A method for removing all intermediate terms from a given equation

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We have learned from DesCartes' geometry by what method the second term might reliably be removed from a given equation; but on the question of removing multiple intermediate terms I have seen nothing hitherto in the analytic arts. On the contrary, I have encountered not a few who believed that the thing could not be done by any art. For this reason I have decided to set down here some things concerning this business, enough at least for those who have some grounding in the analytic art, since the others could scarcely be content with so brief an exposition: reserving the remainder (which they might wish to see here) for some other time.

Thus, in the first place, this must be noted; let some given cubic equation be $x^3 - px^2 + qx - r = 0$, in which x signifies the roots of this equation; and p, q, r represent known quantities. Now in order to remove the second term, let us suppose that $x = y + a$; now with the aid of these 2 equations, a third may be discovered in which the quantity x is absent, and this will be[1]

$$\begin{array}{llll} y^3 & \maltese 3ayy & \maltese 3aay & \maltese a^3 = 0 \\ & -pyy & -2pay & -paa \\ & & \maltese qy & \maltese qa \\ & & & -r \end{array}$$

Now let the second term be made equal to zero (since this is the term we intend to remove) and we shall have $3ay^2 - py^2 = 0$. Whence $a = \frac{p}{3}$: which shows that, for removing the second term in the cubic equation, $x = y + a$ is to be replaced (as we have just done) by $y = x + \frac{p}{3}$.

These things have been fully published, nor are they here spoken of for any other reason than that they serve to illustrate what is to follow, since when these things have been fully understood it is easier to grasp those things I am now about to propose.

Secondly, now let there be two terms to be removed from a given equation, and I say that we must suppose $x^2 = bx + y + a$; if three, $x^3 = cx^2 + bx + y + a$, if four $x^4 = dx^3 + cx^2 + bx + y + a$ and so to infinity. But I shall call these *assumed equations*, in order to distinguish them from equations which may be considered as given. The reason for this is that, for the same reason that with the help of the equation $x = y + a$ only one term at least can be removed (because of course at least only one indeterminant a exists), so by the same reason by the help of $x^2 = bx + y + a$ only two terms can be removed because just two indeterminants a and b are present; and so furthermore with the help of the following $x^3 = cx^2 + bx + y + a$ not more than three can be removed since there are only three indeterminants a, b, c. But since it may be understood by what reason this should follow, I shall show by what reason two terms may be removed from a given equation by means of the assumed [equation] $x^2 = bx + y + a$, and from this it will be easily established by what method one must proceed in this matter however far one might wish (since one may everywhere proceed by the same method). So be it.

Thirdly, [there is] the cubic equation $x^3 - px^2 + qx - r = 0$, from which the two intermediate terms are to be removed: first, the second term is removed (which is certainly of no help, but may at least be omitted here for the sake of time), and then we shall obtain an equation like this $y^3 - qy - r = 0$. Now let the assumed equation (in accordance with our second paragraph) be $y^2 = by + z + a$ and following from this let there be a third equation (by proceeding according to the recognized rules of the analysts) in which the quantity y is absent, and we shall obtain

[*] Editor's note: the original paper says only "by D.T.". This name comes from biographical sources.

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[1] Editor's note: I could not resist printing this equation in the original style. Later equations are in modern notation.

$$\begin{array}{lllll} z^3 & +3az^2 & +3a^2z & +a^3=0 \\ & -2qz^2 & -4qaz & -2qa^2 \\ & & +q^2z & +q^2a \\ & & -qb^2z & -qb^2a \\ & & +3rbz & +3rba \\ & & & -r^2 \\ & & & -qrb \\ & & & +rb^3 \end{array}$$

Now in this cubic equation, the second and third terms are made equal to zero (since we are proposing to remove these two intermediates) and this will give rise to two equations $3az^2-2qz^2=0$ & $3a^2z-4qaz+q^2z-qb^2z+3rbz=0$, by means of which the two intermediates may be determined: thus is found $a=\frac{2q}{3}$ and $b=\frac{3}{q}$ multiplied into $\frac{r}{2}\pm\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}$. Thus if in place of a & b the quantities found are substituted in the equation $y^2=by+z+a$, by this means in a given cubic equation two terms may be removed; or what comes from this, a given cubic equation, by help of this equation $y^2=by+z+a$, will be transformed into another cubic equation where the two intermediate terms will have been removed. And thus the same process is followed for the removal of three, four, five etc terms. For when a given equation, by means of an assumed one, is reworked as another, in which equally high degrees are present (that this thing can be done is clear at a single glance) in this third [equation] three four five etc terms may be placed equal to zero, and hence we shall always have just as many equations as there are indeterminants present, so that these may always be determined by means of these equations.

Fourthly, it must be noted that from this similar rules may be formed, and DesCartes shows [this] in removing the second term from a given equation, when he says that for removing the second term with the aid of the equation $x=y+a$, in a quadratic equation the quantity a must be equal to $\frac{p}{2}$ in a cubic equation $\frac{p}{3}$ and so forth. For I say that for removing two terms by means of the equation $x^2=bx+y+a$ in the cubic equation $x^3+qx+r=0$, a must be supposed equal to $-\frac{2q}{3}$ & $b=\frac{3r}{2q}\pm\sqrt{\frac{9r^2}{4q^2}+\frac{q}{3}}$: In a quadratoquadratic equation $x^4+qx^2+rx+s=0$, a must be supposed to equal $-\frac{2q}{4}$ & $b=\frac{3r}{2q}\pm\sqrt{\frac{9r^2}{4q^2}+\frac{2q}{4}-\frac{2s}{q}}$: and in the equation $x^5+qx^3+rx^2+sx+t=0$, $a=-\frac{2q}{5}$, & $b=\frac{3r}{2q}\pm\sqrt{\frac{9r^2}{4q^2}+\frac{3q}{5}-\frac{2s}{q}}$: finally in the equation $x^6+qx^4+rx^3+sx^2+tx+u=0$, $a=-\frac{2q}{6}$, & $b=\frac{3r}{2q}\pm\sqrt{\frac{9r^2}{4q^2}+\frac{4q}{6}-\frac{2s}{q}}$, and so to infinity. Whoever takes pleasure in such things, having clearly understood this, may easily move this thing further along.

Fifthly, what kind of advance this kind of analysis may be, and of how great a use, expert analysts will easily be able to judge: I myself deduce this corollary at least, that from this indeed a method may be derived for determining analytically the roots of all equations of any degree and I show that by example. For let a cubic equation be $x^3-px^2+qx-r=0$, the second term may be removed from this by means of this $x=\frac{y}{3}+\frac{p}{3}$ and we find

$$\begin{array}{lll} y^3 & -3p^2y & -2p^3=0 \\ & +9qy & +9pq \\ & & -27r \end{array}$$

Now for the sake of brevity, it is supposed that $3p^2-9q=q$ & $2p^3-9pq+27r=r$: whence it will be $y^3-qy-r=0$. Now this equation may be again transformed into another cubic where the two intermediate terms may be absent. and that becomes (just as is shown above) with the help of the equation $y^2=by+z+a$ if the quantity a be made equal to $\frac{2q}{3}$ & $b=\frac{3}{q}$ into $\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}$ (let it be for the sake of brevity $\sqrt{\frac{1}{4}r^2-\frac{1}{27}q^3}=f$ & $\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}=g$) for then is found from this $z^3=-\frac{216f^3g}{q^3}$. Whence $z=-\frac{6f}{q}\sqrt[3]{g}$. Formerly there was $y^2=\frac{3gy}{q}+\frac{2q}{3}+z$, but since the quantity z has now been found, it will be $y=\frac{3g}{2q}+\sqrt{\frac{9g^2}{4q^2}+\frac{2q}{3}-\frac{6f}{q}\sqrt[3]{g}}$. Furthermore formerly it was supposed $x=\frac{p}{3}+\frac{y}{3}$ and since y has now been found it will finally be $x=\frac{p}{3}+\frac{g}{2q}+\sqrt{\frac{g^2}{4q^2}+\frac{2q}{27}-\frac{2f}{3q}\sqrt[3]{g}}$. [This is] The root of the cubic equation $x^3 - px^2 + qx - r = 0$, which root is different from the Cardano expression most particularly in the fact that it includes a single cubic radical sign, where the Cardano expression contains two.

But finally it should be noted that by means of the equations displayed in the second paragraph not only may two, three, four, &c terms from a given equation be removed, but that from various other equations different from these I might also do the same thing. Nay, I can easily show that all possible equations may be handled by this business. But that I might be able to show by this method (according to paragraph five) all possible analytic expressions of roots of every possible equation, I shall explain more fully in its own place. Now in order to confirm these things I have just said, I shall show briefly by what method also another expression of the roots of a cubic equation, by means of the removal of terms, may be revealed, which in the first place answers to the Cardanic expression. For let $y^3 - qy - r = 0$. Suppose $yz = z^2 + a$, and with the help of this, a third may be found, where y is absent within,

$$\begin{array}{llll} z^6 & +3az^4 - rz^3 & +3a^2z^2 + a^3 = 0 \\ & -qz^4 & -qaz^2 & \end{array}$$

In this equation, if the terms where z contains degrees 4 and 2 are made equal to zero, it is found that $a = \frac{q}{3}$ & hence it will be $z^6 - rz^3 + \frac{q^3}{27} = 0$. Whence $z = \sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}$ which when restored into the assumed equation $yz = z^2 + \frac{q}{3}$ will give

$$y = \sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}} + \frac{q/3}{\sqrt[3]{\frac{r}{2} + \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}},$$

the desired root of the equation $y^3 - qy - r = 0$.