' , is isomorphic to the second section.
5. Again by symmetry, the last section through A' is an octahedron.

The 16-cell or the hyperoctahedron

This polytope is the analogue of the octahedron in four dimensions. It is bounded by 16 tetrahedra, 8 of them meeting in every vertex. After drawing five of them as in previous cases, Boole Stott used the same method to obtain the 3-dimensional sections. The first and last sections are clearly tetrahedra (since the 16-cell is bounded by tetrahedra). The other three sections are shown in the next figure:

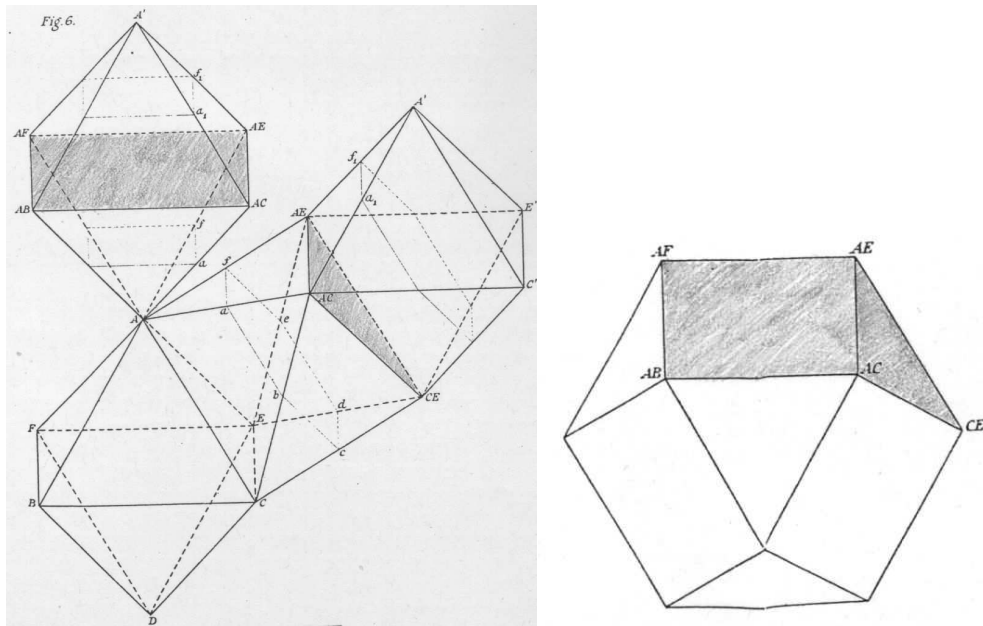


Figure 5.11: Section $H_3 \cap P$ of the 24-cell. [B-S 2].

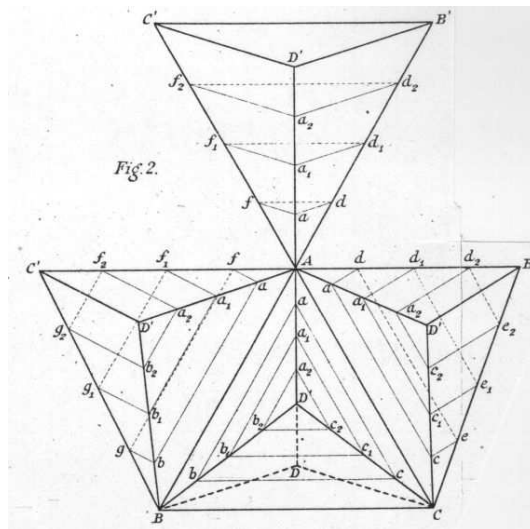


Figure 5.12: Five tetrahedra of the 16-cell. [B-S 2].

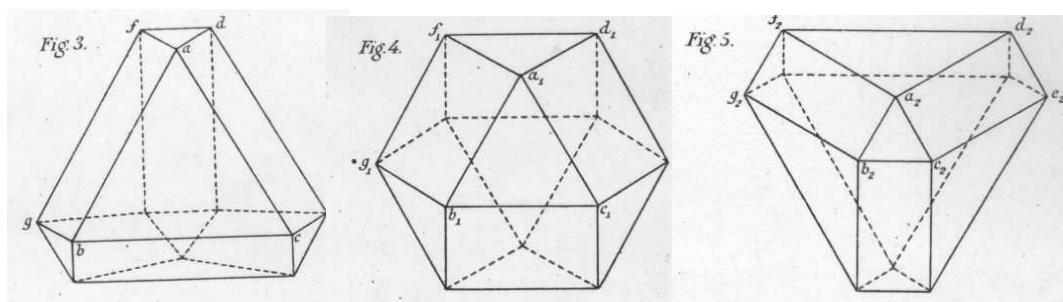


Figure 5.13: 2nd, 3rd and 4th sections of the 16-cell. [B-S 2].

The 120-cell

The 120-cell is bounded by 120 dodecahedra. In making the sections of this polytope, Boole Stott gives drawings of their unfoldings. These sections are also drawn in [B-S 8]. Figure 5.14 shows some of them.

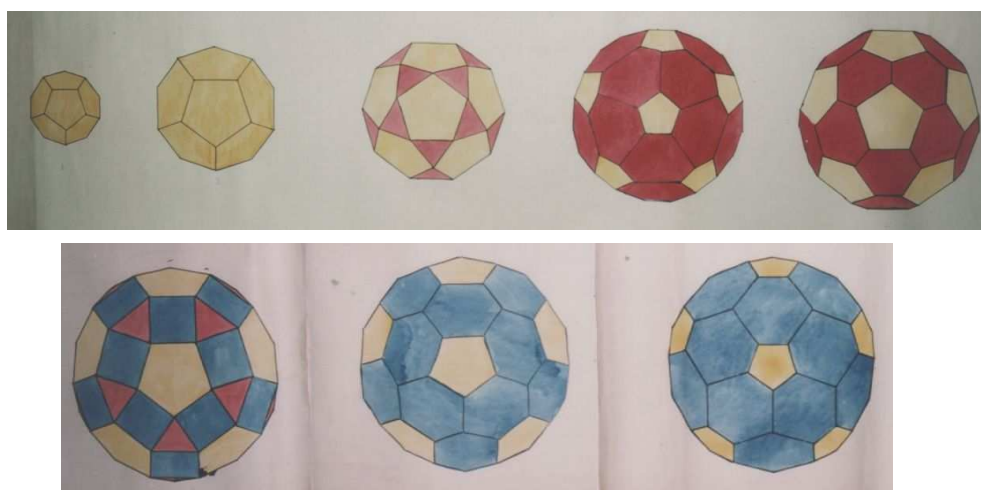


Figure 5.14: Sections of the 120-cell. [B-S 8].

The 600-cell

This is the most complicated polytope. The complete sections are drawn in Boole Stott's paper and constructed in cardboard by her. The models are

currently in exhibition at the University Museum of Groningen. Figure 5.15 illustrate unfoldings and cardboard models of the sections of the 600-cell.⁵



Figure 5.15: Drawings in [B-S 7] and models in Groningen University Museum of Groningen of perpendicular sections of the 600-cell.

⁵Note that the drawing on the right hand side corresponds with the uppermost cardboard model.

5.3.2 1910 paper on semi-regular polytopes

In her paper *Geometrical deduction of semi-regular from regular polytopes and space fillings* [B-S 4], Alicia Boole Stott gave a new construction for the so-called Archimedean solids and her discovery of many of their four dimensional analogues. Inspired by some stereographic photographs of semi-regular polyhedra sent to her by Schoute, she got the idea of deriving semi-regular polyhedra from regular ones. In this way, she discovered many of the Archimedean polytopes (i.e., the generalization of semi-regular solids to any dimension).

Semi-regular polyhedra in 3 dimensions

Definition 11. A *convex semi-regular polyhedron* is a convex polyhedron whose faces are regular polygons of two or more different types, ordered in the same way around each vertex.

There are different kinds of convex semi-regular polyhedra:

1. An *n-gonal prism* is a convex semi-regular polyhedron constructed from two parallel regular polygons with n sides ($n \neq 4$) called *bases* and n squares.
2. An *n-gonal antiprism* is a convex semi-regular polyhedron constructed from two parallel polygons with n sides and $2n$ triangles.
3. An *Archimedean solid* is a convex semi-regular polyhedron which is not a prism or an antiprism. There are 13 different Archimedean solids.

Archimedes (287 BD-212 BC) discovered these thirteen solids. His original manuscripts were lost, but Pappus of Alexandria (290 AD-350 AD) attributes the discovery to him in his Book V.

Five of the Archimedean solids can be obtained by the process of *truncation* (i.e., cutting off all corners) of the five Platonic solids (see Figure 5.16 nr. 1, 2, 3, 4, and 5), namely the *truncated tetrahedron*, *truncated cube*, *truncated octahedron*, *truncated dodecahedron* and the *truncated icosahedron*. Truncating the cube (or octahedron) and the dodecahedron through the middle of an edge, we get the *cuboctahedron* and the *icosidodecahedron* (see Figure 5.16 nr. 6, 7). Truncating all vertices and edges of the last two (see Figure 5.16 nr. 8, 9) will give the *Great rhombicuboctahedron* and the *Great rhombicosidodecahedron*. Truncating the vertices and edges of the cube (or octahedron) and

the dodecahedron (or icosahedron) will give the *Small rhombicuboctahedron* and the *Small rhombicosidodecahedron* (see Figure 5.16 nr. 10, 11). The last two Archimedean solids are obtained by the process *snubbing* (moving the faces of two Platonic solids outwards while giving each face a twist) of the cube and the dodecahedron (see Figure 5.16 nr. 12, 13). The results of this process are called the *snub cube* and the *snub dodecahedron*.

In the following table, v and e denote respectively the number of vertices and edges of an Archimedean solid. Further p_i^* denotes the number of p_i -gonal faces.

| Name | Symbol | v | e | p_3^* | p_4^* | p_5^* | p_6^* | p_8^* | p_{10}^* |
|----------------------|-----------------|-----|-----|---------|---------|---------|---------|---------|------------|
| Truncated tetrah. | (3, 6, 6) | 12 | 18 | 4 | | | 4 | | |
| Truncated cube | (3, 8, 8) | 24 | 36 | 8 | | | | 6 | |
| Truncated octah. | (4, 6, 6) | 24 | 36 | | 6 | | 8 | | |
| Truncated dodec. | (3, 10, 10) | 60 | 90 | 20 | | | | | 12 |
| Truncated icosah. | (5, 6, 6) | 60 | 90 | | | 12 | 20 | | |
| Cuboctahedron | (3, 4, 3, 4) | 12 | 24 | 8 | 6 | | | | |
| Icosidodecahedron | (3, 5, 3, 5) | 30 | 60 | 20 | | 12 | | | |
| Great rhombicuboct. | (4, 6, 8) | 48 | 72 | | 12 | | 8 | 6 | |
| Great rhombicosid. | (4, 6, 10) | 120 | 180 | | 30 | | 20 | | 12 |
| Small rhombicuboct. | (3, 4, 4, 4) | 24 | 48 | 8 | 18 | | | | |
| Small rhombicosidod. | (3, 4, 5, 4) | 60 | 120 | 20 | 30 | 12 | | | |
| Snub cube | (3, 3, 3, 3, 4) | 24 | 60 | 32 | 6 | | | | |
| Snub dodecahedron | (3, 3, 3, 3, 5) | 60 | 150 | 80 | | 12 | | | |

Figure 5.16: Archimedean solids in three dimensions.

Boole Stott's method is based on two operations on polytopes, one is the inverse of the other, called expansion and contraction. This gives a new and very elegant way to construct these solids.

Definition 12. For $i = 0, 1, 2, \dots, n-1$, let L_i denote the set of i -dimensional faces of a regular n -dimensional polytope with O as centre (e.g., L_0 consists of the vertices, L_1 consists of the edges). Let M_i denote the set of centers of the elements of L_i . The e_k -expansion of the polytope is described as follows:

1. Every $l_k \in L_k$ is moved away from O at a certain distance in the direction Om_k , where the point $m_k \in M_k$ is the center of l_k .

2. All these distances are equal and taken so that the vertices in this position now form a new semi-regular polytope.

The operation c_k -contraction is a sort of inverse of *expansion*. For a given semi-regular polytope and a special choice of elements in L_k , the contraction consists of moving towards O those elements until they meet.

These two operations provide an elegant way to construct from a regular polytope a semi-regular one.

Examples of expansion in 3 dimensions: From now on, $\{3, 3\}$, $\{4, 3\}$, $\{3, 4\}$, $\{5, 3\}$ and $\{3, 5\}$ will denote the tetrahedron, cube, octahedron, dodecahedron and icosahedron. The letter ‘t’ in front of a solid will denote the corresponding truncated solid.

1. **e_1 expansion:** This concerns the edges L_1 . This e_1 expansion applied to any Platonic solid has as result the same solid but now truncated. Thus, for any Platonic solid S , we have $e_1(S) = tS$.

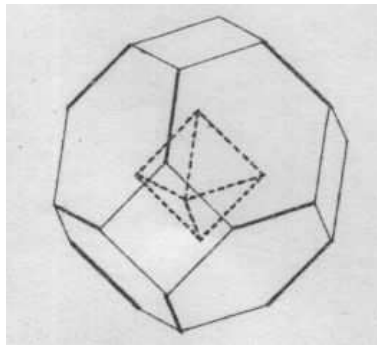


Figure 5.17: Example: $e_1(\{3, 4\}) = t\{3, 4\}$ in [B-S 4].

2. **e_2 expansion:** This concerns the 2-dimensional faces. Boole Stott shows that the e_2 expansion of a Platonic solid and of its dual produces the same Archimedean solid (see Figure 5.18). There are three different solids coming out of this operation, namely:

- i) $e_2(\{3, 3\}) = (3, 4, 3, 4)$

- ii) $e_2(\{4, 3\}) = e_2(\{3, 4\}) = (3, 4, 4, 4)$

- ii) $e_2(\{5, 3\}) = e_2(\{3, 5\}) = (3, 4, 5, 4)$

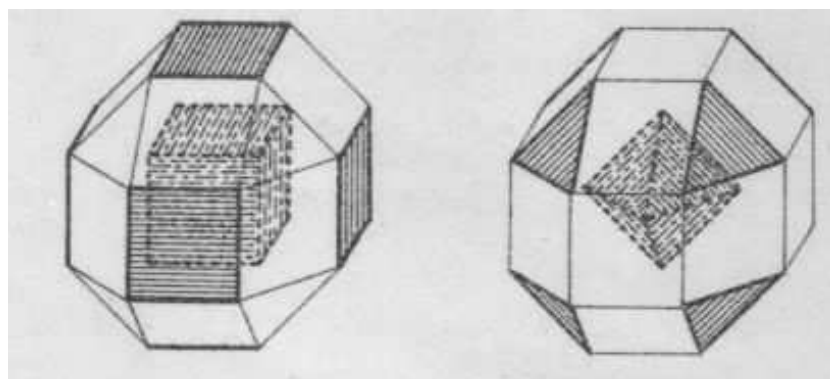


Figure 5.18: Example: $e_2(\{4, 3\})$ and $e_2(\{3, 4\})$. [B-S 4].

Semi-regular polytopes in 4 dimensions

Definition 13. An *Archimedean 4-dimensional polytope* is a convex polytope whose cells (i.e., 3-dimensional faces) are regular or semi-regular convex polyhedra, and such that the group of symmetries of the polytope is transitive on its vertices.

The complete list of Archimedean 4-dimensional polytopes was first given by J. H. Conway in [Con] using a computer search. The list consists of:

1. **45 polytopes:** These are characterized by having n -gonal prisms and Archimedean solids as cells. These polytopes are known through the work of Wythoff in 1918 [Wyth], who gave a method to construct them. However, all 45 had previously been discovered by Boole Stott [B-S 4].
2. **Prismatic polytopes:** These are convex semi-regular 4-dimensional polytopes having as cells two isomorphic convex semi-regular polyhedra (called the bases) and prisms.
 - i) There are 17 of these whose bases are Platonic solids (here we exclude the cube) or Archimedean solids.
 - ii) There are infinitely many prismatic polytopes whose bases are prisms or antiprisms.
3. **Cartesian products:** Boole Stott defined in [B-S 4, pg 4 footnote] the *Cartesian product* of any two polytopes as:

Definition 14. Let $P_1 \subseteq \mathbb{R}^n$ be an n -dimensional polytope and let $P_2 \subseteq \mathbb{R}^m$ be an m -dimensional polytope. The *Cartesian product* of P_1 and P_2 is the polytope:

$$P_1 \times P_2 = \text{conv}(\{(x, y) \in \mathbb{R}^{n+m} \mid x \in P_1, y \in P_2\}),$$

where *conv* denotes the convex hull. There are infinitely many polytopes of this type.

4. **The snub 24-cell:** This consists of 120 tetrahedra and 24 icosahedra. It was first discovered by Thorold Gosset in 1900 [Gos]. Boole Stott collaborated with Coxeter in the investigation of this polytope, and she made models of its sections [Cox1].
5. **The Grand Antiprism:** This consists of twenty 5-gonal antiprisms, and 300 tetrahedra. It was first discovered by John Conway and Michael Guy in 1965 using a computer search.

Boole Stott made very important contributions in her 1910 paper to the semi-regular polytopes in 4 dimensions. She was the first to find all 45 polytopes discussed in the first item of the list above. Her results (using her notation) are summarized in Figure 5.19 and 5.20. The symbols $\{3, 3, 3\}$, $\{4, 3, 3\}$, $\{3, 3, 4\}$, $\{3, 4, 3\}$, $\{5, 3, 3\}$ and $\{3, 3, 5\}$ denote the hypertetrahedron, hypercube, hyperoctahedron, 24-cell, 120-cell and 600-cell respectively, and P_n denotes the n -gonal prism.

| Nr | Symbol | Cells |
|----|------------------------|--|
| 1 | $\{3, 3, 3\}$ | 5 $\{3, 3\}$ |
| 2 | $e_1\{3, 3, 3\}$ | 5 $t\{3, 3\}$ and 5 $\{3, 3\}$ |
| 3 | $e_2\{3, 3, 3\}$ | 5 $(3, 4, 3, 4)$, 10 P_3 and 5 $\{3, 4\}$ |
| 4 | $e_3\{3, 3, 3\}$ | 10 $\{3, 3\}$ and 20 P_3 |
| 5 | $e_1e_2\{3, 3, 3\}$ | 5 $t\{3, 4\}$, 10 P_3 and 5 $t\{3, 3\}$ |
| 6 | $e_1e_3\{3, 3, 3\}$ | 5 $t\{3, 3\}$, 10 P_6 , 10 P_3 and 5 $(3, 4, 3, 4)$ |
| 7 | $e_1e_2e_3\{3, 3, 3\}$ | 10 $t\{3, 4\}$ and 20 P_6 . |
| 8 | $c_0e_1\{3, 3, 3\}$ | 5 $\{3, 4\}$ and 5 $\{3, 3\}$ |
| 9 | $c_0e_1e_2\{3, 3, 3\}$ | 10 $t\{3, 3\}$ |
| 10 | $\{4, 3, 3\}$ | 8 $\{4, 3\}$ |
| 11 | $e_1\{4, 3, 3\}$ | 8 $t\{4, 3\}$ and 16 $\{3, 3\}$ |
| 12 | $e_2\{4, 3, 3\}$ | 8 $(3, 4, 4, 4)$, 32 P_3 and 16 $\{3, 4\}$ |
| 13 | $e_3\{4, 3, 3\}$ | 16 $\{4, 3\}$, 32 P_3 and 16 $\{3, 3\}$ |
| 14 | $e_1e_2\{4, 3, 3\}$ | 8 $t(3, 4, 3, 4)$, 32 P_3 and 16 $t\{3, 3\}$ |
| 15 | $e_1e_3\{4, 3, 3\}$ | 8 $t\{4, 3\}$, 24 P_8 , 32 P_3 and 16 $(3, 4, 3, 4)$ |
| 16 | $e_2e_3\{4, 3, 3\}$ | 8 $(3, 4, 4, 4)$, 24 $\{4, 3\}$, 32 P_6 and 16 $t\{3, 3\}$ |
| 17 | $e_1e_2e_3\{4, 3, 3\}$ | 8 $t(3, 4, 3, 4)$, 24 P_8 , 32 P_6 and 16 $t\{3, 4\}$ |
| 18 | $c_0e_1\{4, 3, 3\}$ | 8 $(3, 4, 3, 4)$, 16 $\{3, 3\}$ |
| 19 | $c_0e_1e_2\{4, 3, 3\}$ | 8 $t\{3, 4\}$, 16 $t\{3, 3\}$ |
| 20 | $\{3, 3, 4\}$ | 16 $\{3, 3\}$ |
| 21 | $e_1\{3, 3, 4\}$ | 16 $t\{3, 3\}$ and 8 $\{3, 4\}$ |
| 22 | $e_2\{3, 3, 4\}$ | 24 $(3, 4, 3, 4)$ and 24 $\{4, 3\}$ |
| 23 | $e_1e_2\{3, 3, 4\}$ | 24 $t\{3, 4\}$ and 24 $\{4, 3\}$ |

Figure 5.19: Boole Stott's semi-regular polytopes (I) in [B-S 4].

| Nr | Symbol | Cells |
|----|------------------------|--|
| 24 | $\{3, 4, 3\}$ | 24 $\{3, 4\}$ |
| 25 | $e_2\{3, 4, 3\}$ | 24 (3, 4, 4, 4), 96 P_3 and 24 (3, 4, 3, 4) |
| 26 | $e_3\{3, 4, 3\}$ | 48 $\{3, 4\}$, 192 P_3 |
| 27 | $e_1e_2\{3, 4, 3\}$ | 24 t(3, 4, 3, 4), 96 P_3 , 24 t{4, 3} |
| 28 | $e_1e_3\{3, 4, 3\}$ | 24 t{3, 4}, 96 P_6 , 96 P_3 and 24 (3, 4, 4, 4) |
| 29 | $e_1e_2e_3\{3, 4, 3\}$ | 24 t(3, 4, 3, 4), 192 P_6 , and 24 t(3, 4, 3, 4) |
| 30 | $c_0e_1e_2\{3, 4, 3\}$ | 48 t{4, 3} |
| 31 | $\{5, 3, 3\}$ | 120 $\{5, 3\}$ |
| 32 | $e_1\{5, 3, 3\}$ | 120 t{5, 3} and 600 $\{3, 3\}$ |
| 33 | $e_2\{5, 3, 3\}$ | 120 (3, 4, 5, 4), 1200 P_3 and 600 $\{3, 4\}$ |
| 34 | $e_3\{5, 3, 3\}$ | 120 $\{5, 3\}$, 720 P_5 , 1200 P_3 and 600 $\{3, 3\}$ |
| 35 | $e_1e_2\{5, 3, 3\}$ | 120 t(3, 5, 3, 5), 1200 P_3 and 600 t{3, 3} |
| 36 | $e_1e_3\{5, 3, 3\}$ | 120 t{5, 3}, 720 P_{10} , 1200 P_3 and 600 (3, 4, 3, 4) |
| 37 | $e_2e_3\{5, 3, 3\}$ | 120 (3, 4, 5, 4), 720 P_5 , 1200 P_6 and 600 t{3, 3} |
| 38 | $e_1e_2e_3\{5, 3, 3\}$ | 120 t(3, 5, 3, 5), 720 P_{10} , 1200 P_6 and 600 t{3, 4} |
| 39 | $c_0e_1\{5, 3, 3\}$ | 120 (3, 5, 3, 5), and 600 $\{3, 3\}$ |
| 40 | $c_0e_2\{5, 3, 3\}$ | 120 $\{3, 5\}$, and 600 $\{3, 4\}$ |
| 41 | $c_0e_1e_2\{5, 3, 3\}$ | 120 t{3, 5}, and 600 t{3, 3} |
| 42 | $\{3, 3, 5\}$ | 600 $\{3, 3\}$ |
| 43 | $e_1\{3, 3, 5\}$ | 600 t{3, 3} and 120 $\{3, 5\}$ |
| 44 | $e_2\{3, 3, 5\}$ | 600 (3, 4, 3, 4), 720 P_5 and 120 (3, 5, 3, 5) |
| 45 | $e_1e_2\{3, 3, 5\}$ | 600 t{3, 4}, 720 P_5 and 120 t{3, 5} |

Figure 5.20: Boole Stott's semi-regular polytopes (II) in [B-S 4].

5.4 Conclusion

Boole Stott's figure represents a fascinating example of an amateur woman mathematician at the turning of the 19th and 20th centuries. Her story tells us several things. The contemporary interest in the fourth dimension plays an essential role in Boole Stott's story, and it is reflected in the work of the figures that influenced her. However, Boole Stott's intuition for four-dimensional space developed in a completely different way to theirs, due to the restriction of educational possibilities for women of the time. This, combined with the particular education received by Boole Stott from her family, seems to have led her to her discoveries. Boole Stott's methods, far from the analytical approach common in her contemporaries, show an extraordinary ability to visualize the fourth dimension.

The way her mathematics influenced others was clearly affected by the fact that many of her discoveries were not properly published, but are only known through some references to her work by Schoute or Coxeter. Still, Boole Stott is vividly remembered through her publications and her collections of models and drawings, which remind us of the beauty and dimensions of her work.

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