Higher-Dimensional Category Theory: Opetopic Foundations

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This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. The statements made in the 'Related Work' section of the Introduction, concerning which ideas are original or novel, are to the best of my knowledge correct.

This dissertation is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other university.

Eugenia L.-G. Cheng
10th March 2002

Summary

The problem of defining a weak n-category has been approached in various different ways, but so far the relationship between these approaches has not been fully understood. The subject of this thesis is the 'opetopic' theory of n-categories, embracing a group of definitions based on the theory of 'opetopes'.

This approach was first proposed by Baez and Dolan, and further approaches to the theory have been proposed by Hermida, Makkai and Power, and Leinster.

The opetopic definition of n-category has two stages. First, the language for describing k-cells is set up; this, in the language of Baez and Dolan, is the theory of *opetopes*. Then, a concept of universality is introduced, to deal with composition and coherence.

We first exhibit an equivalence between the three theories of opetopes as far as they have been proposed. We then give an explicit description of the category **Opetope** of opetopes. We also give an alternative presentation of the construction of opetopes using the 'allowable graphs' of Kelly and Mac Lane.

The underlying data for an opetopic n-category is given by an opetopic set. The category of opetopic sets is described explicitly by Baez and Dolan; we prove that this category is in fact equivalent to the category of presheaves on **Opetope**.

We then turn our attention to the full definition of (weak) n-categories. We define for each n a category **Opic-n-Cat** of opetopic n-categories and 'lax n-functors'. We then examine low-dimensional cases, and exhibit an equivalence between the opetopic and classical theories for the cases $n \leq 2$, giving in particular an equivalence between the opetopic and classical approaches to bicategories.

Finally we present some further discussion on the subject of universality. There are many ways of characterising universal cells; we propose an alternative characterisation to the one proposed by Baez and Dolan.

Contents

Introduction

The problem of defining a weak n-category has been approached in various different ways ([BD2], [HMP1], [Lei2], [Pen], [Bat], [Tam], [Str2], [May], [Lei6]), but so far the relationship between these approaches has not been fully understood. The subject of this thesis is the 'opetopic' theory of n-categories, embracing a group of definitions based on the theory of 'opetopes'.

This approach was first proposed by Baez and Dolan [BD2], and further approaches to the theory have been proposed by Hermida, Makkai and Power [HMP1] and Leinster [Lei2]. We first exhibit an equivalence between these three theories as far as they have been proposed. We then give an explicit description of the category **Opetope** of opetopes and the category **OSet** of opetopic sets, which give the data for the full definition of opetopic n-category.

We then examine low-dimensional cases, and exhibit an equivalence between the opetopic and classical theories for the cases $n \leq 2$. Finally we propose an alternative approach to characterising universality, a key component of the opetopic theory.

The opetopic definition of n-category has two stages. First, the language for describing k-cells is set up; this, in the language of Baez and Dolan, is the theory of opetopes. Then, a concept of universality is introduced, to deal with composition and coherence.

Any comparison of these approaches must therefore begin at the language for describing k-cells, and this is the subject of the first part of this work. In [BD2] Baez and Dolan give a definition of weak n-categories based on operads, opetopes and opetopic sets. In [HMP1] Hermida, Makkai and Power begin an explicitly analogous definition, based on (generalised) multicategories, multitopes and multitopic sets. The analogous components of the construction can therefore be compared step by step.

In [Lei2], Leinster gives an approach based on $(\mathcal{E}, T)$-multicategories; these structures were defined by Burroni [Bur] and have also been treated by Hermida [Her]. The role that these (even more generalised) multicategories plays is not explicitly analogous to that of operads and multicategories in the opetopic and multitopic versions respectively, so the comparison is more subtle. Leinster does, however, give a construction of 'opetopes' with a role analogous to that of Baez-Dolan opetopes; we are also able to compare these constructions, having established the relationship between the underlying theories.

It must be pointed out that we do not use the opetopic definitions precisely as given in [BD2], but rather, we develop a generalisation along lines which Baez and Dolan began but chose to abandon, for reasons unknown to the present author. Baez and Dolan work with operads having an arbitrary *set* of types (objects), but at the beginning of the paper they use operads having an arbitrary *category* of objects, before restricting to the case where the category of objects is small and discrete.

In fact, the use of a *category* of objects is a crucial aspect of our work. A conspicuous difference between the approach given in [BD2] and those of [HMP1] and [Lei2] is the presence in the first case, and the absence in the others, of symmetric actions. As cells of each dimension are successively constructed, so successive layers of symmetry are added in, apparently increasing the disparity between the symmetric and non-symmetric constructions.

However, the morphisms of the category of objects keep account of these successive layers of symmetry. Abandoning this information destroys the relationship between the approaches; by retaining it, a clear relationship can be seen.

We first compare the theories of opetopes step by step. We begin in Chapter 1 by comparing the different underlying theories of multicategories, and then in Chapter 2 we examine the construction of opetopes. In Section 2.1 we compare the process of constructing $(k+1)$-cells from k-cells, called 'slicing' in [BD2]. In Section 2.3 we apply the results to the construction of k-cell shapes themselves, to show that 'opetopes and multitopes are the same up to isomorphism'. That is, the categories of k-dimensional opetopes, multitopes, and Leinster opetopes are equivalent.

In Chapter 3 we give an explicit description of the category **Opetope** of opetopes, which will enable us, in Chapter 4, to prove that the category of opetopic sets is in fact a presheaf category.

We then turn our attention to the full definition of (weak) n-categories. In Chapter 5 we follow through the effects of our previous modifications to modify the rest of the definition as proposed by Baez and Dolan. We define for each n a category **Opic-n-Cat** of opetopic n-categories and 'lax n-functors'. Lax functors are in fact a more general (lax) notion than that of n-functor given in [BD2]; further questions of strictness are discussed later.

In fact, Hermida, Makkai and Power, and Leinster do not appear to have developed their theories to a full definition of n-category, so further possible comparisons with these approaches are limited; instead, we make a comparison with the classical theory. Any proposed definition of n-category should at least be in some way equivalent to the classical definitions as far as the latter are understood. In Section 5.2 we exhibit such equivalence for the cases $n \leq 2$, the main theorem giving an equivalence between the opetopic and classical approaches to bicategories. In comparing these theories there are two main issues:

1) An opetopic 2-category has m-ary 2-cells for all $m \geq 0$, that is, a 2-cell may have a string of m composable 1-cells as its domain; however a 2-cell in a bicategory has only one 1-cell as its domain.

2) In an opetopic 2-category 1-cell composition is not uniquely defined; however, in a bicategory m-fold composition is uniquely defined for $m = 0, 2$ (identities are considered as 0-fold composites).

So in one direction we must generate sets of m-cells, and in the other we must make some choices to specify nullary and binary composites.

To complete our understanding of opetopic n-categories, we would at least wish to construct an $(n+1)$-category of n-categories, but we do not address this matter here. In fact, in Section 5.2 we do not need 3- or even 2-dimensional structures to make a comparison with the classical theory; we prove an equivalence of *categories*, making a comparison already possible with only the 1-dimensional structure defined above.

We conclude the chapter with a brief discussion about notions of strictness in the opetopic theory. We demonstrate that, while the definition of 'lax n-functor' strictifies easily to 'weak n-functor' and 'strict n-functor', the definition of 'weak n-category' neither laxifies nor strictifies easily.

Finally in Chapter 6 we present some further discussion on the subject of universality. There are many ways of characterising universal cells, just as there are many ways of characterising, say, isomorphisms in a category. We propose an alternative characterisation to the one given in Chapter 5.

The idea is to generalise the familiar result in categories, that f is an isomorphism if and only if composition with f is an isomorphism. Here "composition with f" is a function on homsets; however, a feature of the opetopic definition of n-category is that composition is not uniquely defined, that is, $\_ \circ f$ is not a well-defined operation. One way of dealing with this would be to *choose* composites in order to make $\_ \circ f$ into an operation. This is the process of choosing universal cells, necessitated in Section 5.2. However, to avoid making such choices we instead define "composition with f" as a span of hom-$(n-k)$-categories. This "composition span" gives all possible ways of composing with f. We can then characterise f as universal if its composition span gives an $(n-k)$-equivalence of $(n-k)$-categories. For the purposes of this paper we do not attempt to justify the construction beyond drawing some illustrative diagrams at the first few dimensions.

This concludes the main part of the thesis. Appendices A and C contain some of the more involved calculations deferred from Sections 2.2.2 and 5.2.4 respectively.

In Appendix B we give an alternative presentation of the construction of opetopes, using the 'allowable graphs' of Kelly and Mac Lane. In [KM], Kelly and Mac Lane introduce a notion of graph to study coherence for symmetric monoidal closed categories. These graphs give a precise way of describing the trees used in the slice construction for symmetric multicategories, and hence an alternative way of constructing opetopes. However, since we do not use this approach in the rest of the thesis, we do not include it in the main part of the text.

Terminology

i) Since we are concerned chiefly with *weak* n-categories, we follow Baez and Dolan ([BD2]) and omit the word 'weak' unless emphasis is required; we refer to *strict* n-categories as 'strict n-categories'.

ii) We use the term 'weak n-functor' for an n-functor where functoriality holds up to coherent isomorphisms, and 'lax' functor when the constraints are not necessarily invertible.

iii) In [BD2] Baez and Dolan use the terms 'operad' and 'types' where we use 'multicategory' and 'objects'; the latter terminology is more consistent with Leinster's use of 'operad' to describe a multicategory whose 'objects-object' is 1.

iv) In [HMP1] Hermida, Makkai and Power use the term 'multitope' for the objects constructed in analogy with the 'opetopes' of [BD2]. This is intended to reflect the fact that opetopes are constructed using operads but multitopes using multicategories, a distinction that we have removed by using the term 'multicategory' in both cases. However, we continue to use the term 'opetope' and furthermore, use it in general to refer to the analogous objects constructed in each of the three theories. Note also that Leinster uses the term 'opetope' to describe objects which are analogous but not *a priori* the same; we refer to these as 'Leinster opetopes' if clarification is needed.

v) We follow Leinster and use the term '$(\mathcal{E}, T)$-multicategory' for the notion defined by Burroni ([Bur]) as 'T-category' (in French).

vi) We regard sets as sets or discrete categories with no notational distinction.

Related Work

The material in this thesis is, to the best of my knowledge, original. Where the work is based on definitions in the literature, this is clearly stated. Specifically:

Chapter 1, **The theory of multicategories**, takes as its starting point the definitions of multicategory given in [BD2], [HMP1] and [Lei2] respectively. The definitions are those given in these papers (with a few minor corrections); the relationship between the theories is new material.

Chapter 2, **The theory of opetopes**, again takes as its starting point the definitions given in [BD2], [HMP1] and [Lei2]; however the definition according to [BD2] is modified to include the category of objects, and this modification is followed through the rest of the thesis. The relationship between the theories is new material.

Chapter 3, **The category of opetopes**, is new material.

Chapter 4, **Opetopic sets**, contains definitions given in [BD2] but modified along the lines described earlier. The proof that the category of opetopic sets is a presheaf category is new.

Chapter 5, **Weak n-categories**, also uses definitions given in [BD2] but modified along the same lines as above. The analysis of the cases $n \leq 1$ is outlined in [BD2], but the analysis of $n = 2$ is original.

Chapter 6, **An alternative approach to universality** is original.

Appendix A is a proof deferred from Chapter 2.

Appendix B, **Opetopes via Kelly-Mac Lane graphs** is new material.

Appendix C contains calculations deferred from Chapter 5.

I have written up most parts of this thesis before, in papers available electronically ([Che1], [Che2], [Che3], [Che4], [Che5]), but in many places I have added detail and rigour.

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Chapter 1

The theory of multicategories

Opetopes are described using the language of multicategories. In each of the three theories of opetopes in question, a different underlying theory of multicategories is used. In this chapter we give examine the three underlying theories, and we construct a way of relating these theories to one another; this relationship provides subsequent equivalences between the definitions. We adopt a concrete approach here; certain aspects of the definitions suggest a more abstract approach but this will require further work beyond the scope of this thesis.

1.1 Definitions

In this section we give the definitions of the three theories of multicategories used in this work.

1.1.1 Symmetric multicategories

In [BD2] opetopes are constructed using symmetric multicategories. In this section we define **SymMulticat**, the category of symmetric multicategories with a category of objects. The definition we give here includes one axiom which appears to have been omitted from [BD2].

We write $\mathcal{F}$ for the 'free symmetric strict monoidal category' monad on **Cat**, and $\mathbf{S}_k$ for the group of permutations on k objects; we also write ι for the identity permutation.

Definition 1.1.1. *A* symmetric multicategory Q *is given by the following data*

1) *A category $o(Q) = \mathbb{C}$ of objects. We refer to $\mathbb{C}$ as the* object-category*, the morphisms of $\mathbb{C}$ as* object-morphisms*, and if $\mathbb{C}$ is discrete, we say that Q is* object-discrete.

2) *For each $p \in \mathcal{F}\mathbb{C}^{op} \times \mathbb{C}$, a set $Q(p)$ of arrows. Writing*

$$p = (x_1, \ldots, x_k; x),$$

an element $f \in Q(p)$ is considered as an arrow with source and target given by

$$\begin{aligned} s(f) &= (x_1, \ldots, x_k) \\ t(f) &= x \end{aligned}$$

and we say f has arity *k. We may also write $a(Q)$ for the set of all arrows of Q.*

3) *For each object-morphism $f : x \longrightarrow y$, an arrow $\iota(f) \in Q(x; y)$. In particular we write $1_x = \iota(1_x) \in Q(x; x)$.*

4) *Composition: for any $f \in Q(x_1, \ldots, x_k; x)$ and $g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i)$ for $1 \leq i \leq k$, a composite*

$$f \circ (g_1, \ldots, g_k) \in Q(x_{11}, \ldots, x_{1m_1}, \ldots, x_{k1}, \ldots, x_{km_k}; x)$$

5) *Symmetric action: for each permutation $\sigma \in \mathbf{S}_k$, a map*

$$\begin{array}{rccc} \sigma : & Q(x_1, \ldots, x_k; x) & \longrightarrow & Q(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x) \\ & f & \longmapsto & f\sigma \end{array}$$

satisfying the following axioms:

1) *Unit laws: for any $f \in Q(x_1, \ldots, x_m; x)$, we have*

$$1_x \circ f = f = f \circ (1_{x_1}, \ldots, 1_{x_m})$$

2) *Associativity: whenever both sides are defined,*

$$\begin{aligned} f \circ (g_1 \circ (h_{11}, \ldots, h_{1m_1}), \ldots, g_k \circ (h_{k1}, \ldots, h_{km_k})) = \\ (f \circ (g_1, \ldots, g_k)) \circ (h_{11}, \ldots, h_{1m_1}, \ldots, h_{k1}, \ldots, h_{km_k}) \end{aligned}$$

3) *For any $f \in Q(x_1, \ldots, x_m; x)$ and $\sigma, \sigma' \in \mathbf{S}_k$,*

$$(f\sigma)\sigma' = f(\sigma\sigma')$$

4) *For any $f \in Q(x_1, \ldots, x_k; x)$, $g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i)$ for $1 \leq i \leq k$, and $\sigma \in \mathbf{S}_k$, we have*

$$(f\sigma) \circ (g_{\sigma(1)}, \ldots, g_{\sigma(k)}) = f \circ (g_1, \ldots, g_k) \cdot \rho(\sigma)$$

where $\rho : \mathbf{S}_k \longrightarrow \mathbf{S}_{m_1 + \ldots + m_k}$ is the obvious homomorphism.

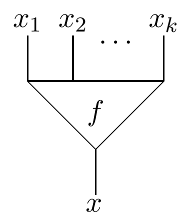
5) *For any $f \in Q(x_1, \ldots, x_k; x)$, $g_i \in Q(x_{i1}, \ldots, x_{im_i}; x_i)$, and $\sigma_i \in \mathbf{S}_{m_i}$ for $1 \leq i \leq k$, we have*

$$f \circ (g_1\sigma_1, \ldots, g_k\sigma_k) = (f \circ (g_1, \ldots, g_k))\sigma$$

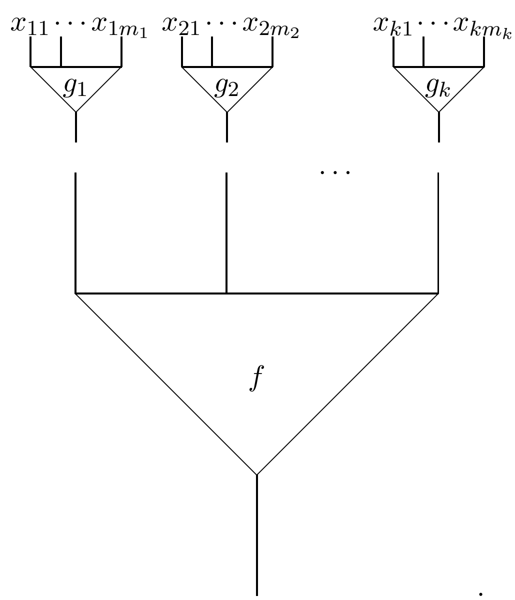
where $\sigma \in \mathbf{S}_{m_1 + \cdots + m_k}$ is the permutation obtained by juxtaposing the σ_i.

6) $\iota(f \circ g) = \iota(f) \circ \iota(g)$

We may draw an arrow $f \in Q(x_1, \ldots, x_k; x)$ as



and a composite $f \circ (g_1, \ldots, g_k)$ as



A symmetric multicategory Q may be thought of as a functor

$$Q : \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbf{Set}$$

with some extra structure.

In a more abstract view, we would expect $\mathcal{F}$ to be a 2-monad on the 2-category **Cat**, which lifts via a generalised form of distributivity to a bimonad on **Prof**, the bicategory of profunctors. Then the Kleisli bicategory for this bimonad should have as objects small categories, and its 1-cells should be essentially profunctors of the form $\mathcal{F}\mathbb{C} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{D}$ in the opposite category. However, the calculations involved in this description are intricate and require further work.

In this abstract view, a symmetric multicategory Q would then be a monad in this bicategory. Arrows and symmetric action (Data 2, 5) are given by the action of Q, identities (Data 3) by the unit of the monad and composition (Data 4) by the multiplication for the monad.

Definition 1.1.2. *Let Q and R be symmetric multicategories with object-categories $\mathbb{C}$ and $\mathbb{D}$ respectively. A* morphism of symmetric multicategories $F : Q \longrightarrow R$ *is given by*

- *A functor* $F = F_0 : \mathbb{C} \longrightarrow \mathbb{D}$
- *For each arrow* $f \in Q(x_1, \ldots, x_k; x)$ *an arrow* $Ff \in R(Fx_1, \ldots, Fx_k; Fx)$

satisfying

- F *preserves identities:* $F(\iota(f)) = \iota(Ff)$ *so in particular* $F(1_x) = 1_{Fx}$
- F *preserves composition: whenever it is defined*
$$F(f \circ (g_1, \ldots, g_k)) = (Ff \circ (Fg_1, \ldots, Fg_k))$$
- F *preserves symmetric action: for each* $f \in Q(x_1, \ldots, x_k; x)$ *and* $\sigma \in \mathbf{S}_k$
$$F(f\sigma) = (Ff)\sigma$$

Composition of such morphisms is defined in the obvious way, and there is an obvious identity morphism $1_Q : Q \longrightarrow Q$. Thus symmetric multicategories and their morphisms form a category **SymMulticat**.

Definition 1.1.3. *A morphism* $F : Q \longrightarrow R$ *is an* equivalence *if and only if the functor* $F_0 : \mathbb{C} \longrightarrow \mathbb{D}$ *is an equivalence, and* F *is full and faithful. That is, given objects* $x_1, \ldots, x_m, x$ *the induced function*

$$F : Q(x_1, \ldots, x_m; x) \longrightarrow R(Fx_1, \ldots, Fx_m; Fx)$$

is an isomorphism.

Note that, given morphisms of symmetric multicategories

$$Q \xrightarrow{F} R \xrightarrow{G} P$$

we have a result of the form 'any 2 gives 3', that is, if any two of F, G and GF are equivalences, then all three are equivalences.

Furthermore, we expect that **SymMulticat** may be given the structure of a 2-category, and that the equivalences in this 2-category would be the equivalences as above. However, we do not pursue this matter here.

1.1.2 Generalised multicategories

In [HMP1] multitopes are constructed using 'generalised multicategories'; in fact we need only a special case of the generalised multicategory defined in [HMP1], that is, the '1-level' case.

Definition 1.1.4. *A* generalised multicategory M *is given by*

- *A set* $o(M)$ *of objects*
- *A set* $a(M)$ *of arrows, with source and target functions*
$$\begin{array}{rcccl} s & : & a(M) & \longrightarrow & o(\mathbb{C})^\star \\ t & : & a(M) & \longrightarrow & o(\mathbb{C}) \end{array}$$
where $A^\star$ *denotes the set of lists of elements of a set* A*. If*
$$s(f) = (x_1, \ldots, x_k)$$
we write $s(f)_p = x_p$ *and* $|s(f)| = \{1, \ldots, k\}$*.*

- *Composition: for any* $f, g \in a(M)$ *with* $t(g) = s(f)_p$*, a composite* $f \circ_p g \in a(M)$ *with*

$$\begin{aligned} t(f \circ_p g) &= t(f) \\ |s(f \circ_p g)| &\cong (|s(f)| \setminus \{p\}) \amalg |s(g)| \end{aligned}$$

and amalgamating maps

$$\begin{aligned} \psi[f, g, p] &: |s(f)| \setminus \{p\} \longrightarrow |s(f \circ_p g)| \\ \phi[f, g, p] &: |s(g)| \longrightarrow |s(f \circ_p g)|. \end{aligned}$$

such that $\psi \amalg \phi$ *gives a bijection as above. Equivalently, writing*

$$\begin{aligned} s(f) &= (x_1, \ldots x_k), \\ s(g) &= (y_1, \ldots, y_j) \end{aligned}$$

and

$$(z_1, \ldots, z_{k+j-1}) = (x_1, \ldots, x_{p-1}, y_1, \ldots y_j, x_{p+1}, \ldots, x_{k+j-1})$$

we have a permutation $\chi = \chi[f, g, p] \in \mathbf{S}_{k+j-1}$ *such that*

$$s(f \circ_p g) = (z_{\chi(1)}, \ldots, z_{\chi(k+j-1)}).$$

- *Identities: for each* $x \in o(M)$ *an arrow* $1_x : x \longrightarrow x \in a(M)$

satisfying the following laws

- *Unit laws: for any* $f \in a(M)$ *with* $s(f)_p = x$ *and* $t(f) = y$*, we have*

$$\begin{gathered} 1_y \circ_1 f = f = f \circ_p 1_x \\ \chi[1_y, f, 1] = \iota = \chi[f, 1_x, p]. \end{gathered}$$

- *Associativity: for any* $f, g, h \in a(M)$ *with* $s(f)_p = t(g)$ *and* $s(g)_q = t(h)$ *we have*

$$(f \circ_p g) \circ_{\bar{q}} h = f \circ_p (g \circ_q h)$$

where $\bar{q} = \phi[f, g, p](q)$*. Furthermore, the composite amalgamation maps must also be equal; that is, the following coherence conditions must be satisfied:*

$$\begin{gathered} \psi[f \circ_p g, h, \bar{q}] \circ \psi[f, g, p] = \psi[f, h \circ_q g, p] \\ \psi[f \circ_p g, h, \bar{q}] \circ \bar{\phi}[f, g, p] = \phi[f, h \circ_q g, p] \circ \psi[g, h, q] \\ \phi[f \circ_p g, h, \bar{q}] = \phi[f, h \circ_q g, p] \circ \phi[g, h, q] \end{gathered}$$

where $\bar{\phi}$ *indicates restriction to the appropriate domain. Note that the conditions concern the source elements of* f*,* g *and* h *respectively.*

- *Commutativity: for any* $f, g, h \in a(M)$ *with* $s(f)_p = t(g)$*,* $s(f)_q = t(h)$*,* $p \neq q$ *we have*

$$(f \circ_p g) \circ_{\bar{q}} h = (f \circ_q h) \circ_{\bar{p}} g$$

where $\bar{q} = \psi[f, g, p]$ *and* $\bar{p} = \psi[f, h, q]$. *As above, the composite amalgamation maps must also be equal; that is, the following coherence conditions must be satisfied:*

$$\psi[f \circ_p g, h, \bar{q}] \circ \bar{\psi}[f, g, p] = \psi[f \circ_q h, g, \bar{p}] \circ \bar{\psi}[f, h, q]$$
$$\psi[f \circ_p g, h, \bar{q}] \circ \phi[f, g, p] = \phi[f \circ_q h, g, \bar{p}]$$
$$\phi[f \circ_p g, h, \bar{q}] = \psi[f \circ_q h, g, \bar{p}] \circ \phi[f, h, q].$$

The conditions concern the source elements of f, g *and* h *respectively.*

Note that the coherence conditions are necessary in case of repeated source elements.

Definition 1.1.5. *A* morphism of generalised multicategories

$$F = (F, \theta) : M \longrightarrow N$$

is given by:

- *for each object* $x \in o(M)$ *an object* $Fx \in o(N)$
- *for each arrow*
$$f : (x_1, \ldots, x_k) \longrightarrow x \in a(M)$$
a transition map $\theta_f = \theta_f^F \in \mathbf{S}_k$ *and an arrow*
$$Ff : (Fx_{\theta^{-1}(1)}, \ldots, Fx_{\theta^{-1}(k)}) \longrightarrow Fx \in a(N)$$

satisfying

- F *preserves identities:* $F(1_x) = 1_{Fx}$
- F *preserves composition: if* $f, g \in a(M)$ *and* $t(g) = s(f)_p$ *then*
$$Ff \circ_{\theta_f(p)} Fg = F(f \circ_p g).$$
Furthermore, the following coherence conditions must be satisfied:
$$\theta_{f \circ_p g} \circ \phi[f, g, p] = \phi[Ff, Fg, \theta_f(p)] \circ \theta_g$$
$$\theta_{f \circ_p g} \circ \psi[f, g, p] = \psi[Ff, Fg, \theta_f(p)] \circ \bar{\theta}_f$$
on the source elements of g *and* f *respectively, where* $\bar{\theta}$ *indicates the restriction of* θ *as appropriate.*

Given morphisms of generalised multicategories $M \xrightarrow{F} N \xrightarrow{G} L$ we have a composite morphism $H = G \circ F : M \longrightarrow L$ where H is the usual composite on objects and arrows, and we put $\theta_f^H = \theta_{Ff}^G \circ \theta_f^F$. There is an identity morphism $1_M : M \longrightarrow M$ which is the usual identity on objects and arrows, with $\theta_f = \iota$ for all $f \in a(M)$.

Thus generalised multicategories and their morphisms form a category **GenMulticat**.

1.1.3 $(\mathcal{E}, T)$-multicategories

In [Lei2] opetopes are constructed using $(\mathcal{E}, T)$-multicategories. These are defined by Burroni in [Bur] as 'T-categories'.

Definition 1.1.6. *Let T be a cartesian monad on a cartesian category $\mathcal{E}$. An $(\mathcal{E}, T)$-multicategory is given by an 'objects-object' C_0 and an 'arrows-object' C_1, with a diagram*

$$TC_0 \xleftarrow{d} C_1 \xrightarrow{c} C_0$$

in $\mathcal{E}$ together with maps $C_0 \xrightarrow{\text{ids}} C_1$ and $C_1 \circ C_1 \xrightarrow{\text{comp}} C_1$ satisfying associative and identity laws. (See [Lei5] for full details.)

We write **CartMonad** for the category of cartesian monads and cartesian monad opfunctors. A cartesian monad opfunctor

$$(U, \phi) : (\mathcal{E}_1, T_1) \longrightarrow (\mathcal{E}_2, T_2)$$

consists of

- a functor $U : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ preserving pullbacks
- a cartesian natural transformation $\phi : UT_1 \longrightarrow T_2U$, that is, a natural transformation whose naturality squares are pullbacks

satisfying certain axioms (see [Str1] and [Lei3] for full definitions).

1.2 Comparisons

We now compare the three theories of multicategories.

1.2.1 Relationship between symmetric and generalised multicategories

We compare symmetric and generalised multicategories by means of a functor

$$\xi : \mathbf{GenMulticat} \longrightarrow \mathbf{SymMulticat}.$$

We begin by constructing this functor, and then show that it is full and faithful.

We construct the functor ξ as follows. Given a generalised multicategory M, we define an object-discrete symmetric multicategory $\xi(M) = Q$ by

- Objects: $o(Q) = \mathbb{C}$ is the discrete category with objects $o(M)$.
- Arrows: for each

  $$p = (x_1, \ldots, x_k; x) \in \mathcal{F}(\mathbb{C})^{\text{op}} \times \mathbb{C}$$

  an element of $Q(p)$ is given by (f, σ) where $\sigma \in \mathbf{S}_k$ and

  $$f : (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \longrightarrow x \in a(M).$$

- Composition: by commutativity, it is sufficient to define

$$\alpha \circ_p \beta = \alpha \circ (1_{x_1}, \ldots, 1_{x_{p-1}}, \beta, 1_{x_{p+1}}, \ldots, 1_{x_k})$$

where

$$\begin{array}{rrcl} & \alpha & = & (f,\sigma) \in Q(x_1, \ldots, x_k; x) \\ \text{and} & \beta & = & (g,\tau) \in Q(y_1, \ldots, y_j; x_p). \end{array}$$

Now given such α and β, we have in M arrows

$$\begin{array}{rrclcl} & f & : & (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) & \longrightarrow & x \\ \text{and} & g & : & (y_{\tau(1)}, \ldots, y_{\tau(j)}) & \longrightarrow & x_p \end{array}$$

giving a composite in M

$$f \circ_{\bar{p}} g : (z_{\chi(1)}, \ldots, z_{\chi(k+j-1)}) \longrightarrow x$$

where $\bar{p} = \sigma^{-1}(p)$, $\chi = \chi(f, g, \bar{p})$ and

$$(z_1, \ldots, z_{k+j-1}) = (x_{\sigma(1)}, \ldots, x_{\sigma(\bar{p}-1)}, y_{\tau(1)}, \ldots, y_{\tau(j)}, x_{\sigma(\bar{p}+1)}, \ldots, x_{\sigma(k)}).$$

We seek a composite in Q with source

$$(a_1, \ldots, a_{k+j-1}) = (x_1, \ldots, x_{p-1}, y_1, \ldots, y_j, x_{p+1}, \ldots, x_k)$$

so the composite should be of the form $(f \circ_{\bar{p}} g, \gamma)$, where $f \circ_{\bar{p}} g$ has source

$$(a_{\gamma(1)}, \ldots, a_{\gamma(k+j-1)})$$

in M. So we define a permutation $\gamma \in \mathbf{S}_{j+k-1}$ by $a_{\gamma(i)} = z_{\chi(i)}$ and we define the composite to be

$$(f, \sigma) \circ_p (g, \tau) = (f \circ_{\bar{p}} g, \gamma).$$

Note that γ is determined by σ, τ and χ.

- For each $x \in \mathbb{C} = o(M)$, $1_x \in Q(x; x)$ is given by $(1_x, \iota)$.

- For each permutation $\sigma \in \mathbf{S}_k$, we have a map

$$\begin{array}{rccc} \sigma : & Q(x_1, \ldots, x_k; x) & \longrightarrow & Q(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; x) \\ & (f, \tau) & \longmapsto & (f, \sigma^{-1}\tau) \end{array}.$$

Note that f has source $(x_{\tau(1)} \ldots, x_{\tau(k)})$ in M, and $(f, \sigma^{-1}\tau)$ on the right hand side exhibits the ith source of f to be $x_{\sigma(\sigma^{-1}\tau)(i)} = x_{\tau(i)}$ as required.

We check that this definition satisfies the conditions for a symmetric multicategory:

1) Unit laws follow from unit laws of **GenMulticat**

2) Associativity follows from associativity in **GenMulticat** and the coherence conditions for amalgamating maps

3) $((f,\tau)\sigma)\sigma' = (f,\sigma^{-1}\tau)\sigma' = (f,\sigma'^{-1}\sigma^{-1}\tau) = (f,\tau)(\sigma\sigma')$

4) Given

$$\begin{array}{rcl}(f,\tau) & \in & Q(x_1,\dots,x_k;x), \\ (g,\mu) & \in & Q(y_1,\dots,y_j,x_p)\end{array}$$

and $\sigma \in \mathbf{S}_k$ we check that

$$(f,\tau)\sigma \circ_{\bar p} (g,\mu) = ((f,\tau)\circ_p (g,\mu))\cdot\rho(\sigma)$$

where $\bar p = \sigma^{-1}(p)$ and ρ is the homomorphism indicated in Section 1.1.1. The required result then follows by simultaneous composition. Note that it is sufficient to check that both expressions in question have the same first component and source (in Q), so we write γ,γ' for the permutations in the second component, without specifying what they are. Now

$$(f,\tau)\sigma \circ_{\bar p} (g,\mu) = (f,\sigma^{-1}\tau)\circ_{\bar p} (g,\mu) = (f \circ_{\tau^{-1}(p)} g, \gamma)$$

with source

$$(x_{\sigma(1)},\dots,x_{\sigma(\bar p-1)},y_1,\dots,y_j,x_{\sigma(\bar p+1)},\dots,x_{\sigma(k)})$$

and

$$((f,\tau)\circ_p (g,\mu))\cdot\rho(\sigma) = (f\circ_{\tau^{-1}(p)} g,\gamma')$$

with source

$$(z_{\rho\sigma(1)},\dots,z_{\rho\sigma(k+j-1)})$$

where

$$(z_1,\dots,z_{k+j-1}) = (x_1,\dots,x_{p-1},y_1,\dots,y_j,x_{p+1},\dots,x_k).$$

The action of $\rho(\sigma)$ is that of σ on the x_i but with $(y_1,\dots,y_j)$ substituted for x_p. So

$$(z_{\rho\sigma(1)},\dots,z_{\rho\sigma(k+j-1)}) = \\ (x_{\sigma(1)},\dots,x_{\sigma(\bar p-1)},y_1,\dots,y_j,x_{\sigma(\bar p+1)},\dots,x_{\sigma(k)})$$

as required.

5) Given (f,τ) and (g,μ) as above, and $\sigma\in\mathbf{S}_j$ we check that

$$(f,\tau)\circ_p (g,\mu)\sigma = ((f,\tau)\circ_p (g,\mu))\sigma'$$

where $\sigma'\in\mathbf{S}_{k+j-1}$ is given by inserting σ at the pth place.

Now, on the left hand side we have

$$\begin{array}{rcl}(f,\tau)\circ_p (g,\mu)\sigma & = & (f,\tau)\circ_p (g,\sigma^{-1}\mu) \\ & = & (f\circ_{\tau^{-1}(p)} g,\gamma),\end{array}$$

say, with source

$$(x_1,\dots,x_{p-1},y_{\sigma(1)},\dots y_{\sigma(j)},x_{p+1},\dots,x_k).$$

This agrees with the right hand side.

6) Since all object-morphisms are identities, this axiom is trivially satisfied.

So $\xi(M)$ is a symmetric multicategory.

Next we define ξ on morphisms of generalised multicategories. Given a morphism $F : M \longrightarrow N$ in **GenMulticat** we define a morphism

$$\xi F : \xi M \longrightarrow \xi N$$

in **SymMulticat** as follows.

- On objects: given $x \in o(\xi M) = o(M)$, put

$$(\xi F)(x) = Fx \in o(N) = o(\xi N)$$

- On arrows: given $(f, \sigma) \in \xi M(x_1, \ldots, x_k; x)$, put

$$\xi F(f, \sigma) = (Ff, \sigma {\theta_f}^{-1})$$

and check that

$$(Ff, \sigma {\theta_f}^{-1}) \in \xi N(Fx_1, \ldots, Fx_k; Fx).$$

First note that

$$t(Ff, \sigma {\theta_f}^{-1}) = t(Ff) = F(t(f)) = Fx.$$

Now

$$s(f) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$

in M, so by the action of (F, θ) we have

$$s(Ff) = (Fx_{\sigma {\theta_f}^{-1}(1)}, \ldots, Fx_{\sigma {\theta_f}^{-1}(k)})$$

in N, and so

$$(Ff, \sigma {\theta_f}^{-1}) \in \xi N(Fx_1, \ldots, Fx_k; Fx)$$

as required.

We check that this definition satisfies the laws for a morphism of symmetric multicategories:

- ξF preserves identities: since $\theta_{1_x} \in \mathbf{S}_1 = \{\iota\}$, we have

$$\xi F(1_x, \iota) = (F(1_x), \iota) = (1_{Fx}, \iota).$$

- ξF preserves composition: we check that $\xi F(\alpha \circ_p \beta) = \xi F\alpha \circ_p \xi F\beta$, and the result then follows by simultaneous composition. Put

$$\begin{array}{rrcl} & \alpha & = & (f, \sigma) \in Q(x_1, \ldots, x_k; x) \\ \text{and} & \beta & = & (g, \tau) \in Q(y_1, \ldots, y_j; y). \end{array}$$

Then

$$\begin{aligned}
\xi F(\alpha \circ_p \beta) &= \xi F(f \circ_{\sigma^{-1}(p)} g \ , \ \gamma) \\
&= (\ F(f \circ_{\sigma^{-1}(p)} g) \ , \ \gamma {\theta_f}^{-1} \ ) \\
&= (\ Ff \circ_{\theta_f \sigma^{-1}(p)} Fg \ , \ \gamma {\theta_f}^{-1} \ )
\end{aligned}$$

and this has source

$$s(F\alpha \circ_p F\beta) = (Fx_1, \dots, Fx_{p-1}, Fy_1, \dots, Fy_j, Fx_{p+1}, \dots, Fx_k).$$

For the right hand side, we have

$$\xi F\alpha = (Ff, \sigma\theta_f^{-1})$$

$$\xi F\beta = (Fg, \tau\theta_g^{-1})$$

and so the first component of $\xi F\alpha \circ_p \xi F\beta$ is also $Ff \circ_{\theta_f \sigma^{-1}(p)} Fg$. So since $\xi F(\alpha \circ_p \beta)$ and $\xi F\alpha \circ_p \xi F\beta$ agree in the first component and source, we have the result required.

- ξF preserves symmetric action:

$$\begin{aligned}
\xi F(\ (f,\tau)\sigma\ ) &= \xi F(f, \sigma^{-1}\tau) \\
&= (Ff \ , \ \sigma^{-1}\tau{\theta_f}^{-1}) \\
&= (Ff \ , \ \tau{\theta_f}^{-1})\sigma \\
&= (\xi F(f,\tau))\sigma
\end{aligned}$$

So ξF is a morphism of symmetric multicategories.

We check that ξ is functorial. Clearly $\xi 1_M = 1_{\xi M}$. Now consider morphisms of generalised multicategories

$$M \xrightarrow{F} N \xrightarrow{G} L$$

so we need to show

$$\xi(G \circ F) = \xi G \circ \xi F.$$

- On objects

$$\begin{aligned}
\xi(G \circ F)(x) &= (G \circ F)(x) \\
&= (\xi G \circ \xi F)(x)
\end{aligned}$$

- On arrows

$$\begin{aligned}
\xi(G \circ F)(f,\sigma) &= (\ (G \circ F)(f) \ , \ \sigma({\theta^{GF}}_f)^{-1} \ ) \\
&= (\ GFf \ , \ \sigma({\theta^G}_{Ff} \circ {\theta^F}_f)^{-1} \ ) \\
&= (\ GFf \ , \ \sigma({\theta^F}_f)^{-1}({\theta^G}_{Ff})^{-1} \ ) \\
&= \xi G(\ Ff \ , \ \sigma({\theta^F}_f)^{-1} \ ) \\
&= (\xi G \circ \xi F)(f,\tau)\sigma
\end{aligned}$$

So ξ is a functor as required.

Proposition 1.2.1. *The functor* $\xi : \mathbf{GenMulticat} \longrightarrow \mathbf{SymMulticat}$ *is full and faithful.*

Proof. Given any morphism

$$G : \xi M \longrightarrow \xi N$$

of symmetric multicategories, we show that there is a unique morphism

$$H = (H, \theta) : M \longrightarrow N$$

of generalised multicategories such that

$$\xi H = G.$$

Suppose first that such an H exists.

- On objects: for each object $x \in o(M) = o(\xi M)$ we must have

$$Hx = (\xi H)x = Gx.$$

- On arrows: given an arrow $f \in M(x_1, \ldots, x_k; x)$, we certainly have

$$\begin{array}{rl} & (f, \iota) \in \xi M(x_1, \ldots, x_k; x) \\ \text{and} & G(f, \iota) = (\bar{f}, \sigma) \in \xi N(Gx_1, \ldots Gx_k; Gx), \end{array}$$

  say, where $\bar{f}$ is a morphism in N with source

$$s(\bar{f}) = (Gx_{\sigma(1)}, \ldots, Gx_{\sigma(k)}).$$

  Now $(\xi H)(f, \iota) = (Hf, \theta_f^{-1})$ but we must have

$$\begin{array}{rcl} (\xi H)(f, \iota) & = & G(f, \iota) \\ & = & (\bar{f}, \sigma) \end{array}$$

  so we must have $Hf = \bar{f}$ and $\theta_f = \sigma^{-1}$.

So we define H as above and check that this satisfies the axioms for a morphism of generalised multicategories.

- H preserves identities

We have

$$G(1_x, \iota) = (1_{Gx}, \iota)$$

so

$$H(1_x) = 1_{Gx} = 1_{Hx}.$$

- H preserves composition

We need to show
$$Hf \circ_{\theta_f(p)} Hg = H(f \circ_p g)$$
and that the coherence conditions are satisfied. Now, G preserves the composition of ξM so
$$G\alpha \circ_p G\beta = G(\alpha \circ_p \beta).$$
Now we have
$$\begin{aligned} G\alpha \circ_p G\beta &= (\bar{f}, \theta_f^{-1}) \circ_p (\bar{g}, \theta_g^{-1}) \\ &= (\bar{f} \circ_{\theta_f(p)} \bar{g}, \gamma), \text{ say} \end{aligned}$$
and
$$\begin{aligned} G(\alpha \circ_p \beta) &= G(f \circ_p g, \gamma') \\ &= (\overline{f \circ_p g}, \gamma''), \text{ say.} \end{aligned}$$
So these must be equal on both components. Comparing first components, we have
$$\overline{f \circ_p g} = \bar{f} \circ_{\theta_f(p)} \bar{g}$$
but by definition we have
$$\begin{aligned} \overline{f \circ_p g} &= H(f \circ_p g) \\ \text{and } \bar{f} \circ_{\theta_f(p)} \bar{g} &= Hf \circ_{\theta_f(p)} Hg \end{aligned}$$
so
$$Hf \circ_{\theta_f(p)} Hg = H(f \circ_p g)$$
as required. Furthermore, equality of the second components gives precisely the coherence condition we require, since γ is formed from θ_f, θ_g and the amalgamation map $\chi(\bar{f}, \bar{g}, \theta_f(p))$, and γ'' is formed from $\chi(f, g, p)$ and $\theta_{f \circ_p g}$.

So H is a morphism of generalised multicategories; by construction it is unique such that $\xi H = G$, so ξ is indeed full and faithful. $\Box$

We now give necessary and sufficient conditions for a symmetric multicategory to be in the image of ξ.

Definition 1.2.2. *We say that a symmetric multicategory Q is* freely symmetric *if and only if for every arrow $\alpha \in Q$ and permutation σ*
$$\alpha\sigma = \alpha \Rightarrow \sigma = \iota.$$

Proposition 1.2.3. *Let Q be a symmetric multicategory. Then $Q \cong \xi(M)$ for some generalised multicategory M if and only if Q is object-discrete and freely symmetric.*

Proof. Suppose $Q \cong \xi(M)$. Then by the definition of ξ, Q is object-discrete, with object-category $\mathbb{C} \cong o(M)$. To show that Q is freely symmetric, write $p = (x_1, \ldots, x_k; x)$, so
$$\begin{aligned} Q(p) = \{(f, \tau) \quad | \quad & f \in a(M),\ \tau \in \mathbf{S}_k \\ & f : x_{\tau(1)}, \ldots, x_{\tau(k)} \longrightarrow x \in M\} \end{aligned}$$

and consider $\alpha = (f, \tau) \in Q(p)$. Now $(f, \tau)\sigma = (f, \sigma^{-1}\tau)$ so

$$\begin{aligned} \alpha\sigma = \alpha \quad &\Rightarrow \quad \sigma^{-1}\tau = \tau \\ &\Rightarrow \quad \sigma = \iota \end{aligned}$$

as required.

Conversely, suppose that Q is object-discrete and freely symmetric. So, given an arrow α of arity k, we have distinct arrows $\alpha\sigma$ for each $\sigma \in \mathbf{S}_k$. We define an equivalence relation $\sim$ on $a(Q)$, by

$$\alpha \sim \beta \iff \beta = \alpha\sigma \text{ for some permutation } \sigma$$

and we specify a representative of each equivalence class.

Now let M be a generalised multicategory whose objects are those of Q, and whose arrows are the chosen representatives of the equivalence classes of $\sim$. Composition is inherited, with amalgamation maps re-ordering the sources as necessary. So associativity and commutativity are inherited; the coherence conditions for amalgamation maps are satisfied since Q is freely symmetric. Observe that for each $x \in \mathbb{C}$, the equivalence class of 1_x is $\{1_x\}$, so M inherits identities.

So M is a generalised multicategory, and $\xi(M) \cong Q$. Note that a different choice of representatives would give an equivalent generalised multicategory. $\square$

Definition 1.2.4. *We call a symmetric multicategory* tidy *if it is freely symmetric with a category of objects equivalent to a discrete one. We write* **TidySymMulticat** *for the full subcategory of* **SymMulticat** *whose objects are tidy symmetric multicategories.*

Lemma 1.2.5. *A symmetric multicategory is tidy if and only if it is equivalent to one in the image of ξ.*

Proof. We show that Q is tidy if and only if $Q \simeq R$ where R is freely symmetric and object-discrete. The result then follows by Proposition 1.2.3.

Suppose Q is tidy. We construct R as follows. Let $\mathbb{C}$ be the category of objects of Q, with $\mathbb{C}$ equivalent to a discrete category S, say, by

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} S.$$

Then R is given by

- $o(R) = S$.
- $R(d_1, \ldots, d_n; d) = Q(Gd_1, \ldots, Gd_n; Gd)$.
- identities, composition and symmetric action induced from Q.

Then certainly $Q \simeq R$ and R is freely symmetric and object-discrete; the converse is clear. $\square$

We will later see (Section 2.3) that only tidy symmetric multicategories are needed for the construction of opetopes. We now include another result that will be useful in the next section.

Lemma 1.2.6. *If Q is a tidy symmetric multicategory then* $\mathrm{elt}\,Q$ *is equivalent to a discrete category.*

Proof. This may be proved by direct calculation; it is also seen in Proposition 2.2.2. $\square$

Note that we write $\mathrm{elt}\,Q$ for the category of elements of Q, where Q is here considered as a functor $Q : \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbf{Set}$ with certain extra structure.

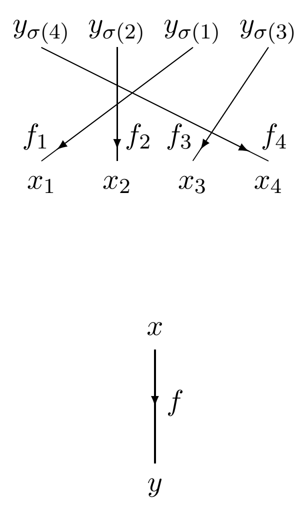
So $\mathrm{elt}\,Q$ has as objects pairs (p, g) with $p \in \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ and $g \in Q(p)$; a morphism $\alpha : (p, g) \longrightarrow (p', g')$ is an arrow $\alpha : p \longrightarrow p' \in \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$ such that

$$\begin{array}{rccc} Q(\alpha): & Q(p) & \longrightarrow & Q(p') \\ & g & \longmapsto & g'\,. \end{array}$$

For example, an arrow

$$(\sigma, f_1, f_2, f_3, f_4; f) : (x_1, x_2, x_3, x_4; x) \longrightarrow (y_1, y_2, y_3, y_4; y) \in \mathcal{F}\mathbb{C}^{\mathrm{op}} \times \mathbb{C}$$

may be represented by the following diagram



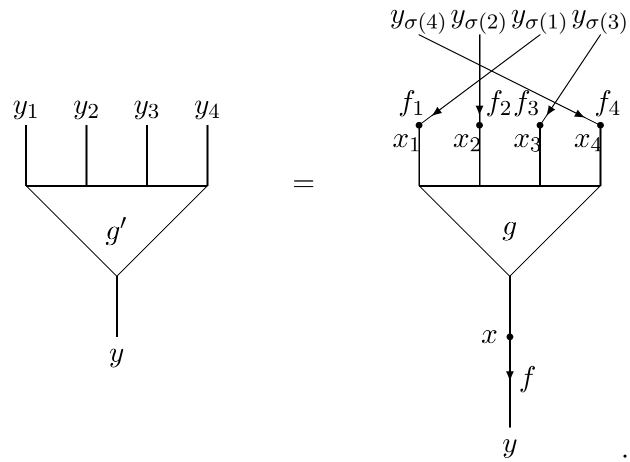
Then, given any arrow $g \in Q(x_1, \ldots x_m; x)$, we have an arrow

$$\alpha(g) = g' \in Q(y_1, \ldots, y_m; y)$$

given by

$$g' = (\iota(f) \circ g \circ (\iota(f_1), \ldots, \iota(f_m))\sigma).$$

So continuing the above example we may have:



Note that we may write an object $(p, g) \in \mathrm{elt}(Q)$ simply as g, since p is uniquely determined by g.

1.2.2 Relationship between symmetric multicategories and cartesian monads

The respective roles of multicategories in the Baez-Dolan and Leinster approaches are not explicitly analogous. In this section we exhibit instead a correspondence between certain symmetric multicategories and certain cartesian monads, by constructing a functor

$$\zeta : \mathbf{TidySymMulticat} \longrightarrow \mathbf{CartMonad}.$$

This is enough since we will see that all the symmetric multicategories involved in the construction of opetopes are tidy.

We begin by defining the action of ζ on objects; so for any tidy symmetric multicategory Q, we construct a cartesian monad $\zeta(Q) = (\mathcal{E}_Q, T_Q)$, say. Informally, the idea behind this construction is that T_Q should encapsulate information about the arrows of Q. The functor part is constructed to give the arrows themselves, the unit to give the identities, and multiplication the reduction laws (composites).

Write $o(Q) = \mathbb{C}$. Q is tidy, so $\mathbb{C} \simeq S$, say, where S is a discrete category. For various of the constructions which follow, we assume that we have chosen a specific functor $S \xrightarrow{\sim} \mathbb{C}$. However, when isomorphism classes are taken subsequently, we observe that the construction in question does not depend on the choice of this functor.

Put $\mathcal{E}_Q = \mathbf{Set}/S$ and observe immediately that this is cartesian. (This is sufficient here, though of course $\mathbf{Set}/S$ has much more structure than this.)

Informally, an element $(X, f) = (X \xrightarrow{f} S)$ of $\mathbf{Set}/S$ may be thought of as a system for labelling Q-objects with 'compatible' elements of X; each 'label' is compatible with an isomorphism class of Q-objects. Then the action of T_Q assigns compatible labels to the source elements of Q-arrows in every way possible; the target is not affected. The resulting set of 'source-labelled Q-arrows' is itself made into a set of labels by regarding each arrow as a 'label' for its target.

We now give the formal definition of the functor $T_Q : \mathcal{E}_Q \longrightarrow \mathcal{E}_Q$. For the action on object-categories, consider $(X, f) = (X \xrightarrow{f} S) \in \mathbf{Set}/S$. We have the following composite functor

$$\mathrm{elt}\, Q \xrightarrow{s} \mathcal{F}\mathbb{C}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{F}S^{\mathrm{op}}$$

where $\mathcal{F}$ denotes the free symmetric strict monoidal category monad on **Cat**, and s and t the source and target functions respectively. Consider the pullback

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \mathcal{F}X^{\mathrm{op}} \arrow[d, "\mathcal{F}f^{\mathrm{op}}"] \\ \mathrm{elt}\, Q \arrow[r] & \mathcal{F}S^{\mathrm{op}} \end{tikzcd} .$$

Since Q is tidy, $\mathrm{elt}\, Q$ is equivalent to a discrete category, and so too is the above pullback; so we have

$$\mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \simeq X',$$

say, where X' is discrete. Put $T_Q(X, f) = (X', f')$ where f' is the composite

$$X' \xrightarrow{\sim} \mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \longrightarrow \mathrm{elt}\, Q \xrightarrow{t} \mathbb{C} \xrightarrow{\sim} S.$$

This is well-defined since if $(\alpha, \underline{x}) \cong (\alpha', \underline{x}') \in \mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}}$ then certainly $\alpha \cong \alpha' \in \mathrm{elt}\, Q$ and so $t(\alpha) \cong t(\alpha') \in \mathbb{C}$.

We now define the action of T_Q on morphisms. A morphism

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

in $\mathbf{Set}/S$ induces a functor

$$\mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \longrightarrow \mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}Y^{\mathrm{op}}$$

which, by construction, makes the following diagram commute:

$$\begin{tikzcd} \mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \arrow[rr] \arrow[dr] & & \mathrm{elt}\, Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}Y^{\mathrm{op}} \arrow[dl] \\ & \mathrm{elt}\, Q & \end{tikzcd}$$

giving a morphism

$$\begin{tikzcd} X' \arrow[rr, "h'"] \arrow[dr, "f'"'] & & Y' \arrow[dl, "g'"] \\ & S & \end{tikzcd}$$

in **Set**$/S$. We define T_Q on morphisms by $T_Q(h) = h'$. T_Q is clearly functorial; we now show that it inherits a cartesian monad structure from the identities and composition of Q. For convenience we write $\mathcal{E}_Q = \mathcal{E}$ and $T_Q = T$.

- unit

We seek a natural transformation $\eta : 1_{\mathcal{E}} \Longrightarrow T$, so with the above notation we need components

$$\eta_{(X,f)} : (X, f) \longrightarrow (X', f').$$

Given $(X, f) \in \mathbf{Set}/S$, we have a functor $X \longrightarrow \mathrm{elt}\,Q$ given by the composite

$$X \xrightarrow{f} S \xrightarrow{\sim} \mathbb{C} \xrightarrow{1_{-}} \mathrm{elt}\,Q.$$

We also have a functor $X \longrightarrow \mathcal{F}X^{\mathrm{op}}$ given by the unit of the monad $\mathcal{F}$. These induce a functor

$$X \longrightarrow \mathrm{elt}\,Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}}$$

and we define the component $\eta_{(X,f)}$ to be the composite

$$X \longrightarrow \mathrm{elt}\,Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \xrightarrow{\sim} X'.$$

Explicitly, $\eta_{(X,f)}$ acts as follows. We have $\eta_{(X,f)}(x) = [(1_c, x)]$, the isomorphism class of

$$(1_c, x) \in \mathrm{elt}\,Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}}.$$

So $(1_c, x)$ is an "identity labelled by x", where $c \in \mathbb{C}$ is any object in the isomorphism class fx. We can see explicitly that this is well defined since if $c \cong c'$ we have $1_c \cong 1_{c'} \in \mathrm{elt}\,Q$ and thus

$$[(1_c, x)] = [(1_{c'}, x)].$$

The following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "\eta_{(X,f)}"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

so $\eta_{(X,f)}$ is indeed a morphism $(X, f) \longrightarrow T(X, f) \in \mathbf{Set}/S$.

Next we show that the components $\eta_{(X,f)}$ satisfy naturality; so we show that for any morphism $h : (X, f) \longrightarrow (Y, g) \in \mathbf{Set}/S$ the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\eta_{(X,f)}"] \arrow[d, "h"'] & X' \arrow[d, "h'"] \\
Y \arrow[r, "\eta_{(Y,g)}"'] & Y'
\end{tikzcd}$$

This follows from the construction of η, and naturality of the unit for $\mathcal{F}$; alternatively, we see that on elements, the right-ish leg gives

$$x \longmapsto [(1_c, x)] \longmapsto [(1_c, hx)]$$

with c in the isomorphism class fx, and the left-ish leg gives

$$x \longmapsto hx \longmapsto [(1_{c'}, hx)]$$

with c' in the isomorphism class ghx. But $gh = f$ since $h : (X, f) \longrightarrow (Y, g)$, so $c' \cong c$ and $[(1_{c'}, hx)] = [(1_c, hx)]$.

It also follows from the construction of η that the square is a pullback; it is similarly easily seen by considering elements.

- multiplication

We seek a natural transformation $\mu : T^2 \Longrightarrow T$. Consider $(X, f) \in$ **Set**$/S$. Then by definition

$$\begin{aligned} X' &\simeq \operatorname{elt} Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} = A, \text{ say} \\ \text{and} \quad X'' &\simeq \operatorname{elt} Q \times_{\mathcal{F}S^{\mathrm{op}}} \mathcal{F}X'^{\mathrm{op}} = B, \text{ say.} \end{aligned}$$

We construct a commutative square

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & \mathcal{F}X^{\mathrm{op}} \arrow[d] \\
\operatorname{elt} Q \arrow[r] & \mathcal{F}S^{\mathrm{op}}
\end{tikzcd}$$

and use the universal property of the pullback A to induce a morphism $B \longrightarrow A$, and hence $X'' \longrightarrow X'$.

The morphism $B \longrightarrow \mathcal{F}X^{\mathrm{op}}$ along the top is given by

$$\operatorname{elt} Q \times \mathcal{F}X'^{\mathrm{op}} \xrightarrow{p_2} \mathcal{F}X'^{\mathrm{op}} \xrightarrow{\mathcal{F}p_2} \mathcal{F}\mathcal{F}X^{\mathrm{op}} \xrightarrow{\mu} \mathcal{F}X^{\mathrm{op}}$$

where p_1 and p_2 denote the first and second projections respectively. The morphism $B \longrightarrow \operatorname{elt} Q$ on the left is given by

$$\operatorname{elt} Q \times \mathcal{F}X'^{\mathrm{op}} \xrightarrow{(1, \mathcal{F}p_1)} \operatorname{elt} Q \times \mathcal{F}(\operatorname{elt} Q)^{\mathrm{op}} \longrightarrow \operatorname{elt} Q$$

where the second morphism is composition in Q. Then, by definition of X' and naturality of μ, the above square commutes, inducing a map

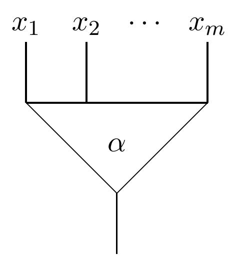
$$B \longrightarrow A$$

and hence, on isomorphism classes, a map

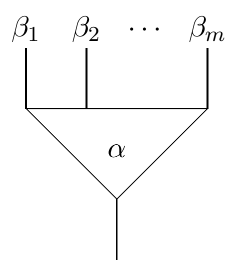
$$\begin{tikzcd} X'' \arrow[rr, "\mu_{(X,f)}"] \arrow[dr] & & X' \arrow[dl] \\ & S & \end{tikzcd}$$

in **Set**$/S$ as required.

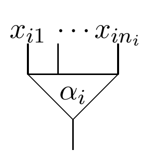
Informally, (X, f) is a system for labelling Q-objects, and $T(X, f) = (X', f')$ gives source-labelled Q-arrows. A typical element of X' may be thought of as the isomorphism class of



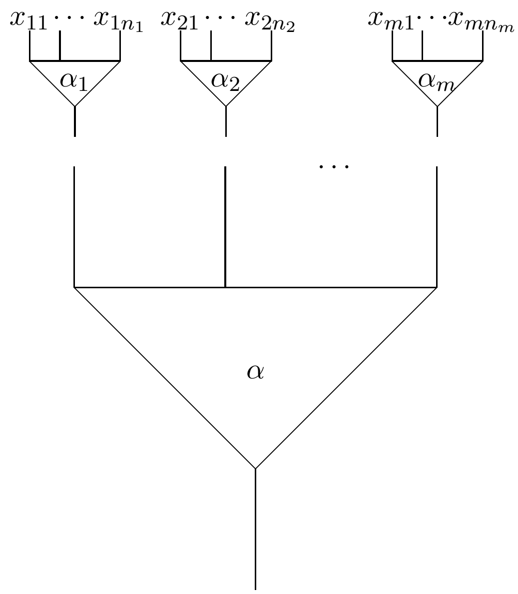
where $\alpha \in \mathrm{elt}\, Q$ and $s(\alpha) \cong (fx_1, \ldots, fx_n)$. Then f' takes this element to $[t(\alpha)]$. So a typical element θ of $T^2(X, f) = (X'', f'')$ is the isomorphism class of



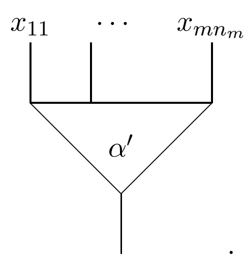
where $\beta_i \in X'$ and $s(\alpha) \cong (f'(\beta_1), \ldots, f'(\beta_m))$. Writing β_i as the isomorphism class of



we can draw θ as (the isomorphism class of)



where $\alpha, \alpha_1, \ldots, \alpha_m \in \mathrm{elt}\, Q$ and $s(\alpha) \cong (t(\alpha_1), \ldots t(\alpha_m))$. So, via the relevant object-isomorphisms, we may compose the underlying Q-arrows to give α', say, which is defined up to isomorphism. We then concatenate the X-labels (via the multiplication for $\mathcal{F}$) to give



Finally, we take the isomorphism class of this to give $\mu_{(X,f)}(\theta) \in X'$, and $f''(\mu_{(X,f)}(\theta)) = [t(\alpha')] = [t(\alpha)] \in S$.

It follows that μ defined in this way is a cartesian natural transformation.

- T preserves pullbacks

First observe that a commutative square in **Set**$/S$ is a pullback if and only if applying the forgetful functor **Set**$/S \longrightarrow$ **Set** gives a pullback in **Set**. Then T preserves pullbacks since $\mathcal{F}$ preserves pullbacks.

So $T_Q = (T, \eta, \mu)$ is a cartesian monad on $\mathcal{E}_Q = \mathcal{E}$ and we may define $\zeta(Q) = (\mathcal{E}_Q, T_Q)$.

We now define the action of ζ on morphisms. Let

$$F : Q \longrightarrow R$$

be a morphism of tidy symmetric multicategories. We construct a cartesian monad opfunctor

$$(U_F, \phi_F) : (\mathcal{E}_Q, T_Q) \longrightarrow (\mathcal{E}_R, T_R)$$

that is

- a functor $U = U_F : \mathbf{Set}/S_Q \longrightarrow \mathbf{Set}/S_R$ preserving pullbacks

- a cartesian natural transformation $\phi = \phi_F : UT_Q \longrightarrow T_R U$

satisfying certain axioms.

We define U as follows. On objects, we have a functor

$$F : o(Q) \longrightarrow o(R)$$

giving a morphism on isomorphism classes

$$\bar{F} : S_Q \longrightarrow S_R.$$

This induces a functor

$$\mathbf{Set}/S_Q \longrightarrow \mathbf{Set}/S_R$$

by composition with $\bar{F}$, which clearly preserves pullbacks; we define U to be this functor.

We now construct the components of ϕ. Given $(X, f) \in \mathbf{Set}/S_Q$ write

$$\begin{array}{rl} & T_Q(X,f) = (X^Q, f^Q) \\ \text{and} & X^Q \simeq \mathrm{elt}\, Q \times_{\mathcal{F}S_Q{}^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}}. \end{array}$$

We seek

$$\phi_{(X,f)} : (X^Q, \bar{F} \circ f^Q) \longrightarrow (X^R, (\bar{F} \circ f)^R) \in \mathbf{Set}/S_R$$

that is, a morphism $X^Q \longrightarrow X^R$ such that the outside of the following diagram commutes

$$\begin{tikzcd}
X^Q \arrow[r, dotted] \arrow[d, "\sim"'] & X^R \arrow[d, "\sim"] \\
\mathrm{elt}\, Q \times_{\mathcal{F}S_Q{}^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \arrow[r, "{(F,1)}"] \arrow[d] & \mathrm{elt}\, R \times_{\mathcal{F}S_R{}^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \arrow[d] \\
\mathrm{elt}\, Q \arrow[r, "F"] \arrow[d, "t"'] & \mathrm{elt}\, R \arrow[d, "t"] \\
o(Q) \arrow[r, "F"] \arrow[d, "\sim"'] & o(R) \arrow[d, "\sim"] \\
S_Q \arrow[r, "\bar{F}"'] & S_R
\end{tikzcd}$$
.

The map $X^Q \longrightarrow X^R$ is induced by $(F,1)$ on isomorphism classes as shown in the diagram, since the pullback

$$\mathrm{elt}\, R \times_{\mathcal{F}S_R{}^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}}$$

is along the morphism $\bar{F} \circ f$. We define $\phi_{(X,f)}$ to be this map. Observe that all squares in the diagram commute, so $\phi_{(X,f)}$ is a morphism in $\mathbf{Set}/S_R$ as required.

We now check that these components satisfy naturality. Given any morphism $h : (X,f) \longrightarrow (Y,g) \in \mathbf{Set}/S_Q$, we have the following diagram

$$\begin{tikzcd}
\mathrm{elt}\, Q \times_{\mathcal{F}S_Q{}^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \arrow[r, "{(F,1)}"] \arrow[d, "{(1,\mathcal{F}h)}"'] & \mathrm{elt}\, R \times_{\mathcal{F}S_R{}^{\mathrm{op}}} \mathcal{F}X^{\mathrm{op}} \arrow[d, "{(1,\mathcal{F}h)}"] \\
\mathrm{elt}\, Q \times_{\mathcal{F}S_Q{}^{\mathrm{op}}} \mathcal{F}Y^{\mathrm{op}} \arrow[r, "{(F,1)}"'] & \mathrm{elt}\, R \times_{\mathcal{F}S_R{}^{\mathrm{op}}} \mathcal{F}Y^{\mathrm{op}}
\end{tikzcd}$$

Considering this componentwise, it clearly commutes and is a pullback. The result on isomorphism classes follows.

Finally, by functoriality of F, (U,ϕ) satisfies the axioms for a monad opfunctor. So (U,ϕ) is a cartesian monad opfunctor and the construction is clearly functorial. This completes the definition of ζ.

We observe immediately that the construction of $(\mathcal{E}_Q, T_Q)$ uses only the isomorphism classes of objects and arrows of Q. So

$$(\mathcal{E}_{Q_1}, T_{Q_1}) \cong (\mathcal{E}_{Q_2}, T_{Q_2}) \iff Q_1 \simeq Q_2.$$

Recall (1.1.1) that we expect that a symmetric multicategory Q may be given as a monad in a certain bicategory, in which case the identities are given by the unit, and composition laws by multiplication. In this abstract framework there should be a morphism from the underlying bicategory to the 2-category **Cat**, taking the monad Q to the monad T_Q, but this is somewhat beyond the scope of this thesis.

Chapter 2

The theory of opetopes

In this chapter we give the analogous constructions of opetopes in each theory, and show in what sense they are equivalent. That is, we show that the respective categories of k-opetopes are equivalent.

2.1 Slicing

We first discuss the process by which $(k+1)$-cells are constructed from k-cells. In [BD2], the 'slice' construction is used, giving for any symmetric multicategory Q the slice multicategory Q^+. In [HMP1] the 'multicategory of function replacement' is used but this has a more far-reaching role than that of the Baez-Dolan slice. For comparison with the Baez-Dolan theory, we construct a 'slice' which is analogous to the Baez-Dolan slice and is a special case of a multicategory of function replacement.

In [Lei2] the 'free $(\mathcal{E}, T)$-operad' construction is used, giving, for any 'suitable' monad $(\mathcal{E}, T)$, the free $(\mathcal{E}, T)$-operad monad $(\mathcal{E}', T')$.

2.1.1 Slicing a symmetric multicategory

Let Q be a symmetric multicategory with a category $\mathbb{C}$ of objects, so Q may be considered as a functor $Q : \mathcal{F}\mathbb{C}^{\text{op}} \times \mathbb{C} \longrightarrow \textbf{Set}$ with certain extra structure. The slice multicategory Q^+ is given by:

- Objects: put $o(Q^+) = \text{elt}(Q)$

So the category $o(Q^+)$ has as objects pairs (p, g) with $p \in \mathcal{F}\mathbb{C}^{\text{op}} \times \mathbb{C}$ and $g \in Q(p)$; a morphism $\alpha : (p, g) \longrightarrow (p', g')$ is an arrow $\alpha : p \longrightarrow p' \in \mathcal{F}\mathbb{C}^{\text{op}} \times \mathbb{C}$ such that

$$\begin{array}{rccc} Q(\alpha): & Q(p) & \longrightarrow & Q(p') \\ & g & \longmapsto & g' \end{array}$$

Then, given any arrow

$$g \in Q(x_1, \ldots x_m; x)$$

we have an arrow $\alpha(g) = g' \in Q(y_1, \ldots, y_m; y)$ given by

$$g' = (\iota(f) \circ g \circ (\iota(f_1), \ldots, \iota(f_m))\sigma)$$

(see Section 1.2.1).

- Arrows: $Q^+(f_1, \ldots, f_n; f)$ is given by the set of 'configurations' for composing $f_1, \ldots, f_n$ as arrows of Q, to yield f.

Writing $f_i \in Q(x_{i1}, \ldots x_{im_i}; x_i)$ for $1 \leq i \leq n$, such a configuration is given by (T, ρ, τ) where

1) T is a planar tree with n nodes. Each node is labelled by one of the f_i, and each edge is labelled by an object-morphism of Q in such a way that the (unique) node labelled by f_i has precisely m_i edges going in from above, labelled by $a_{i1}, \ldots, a_{im_i} \in \text{arr}(\mathbb{C})$, and the edge coming out is labelled $a_i \in a(\mathbb{C})$, where $\text{cod}(a_{ij}) = x_{ij}$ and $\text{dom}(a_i) = x_i$.

2) $\rho \in \mathbf{S}_k$ where k is the number of leaves of T.

3) $\tau : \{\text{nodes of } T\} \longrightarrow [n] = \{1, \ldots, n\}$ is a bijection such that the node N is labelled by $f_{\tau(N)}$. (This specification is necessary to allow for the possibility $f_i = f_j,\ i \neq j$.)

Note that (T, ρ) may be considered as a 'combed tree', that is, a planar tree with a 'twisting' of branches at the top given by ρ.

The arrow resulting from this composition is given by composing the f_i according to their positions in T, with the a_{ij} acting as arrows $\iota(a_{ij})$ of Q, and then applying ρ according to the symmetric action on Q. This construction uniquely determines an arrow $(T, \rho, \tau) \in Q^+(f_1, \ldots, f_n; f)$.

- Composition

When it can be defined, $(T_1, \rho_1, \tau_1) \circ_m (T_2, \rho_2, \tau_2) = (T, \rho, \tau)$ is given by

1) (T, ρ) is the combed tree obtained by replacing the node ${\tau_1}^{-1}(m)$ by the tree (T_2, ρ_2), composing the edge labels as morphisms of $\mathbb{C}$, and then 'combing' the tree so that all twists are at the top.

2) τ is the bijection which inserts the source of T_2 into that of T_1 at the mth place.

- Identities: given an object-morphism

$$\alpha = (\sigma, f_1, \ldots, f_m; f) : g \longrightarrow g',$$

$\iota(\alpha) \in Q^+(g; g')$ is given by a tree with one node, labelled by g, twist σ, and edges labelled by the f_i and f as in the example above.

- Symmetric action: $(T, \rho, \tau)\sigma = (T, \rho, \sigma^{-1}\tau)$

This is easily seen to satisfy the axioms for a symmetric multicategory.

Note that, given a labelled tree T with n nodes and k leaves, there is an arrow $(T, \rho, \tau) \in a(Q^+)$ for every permutation $\rho \in \mathbf{S}_k$ and every bijection $\tau : \{\text{nodes of } T\} \longrightarrow [n]$. Suppose

$$\begin{array}{rrcl} & s(T, \rho, \tau) & = & (f_1, \ldots, f_n) \\ \text{and} & t(T, \rho, \tau) & = & f. \end{array}$$

Then, given any $\rho_1 \in \mathbf{S}_k,\ \tau : \{\text{nodes of } T\} \longrightarrow [n]$, we have

$$\begin{array}{rrcl} & s(T, \rho_1\rho, \tau) & = & (f_1, \ldots, f_n) \\ \text{and} & t(T, \rho_1\rho, \tau) & = & f\rho_1 \end{array}$$

whereas

$$\begin{array}{rrcl} & s(T, \rho, \tau_1\tau) & = & (f_{\tau_1{}^{-1}(1)}, \ldots f_{\tau_1{}^{-1}(n)}) \\ \text{and} & t(T, \rho, \tau_1\tau) & = & f. \end{array}$$

We observe immediately that Q^+ is freely symmetric, since

$$\begin{array}{rcl} (T, \rho, \tau)\sigma = (T, \rho, \tau) & \Rightarrow & \sigma^{-1}\tau = \tau \\ & \Rightarrow & \sigma = \iota. \end{array}$$

However Q^+ is not in general object-discrete; we will later see (Proposition 2.2.2) that Q^+ is tidy if Q is tidy.

2.1.2 Slicing a generalised multicategory

Given a generalised multicategory M, we define a slice multicategory M_+. We use the 'multicategory of function replacement' as defined in [HMP1], which plays a role similar to (but more far-reaching than) that of the Baez-Dolan slice. The slice defined in this section is only a special case of a multicategory of function replacement, but it is sufficient for the construction of multitopes. Moreover, for the purpose of comparison it is later helpful to be able to use this closer analogy of the Baez-Dolan slice.

We first explain how this slice arises from the multicategory of function replacement as defined in [HMP1], and then give an explicit construction of the slice multicategory that is analogous to the symmetric case. This latter construction is the one we continue to use in the rest of the work.

Using the terminology of [HMP1], the slice is defined as follows. Let $\mathcal{L}$ be the language with objects $o(M)$ and arrows $a(M)$, and let $\mathbb{F}$ be the free generalised multicategory on $\mathcal{L}$. So the objects of $\mathbb{F}$ are the objects of M, and the arrows of $\mathbb{F}$ are formal composites of arrows of M. We define a morphism of generalised multicategories $h : \mathbb{F} \longrightarrow M$ as the identity on objects, and on arrows the action of composing the formal composite to yield an arrow of M. Then we define M_+ to be the multicategory of function replacement on $(\mathcal{L}, \mathbb{F}, h)$.

Explicitly, the slice multicategory M_+ is a generalised multicategory given by:

- Objects: $o(M_+) = a(M)$.
- Arrows: $a(M_+)$ is given by configurations for composing arrows of M.

Such a configuration is given by $T = (T, \rho_T, \tau_T)$, where:

i) T is a planar tree with n nodes labelled by $f_1, \dots f_n \in a(M)$, and edges labelled by objects of M in such a way that, writing

$$s(f_i) = (x_{i1}, \dots, x_{im_i}),$$

the node labelled by f_i has m edges coming in, labelled by $x_{i1}, \dots, x_{im_i}$ from left to right, and one edge going out, labelled by $t(f_i)$.

ii) $\rho_T \in \mathbf{S}_k$, where k is the number of leaves of T. The composition in M given by T has specified amalgamation maps giving information about the ordering of the source; ρ_T is the permutation induced on the source.

iii) $\tau_T : \{\text{nodes of } T\} \longrightarrow [n]$ is a bijection so that the node N is labelled by $f_{\tau_T(N)}$. In fact, specifying τ_T corresponds to specifying amalgamation maps in the free multicategory $\mathbb{F}$, and this defines the amalgamation maps of M_+.

Note that whereas in the symmetric case ρ and τ may be chosen freely for any given T, in this case precisely one ρ_T and τ_T is specified for each T. The source and target of such an arrow T are given by $s(T) = (f_1, \dots f_n)$ and $t(T) = f \in a(M)$, the result of composing the f_i according to their positions in T. Here, the tree T may be thought of as a combed tree as in the symmetric case, but with all edges labelled by identities.

- Composition

When it can be defined, we have $T_1 \circ_m T_2 = T$ as follows:

i) T is the combed labelled tree obtained from (T_1, τ_{T_1}) by replacing the node ${\tau_{T_1}}^{-1}(m)$ by the combed tree (T_2, τ_{T_2}), combing the tree and then forgetting the twist at the top.

ii) The amalgamation maps are defined to reorder the source as necessary according to τ_{T_1}, τ_{T_2} and τ_T.

- Identities: 1_f is the tree with one node, labelled by f.

This definition is easily seen to satisfy the axioms for a generalised multicategory. Note that a different choice of amalgamation maps for $\mathbb{F}$ gives rise to different bijections τ_T and hence different amalgamation maps in M_+, resulting in an isomorphic slice multicategory.

2.1.3 Slicing a $(\mathcal{E}, T)$-multicategory

In [Lei2] the 'free $(\mathcal{E}, T)$-operad' construction is used to construct $(k+1)$-cells from k-cells; this gives, for any *suitable* monad $(\mathcal{E}, T)$, the 'free $(\mathcal{E}, T)$-operad' monad $(\mathcal{E}, T)' = (\mathcal{E}', T')$. In order to compare this construction with the Baez-Dolan slice, we examine the monad $\zeta(Q)'$. First we must show that $\zeta(Q)'$ can actually be constructed, that is, that $\zeta(Q) = (\mathcal{E}_Q, T_Q)$ is a suitable monad.

First recall ([Lei2]) that a cartesian monad $(\mathcal{E}, T)$ is *suitable* if it satisfies:

i) $\mathcal{E}$ has disjoint finite coproducts which are stable under pullback

ii) $\mathcal{E}$ has colimits of nested sequences; these commute with pullbacks and have monic coprojections

iii) T preserves colimits of nested sequences.

Here a nested sequence is a string of composable monics.

Proposition 2.1.1. *Let Q be a tidy symmetric multicategory. Then $(\mathcal{E}_Q, T_Q)$ is a suitable monad.*

Proof. Certainly $\mathcal{E}_Q$ is a suitable category, and we have already shown that $(\mathcal{E}_Q, T_Q)$ is cartesian. So it remains to show that T_Q preserves colimits of nested sequences.

First observe that a morphism h in $\mathbf{Set}/S$ is monic if and only if h is monic as a morphism in $\mathbf{Set}$, that is, injective. Given a nested sequence

$$(A_0, f_0) \overset{i_0}{\rightarrowtail} (A_1, f_1) \overset{i_1}{\rightarrowtail} (A_2, f_2) \cdots \in \mathbf{Set}/S$$

we have a nested sequence

$$A_0 \overset{i_0}{\rightarrowtail} A_1 \overset{i_1}{\rightarrowtail} A_2 \cdots \in \mathbf{Set}.$$

Since $\mathbf{Set}$ is suitable, this nested sequence has a colimit A whose coprojections are monics. Then the morphisms $f_0, f_1, \ldots$ define a cone with vertex S, inducing a unique morphism $A \xrightarrow{f} S$ making everything commute; (A, f) is then a colimit for the nested sequence in $\mathbf{Set}/S$. So (A, f) is a colimit for the nested sequence in $\mathbf{Set}/S$ exactly when A is a colimit for the nested sequence in $\mathbf{Set}$.

Having made these observations, it is easy to check that T_Q preserves such colimits. □

2.2 Comparisons

In this section we compare the slice constructions and make precise the sense in which they correspond to one another. Recall (Sections 1.2.1, 1.2.2) that we have defined functors

$$\mathbf{GenMulticat} \xrightarrow{\xi} \mathbf{TidySymMulticat} \xrightarrow{\zeta} \mathbf{CartMonad}.$$

We now show that these functors 'commute' with slicing, up to equivalence (for ξ) and isomorphism (for ζ).

2.2.1 Generalised and symmetric multicategories

We will eventually prove (Corollary 2.2.3) that for any generalised multicategory M

$$\xi(M_+) \simeq \xi(M)^+.$$

We prove this by constructing, for any morphism of symmetric multicategories $\phi : Q \longrightarrow \xi(M)$ a morphism $\phi^+ : Q^+ \longrightarrow \xi(M_+)$ such that

$$\phi \text{ is an equivalence} \Rightarrow \phi^+ \text{ is an equivalence.}$$

The result then follows by considering the case $\phi = 1$.

We begin by constructing ϕ^+. Recall

$$\begin{array}{rcll} o(Q^+) & = & a(Q) & \\ a(Q^+) & = & \{(T,\rho,\tau): & T \text{ a labelled tree with } n \text{ nodes, } k \text{ leaves} \\ & & & \rho \in \mathbf{S}_k, \\ & & & \tau : \{\text{nodes of T}\} \stackrel{\sim}{\longrightarrow} [n] \\ & & & \text{edges labelled by morphisms of } \mathbb{C}\} \\ o(\xi(M_+)) & = & a(M) & \\ a(\xi(M_+)) & = & \{(T,\sigma) \ : & T \text{ a labelled tree with } n \text{ nodes} \\ & & & \sigma \in \mathbf{S}_n \\ & & & \text{edges labelled by identities}\}. \end{array}$$

The idea is that given a way of composing arrows $f_1, \ldots, f_n$ of Q to an arrow f, we have a way of composing arrows $g_1, \ldots, g_n$ of M to an arrow g, where

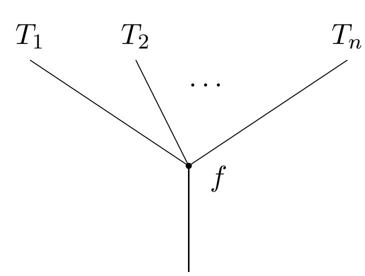
$$\begin{array}{rcl} \phi(f_i) & = & (g_i, \sigma_i) \\ \text{and } \phi(f) & = & (g, \sigma). \end{array}$$

Observe that since ξM is object-discrete, we have $\phi a = 1$ for all object-morphisms $a \in \mathbb{C}$.

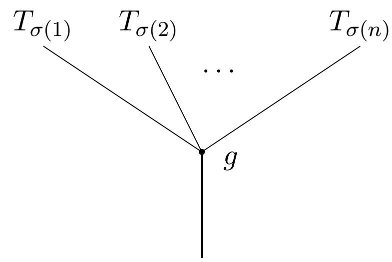
So we define ϕ^+ as follows:

- On objects: if $\phi(f) = (g, \sigma),\ g \in a(M)$ then put $\phi^+(f) = g$.
- On object-morphisms: since $\xi(M^+)$ is object-discrete, we must have $\phi^+(\alpha) = 1$ for all object-morphisms α.
- On arrows: put $\phi^+ : (T, \rho, \tau) \longmapsto (\bar{T}, \tau \circ {\tau_{\bar{T}}}^{-1})$, where $\bar{T}$ is the labelled planar tree obtained as follows. Given a node with label f say, and $\phi(f) = (g, \sigma)$:
  - i) replace the label with g
  - ii) 'twist' the inputs of the node according to σ
  - iii) proceed similarly with all nodes, make all edge labels identities, then comb and ignore the twist at the top of the resulting tree (since the twist in M_+ is determined by the tree).

For example, suppose T is given by



where the T_i are subtrees of T, and $\phi(f) = (g, \sigma)$. Then steps (i) and (ii) above give



and $\bar{T}$ is then defined inductively on the subtrees. Node N in $\bar{T}$ is considered to be the image of node N in T under the operation $T \longrightarrow \bar{T}$.

Writing

$$\begin{aligned} s(T, \rho, \tau) &= (f_1, \dots, f_n) \\ \text{and} \quad t(T, \rho, \tau) &= f \end{aligned}$$

we check that

$$\begin{aligned} s(\phi^+(T, \rho, \tau)) &= (\phi^+(f_1), \dots, \phi^+(f_n)) \\ \text{and} \quad t(\phi^+(T, \rho, \tau)) &= \phi^+(f). \end{aligned}$$

Writing $s(\bar{T}, \tau \circ \tau_{\bar{T}}^{-1}) = (g_1, \dots, g_n)$ in $\xi(M)$, we have, in M_+

$$s(\bar{T}) = (g_{\tau \circ \tau_{\bar{T}}^{-1}(1)}, \dots, g_{\tau \circ \tau_{\bar{T}}^{-1}(n)})$$

so node N is labelled in $\bar{T}$ by $g_{\tau \circ \tau_{\bar{T}}^{-1}(\tau_{\bar{T}}(N))} = g_{\tau(N)}$ and in T by $f_{\tau(N)}$. So by definition of $\bar{T}$ we have

$$\phi^+(f_{\tau(N)}) = g_{\tau(N)}$$

so $\phi^+(f_i) = g_i$ for each i and

$$s(\bar{T}, \tau \circ \tau_{\bar{T}}^{-1}) = (\phi^+(f_1), \dots, \phi^+(f_n))$$

as required. Also, $t(\bar{T}, \tau \circ \tau_{\bar{T}}^{-1}) = \phi^+(f)$ by functoriality of ϕ and definition of composition in $\xi(M)$.

We have shown that ϕ^+ is functorial on the object-category $o(Q^+)$; we need to check the remaining conditions for ϕ^+ to be a morphism of symmetric multicategories. We may now assume that all edge labels are identities since they all become identities under the action of ϕ^+.

- ϕ^+ preserves identities:

$1_f \in a(Q^+)$ is (T, ι, ι) where T has one node, labelled by f. So we have $\phi^+(1_f) = T$ where T has one node, labelled by $\phi^+(f)$, and $\phi^+(1_f) = 1_{\phi^+}(f)$.

- ϕ^+ preserves composition: We need to show

$$\phi^+(\alpha \circ_m \beta) = \phi^+(\alpha) \circ_m \phi^+(\beta).$$

Now, the underlying trees are the same by functoriality of ϕ, the permutation of leaves is the same by coherence for amalgamation maps of M, and the node ordering is the same by definition of ϕ^+.

- ϕ^+ preserves symmetric action:

$$\begin{aligned}\phi^+((T,\rho,\tau)\sigma)) &= \phi^+(T,\rho,\sigma^{-1}\tau) \\ &= (\bar{T},\sigma^{-1}\tau \circ {\tau_{\bar{T}}}^{-1}) \\ &= (\bar{T},\tau \circ {\tau_{\bar{T}}}^{-1})\sigma \\ &= (\phi^+(T,\rho,\tau))\sigma.\end{aligned}$$

So ϕ^+ is a morphism of symmetric multicategories.

Proposition 2.2.1. *Let Q be a symmetric multicategory, M a generalised multicategory and $\phi : Q \longrightarrow \xi(M)$ a morphism of symmetric multicategories. If ϕ is an equivalence then ϕ^+ is an equivalence.*

This enables us to prove the following proposition:

Proposition 2.2.2. *If Q is tidy then Q^+ is tidy.*

Proof of Proposition 2.2.1. First we observe that given any such morphism ϕ, Q is freely symmetric:

$$\begin{aligned}\alpha\sigma = \alpha \quad &\Rightarrow \quad \phi(\alpha\sigma) = \phi(\alpha)\sigma = \phi(\alpha) \ \in \xi(M) \\ &\Rightarrow \quad \sigma = \iota,\end{aligned}$$

the second implication following from $\xi(M)$ being freely symmetric.

Now, given that ϕ is full, faithful and essentially surjective on the category of objects, and full and faithful, we prove the proposition in the following steps:

i) ϕ^+ is surjective on objects

ii) ϕ^+ is full on the category of objects

iii) ϕ^+ is faithful on the category of objects

iv) ϕ^+ is full

v) ϕ^+ is faithful

Proof of (i). Recall the action of ϕ^+ on objects: let $f \in o(Q^+) = a(Q)$ with $\phi(f) = (g,\sigma)$ then $\phi^+ : f \longmapsto g$. Now, given any $g \in o(\xi(M_+)) = a(M)$, we have $(g,\iota) \in a(\xi(M))$. ϕ is full and surjective, so there exists $f \in a(Q)$ such that $\phi(f) = (g,\sigma)$ and $\phi^+(f) = g$. $\square$

Proof of (ii). $\xi(M_+)$ is object-discrete so we only need to show that if $\phi^+(f_1) = \phi^+(f_2)$ then there is a morphism $f_1 \longrightarrow f_2$ in $o(Q^+)$. Now

$$\begin{array}{rcll} \phi^+(f_1) = \phi^+(f_2) \ \Rightarrow \ \phi(f_1) & = & \phi(f_2)\sigma & \text{for some permutation } \sigma \\ & = & \phi(f_2\sigma). & \end{array}$$

Suppose

$$\begin{array}{rcccc} f_1 & : & a_1, \ldots, a_n & \longrightarrow & a \\ \text{and } f_2\sigma & : & b_1, \ldots, b_n & \longrightarrow & b. \end{array}$$

Then we must have $\phi(a_i) = \phi(b_i)$ for all i, and $\phi(a) = \phi(b)$. So there exist morphisms

$$\begin{array}{rcccc} g_i & : & b_i & \longrightarrow & a_i \\ \text{and } g & : & a & \longrightarrow & b \end{array}$$

and we have

$$f_2\sigma = g \circ f_1 \circ (g_1, \ldots, g_n)$$

giving a morphism $f_1 \longrightarrow f_2$ as required. □

Proof of (iii). An arrow $\alpha : f_1 \longrightarrow f_2$ is uniquely of the form $(\sigma, g_1, \ldots, g_n; g)$ with

$$\begin{array}{rcccc} g_i & : & s(f_2)_{\sigma(i)} & \longrightarrow & s(f_1)_i \\ \text{and } g & : & t(f_1) & \longrightarrow & t(f_2) \end{array}$$

as arrows of $\mathbb{C}$. Since ϕ is faithful on the category of objects and $\xi(M)$ is object-discrete, there can only be one such map. □

Proof of (iv). Given $f_1, \ldots, f_n, f \in o(Q^+)$ and

$$(T, \sigma) : (\phi^+(f_1), \ldots, \phi^+(f_n)) \longrightarrow \phi^+(f) \in \xi(M_+)$$

we seek

$$(T', \rho, \tau) : (f_1, \ldots, f_n) \longrightarrow f \in Q^+$$

such that

$$\phi^+(T', \rho, \tau) = (T, \sigma)$$

i.e. such that $\bar{T}' = T$ and $\tau \circ {\tau_{\bar{T}}}^{-1} = \sigma$.

Write $\phi(f) = (g, \alpha)$ and for each i, $\phi(f_i) = (g_i, \alpha_i)$. Then $\phi^+(f_i) = g_i$ and $\phi^+(f) = g$. (T, σ) is a configuration for composing the g_i to yield g, so we certainly have a configuration for composing the (g_i, α_i) to yield g_i as follows: replace node label g_i by (g_i, α_i) and insert a twist ${\alpha_i}^{-1}$ above the node, then comb and add the necessary twist at the top.

This gives a configuration for composing the f_i as follows. We have

$$t(g_i, \alpha_i) = s(g_k, \alpha_k)_m \Rightarrow \phi(t(f_i)) = \phi(s(f_k)_m).$$

Now ϕ is faithful on the category of objects, so there exists a morphism

$$t(f_i) \longrightarrow s(f_k)_m$$

and we label the edge joining $t(f_i)$ and $s(f_i)_m$ with this object-morphism. So this gives a configuration for composing the f_i, to yield h, say, with $\phi(h) = \phi(f)$. That is, we have a morphism

$$(f_1, \ldots, f_n) \xrightarrow{\theta} h$$

such that $\phi^+(\theta) = (T, \sigma)$.

Now ϕ is full on the category of objects, so if $\phi(h) = \phi(f)$ then there is a morphism $\alpha : h \longrightarrow f$ in $o(Q^+)$. So we have

$$(f_1, \ldots, f_n) \xrightarrow{\theta} h \xrightarrow{\iota(\alpha)} f$$

and $\phi^+(\iota(\alpha))$ is the identity since $\xi(M_+)$ is object-discrete. So

$$\phi^+(\iota(\alpha) \circ \theta) = \phi^+(\theta) = (T, \sigma)$$

as required. $\square$

Proof of (v). Suppose $\phi^+(\alpha) = \phi^+(\beta)$. Then, writing

$$\begin{array}{rcccc} \alpha = (T_1, \rho_1, \tau_1) & : & (f_1, \ldots, f_n) & \longrightarrow & f \\ \beta = (T_2, \rho_2, \tau_2) & : & (f_1, \ldots, f_n) & \longrightarrow & f \end{array}$$

we have $\bar{T_1} = \bar{T_2} = \bar{T}$, say, and $\tau_1 \circ {\tau_{\bar{T_1}}}^{-1} = \tau_2 \circ {\tau_{\bar{T_2}}}^{-1}$ so $\tau_1 = \tau_2$. So given any node N in $\bar{T}$, its pre-image in T_1 has the same label f_i as its pre-image in T_2. The same is true of edge labels, since ϕ is faithful on the category of objects.

Then the tree T_1 may be obtained from $\bar{T}$ as follows. Suppose $\phi(f_i) = (g_i, \sigma)$ and $\phi(f) = g$. Then for the node labelled by g_i, apply the twist σ^{-1} to the edges above it, and then relabel the node with f_i. This process may also be applied to obtain the tree T_2. Since the process is the same in both cases, we have $T_1 = T_2 = T$, say.

Finally, suppose f' is the arrow obtained from composing according to T. Then by the action of α, $f = f'\rho_1$, and by the action of β, $f = f'\rho_2$. Then, since Q is freely symmetric, $\rho_1 = \rho_2$, so $\alpha = \beta$ as required. $\square$

Proof of Proposition 2.2.2. Given a tidy symmetric multicategory Q we need to show that Q^+ is also tidy.

Recall (Lemma 1.2.5) that a symmetric multicategory Q is tidy if and only if it is equivalent to one in the image of ξ, ξM say, with equivalence given by

$$\phi : Q \longrightarrow \xi(M).$$

Then by Proposition 2.2.1 ϕ^+ is an equivalence

$$\phi^+ : Q^+ \longrightarrow \xi(M_+)$$

so Q^+ is tidy as required. $\square$

Corollary 2.2.3. *Let M be a generalised multicategory. Then*

$$\xi(M)^+ \simeq \xi(M_+)$$

as symmetric multicategories with a category of objects.

Proof. Put $Q = \xi(M)$, $\phi = 1$ in Proposition 2.2.1. $\square$

2.2.2 Symmetric multicategories and cartesian monads

We now compare Leinster slicing with Baez-Dolan slicing. Since $\zeta(Q) = (\mathcal{E}_Q, T_Q)$ is suitable (Proposition 2.1.1), we can form $\zeta(Q)' = (\mathcal{E}_Q{}', T_Q{}')$, the free $(\mathcal{E}_Q, T_Q)$-operad monad. Also, Q^+ is tidy since Q is tidy (Proposition 2.2.2), so we can form the monad $\zeta(Q^+) = (\mathcal{E}_{Q^+}, T_{Q^+})$. For the comparison, we have the following result.

Proposition 2.2.4. *Let Q be a tidy symmetric multicategory. Then*

$$\zeta(Q)' \cong \zeta(Q^+)$$

that is

$$(\mathcal{E}_Q{}', T_Q{}') \cong (\mathcal{E}_{Q^+}, T_{Q^+})$$

in the category **CartMonad**.

This proof is somewhat technical and we defer it to Appendix A. Informally, the idea is as follows. T_{Q^+} takes a set A of 'labels for arrows of Q' and returns the set A_2 of configurations for composing labelled arrows according to their underlying arrows. On the other hand, T_Q' takes a diagram of the form

$$\begin{tikzcd} & \begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \arrow[dl] \arrow[dr] & \\ T_Q \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} \end{tikzcd}$$

and forms the free $(\mathcal{E}_Q, T_Q)$ multicategory on it, with underlying graph

$$\begin{tikzcd} & \begin{pmatrix} A_1 \\ \downarrow \\ S \end{pmatrix} \arrow[dl] \arrow[dr] & \\ T_Q \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} \end{tikzcd} .$$

So T'_Q gives the set A_1 of all formal composites of arrows labelled in A according to the structure of T_Q, which is precisely the set of configurations as above.

Recall that

$$\zeta(Q_1) \cong \zeta(Q_2) \iff Q_1 \simeq Q_2.$$

We immediately deduce the following result, comparing all three processes of slicing.

Corollary 2.2.5. *Let M be a generalised multicategory. Then*

$$\zeta\xi(M_+) \cong \zeta(\xi(M)^+) \cong \zeta\xi(M)'.$$

2.3 Opetopes and multitopes

In this section we compare the construction of opetopes and multitopes, applying the results we have already established. Opetopes and multitopes are constructed by iterating the slicing process. Note that the 'opetopes' defined in [Lei2] are not *a priori* the same as those defined in [BD2]; we refer to the former as 'Leinster opetopes'.

2.3.1 Opetopes

For any symmetric multicategory Q we write

$$Q^{k+} = \begin{cases} Q & k = 0 \\ (Q^{(k-1)+})^+ & k \geq 1 \end{cases}$$

Let I be the symmetric multicategory with precisely one object, precisely one (identity) object-morphism, and precisely one (identity) arrow. A *k-dimensional opetope*, or simply *k-opetope*, is defined in [BD2] to be an object of I^{k+}. We write $\mathbb{C}_k = o(I^{k+})$, the category of k-opetopes.

2.3.2 Multitopes

Multitopes are defined in [HMP1] using the multicategory of function replacement. We give the same construction here, but state it in the language of slicing; this makes the analogy with Section 2.3.1 clear.

For any generalised multicategory M we write

$$M_{k+} = \begin{cases} M & k = 0 \\ (M_{(k-1)+})_+ & k \geq 1 \end{cases}$$

Let J be the generalised multicategory with precisely one object and precisely one (identity) morphism. Then a *k-multitope* is defined to be an object of J_{k+}. We write $P_k = o(J_{k+})$, the set of k-multitopes; we will also regard this as a discrete category.

2.3.3 Leinster opetopes

In [Lei2], k-opetopes are defined by a sequence $(\mathbf{Set}/S_k, T_k)$ of cartesian monads given by iterating the slice as follows.

For any cartesian monad ($\mathcal{E}$,T) write

$$(\mathcal{E}, T)^{k'} = \begin{cases} (\mathcal{E}, T) & k = 0 \\ ((\mathcal{E}, T)^{(k-1)'})' & k \geq 1 \end{cases}$$

Put $(\mathcal{E}_0, T_0) = (\mathbf{Set}, id)$ and for $k \geq 1$ put $(\mathcal{E}_k, T_k) = (\mathbf{Set}, id)^{k'}$. It follows that for each k, $(\mathcal{E}_k, T_k)$ is of the form $(\mathbf{Set}/S_k, T_k)$ where $S_0 = 1$ and S_{k+1} is given by

$$\begin{pmatrix} S_{k+1} \\ \downarrow \\ S_k \end{pmatrix} = T_k \begin{pmatrix} S_k \\ \downarrow 1 \\ S_k \end{pmatrix}$$

Then *Leinster k-opetopes* are defined to be the elements of S_k; as above, we will regard S_k as a discrete category.

2.3.4 Comparisons

We first compare opetopes and multitopes.

Proposition 2.3.1. *For each $k \geq 0$*

$$\xi(J_{k+}) \simeq I^{k+}.$$

Proof. By induction. First observe that $\xi(J) \cong I$ and write ϕ for this isomorphism. So for each $k \geq 0$ we have

$$\phi^{k+} : I^{k+} \longrightarrow \xi(J_{k+}),$$

where

$$\phi^{k+} = \begin{cases} \phi & k = 0 \\ (\phi^{(k-1)+})^+ & k \geq 1 \end{cases}$$

Now I is (trivially) tidy, so by Proposition 2.2.2, I^{k+} is tidy for each $k \geq 0$. So by Proposition 2.2.1, ϕ^{k+} is an equivalence for all $k \geq 0$. $\square$

We now compare opetopes and Leinster opetopes.

Proposition 2.3.2. *For each $k \geq 0$*

$$\zeta(I^{k+}) \cong (\mathbf{Set}, \mathrm{id})^{k'} = (\mathbf{Set}/S_k, T_k).$$

Proof. By induction. For $k = 0$ we need to show

$$(\mathcal{E}_{I^{k+}}, T_{I^{k+}}) \cong (\mathbf{Set}, id).$$

Now $\mathcal{E}_I = \mathbf{Set}/S_I$ where $S_I \simeq o(I) = 1$. So $\mathcal{E}_I \cong \mathbf{Set}/1 \cong \mathbf{Set}$. Given any $\begin{pmatrix} X \\ \downarrow ! \\ 1 \end{pmatrix} \in \mathbf{Set}/1$, $T_I \begin{pmatrix} X \\ \downarrow \\ 1 \end{pmatrix}$ is equivalent to the pullback

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \mathcal{F}X^{\text{op}} \arrow[d] \\
\operatorname{elt} I \arrow[r, "t"'] & \mathcal{F}1^{\text{op}}
\end{tikzcd}$$

But I has only one arrow, which is unary (the identity), so

$$T_I \begin{pmatrix} X \\ \downarrow \\ 1 \end{pmatrix} \cong \begin{pmatrix} X \\ \downarrow \\ 1 \end{pmatrix}$$

and

$$(\mathcal{E}_I, T_I) \cong (\mathbf{Set}, id)$$

as required.

Now suppose $\zeta(I^{(k-1)+}) \cong (\mathbf{Set}, id)^{(k-1)'}$. Then by Proposition 2.2.4 we have

$$\zeta(I^{k+}) \cong \zeta(I^{(k-1)+})' \cong (\mathbf{Set}, id)^{k'}$$

so by induction the result is true for all $k \geq 0$. □

Then on objects, the above equivalences give the following result.

Corollary 2.3.3. *For each $k \geq 0$*

$$P_k \simeq \mathbb{C}_k \simeq S_k.$$

We eventually aim to define a category **Opetope** of opetopes of all dimensions, whose morphisms are 'face maps' of opetopes. In [HMP1] Hermida, Makkai and Power explicitly define **Multitope**, the category of multitopes; Baez and Dolan do not give this explicit construction. In Chapter 3 we give an explicit construction of **Opetope**. Assuming the underlying idea is the same, this would be equivalent to the category **Multitope**, but we do not attempt to prove it in this thesis.

Chapter 3

The category of opetopes

In this chapter we give an explicit construction of the category **Opetope** of opetopes. This construction will enable us, in Chapter 4, to prove that the category of opetopic sets is in fact a presheaf category.

In Chapter 2 we constructed, for each $k \geq 0$, a category $\mathbb{C}_k$ of k-opetopes. For the category **Opetope** of opetopes of all dimensions, the idea is that each category $\mathbb{C}_k$ should be a full subcategory of **Opetope**; furthermore there should be 'face maps' exhibiting the constituent m-opetopes, or 'faces' of a k-opetope, for $m \leq k$. We refer to the m-opetope faces as m-faces.

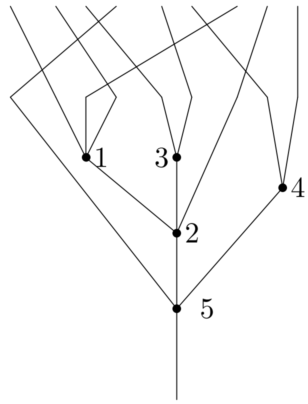
The $(k-1)$-faces of a k-opetope α should be the $(k-1)$-opetopes of its source and target; these should all be distinct. Then each of these faces has its own $(k-2)$-faces, but all these $(k-2)$-opetopes should not necessarily be considered as distinct $(k-2)$-faces in α. For α is a configuration for composing its $(k-1)$-faces at their $(k-2)$-faces, so the $(k-2)$-faces should be identified with one another at places where composition is to occur. That is, the composite face maps from these $(k-2)$-opetopes to α should therefore be equal. Some further details are then required to deal with isomorphic copies of opetopes.

Recall that a 'configuration' for composing $(k-1)$-opetopes is expressed as a tree (see Section 2.1.1) whose nodes are labelled by the $(k-1)$-opetopes in question, with the edges giving their inputs and outputs. So composition occurs along each edge of the tree, via an object-morphism label, and thus the tree tells us which $(k-1)$-opetopes are identified.

In order to express this more precisely, we first give a more formal description of trees (Section 3.1.1). In fact, this leads to an abstract description of trees as certain Kelly-Mac Lane graphs. However, as this is not used in the rest of the work, we include it Appendix B.

3.1 Background on trees

Recall the trees introduced in Section 2.1.1 to describe the morphisms of a slice multicategory. These are 'labelled combed trees' with ordered nodes. In fact, we will first consider the *unlabelled* version of such trees, since the labelled version follows easily. For example the following is a tree:



Explicitly, a tree $T = (T, \rho, \tau)$ consists of

i) A planar tree T

ii) A permutation $\rho \in \mathbf{S}_l$ where $l =$ number of leaves of T

iii) A bijection $\tau : \{\text{nodes of } T\} \longrightarrow \{1, 2, \ldots, k\}$ where $k =$ number of nodes of T; equivalently an ordering on the nodes of T.

Note that there is a 'null tree' with no nodes

.

3.1.1 Formal description of trees

In this section we give a formal description of the above trees, characterising them as connected graphs with no closed loops (in the conventional sense of 'graph'). This will enable us, in Section 3.2, to determine which faces of faces are identified in an opetope.

Note that the material in this section will be useful in Appendix B. It enables us, in Section B.2.2, to express a tree as a Kelly-Mac Lane graph; it also enables us, in Section B.2.5, to show that all *allowable* graphs of the correct shape arise in this way.

We consider a tree with k nodes $N_1, \ldots, N_k$ where N_i has m_i inputs and one output. Let N be a node with $(\sum\limits_i m_i) - k + 1$ inputs; N will be used to represent the leaves and root of the tree.

Then a tree is given by a bijection

$$\coprod_i \{\text{inputs of } N_i\} \coprod \{\text{output of } N\} \longrightarrow \coprod_i \{\text{output of } N_i\} \coprod \{\text{inputs of } N\}$$

since each input of a node is either connected to a unique output of another node, or it is a leaf, that is, input of N. Similarly each output of a node is either attached to an input of another node, or it is the root, that is, output of N.

We express this formally as follows.

Lemma 3.1.1. *Let T be a tree with nodes $N_1, \ldots, N_k$, where N_i has inputs $\{x_{i1}, \ldots, x_{im_i}\}$ and output x_i. Let N be a node with inputs $\{z_1, \ldots, z_l\}$ and output z, with*

$$l = (\sum_{i=1}^{k} m_i) - k + 1.$$

Then T is given by a bijection

$$\alpha \ : \ \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}.$$

Proof. We construct the bijection α.

Consider x_{ij} on the left hand side. This is the jth input of N_i, which is either

i) joined to the output of a unique N_r, in which case $\alpha(x_{ij}) = x_r$, or

ii) the pth leaf of the tree, in which case $\alpha(x_{ij}) = z_p$.

Finally, z is the root of the tree, so must be the output of a unique N_r, so $\alpha(z) = x_r$.

For the inverse, consider x_r on the right hand side. This is the output of the rth node, so is either

i) joined to the jth input of a unique N_i, in which case $\alpha^{-1}(x_r) = \alpha(x_{ij})$, or

ii) is the root of the tree, in which case $\alpha^{-1}(x_r) = z$.

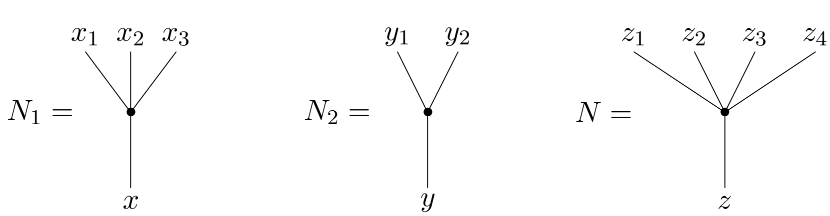
Each z_r is a leaf of the tree, so must be the jth input of a unique N_i, so $\alpha^{-1}(z_r) = x_{ij}$.

α^{-1} thus defined is inverse to α, so α is a bijection.

Note that if $k = 0$ we have the null tree with no nodes; then $l = 1$ and N has one input z_1. Then the bijection α is given by $\alpha(z) = z_1$.

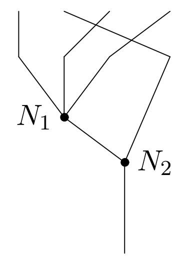
□

For example, consider



.

Then a tree



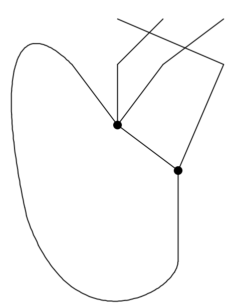
is given by the following bijection:

$$\begin{array}{rcl}\{x_1,x_2,x_3,y_1,y_2,z\} & \longrightarrow & \{x,y,z_1,z_2,z_3,z_4\}\\ x_1 & \longmapsto & z_1\\ x_2 & \longmapsto & z_3\\ x_3 & \longmapsto & z_4\\ y_1 & \longmapsto & x\\ y_2 & \longmapsto & z_2\\ z & \longmapsto & y.\end{array}$$

For the converse, every such bijection gives a graph, but it is not necessarily a tree. For example

$$\begin{array}{rcl} x_1 & \longmapsto & y\\ x_2 & \longmapsto & z_3\\ x_3 & \longmapsto & z_4\\ y_1 & \longmapsto & x\\ y_2 & \longmapsto & z_2\\ z & \longmapsto & z_1\end{array}$$

gives the following graph:



.

So we need to ensure that the resulting graph has no closed loops; the use of the 'formal' node N then ensures connectedness. We express this formally as follows.

Lemma 3.1.2. *Let $N_1,\ldots,N_k,N$ be nodes where N_i has inputs $\{x_{i1},\ldots,x_{im_i}\}$ and output x_i, and N has inputs $\{z_1,\ldots,z_l\}$ and output z, with $l=(\sum\limits_{i=1}^{k} m_i)-k+1$. Let α be a bijection*

$$\coprod_i\{x_{i1},\ldots,x_{im_i}\}\coprod\{z\}\longrightarrow\coprod_i\{x_i\}\coprod\{z_1,\ldots,z_l\}.$$

Then α defines a graph with nodes $N_1, \ldots, N_k$.

Lemma 3.1.3. *Let α be a graph as above. Then α has a closed loop if and only if there is a non-empty sequence of indices*

$$\{t_1, \ldots, t_n\} \subseteq \{1, \ldots, k\}$$

such that for each $2 \leq j \leq n$

$$\alpha(x_{t_j b_j}) = x_{t_{j-1}}$$

for some $1 \leq b_j \leq m_j$, and

$$\alpha(x_{t_1 b_1}) = x_{t_n}$$

for some $1 \leq b_1 \leq m_1$.

Proof. A closed loop in α is a sequence of nodes

$$\{N_{t_1}, \ldots, N_{t_n}\}$$

such that for each $2 \leq j \leq n$, N_{t_j} is joined to $N_{t_{j-1}}$, and also N_{t_1} is joined to N_{t_n}.

That is, for each $2 \leq j \leq n$, some leaf of N_{t_j} is joined to the root of $N_{t_{j-1}}$, and also some leaf of N_{t_1} is joined to N_{t_n}. This is precisely the case described formally in the Lemma, with the b_j giving the leaves in question. □

For example in the above case we have

$$\begin{array}{rrcl} \alpha: & x_{11} & \longmapsto & x_2 \\ & x_{12} & \longmapsto & z_3 \\ & x_{13} & \longmapsto & z_4 \\ & x_{21} & \longmapsto & x_1 \\ & x_{22} & \longmapsto & z_2 \\ & z & \longmapsto & z_1 \end{array}$$

which has a loop given by indices $\{1, 2\}$, since

$$\alpha(x_{21}) = x_1 \quad \text{and} \quad \alpha(x_{11}) = x_2.$$

Note that a graph with no nodes cannot satisfy the above condition since the sequence $\{N_{t_1}, \ldots, N_{t_n}\}$ is required to be non-empty.

Corollary 3.1.4. *A tree with nodes $N_1, \ldots, N_k$ is precisely a bijection α as in Lemma 3.1.2, such that there is no sequence of indices as in Lemma 3.1.3.*

Proof. α defines a graph; this is a tree if and only if there is no closed loop. Note that if $k = 0$ we have a bijection

$$\alpha : \{z\} \longrightarrow \{z_1\}$$

that is, the null tree. □

3.1.2 Labelled trees

For the construction of opetopes we require the 'labelled' version of the trees presented in Section 3.1. A tree labelled in a category $\mathbb{C}$ is a tree as above, with each edge labelled by a morphism of $\mathbb{C}$ considered to be pointing 'down' towards the root.

Proposition 3.1.5. *Let $N_1, \ldots, N_k, N$ be nodes where N_i has inputs*

$$\{x_{i1}, \ldots, x_{im_i}\}$$

and output x_i, and N has inputs $\{z_1, \ldots, z_l\}$ and output z, with

$$l = (\sum_{i=1}^{k} m_i) - k + 1.$$

Then a labelled tree with these nodes is given by a bijection

$$\alpha : \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}$$

satisfying the conditions as above, together with, for each

$$y \in \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\}$$

a morphism $f \in \mathbb{C}$ giving the label of the edge joining y and $\alpha(y)$. Then y is considered to be labelled by the object $\mathrm{cod}(f)$ *and $\alpha(y)$ by the object* $\mathrm{dom}(f)$.

Proof. Follows immediately from Corollary 3.1.4 and the definition. □

3.2 The category of opetopes

In Section 2.3.1 we constructed for each $k \geq 0$ the category $\mathbb{C}_k$ of k-opetopes. We now construct a category **Opetope** of opetopes of all dimensions whose morphisms are, essentially, face maps. Each category $\mathbb{C}_k$ is to be a full subcategory of **Opetope**, and there are no morphisms from an opetope to one of lower dimension.

We construct the category **Opetope** $= \mathcal{O}$ as follows. Write $\mathcal{O}_k = \mathbb{C}_k$. For the objects:

$$\mathrm{ob}\ \mathcal{O} = \coprod_{k \geq 0} \mathcal{O}_k.$$

The morphisms of $\mathcal{O}$ are given by generators and relations as follows.

- Generators

1) For each morphism $f : \alpha \longrightarrow \beta \in \mathcal{O}_k$ there is a morphism

$$f : \alpha \longrightarrow \beta \in \mathcal{O}.$$

2) Let $k \geq 1$ and consider $\alpha \in \mathcal{O}_k = o(I^{k+}) = \mathrm{elt}(I^{(k-1)+})$. Write $\alpha \in I^{(k-1)+}(x_1, \ldots, x_m; x)$. Then for each $1 \leq i \leq m$ there is a morphism
$$s_i : x_i \longrightarrow \alpha \in \mathcal{O}$$
and there is also a morphism
$$t : x \longrightarrow \alpha \in \mathcal{O}.$$
We write G_k for the set of all generating morphisms of this kind.

Before giving the relations on these morphisms we make the following observation about morphisms in $\mathcal{O}_k$. Consider
$$\begin{array}{rcl} \alpha & \in & I^{(k-1)+}(x_1, \ldots, x_m; x) \\ \beta & \in & I^{(k-1)+}(y_1, \ldots, y_m; y) \end{array}$$
A morphism $\alpha \xrightarrow{g} \beta \in \mathcal{O}_k$ is given by a permutation σ and morphisms
$$\begin{array}{ccccc} x_i & \xrightarrow{f_i} & y_{\sigma(i)} & & \\ x & \xrightarrow{f} & y & \in & \mathcal{O}_{k-1} \end{array}$$
So for each face map γ there is a unique 'restriction' of g to the specified face, giving a morphism γg of $(k-1)$-opetopes.

Note that, to specify a morphism in the category $\mathcal{F}\mathcal{O}_{k-1}{}^{\mathrm{op}} \times \mathcal{O}_{k-1}$ the morphisms f_i above should be in the direction $y_{\sigma(i)} \longrightarrow x_i$, but since these are all unique isomorphisms the direction does not matter; the convention above helps the notation. We now give the relations on the above generating morphisms.

- Relations

1) For any morphism
$$\alpha \xrightarrow{g} \beta \ \in \mathcal{O}_k$$
and face map
$$x_i \xrightarrow{s_i} \alpha$$
the following diagrams commute
$$\begin{tikzcd} x_i \arrow[r, "s_i"] \arrow[d, "s_i(g)"'] & \alpha \arrow[d, "g"] \\ y_{\sigma(i)} \arrow[r, "s_{\sigma(i)}"'] & \beta \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "t"] \arrow[d, "t(g)"'] & \alpha \arrow[d, "g"] \\ y \arrow[r, "t"'] & \beta \end{tikzcd}$$
We write these generally as
$$\begin{tikzcd} x \arrow[r, "\gamma"] \arrow[d, "\gamma g"'] & \alpha \arrow[d, "g"] \\ y \arrow[r, "\gamma'"'] & \beta \end{tikzcd}$$

2) Faces are identified where composition occurs: consider $\theta \in \mathcal{O}_k$ where $k \geq 2$. Recall that θ is constructed as an arrow of a slice multicategory, so is given by a labelled tree, with nodes labelled by its $(k-1)$-faces, and edges labelled by object-morphisms, that is, morphisms of $\mathcal{O}_{k-2}$.

   So by the formal description of trees (Section 3.1.1), θ is a certain bijection, and the elements that are in bijection with each other are the $(k-2)$-faces of the $(k-1)$-faces of θ; they are given by composable pairs of face maps of the second kind above. That is, the node labels are given by face maps $\alpha \xrightarrow{\gamma} \theta$ and then the inputs and outputs of those are given by pairs

$$x \xrightarrow{\gamma_1} \alpha \xrightarrow{\gamma_2} \theta$$

   where $\gamma_2 \in G_k$ and $\gamma_1 \in G_{k-1}$. Now, if

$$\begin{array}{rcccccc} & x & \xrightarrow{\gamma_1} & \alpha & \xrightarrow{\gamma_2} & \theta \\ \text{and} & y & \xrightarrow{\gamma_3} & \beta & \xrightarrow{\gamma_4} & \theta \end{array}$$

   correspond under the bijection, there must be a unique object-morphism

$$f : x \longrightarrow y$$

   labelling the relevant edge of the tree. Then for the composites in $\mathcal{O}$ we have the relation: the following diagram commutes

$$\begin{tikzcd} x \arrow[r, "\gamma_1"] \arrow[d, "f"'] & \alpha \arrow[dr, "\gamma_2"] & \\ & & \theta. \\ y \arrow[r, "\gamma_3"] & \beta \arrow[ur, "\gamma_4"'] & \end{tikzcd}$$

3) Composition in $\mathcal{O}_k$ is respected, that is, if $g \circ f = h \in \mathcal{O}_k$ then $g \circ f = h \in \mathcal{O}$.

4) Identities in $\mathcal{O}_k$ are respected, that is, given any morphism $x \xrightarrow{\gamma} \alpha \in \mathcal{O}$ we have $\gamma \circ 1_x = \gamma$.

Note that only the relation (2) is concerned with the identification of faces with one another; the other relations are merely dealing with isomorphic copies of opetopes.

We immediately check that the above relations have not identified any morphisms of $\mathcal{O}_k$.

Lemma 3.2.1. *Each $\mathcal{O}_k$ is a full subcategory of $\mathcal{O}$.*

Proof. Clear from definitions. □

We now check that the above relations have not identified any $(k-1)$-faces of k-opetopes.

Proposition 3.2.2. *Let $x \in \mathcal{O}_{k-1}$, $\alpha \in \mathcal{O}_k$ and $\gamma_1, \gamma_2 \in G_k$ with*

$$\gamma_1,\ \gamma_2 : x \longrightarrow \alpha$$

Then $\gamma_1 = \gamma_2 \in \mathcal{O} \implies \gamma_1 = \gamma_2 \in G_k$.

We prove this by expressing all morphisms from $(k-1)$-opetopes to k-opetopes in the following "normal form"; this is a simple exercise in term rewriting (see [JWK]).

Lemma 3.2.3. *Let $x \in \mathcal{O}_{k-1}$, $\alpha \in \mathcal{O}$. Then a morphism*

$$x \longrightarrow \alpha \ \in \mathcal{O}$$

is uniquely represented by

$$x \xrightarrow{\gamma} \alpha$$

or a pair

$$x \xrightarrow{f} y \xrightarrow{\gamma} \alpha$$

where $f \in \mathcal{O}_{k-1}$ and $\gamma \in G_k$.

Proof. Any map $x \longrightarrow \alpha$ is represented by terms of the form

$$x \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m \xrightarrow{\gamma} \alpha_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{j-1}} \alpha_j \xrightarrow{g_j} \alpha$$

where each $f_i \in \mathcal{O}_{k-1}$ and each $g_r \in \mathcal{O}_k$. Equalities are generated by equalities in components of the following forms:

1) $$\xrightarrow{\quad\gamma\quad}\ \xrightarrow{\quad g\quad} = \xrightarrow{\quad\gamma g\quad}\ \xrightarrow{\quad\gamma'\quad}$$

2) $$\xrightarrow{\quad f\quad}\ \xrightarrow{\quad f'\quad} = \xrightarrow{\quad f' \circ f\quad} \qquad \in \mathcal{O}_{k-1}$$

3) $$\xrightarrow{\quad g\quad}\ \xrightarrow{\quad g'\quad} = \xrightarrow{\quad g' \circ g\quad} \qquad \in \mathcal{O}_k$$

4) $$\xrightarrow{\quad 1\quad}\ \xrightarrow{\quad\gamma\quad} = \xrightarrow{\quad\gamma\quad}$$

where $\gamma \in G_k$ and γg and γ' are as defined above. That is, equalities in terms are generated by equations $t = t'$ where t' is obtained from t by replacing a component of t of a left hand form above, with the form in the right hand side, or vice versa.

We now orient the equations in the term rewriting style in the direction

$$\Longrightarrow$$

from left to right in the above equations. We then show two obvious properties:

1) Any reduction of t by $\Longrightarrow$ terminates in at most $2j+m$ steps.

2) If we have

$$\begin{tikzcd}[column sep=small] & t \arrow[dl] \arrow[dr] & \\ t' & & t'' \end{tikzcd}$$

then there exists t''' with

$$\begin{tikzcd}[column sep=small] & t \arrow[dl] \arrow[dr] & \\ t' \arrow[dr, dashed] & & t'' \arrow[dl, dashed] \\ & t''' & \end{tikzcd}$$

where the dotted arrows indicate a chain of equations (in this case of length at most 2).

The first part is clear from the definitions; for the second part the only non-trivial case is for a component of the form

$$\xrightarrow{\quad \gamma \quad} \; \xrightarrow{\quad g_1 \quad} \; \xrightarrow{\quad g_2 \quad} .$$

This reduces uniquely to

$$\xrightarrow{\gamma(g_2 \circ g_1)} \; \xrightarrow{\quad \gamma' \quad}$$

since ‘restriction’ is unique, as discussed earlier.

It follows that, for any terms t and s, $t = s$ if and only if t and s reduce to the same normal form as above. $\square$

Proof of Proposition 3.2.2 . γ_1 and γ_2 are in normal form. $\square$

Some low-dimensional examples of face maps are given in Appendix B.

Chapter 4

Opetopic Sets

In this chapter we examine the theory of opetopic sets. An opetopic set is to be the data for an n-category. The idea is that the category of opetopic sets should be the category of presheaves on the category of opetopes. However, in [BD2] the category of opetopes is not described fully, so opetopic sets are defined directly instead, and no equivalence with a presheaf category is proved. We are now able to prove such an equivalence using the construction of the previous chapter.

We begin by following through our modifications to the opetopic theory to include the theory of opetopic sets. We then use results of [Kel1] to prove that the category of opetopic sets is indeed equivalent to the category of presheaves on $\mathcal{O}$, the category of opetopes defined in Chapter 3.

Recall that, by the equivalences proved in the Chapter 2, we have equivalent categories of opetopes, multitopes and Leinster opetopes. So we may define equivalent categories of opetopic sets by taking presheaves on any of these three categories. In the following definitions, although the opetopes we consider are the 'symmetric multicategory' kind, the concrete description of an opetopic set is not *precisely* as a presheaf on the category of these opetopes. The sets given in the data are indexed not by opetopes themselves but by *isomorphism classes* of opetopes; so at first sight this resembles a presheaf on the category of Leinster opetopes. However, we do not pursue this matter here, since the equivalences proved in Chapter 2 are sufficient for the purposes of this thesis.

We adopt this presentation in order to avoid naming the same cells repeatedly according to the symmetries; that is, we do not keep copies of cells that are isomorphic by the symmetries.

4.1 Definitions

In [BD2], weak n-categories are defined as opetopic sets satisfying certain universality conditions. However, opetopic sets are defined using only symmetric multicategories with a *set* of objects; in the light of the results of the previous chapters, we seek a definition using symmetric multicategories with a *category* of objects. The definitions we give here are those given in [BD2] but with modifications as demanded by the results of the previous chapters.

The underlying data for an opetopic n-category are given by an opetopic set. Recall that, in [BD2], a Q-opetopic set X is given by, for each $k \geq 0$, a symmetric multicategory $Q(k)$ and a set $X(k)$ over $o(Q(k))$, where

$$\begin{aligned} Q(0) &= Q \\ \text{and} \quad Q(k+1) &= {Q(k)_{X(k)}}^{+}. \end{aligned}$$

An opetopic set is then an I-opetopic set, where I is the symmetric multicategory with one object and one (identity) arrow.

The idea is that the category of opetopic sets should be equivalent to the presheaf category

$$[\mathbf{Opetope}^{\mathrm{op}}, \mathbf{Set}]$$

and we use this to motivate our generalisation of the Baez-Dolan definitions.

We have constructed (Section 2.3.1) categories $\mathbb{C}(k)$ of k-opetopes, and each $\mathbb{C}(k)$ is a full subcategory of **Opetope**. A functor

$$\mathbf{Opetope}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

may be considered as assigning to each opetope a set of 'labels'.

Recall that for each k, $\mathbb{C}(k)$ is equivalent to a discrete category. So it is sufficient to specify 'labels' for each isomorphism class of opetopes. (In fact, we are thus considering labels for 'Leinster opetopes' but we do not pursue this idea any further here.)

Recall (Section 1.2.1) that we call a symmetric multicategory Q *tidy* if it is freely symmetric with a category of objects $\mathbb{C}$ equivalent to a discrete category. Throughout this chapter we say 'Q has object-category $\mathbb{C}$ equivalent to S discrete' to mean that S is the set of isomorphism classes of $\mathbb{C}$, so $\mathbb{C}$ is equipped with a morphism $\mathbb{C} \xrightarrow{\sim} S$. We begin by defining the construction used for 'labelling' as discussed above. The idea is to give a set of labels as a set over the isomorphism classes of objects of Q, and then to 'attach' the labels using the following pullback construction.

Definition 4.1.1. *Let Q be a tidy symmetric multicategory with category of objects $\mathbb{C}$ equivalent to S discrete. Given a set X over S, that is, equipped with a function $f : X \longrightarrow S$, we define the* pullback multicategory Q_X *as follows.*

- *Objects: $o(Q_X)$ is given by the pullback*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "\sim"] \\ X \arrow[r, "f"'] & S \end{tikzcd} .$$

Observe that the morphism on the left is an equivalence, so $o(Q_X)$ is equivalent to X discrete. Write h for this morphism.

- *Arrows: given objects* $a_1, \ldots a_k, a \in o(Q_X)$ *we have*

$$Q_X(a_1, \ldots, a_k; a) \cong Q(fh(a_1), \ldots, fh(a_k); fh(a)).$$

- *Composition, identities and symmetric action are then inherited from* Q.

We observe immediately that since Q is tidy, Q_X is tidy. Also note that if Q is object-discrete this definition corresponds to the definition of pullback symmetric multicategory given in [BD2].

We are now ready to describe the construction of opetopic sets.

Definition 4.1.2. *Let* Q *be a tidy symmetric multicategory with object-category* $\mathbb{C}$ *equivalent to* S *discrete. A* Q-opetopic set X *is defined recursively as a set* $X(0)$ *over* S *together with a* ${Q_X}^+$*-opetopic set* X_1.

So a Q-opetopic set consists of, for each $k \geq 0$:

- a tidy symmetric multicategory $Q(k)$ with object-category $\mathbb{C}(k)$ equivalent to $S(k)$ discrete
- a set $X(k)$ and function $X(k) \xrightarrow{f_k} S(k)$

where

$$\begin{array}{rcl} Q(0) & = & Q \\ \text{and} \quad Q(k+1) & = & {Q(k)_{X(k)}}^+. \end{array}$$

We refer to X_1 as the underlying ${Q(k)_{X(k)}}^+$-opetopic set of X.

We now define morphisms of opetopic sets. Suppose we have opetopic sets X and X' with notation as above, together with a morphism of symmetric multicategories

$$F : Q \longrightarrow Q'$$

and a function

$$F_0 : X(0) \longrightarrow X'(0)$$

such that the following diagram commutes

$$\begin{tikzcd} X(0) \arrow[r, "f_0"] \arrow[d, "F_0"'] & S(0) \arrow[d, "F"] \\ X'(0) \arrow[r, "f'_0"'] & S'(0) \end{tikzcd}$$

where the morphism on the right is given by the action of F on objects. This induces a morphism

$$Q_{X(0)} \longrightarrow Q'_{X'(0)}$$

and so a morphism

$${Q_{X(0)}}^+ \longrightarrow {Q'_{X'(0)}}^+.$$

We make the following definition.

Definition 4.1.3. *A* morphism of Q-opetopic sets

$$F : X \longrightarrow X'$$

is given by:

- *an underlying morphism of symmetric multicategories and function F_0 as above*

- *a morphism $X_1 \longrightarrow X'_1$ of their underlying opetopic sets, whose underlying morphism is induced as above.*

So F consists of

- a morphism $Q \longrightarrow Q'$

- for each $k \geq 0$ a function $F_k : X(k) \longrightarrow X'(k)$ such that the following diagram commutes

$$\begin{tikzcd}
X(k) \arrow[r, "f_k"] \arrow[d, "F_k"'] & S(k) \arrow[d] \\
X'(k) \arrow[r, "f'_k"'] & S'(k)
\end{tikzcd}$$

where the map on the right hand side is induced as appropriate.

Note that the above notation for a Q-opetopic set X and morphism F will be used throughout this chapter, unless otherwise specified.

Definition 4.1.4. *An* opetopic set *is an I-opetopic set. A morphism of opetopic sets is a morphism of I-opetopic sets. We write* **OSet** *for the category of opetopic sets and their morphisms.*

Eventually, a weak n-category is defined as an opetopic set with certain properties. The idea is that k-cells have underlying shapes given by the objects of I^{k+}. These are 'unlabelled' cells. To make these into fully labelled k-cells, we first give labels to the 0-cells, via the function $X(0) \longrightarrow S(0)$, and then to 1-cells via $X(1) \longrightarrow S(1)$, and so on. This idea may be captured in the following 'schematic' diagram.

$$
\begin{tikzcd}
 & & Q(0)_{X(0)} \arrow[r] \arrow[d, "+"] & Q(0) = I \arrow[d, "+"] \\
 & Q(1)_{X(1)} \arrow[r] \arrow[d, "+"] & Q(1) = {Q(0)_{X(0)}}^{+} & I^{+} \arrow[d, "+"] \\
Q(2)_{X(2)} \arrow[r] \arrow[d, "+"] & Q(2) = {Q(1)_{X(1)}}^{+} & & I^{2+} \arrow[d, "+"] \\
{Q(2)_{X(2)}}^{+} & & & I^{3+} \arrow[d, "+"] \\
 & \vdots & & {}
\end{tikzcd}
$$

Bearing in mind our modified definitions, we use the Baez-Dolan terminology as follows.

Definitions 4.1.5.

- *A k-*dimensional cell *(or k-*cell*) is an element of $X(k)$ (i.e. an isomorphism class of objects of $Q(k)_{X(k)}$).*

- *A k-*frame *is an isomorphism class of objects of $Q(k)$ (i.e. an isomorphism class of arrows of $Q(k-1)_{X(k-1)}$).*

- *A k-*opening *is an isomorphism class of arrows of $Q(k-1)$, for $k \geq 1$.*

So a k-opening may acquire $(k-1)$-cell labels and become a k-frame, which may itself acquire a label and become a k-cell. We refer to such a cell and frame as being *in* the original k-opening.

On objects, the above schematic diagram becomes:

$$
\begin{tikzcd}
 & & & \text{0-cells} \arrow[r] \arrow[d, "+"] & \text{0-opetopes} \arrow[d, "+"] \\
 & & \text{1-cells} \arrow[r] \arrow[d, "+"] & \text{1-frames} \arrow[r, dotted] \arrow[d, dotted, "+"] & \text{1-opetopes} \arrow[d, "+"] \\
 & \text{2-cells} \arrow[r] \arrow[d, "+"] & \text{2-frames} \arrow[r, dotted] \arrow[d, dotted, "+"] & \text{2-openings} \arrow[r, dotted] \arrow[d, dotted] & \text{2-opetopes} \arrow[d, "+"] \\
 {} \arrow[r] & \text{3-frames} \arrow[r, dotted] \arrow[d, dotted, "+"] & \text{3-openings} \arrow[r, dotted] \arrow[d, dotted] & \cdot \arrow[r, dotted] \arrow[d, dotted] & \text{3-opetopes} \arrow[d, "+"] \\
 & {} & {} & {} & {}
\end{tikzcd}
$$

$$
\xrightarrow[\textit{labels for 2-cells}]{\hspace{3em}} \quad \xrightarrow[\textit{labels for 1-cells}]{\hspace{3em}} \quad \xrightarrow[\textit{labels for 0-cells}]{\hspace{3em}}
$$

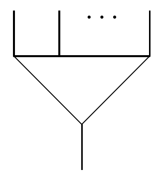
Horizontal arrows represent the process of labelling, as shown; vertical arrows represent the process of 'moving up' dimensions. Starting with a k-opetope, we have from right to left the progressive labelling of 0-cells, 1-cells, and so on, to form a k-cell at the far left, the final stages being:

$$
\begin{tikzcd}
k\text{-opening} \arrow[d, "\text{labels for constituent } (k-1)\text{-cells}"] \\
k\text{-frame} \arrow[d, "\text{label for } k\text{-cell itself}"] \\
k\text{-cell}
\end{tikzcd}
$$

A k-opening acquires labels as an arrow of $Q(k-1)$, becoming a k-frame as an arrow of $Q(k-1)_{X(k-1)}$. That is, it has $(k-1)$-cells as its source and a $(k-1)$-cell as its target.

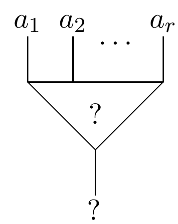
Definition 4.1.6. *A* k-niche *is a k-opening (i.e. arrow of $Q(k-1)$) together with labels for its source only.*

We may represent these notions as follows. Let f be an arrow of $Q(k-1)$, so f specifies a k-opening which we might represent as

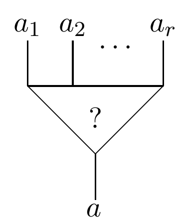


.

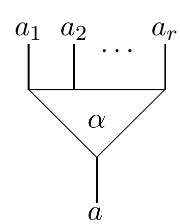
Then a niche in f is represented by



where $a_1, \ldots a_r$ are 'valid' labels for the source elements of f; a k-frame is represented by



where a is a 'valid' label for the target of f. Finally a k-cell is represented by



.

Since all symmetric multicategories in question are tidy, we may in each case represent the same isomorphism class by any symmetric variant of the above diagrams. Also, we refer to k-cells as labelling k-opetopes, rather than isomorphism classes of k-opetopes.

4.2 OSet is a presheaf category

In this section we prove that the category of opetopic sets is a presheaf category, and moreover, that it is equivalent to the presheaf category

$$[\mathcal{O}^{\mathrm{op}}, \mathbf{Set}].$$

To prove this we use [Kel1], Theorem 5.26, in the case $\mathcal{V} = \mathbf{Set}$. This theorem is as follows.

Theorem 4.2.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-category. In order that $\mathcal{C}$ be equivalent to $[\mathcal{E}^{\mathrm{op}}, \mathcal{V}]$ for some small category $\mathcal{E}$ it is necessary and sufficient that $\mathcal{C}$ be cocomplete, and that there be a set of small-projective objects in $\mathcal{C}$ constituting a strong generator for $\mathcal{C}$.*

We see from the proof of this theorem that if E is such a set and $\mathcal{E}$ is the full subcategory of $\mathcal{C}$ whose objects are the elements of E, then

$$\mathcal{C} \simeq [\mathcal{E}^{\mathrm{op}}, \mathcal{V}].$$

We prove the following propositions; the idea is to "realise" each isomorphism class of opetopes as an opetopic set; the set of these opetopic sets constitutes a strong generator as required.

Proposition 4.2.2. **OSet** *is cocomplete.*

Proposition 4.2.3. *There is a full and faithful functor*

$$G : \mathcal{O} \longrightarrow \mathbf{OSet}.$$

Proposition 4.2.4. *Let $\alpha \in \mathcal{O}$. Then $G(\alpha)$ is small-projective in* **OSet**.

Proposition 4.2.5. *Let*

$$E = \coprod\{G(\alpha) \mid \alpha \in \mathcal{O}\} \subseteq \mathbf{OSet}.$$

Then E is a strongly generating set for **OSet**.

Corollary 4.2.6. **OSet** *is a presheaf category.*

Corollary 4.2.7.

$$\mathbf{OSet} \simeq [\mathcal{O}^{op}, \mathbf{Set}].$$

Proof of Proposition 4.2.2. Consider a diagram

$$D : \mathbb{I} \longrightarrow \mathbf{OSet}$$

where $\mathbb{I}$ is a small category. We seek to construct a limit Z for D; the set of cells of Z of shape α is given by a colimit of the sets of cells of shape α in each $D(I)$.

We construct an opetopic set Z as follows. For each $k \geq 0$, $Z(k)$ is a colimit in **Set**:

$$Z(k) = \int^{I \in \mathbb{I}} D(I)(k).$$

Now for each k we need to give a function

$$F(k) : Z(k) \longrightarrow o(Q(k))$$

where

$$Q(k) = Q(k-1)_{Z(k-1)}{}^{+}$$
$$Q(0) = I.$$

That is, for each $\alpha \in Z(k)$ we need to give its frame. Now

$$Z(k) = \left. \coprod_{I \in \mathbb{I}} D(I)(k) \middle/ \sim \right.$$

where $\sim$ is the equivalence relation generated by

$$D(u)(\alpha_{I'}) \sim \alpha_I \quad \text{for all } u : I \longrightarrow I' \in \mathbb{I} \text{ and } \alpha_I \in D(I)(k).$$

So $\alpha \in Z(k)$ is of the form $[\alpha_I]$ for some $\alpha_I \in D(I)(k)$ where $[\alpha_I]$ denotes the equivalence class of α_I with respect to $\sim$.

Now suppose the frame of α_I in $D(I)$ is

$$(\beta_1, \ldots, \beta_j) \xrightarrow{?} \beta$$

where $\beta_i, \beta \in D(I)(k-1)$ label some k-opetope x. We set the frame of $[\alpha_I]$ to be

$$(\ [\beta_1], \ldots, [\beta_j]\ ) \xrightarrow{?} [\beta]$$

labelling the same opetope x. This is well-defined since a morphism of opetopic sets preserves frames of cells, so the frame of $D(u)(\alpha_I)$ is

$$(\ D(u)(\beta_1)\ , \ldots,\ D(u)(\beta_j)\ ) \xrightarrow{?} D(u)(\beta)$$

also labelling k-opetope x. It follows from the universal properties of the colimits in **Set** that Z is a colimit for D, with coprojections induced from those in **Set**. Then, since **Set** is cocomplete, **OSet** is cocomplete. $\Box$

Proof of Proposition 4.2.3. Let α be a k-opetope. We express α as an opetopic set $G(\alpha) = \hat{\alpha}$ as follows, using the usual notation for an opetopic set. The idea is that the m-cells are given by the m-faces of α.

For each $m \geq 0$ set

$$X(m) = \{\ [(x, f)]\ \mid \quad x \in \mathcal{O}_m \text{ and } x \xrightarrow{f} \alpha \in \mathcal{O} \text{ where } [\ ] \text{ denotes isomorphism class in } \mathcal{O}/\alpha\}.$$

So in particular we have

$$X(k) = \{[(\alpha, 1)]\}$$

and for all $m > k$, $X(m) = \emptyset$. It remains to specify the frame of $[(x, f)]$. The frame is an object of

$$Q(m) = Q(m-1)_{X(m-1)}{}^{+}$$

so an arrow of

$$Q(m-2)_{X(m-2)}{}^{+}$$

labelled with elements of $X(m-1)$. Now such an arrow is a configuration for composing arrows of $Q(m-2)_{X(m-2)}$; for the frame as above, this is given by the opetope x as a labelled tree. Then the $(m-1)$-cell labels are given as follows. Write

$$x : y_1, \ldots, y_j \longrightarrow y$$

say, and so we have for each i a morphism

$$y_i \longrightarrow x$$

and a morphism

$$y \longrightarrow x \in \mathcal{O}.$$

Then the labels in $X(m-1)$ are given by

$$[y_i \longrightarrow x \xrightarrow{f} \alpha] \in X(m-1)$$

and

$$[y \longrightarrow x \xrightarrow{f} \alpha] \in X(m-1).$$

Now, given a morphism

$$h : \alpha \longrightarrow \beta \in \mathcal{O}$$

we define

$$\hat{h} : \hat{\alpha} \longrightarrow \hat{\beta} \in \mathbf{OSet}$$

by

$$[(x,f)] \mapsto [(x, h \circ f)]$$

which is well-defined since if $(x,f) \cong (x', f')$ then $(x, hf) \cong (x', hf')$ in $\mathcal{O}/\alpha$. This is clearly a morphism of opetopic sets.

Observe that any morphism $\hat{\alpha} \longrightarrow \hat{\beta}$ must be of this form since the faces of α must be preserved. Moreover, if $\hat{h} = \hat{g}$ then certainly $[(\alpha, h)] = [(\alpha, g)]$. But this gives $(\alpha, h) = (\alpha, g)$ since there is a unique morphism $\alpha \longrightarrow \alpha \in \mathcal{O}$ namely the identity. So G is full and faithful as required. $\Box$

Proof of Proposition 4.2.4. For any $\alpha \in \mathcal{O}_k$ we show that $\hat{\alpha}$ is small-projective, that is that the functor

$$\psi = \mathbf{OSet}(\hat{\alpha}, -) : \mathbf{OSet} \longrightarrow \mathbf{Set}$$

preserves small colimits. First observe that for any opetopic set X

$$\begin{array}{rcl} \psi(X) = \mathbf{OSet}(\hat{\alpha}, X) & \cong & \{k\text{-cells in } X \text{ whose underlying } k\text{-opetope is } \alpha\} \\ & \subseteq & X(k) \end{array}$$

and the action on a morphism $F : X \longrightarrow Y$ is given by

$$\begin{array}{rccc} \psi(F) = \mathbf{OSet}(\hat{\alpha}, F) : & \mathbf{OSet}(\hat{\alpha}, X) & \longrightarrow & \mathbf{OSet}(\hat{\alpha}, Y) \\ & x & \mapsto & F(x). \end{array}$$

So ψ is the 'restriction' to the set of cells of shape α. This clearly preserves colimits since the cells of shape α in the colimit are given by a colimit of the sets cells of shape α in the original diagram. $\Box$

Proof of Proposition 4.2.5. First note that

$$\hat{\alpha} = \hat{\beta} \iff \alpha \cong \beta \in \mathcal{O}$$

so

$$E \cong \coprod_k S_k$$

where for each k, S_k is the set of k-dimensional Leinster opetopes. Since each S_k is a set it follows that E is a set.

We need to show that, given a morphism of opetopic sets $F : X \longrightarrow Y$, we have

$$\mathbf{OSet}(\hat{\alpha}, F) \text{ is an isomorphism for all } \hat{\alpha} \implies F \text{ is an isomorphism.}$$

Now, we have seen above that

$$\mathbf{OSet}(\hat{\alpha}, X) \cong \{\text{cells of } X \text{ of shape } \alpha\}$$

so

$$\mathbf{OSet}(\hat{\alpha}, F) = F|_\alpha = F \text{ restricted to cells of shape } \alpha.$$

So

$$\begin{aligned} &\mathbf{OSet}(\hat{\alpha}, F) \text{ is an isomorphism for all } \hat{\alpha} \\ &\qquad \iff F|_\alpha \text{ is an isomorphism for all } \alpha \in \mathcal{O} \\ &\qquad \iff F \text{ is an isomorphism.} \end{aligned}$$

$\Box$

Proof of Corollary 4.2.6. Follows from Propositions 4.2.2, 4.2.3, 4.2.4, 4.2.5 and [Kel1] Theorem 5.26. $\Box$

Proof of Corollary 4.2.7. Let $\mathcal{E}$ be the full subcategory of **OSet** whose objects are those of E. Since G is full and faithful, $\mathcal{E}$ is the image of G and we have

$$\mathcal{O} \simeq \mathcal{E}$$

and hence

$$\mathbf{OSet} \simeq [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}] \simeq [\mathcal{O}^{\mathrm{op}}, \mathbf{Set}].$$

$\Box$

Chapter 5

Weak n-categories

In this chapter we consider the complete definition of n-category. We begin by completing our modifications to the Baez-Dolan definition; we then seek to shed some light on the definition by examining the case $n = 2$ together with some preliminary examples.

5.1 Definitions

In [BD2], weak n-categories are defined as opetopic sets satisfying certain universality conditions. Thus far we have examined only the theory of opetopes and opetopic sets. It now remains to discuss the notion of universality.

5.1.1 Universality

In the definition of opetopic n-category, it is universality that deals with composition, constraints, axioms and coherence. We now modify the Baez-Dolan definition of universality in the context of the modifications discussed so far in this work. Furthermore, with clarity in mind we state the definition in a terser form than in [BD2].

In Section 5.1.2 we will have the following definition: *An* opetopic n-category *is an opetopic set in which*

i) Every niche has an n-universal occupant.

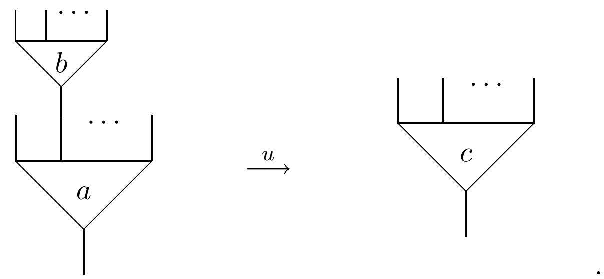
ii) Every composite of n-universals is n-universal.

We use the word 'composite' in the following sense. Let a, b and c be k-cells in an opetopic set X, with $k \geq 1$. Given a universal $(k+1)$-cell

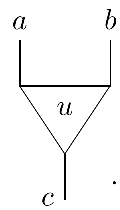
$$u : (a, b) \longrightarrow c$$

we say that c is a composite of a and b. Furthermore, we say that u and b give a factorisation of c through a (and also u and a give a factorisation of c through b).

If a and b are pasted at the target of b, say, we may represent this as



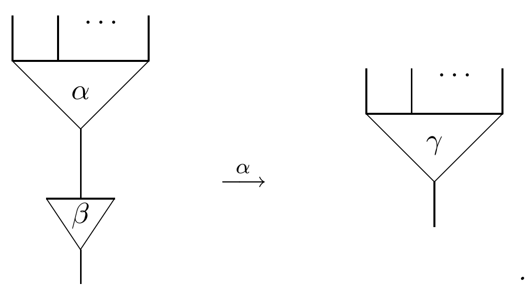
Alternatively, regarding a, b and c as objects of a symmetric multicategory at the next dimension up, we may represent this as



We will define n-universality for k-cells and for k-cell factorisations. The definition is by descending induction on k.

Definition 5.1.1. *A k-cell α is* n-universal *if either $k > n$ and α is unique in its niche, or $k \leq n$ and (1) and (2) below are satisfied:*

(1) Given any k-cell γ in the same niche as α, there is a factorisation $u : (\beta, \alpha) \longrightarrow \gamma$



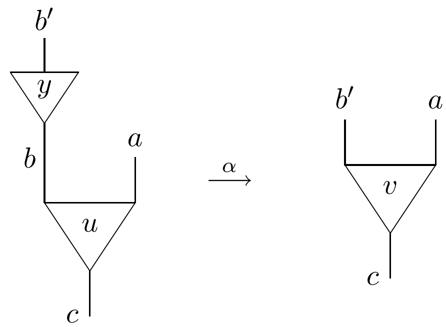
(2) Any such factorisation is n-universal.

Definition 5.1.2. *A factorisation $u : (b, a) \longrightarrow c$ of k-cells is* n-universal *if $k > n$, or $k \leq n$ and (1) and (2) below are satisfied:*

(1) Given any k-cell b' in the same frame as b, and any $(k+1)$-cell

$$v : (b', a) \longrightarrow c$$

with b' and a pasted in the same configuration as b and a in the source of u, there is a factorisation of $(k+1)$-cells $(u, y) \longrightarrow v$



(2) *Any such factorisation is itself n-universal.*

If n is clear from the context then we simply say 'universal'.

Note that in the terminology of [BD2], the definition of 'universal factorisation' given above corresponds to a special case of 'balanced punctured niche'. Furthermore, in each of the above definitions, each clause (1) and (2) corresponds to the assertion that a certain punctured niche is balanced.

Although we have still only defined 'opetopic n-category' in passing, the following examples concerning particular cases in opetopic n-categories may help to clarify the above definitions.

Examples 5.1.3.

1) In an opetopic n-category the (unique) universal 1-ary $(n+1)$-cells have the form $x \longrightarrow x$, since we have such universals given by the targets of universal nullary $(n+2)$-cells

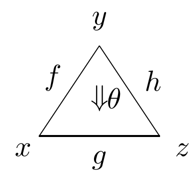
$$(\cdot) \longrightarrow (x \to x).$$

2) In an opetopic n-category, a factorisation of n-cells is universal if and only if it is unique. To see this, consider such a universal factorisation $u : (b, a) \longrightarrow c$. Now any $(n+1)$-cell is unique in its niche and hence universal, so any $(n+1)$-cell $v : (b', a) \longrightarrow c$ is a factorisation. But then, by universality of the first factorisation, we have a (necessarily universal) $(n+1)$-cell $y : b' \longrightarrow b$ giving $b = b'$ and $u = v$, i.e. the factorisation is unique.

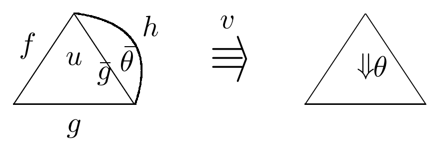
3) In a 1-category, a 1-cell $x \xrightarrow{f} y$ is universal if and only if for any 1-cell $x \xrightarrow{g} z$ there is a *unique* factorisation

$$\begin{tikzcd} & y \arrow[dr, "\bar{g}"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[rr, phantom, shift left=3, "\Downarrow u"] & & z \end{tikzcd}.$$

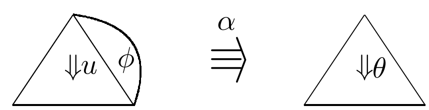
4) In a 2-category, a 1-cell $x \xrightarrow{f} y$ is universal if and only if for any 1-cell $x \xrightarrow{g} z$ there is a factorisation as above; however, we do not demand that such a factorisation be unique, but only universal. That is, given a 2-cell



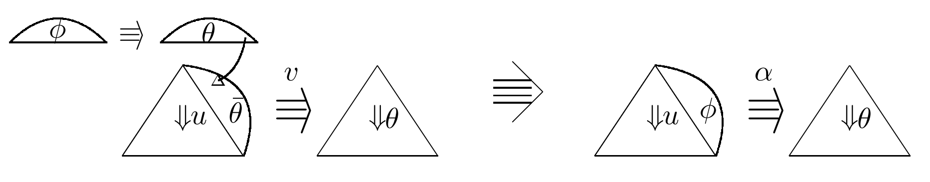
there is a *unique* factorisation



5) In a 3-category, f as above is 3-universal if and only if any such factorisation v as above is universal (rather than unique). That is, given any 3-cell



there is a *unique* factorisation



Definitions 5.1.4.

- *An* n-coherent Q-algebra *is a* Q*-opetopic set in which*
  - *i) Every niche has a universal cell in it (or universal 'occupant').*
  - *ii) Composites of universals are universal.*
- *A* morphism of n-coherent Q-algebras *is simply a morphism of their underlying* Q*-opetopic sets.*

Observe that an n-coherent Q-algebra is specified uniquely up to isomorphism by the sets $X(k)$ and functions f_k for $k \leq n+1$, since for $k \geq n+2$ the sets $X(k)$ and functions f_k are induced. A morphism of such is then uniquely determined by the functions F_k for $k \leq n$.

In [BD2] a morphism of n-coherent Q-algebras is required to preserve universality, yielding a stronger notion. We will later see that for $n = 2$ this gives weak rather than lax functors of bicategories. For the time being we consider the lax case only; we discuss strictness in Section 5.2.5.

5.1.2 Opetopic n-categories

We are now ready to state the definition of n-category. The statement here is exactly as in [BD2]; the differences have all been absorbed into the preliminary definitions. However, e note that the exact relationship between our complete modified definition and the exact Baez-Dolan original remains unclear.

Definitions 5.1.5.

- *An* opetopic n-category *is an n-coherent I-algebra.*
- *A* lax n-functor *is a morphism of n-coherent I-algebras.*

We write **Opic-n-Cat** *for the category of opetopic n-categories and lax n-functors.*

So an opetopic n-category is an opetopic set in which

i) Every niche has an n-universal occupant.

ii) Every composite of n-universals is n-universal.

We now restate, in this modified context, a useful proposition from [BD2]. This is a generalisation of the fact that in a category $\mathcal{C}$, for any objects a, b we have a 'homset' $\mathcal{C}(a,b)$ of morphisms $a \longrightarrow b$. Similarly, in a bicategory $\mathcal{B}$, we have 'hom-categories' $\mathcal{B}(a,b)$ whose objects are 1-cells and morphisms 2-cells; so we also have, for any 2-cells α, β, homsets $\mathcal{B}(\alpha,\beta)$.

Thus in an n-category we expect to have 'hom-$(n-m)$-categories' of m-cells. However, since here the domain of an m-cell is not necessarily just a single $(m-1)$-cell, instead of having just a pair of $(m-1)$-cells as above, we need an m-frame to give the domain and codomain specifying the hom-category.

Proposition 5.1.6. *Let X be an n-coherent Q-algebra. Then for $m \leq n$ any m-frame determines an opetopic $(n-m)$-category.*

The idea is first to restrict X to cells of dimension m and above; this is clearly still $(n-m)$-coherent. We can then restrict to only those cells in the given frame α by 'pulling back' along the morphism

$$1 \xrightarrow{\alpha} S(m).$$

So we follow Baez-Dolan and use the following construction of 'pullback opetopic set'. Let Q and Q' be tidy symmetric multicategories with object-categories $\mathbb{C}$ and $\mathbb{C}'$ respectively, with $\mathbb{C} \simeq S$ and $\mathbb{C}' \simeq S'$ discrete. Let X be a Q-opetopic set. Suppose we have a morphism $S' \longrightarrow S$. Then we may construct a pullback opetopic set X' by induction as follows. Let $X'(0)$ be given by the pullback

$$\begin{tikzcd}
X'(0) \arrow[r] \arrow[d] & X(0) \arrow[d] \\
S'(0) \arrow[r] & S(0)
\end{tikzcd} .$$

Now we have equivalences

$$o({Q_{X(0)}}^+) \xrightarrow{\sim} S(1),$$
$$o({Q'_{X(0)}}^+) \xrightarrow{\sim} S'(1)$$

where $S(1)$ and $S'(1)$ are discrete. So the morphism

$$X'(0) \longrightarrow X(0)$$

induces a morphism

$$S'(1) \longrightarrow S(1)$$

and we may form a pullback opetopic set of X_1 along this morphism; we set this to be X'_1, the underlying ${Q'_{X'(0)}}^+$-opetopic set of X'.

Proposition 5.1.7. (see [BD2], Proposition 45) *If X is n-coherent then X' is n-coherent.*

Proof. It is easy to check that a cell in X' is universal if and only if the corresponding cell in X is universal, and that a factorisation in X' is universal if and only if the corresponding factorisation in X is universal. $\square$

Proof of Proposition 5.1.6. Let α be an m-frame in X with $m \leq n$, so $\alpha \in S(m)$. Now X determines an $(n-m)$-coherent $Q(m)$-algebra, and we have a morphism

$$o(I) = 1 \xrightarrow{\alpha} S(m)$$

so we may form a pullback I-opetopic set along this morphism.

By Proposition 5.1.7 this is $(n-m)$-coherent, i.e. it is an opetopic $(n-m)$-category. $\square$

Examples 5.1.8.

1) In an n-category X, every 1-frame determines an $(n-1)$-category.

A 1-frame in X is given by

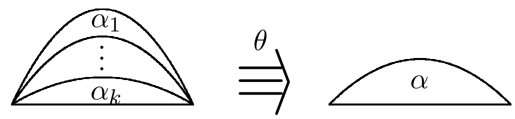
$$\begin{tikzcd} a \arrow[r, "?"] & b \end{tikzcd} .$$

We denote the induced $(n-1)$-category by $\mathrm{Hom}(a,b)$ or $X(a,b)$; its cells are of the form shown below.

0-cells $\begin{tikzcd} a \arrow[r, "f"] & b \end{tikzcd}$

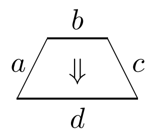
1-cells $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[r, swap, "g"] & b \end{tikzcd}$

2-cells (k-ary)



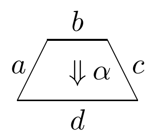
⋮

2) Given a 2-frame

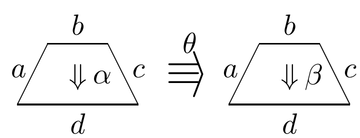


say, we have an $(n-2)$-category whose cells are of the form shown below.

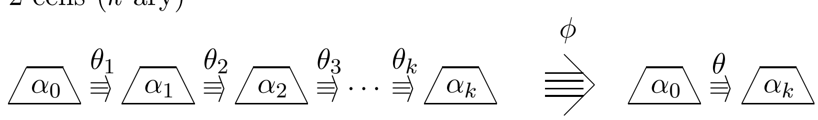
0-cells



1-cells



2-cells (k-ary)



⋮

3) Given an $(n-1)$-frame we have a 1-category whose objects are $(n-1)$-cells and arrows are 1-ary n-cells.

5.2 The theory of bicategories

Any proposed definition of n-category should at least be in some way equivalent to the classical definitions as far as the latter are understood. In [BD2] Baez and Dolan examine the case $n = 1$ but do not explain how their definition is equivalent to the classical definition of bicategories in the case $n = 2$. This is perhaps because, without the modifications described in this thesis, such an equivalence does not arise. In this section we establish an equivalence between the (modified) opetopic and the classical approaches to bicategories. We begin with some examples to help clarify and motivate the later arguments; our general aim is to shed some light on the inescapable loops in the definition of universality, as well as to compare the resulting structures with the classical ones. We conclude with an informal discussion on the subject of strictness.

Note that for $n \leq 1$ the difference between our definition and the original Baez-Dolan definition is not yet apparent. The result for $n = 1$ is described in [BD2] (Example 42); we include it here (with more detail) for completeness.

5.2.1 Opetopic 0-categories

An opetopic 0-category X is determined, up to isomorphism, by the set $X(0)$. For, given any 0-cell $a \in A$, the following nullary 2-niche

$$\begin{array}{c} \cdot \\ \Downarrow \\ a \longrightarrow a \end{array}$$

must have a unique occupant, and so the unique occupant of the following 1-niche

$$a \xrightarrow{?} ?$$

must have a as its target, and we can call the 1-cell 1_a, giving

$$X(1) \cong \{a \longrightarrow a : a \in A\}.$$

Proposition 5.2.1. *There is an equivalence*

$$\textbf{Opic-}0\textbf{-Cat} \xrightarrow{\sim} \textbf{Set}$$

surjective in the direction shown.

Proof. We construct such a functor, ζ. Let X be an opetopic 0-category. We put

$$\zeta(X) = X(0).$$

A morphism $f : X \longrightarrow Y$ of opetopic 0-categories is uniquely specified by the function $f_0 : X(0) \longrightarrow Y(0)$ so we put

$$\zeta(f) = f_0.$$

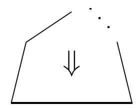
Conversely, given a set A, we have an opetopic 0-category X such that $\zeta(X) = A$; X is defined by

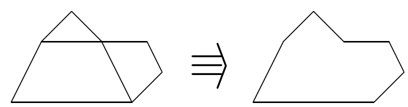
$$\begin{array}{rcl} X(0) & = & A \\ X(1) & = & \{a \xrightarrow{1_a} a : a \in A\}. \end{array}$$

So ζ is surjective, and it is clearly full and faithful, giving an equivalence as required. $\Box$

5.2.2 Opetopic 1-categories

We first clarify our notation. We draw

- 1-cells as arrows $\longrightarrow$
- 2-cells as 

- 3-cells as  ...

These represent isomorphism classes of objects in the appropriate symmetric multicategory. We give below some typical examples of openings, niches, frames and cells.

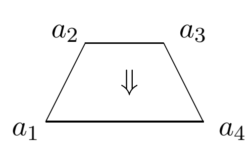
1-opening $\longrightarrow$

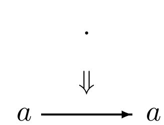
1-niche $a \xrightarrow{?} ?$

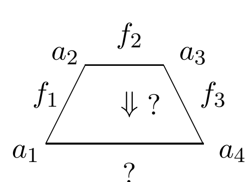
1-frame $a \xrightarrow{?} b$

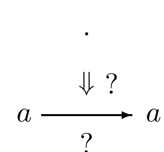
1-cell $a \xrightarrow{f} b$

3-ary nullary

2-opening 

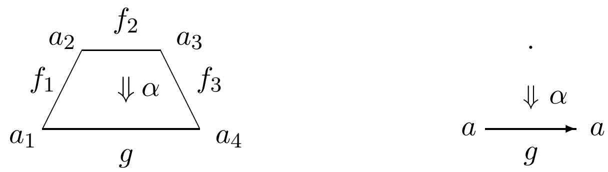


2-niche 



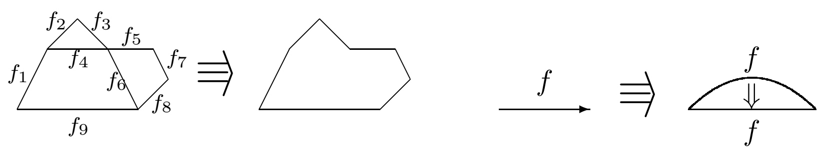
2-frame

2-cell

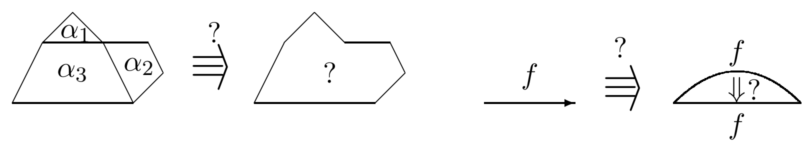


Where confusion is unlikely, we may omit some lower-dimensional labels once the higher-dimensional ones are in place, as in the following examples.

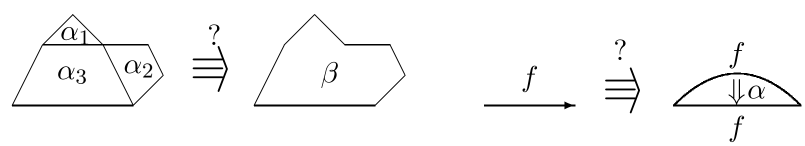
3-opening



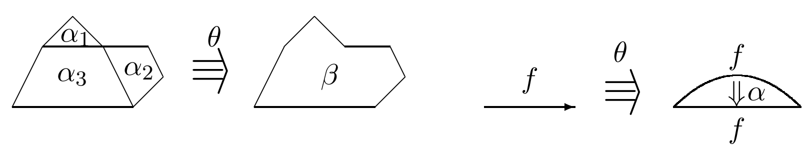
3-niche



3-frame



3-cell



We begin by constructing a functor

$$\zeta : \textbf{Opic-1-Cat} \longrightarrow \textbf{Cat};$$

we will eventually show that this functor is an equivalence.

- On objects

Given an opetopic 1-category X we define a category $\mathcal{C} = \mathcal{C}_X$ as follows. First set ob $\mathcal{C} = X(0)$. Then, given objects $a, b \in X(0)$, let $\mathcal{C}(a, b)$ be the preimage of $a \xrightarrow{?} b$ under f_1. (Recall that we have a 0-category $\mathrm{Hom}(a,b)$, that is, a set.)

Composition and identities in $\mathcal{C}$ are defined according to the 2-cells in X as follows. For composition consider 1-cells $a \xrightarrow{f} b$, $b \xrightarrow{g} c$. We have the following 2-niche

$$\begin{tikzcd}[column sep=small] & {} \arrow[dr, dash, "g"] \arrow[d, phantom, "\Downarrow ?"] & \\ {} \arrow[ur, dash, "f"] \arrow[rr, dash, "?"'] & {} & {} \end{tikzcd}$$

which has a unique occupant; we write it as

$$\begin{tikzcd}[column sep=small] & {} \arrow[dr, dash, "g"] \arrow[d, phantom, "\Downarrow u"] & \\ {} \arrow[ur, dash, "f"] \arrow[rr, dash, "gf"'] & {} & {} \end{tikzcd} .$$

For identities we have already observed (Examples 5.1.3) that in an opetopic n-category the universal 1-ary $(n+1)$-cells are of the form $a \longrightarrow a$. Explicitly, for $n = 1$ we have for any $a \in X(0)$ a nullary 2-niche

$$\begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow ?"] & \\ a \arrow[rr, "?"'] & {} & a \end{tikzcd}$$

which must have a unique occupant. So we write it as

$$\begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow u"] & \\ a \arrow[rr, "1_a"'] & {} & a \end{tikzcd}$$

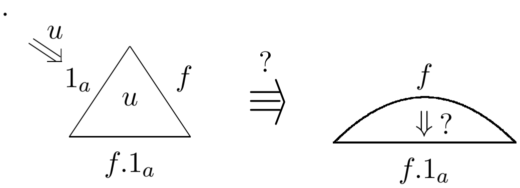
and check that this does indeed act as the identity with respect to the composition defined above. We seek the unique occupant of the niche

$$\begin{tikzcd}[column sep=small] & {} \arrow[dr, dash, "f"] \arrow[d, phantom, "\Downarrow ?"] & \\ {} \arrow[ur, dash, "1_a"] \arrow[rr, dash, "?"'] & {} & {} \end{tikzcd} ,$$

that is

$$\begin{tikzcd}[column sep=small] & {} \arrow[dr, dash, "f"] \arrow[d, phantom, "\Downarrow u"] & \\ {} \arrow[ur, dash, "1_a"] \arrow[rr, dash, "f.1_a"'] & {} & {} \end{tikzcd} .$$

Certainly we have the following 3-niche

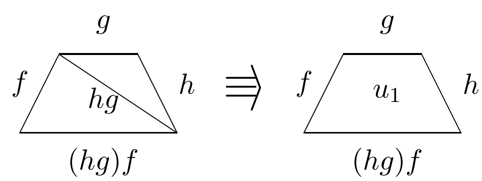


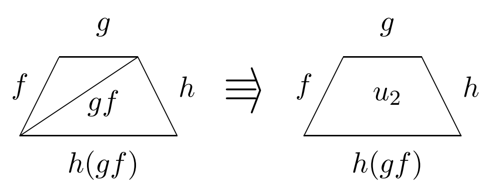
with a unique occupant. So by Example 5.1.3(1), we have $f.1_a = f$ as required. Similarly $1_a.f = f$.

It remains to check that associativity holds. Given 1-cells

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

we have the following universal 3-cells





But u_1 and u_2 are occupants of the same 2-niche; by uniqueness they must be the same, giving

$$(hg)f = h(gf)$$

as required. So we have defined a category, and we set

$$\zeta(X) = \mathcal{C}_X.$$

Observe that we find composites and identities by considering universal 2-cells, and we check axioms by considering universal 3-cells.

- On morphisms

Given a morphism of opetopic 1-categories $F : X \longrightarrow Y$ we seek to define a functor $F : \mathcal{C}_X \longrightarrow \mathcal{C}_Y$. We define the action of F on objects and arrows by the functions

$$\begin{array}{rrcl} & F_0 & : & X(0) \longrightarrow Y(0) \\ \text{and} & F_1 & : & X(1) \longrightarrow Y(1). \end{array}$$

We check functoriality. By definition of morphisms of opetopic 1-categories, the following diagram commutes

$$\begin{tikzcd}
X(1) \arrow[r] \arrow[d, "F_1"'] & o({I_{X(0)}}^{+}) = o(Q(1)) \arrow[d] \\
Y(1) \arrow[r] & o({I_{Y(0)}}^{+}) = o(R(1))
\end{tikzcd}$$

giving

$$\begin{array}{rrcl} & F(\text{dom } f) & = & \text{dom (Ff)} \\ \text{and} & F(\text{cod } f) & = & \text{cod (Ff).} \end{array}$$

Now the function

$$F_2 : X(2) \longrightarrow Y(2)$$

makes the following diagram commute

$$\begin{tikzcd}
X(2) \arrow[r] \arrow[d, "F_2"'] & o({Q(1)_{X(1)}}^{+}) \arrow[d] \\
Y(2) \arrow[r] & o({R(1)_{Y(1)}}^{+})
\end{tikzcd}$$

so under the action of F_2 the following (universal) 2-cell in X

$$\begin{tikzcd}
& \cdot \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow u"] & \\
\cdot \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & \cdot
\end{tikzcd}$$

gives the following 2-cell in Y

$$\begin{tikzcd}
& \cdot \arrow[dr, "Fg"] \arrow[d, phantom, "Fu"] & \\
\cdot \arrow[ur, "Ff"] \arrow[rr, "F(gf)"'] & {} & \cdot
\end{tikzcd}$$

and so we have $F(g \circ f) = Fg \circ Ff$ by uniqueness of 2-niche occupants. Similarly consider the following nullary 2-cell in X

$$\begin{tikzcd}
& \cdot \arrow[d, phantom, "\Downarrow u"] & \\
a \arrow[rr, "1_a"'] & {} & a
\end{tikzcd} \quad .$$

Under the action of F_2 we have the following 2-cell in Y

$$\begin{tikzcd} & \cdot & \\ Fa \arrow[rr, "\Downarrow Fu", "F(1_a)"'] & & Fa \end{tikzcd}$$

and so we have $F(1_a) = 1_{Fa}$ by uniqueness of 2-niche occupants.

So F is a functor as required. Observe that in the above construction we do not need to stipulate that universality be preserved.

Finally, before showing that ζ is an equivalence, we characterise universal 1-cells as invertibles.

Proposition 5.2.2. *A 1-cell f in X is universal if and only if it is invertible as an arrow of $\mathcal{C}_X$.*

Proof 1 (bare hands). Let $a \xrightarrow{f} b$ be a universal 1-cell in X. We certainly have a 1-cell

$$a \xrightarrow{1_a} a \ .$$

So by clause (1) of the definition of universal 1-cell we have a factorisation, that is a 1-cell

$$b \xrightarrow{g} a$$

and a universal 2-cell

$$\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ a \arrow[ur, "f"] \arrow[rr, "1_a"'] & {} & a \end{tikzcd}$$

so we have $gf = 1_a$.

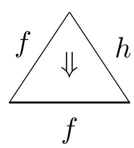
Now consider the 1-cell

$$a \xrightarrow{f} b \ .$$

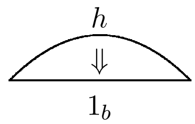
Similarly, we have a universal 2-cell

$$\begin{tikzcd} & b \arrow[dr, "1_b"] \arrow[d, phantom, "\Downarrow u"] & \\ a \arrow[ur, "f"] \arrow[rr, "f"'] & {} & b \end{tikzcd} .$$

Now by clause (1) of the definition of universal 2-cell, if we have a 2-cell



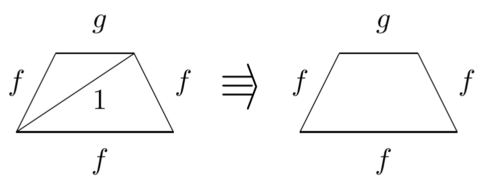
then we have a factorisation, so we certainly have a 2-cell



By uniqueness of 2-niche occupants, this gives

$$hf = f \Rightarrow h = 1_b.$$

Now consider the following 3-cell

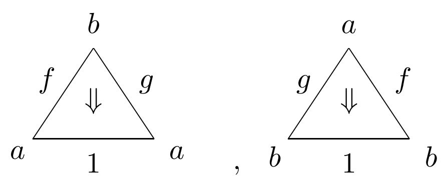


giving $f(gf) = f$. But by associativity we have

$$f(gf) = (fg)f = f$$

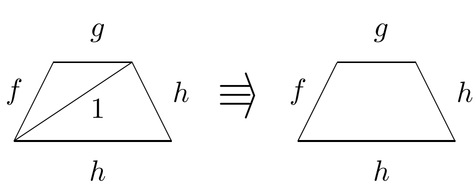
so we have $fg = 1_b$. So if f is universal in X then f is invertible in $\mathcal{C}_X$.

Conversely, suppose f is invertible in $\mathcal{C}_X$, so we have in X 2-cells

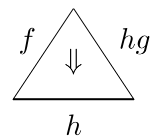


We now show that f is universal:

i) Given any 0-cell $b' \in X(0)$ and 1-cell $a \xrightarrow{h} b'$ we have the following 3-cell

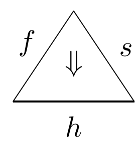


so by associativity the following universal 2-cell

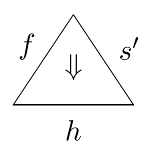


giving a factorisation for h as required.

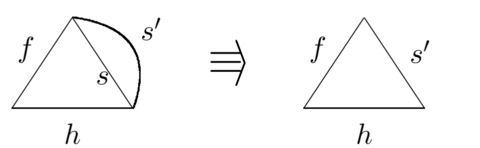
ii) We show that any such factorisation is universal. Let



be such a factorisation. Then given any other 2-cell



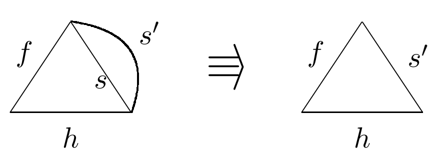
we need to exhibit a factorisation



Now

$$h = sf \Rightarrow hg = sfg = s$$

so we have $s' = hg = s$ and 3-cell



as required. Any such factorisation is then trivially universal.

So if f is invertible then f is universal, and the proposition is proved. $\square$

Although the above calculations may help in understanding the definitions, the proposition may be proved more quickly using the Yoneda Lemma as follows.

Proof 2 (Yoneda). f is universal in X if and only if

1) Given any arrow $b \xrightarrow{g} c$ there is an arrow $b \xrightarrow{\bar{g}} c$ such that $\bar{g}f = g$ and

2) $h_1 f = h_2 f \Rightarrow h_1 = h_2$

i.e. for all $c \in \text{ob}\,\mathcal{C}$ the function

$$\begin{array}{rccc} f^* : & \mathcal{C}(b,c) & \longrightarrow & \mathcal{C}(a,c) \\ & h & \longmapsto & h \circ f \end{array}$$

is an isomorphism. But this is true if and only if f is isomorphism since the Yoneda embedding is full and faithful. $\square$

In Chapter 6 we propose a characterisation of universality that generalises the above Yoneda result.

Proposition 5.2.3. *The functor ζ exhibits an equivalence of categories*

$$\textbf{Opic-}1\textbf{-Cat} \xrightarrow{\sim} \textbf{Cat}$$

surjective in the direction shown.

Proof. We have defined a functor

$$\zeta : \textbf{Opic-}1\textbf{-Cat} \longrightarrow \textbf{Cat}$$

above, and it is clearly full and faithful; we show that it is surjective.

Given any (small) category $\mathcal{C}$, we may construct an opetopic 1-category X with $X(0) = \text{ob}\,\mathcal{C}$ and $X(1) = \text{arr}\,\mathcal{C}$. We see immediately that every 1-niche has a universal occupant $a \xrightarrow{1_a} a$. The set $X(2)$ is defined as follows.

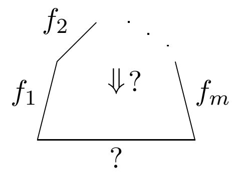
Every nullary 2-niche

$$\begin{array}{c} \cdot \\ \Downarrow ? \\ a \xrightarrow[?]{\hspace{2em}} a \end{array}$$

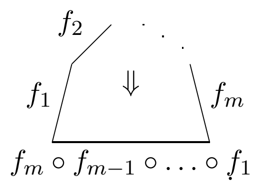
has a unique occupant

$$\begin{array}{c} \cdot \\ \Downarrow \\ a \xrightarrow[1_a]{\hspace{2em}} a \end{array}$$

and every m-ary 2-niche



has a unique occupant



Furthermore, since a 1-cell is universal if and only if it is invertible as an arrow of $\mathcal{C}$, composites of universals are universal.

So X is 1-coherent, and clearly $\zeta(X) = \mathcal{C}$. $\square$

5.2.3 n-cells in an n-category

The definition of universality works from the top down: universal cells are understood via cells in the dimension above, and the starting point is that all cells in dimensions higher than $(n+1)$ are trivial. So in effect, n-cells result from the 'first' step of the induction; we now make some general observations about n-cells, which will be useful later.

Recall (Example 5.1.8(3)) that every $(n-1)$-frame determines an opetopic 1-category. So we have an opetopic 1-category of $(n-1)$-cells and 1-ary n-cells, or, by Proposition 5.2.3, a category.

Let X be an opetopic n-category. First recall that composites of n-cells in X are uniquely determined, since occupants of $(n+1)$-niches are unique. Also, composition of n-cells is strictly associative and a morphism of opetopic n-categories must be strictly functorial on n-cell composites. (In fact, we have a symmetric multicategory of $(n-1)$-cells and n-cells.)

Now consider an n-niche α in X. Then, given any universal occupant u, every occupant f of α factors uniquely as

$$f = g \circ u$$

where g is a 1-ary n-cell. So, for any such universal, we may express the set of occupants of α as

$$g \circ u \quad \text{such that} \quad g \in X(n)_1 \text{ and } s(g) = t(u)$$

where $X(n)_1$ is the set of 1-ary n-cells. Given any other universal occupant u', we then have

$$u' = x \circ u$$

for some (unique) universal x. So we have

$$\{g' \circ u'\} = \{g \circ u\}$$

since $g' \circ u' = g' \circ (x \circ u) = (g' \circ x) \circ u$.

More generally, given any non-empty set U of universal occupants of α, the set of occupants of α may be expressed as

$$\{g \circ u : u \in U, g \in X(n)_1, s(g) = t(u)\} \Big/ \sim \; .$$

Here $\sim$ is the equivalence relation generated by

1) $g \circ u \sim g' \circ u' \iff g = g' \circ x_{uu'}$

2) $1 \circ u \sim u$

where for any $u, u' \in U$, $x_{uu'}$ is the unique universal such that

$$u' = x_{uu'} \circ u.$$

5.2.4 Equivalence between approaches to bicategories

We are now ready to turn our attention to the case $n = 2$. We show how to construct a classical bicategory from an opetopic 2-category, leading to the main theorem of this chapter, which shows how the opetopic and classical theories of bicategories are equivalent.

An important difference between this construction and that for the case $n = 1$ is that an element of choice now arises. The universality condition stipulates that every niche should have a universal occupant, but does not *specify* such universals. This approach differs from the approach of Leinster ([Lei5]), for example, in which contractibility is defined as a property but specific contractions are then given.

This approach also differs from the classical approach to bicategories, in which binary and nullary composites of 1-cells are specified, even though m-fold composites are not, for $m > 2$. (Note that 1-cell identities are considered as 'nullary composites'.) Leinster refers to this theory as being 'biased' towards binary composites; in [Lei2], he introduces the notion of *unbiased bicategory*. The theory of bicategories is made 'unbiased' by specifying m-fold composites for all m. This theory turns out to be equivalent to the classical one ([Lei5]). Leinster also comments that, provided at least one choice has been made for each of $k = 0$ and some $k \geq 2$, an equivalent theory of bicategories may be formed.

Another way of eliminating bias from a bicategory might be to choose *no specified composites*. We will later see that this is how the opetopic approach may be interpreted. Once we have shown that this theory is equivalent to the classical one, it is easy to see which choices give rise to a theory of bicategories, and it follows immediately that all such theories are equivalent.

Theorem 5.2.4. *Write* **Bicat** *for the category of bicategories and morphisms (lax functors). Then*

$$\mathbf{Opic\text{-}2\text{-}Cat} \simeq \mathbf{Bicat}.$$

Given an opetopic 2-category X, we seek to construct a bicategory $\mathcal{B}$ (using the definition given in [Lei1]). To do this we need to make some choices of universal 2-cells. The general idea is

- the 0-cells of $\mathcal{B}$ are the 0-cells of X
- the 1-cells of $\mathcal{B}$ are the 1-cells of X
- the 2-cells of $\mathcal{B}$ are the 1-ary 2-cells of X.

We then choose a universal occupant for each 0-ary and 2-ary 2-niche in X. Then

- 1-cell composition in $\mathcal{B}$ is given by the chosen 2-ary universal 2-cells in X
- 1-cell identities in $\mathcal{B}$ are given by the chosen nullary universal 2-cells in X
- constraints are induced from composites of the chosen universals
- axioms are seen to hold by examining 4-cells.

In fact, we define a category of 'biased opetopic 2-categories' in which these choices have already been made.

Definitions 5.2.5.

- *A* biased opetopic 2-category *is an opetopic 2-category together with a chosen universal occupant for every nullary and 2-ary 2-niche.*
- *A* morphism of biased opetopic 2-categories *is simply a morphism of the underlying 2-categories.*

We write $\mathbf{Opic\text{-}2\text{-}Cat}_b$ *for the category of biased opetopic 2-categories and morphisms.*

Note that the choice of universal 2-cells is free, that is, the chosen cells are not required to satisfy any axioms. Furthermore, no preservation condition is imposed on the morphisms in this category.

Proposition 5.2.6. *There is an equivalence*

$$\mathbf{Opic\text{-}2\text{-}Cat}_b \xrightarrow{\sim} \mathbf{Opic\text{-}2\text{-}Cat}$$

surjective in the direction shown.

Proof. Clear from the definitions. □

So in fact, we prove the following proposition:

Proposition 5.2.7. *There is an equivalence*

$$\textbf{Opic-}2\textbf{-Cat}_b \xrightarrow{\sim} \textbf{Bicat}$$

surjective in the direction shown.

Finally we will make some comments about the choices made in forming a biased opetopic 2-category.

For the longer calculations in this section, and for an explanation of the 'shorthand' used in manipulating 2-cells, we refer the reader to Appendix C.

Proof of Proposition 5.2.7. We construct a functor

$$\zeta : \textbf{Opic-}2\textbf{-Cat}_b \longrightarrow \textbf{Bicat}$$

and show that it is surjective, full and faithful.

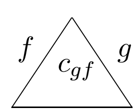
- We define the action of ζ on objects.

Let X be a biased opetopic 2-category. So in addition to the usual data, we have

i) for each object $A \in X(0)$ a chosen universal 2-cell

$$A \xrightarrow{\Downarrow\, \iota_A} A$$

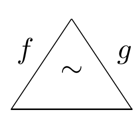
ii) for each pair f, g of composable 1-cells, a chosen universal 2-cell



.

We may indicate these chosen 2-cells by $\sim$ as in

$$A \xrightarrow{\Downarrow\, \sim} A \quad ,$$



.

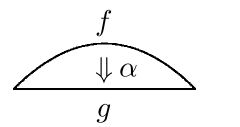
We now define a bicategory $\mathcal{B} = \mathcal{B}_X$ as follows. First set

$$\mathrm{ob}(\mathcal{B}) = X(0).$$

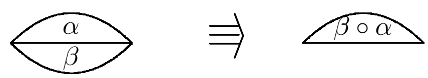
Recall (Proposition 5.1.6) that given objects $A, B \in X(0)$, we have an opetopic 1-category $\mathrm{Hom}(A, B)$. Let $\mathcal{B}(A, B)$ be the category corresponding to $\mathrm{Hom}(A, B)$ according to Proposition 5.2.3. So we have 1-cells given by 1-cells of X

$$a \xrightarrow{f} b$$

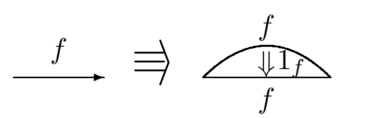
and 2-cells given by 1-ary 2-cells of X



2-cell composites are given by the (unique) 3-cell occupants, for example



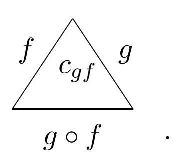
and 2-cell identities by nullary 3-cells



Now for any objects $A, B, C \in \mathrm{ob}\ \mathcal{B}$ we need a functor

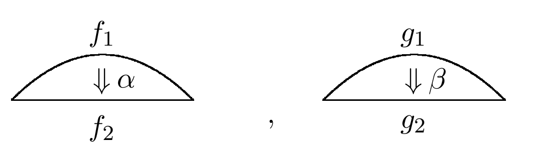
$$\begin{array}{rccc} c_{ABC}: & \mathcal{B}(B,C) \times \mathcal{B}(A,B) & \longrightarrow & \mathcal{B}(A,C) \\ & (g, f) & \longmapsto & g \circ f = gf \\ & (\beta, \alpha) & \longmapsto & \beta * \alpha. \end{array}$$

We define $g \circ f$ to be the target 1-cell of the chosen universal c_{gf}, so we have

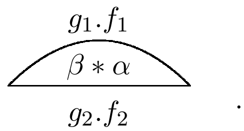


Note that for each composable pair f, g, we have specified a 2-cell c_{gf}; this is crucially stronger than merely specifying a 1-cell $g \circ f$.

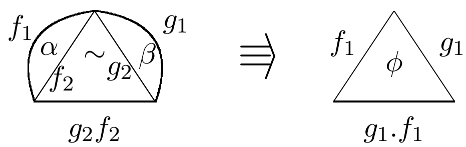
We now show how horizontal 2-cell composition is induced. Consider 2-cells



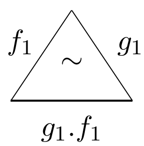
we seek a 2-cell



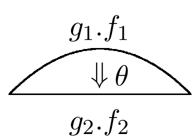
We have a 3-cell



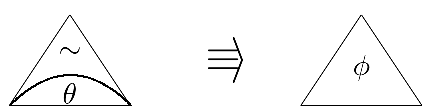
unique in its niche, and a universal 2-cell



inducing, by definition of universality, a 2-cell



unique such that there is a 3-cell



Put $\beta * \alpha = \theta$. We check functoriality, that is

i) $1_g * 1_f = 1_{gf}$

ii) $(\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1) = (\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1)$ (middle 4 interchange)

(see Appendix, Lemma C.2.1).

Next we need, for each object A, a 1-cell $A \xrightarrow{I_A} A$. We define this to be the target of the chosen universal ι_A, so we have

$$\begin{array}{c} \cdot \\ \Downarrow \iota_A \\ A \xrightarrow[I_A]{} A \end{array} .$$

Note that, as before, we have specified a universal 2-cell, not just the 1-cell I_A.

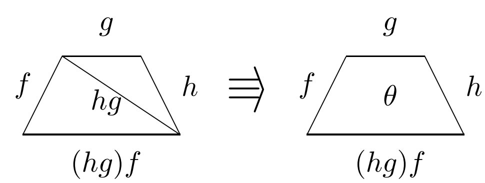
We now seek natural isomorphisms a, r, l. Each of these is induced uniquely from the chosen universals ι and c. For a, consider 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

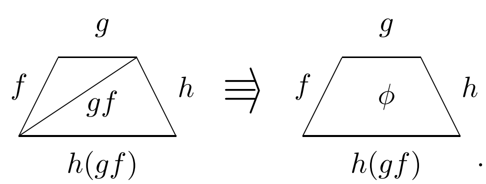
We seek a natural isomorphism

$$a_{hgf} : (hg)f \xrightarrow{\sim} h(gf).$$

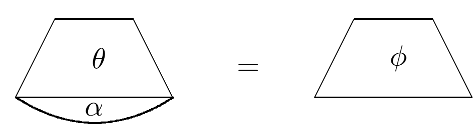
We have



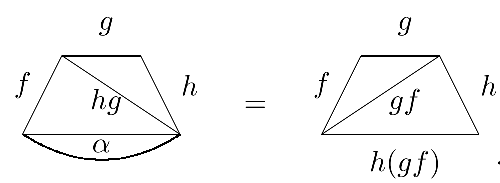
and



θ and ϕ are composites of universals, so universal. Universality of θ induces a unique 2-cell α such that

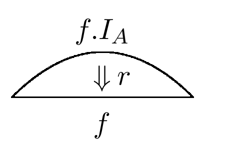


so

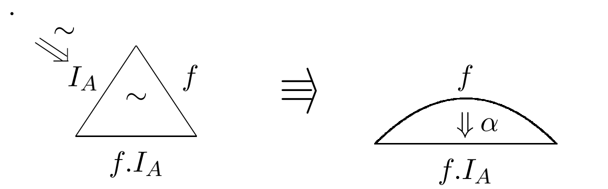


Put $a_{hgf} = \alpha$. We see from universality of ϕ that a_{hgf} is an isomorphism; we check that it satisfies naturality (see Appendix, Lemma C.2.2).

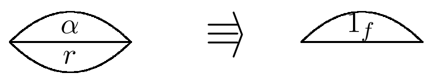
Next we seek a natural transformation r, so we need for any 1-cell $A \xrightarrow{f} B$ a 2-cell



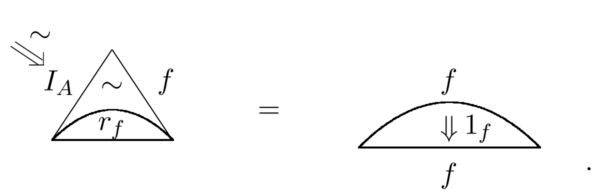
Now we have a 3-cell



and the target 2-cell α is universal since it is the composite of universals. (Note that this is not the same α as above.) So α induces



so



Since α is universal it is an isomorphism with r_f as its inverse; so r_f is also an isomorphism. We also check naturality (see Appendix, Lemma C.2.3). The construction of and result for l follow similarly.

Finally we check the axioms for a bicategory (see Appendix, Lemma C.2.4). So we have defined a bicategory $\mathcal{B}_X$ and we put $\zeta(X) = \mathcal{B}_X$.

- We define the action of ζ on morphisms.

Let $F : X \longrightarrow X'$ be a morphism of opetopic 2-categories, so for each k we have

$$\begin{tikzcd}
X(k) \arrow[r, "f_k"] \arrow[d, "F_k"'] & S(k) \arrow[d] \\
X'(k) \arrow[r, "f'_k"'] & S'(k)
\end{tikzcd} .$$

We construct from F a lax functor

$$(F, \phi) : \mathcal{B}_X \longrightarrow \mathcal{B}_{X'}.$$

The action of F on objects is given by the function

$$F_0 : X(0) \longrightarrow X'(0);$$

we also need, for any objects $A, B \in \text{ob}\, \mathcal{B}_X$ a functor

$$F_{AB} : \mathcal{B}_X(A, B) \longrightarrow \mathcal{B}_{X'}(FA, FB).$$

Now for any $A, B \in \mathrm{ob}\, \mathcal{B}_X$ we have an opetopic 1-category $\mathrm{Hom}(A, B)$, and restricting F to this gives a morphism of opetopic 1-categories

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(FA, FB)$$

so by Proposition 5.2.3 we have a functor F_{AB} as required.

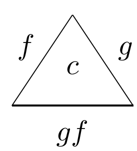
Next we seek a natural transformation ϕ_{ABC}, so for any 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

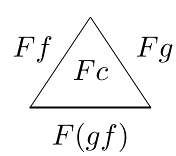
we need a 2-cell

$$\phi_{gf} : Fg \circ Ff \longrightarrow F(g \circ f).$$

We have in X a chosen universal 2-cell

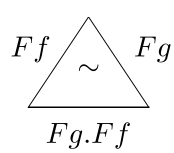


so under the action of F we have in X' a 2-cell

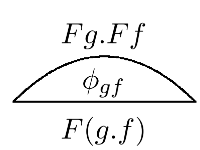


.

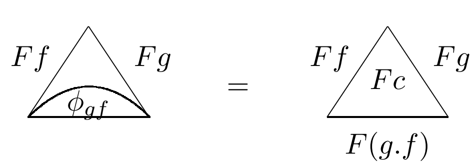
But in X' we have a chosen universal 2-cell



which, by definition of universality, induces a 2-cell



unique such that



;

we check that this satisfies naturality (see Appendix, Lemma C.2.5).

We now seek a natural transformation ϕ_A for each object A, so we seek a 2-cell

$$\begin{tikzcd} {} \arrow[rr, bend left, "I_{FA}"] \arrow[rr, "FI_A"'] \arrow[rr, phantom, "\Downarrow \phi_A"] & & {} \end{tikzcd} .$$

We have in X a chosen universal 2-cell

$$\begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow \iota_A"] & \\ A \arrow[rr, "I_A"'] & & A \end{tikzcd}$$

so applying F gives a 2-cell in X'

$$\begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow F\iota_A"] & \\ FA \arrow[rr, "FI_A"'] & & FA \end{tikzcd} .$$

Now the chosen universal in X'

$$\begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow \iota_{FA}"] & \\ FA \arrow[rr, "I_{FA}"'] & & FA \end{tikzcd}$$

induces, by universality, a 2-cell

$$\begin{tikzcd} {} \arrow[rr, bend left, "I_{FA}"] \arrow[rr, "FI_A"'] \arrow[rr, phantom, "\phi_A"] & & {} \end{tikzcd}$$

unique such that

$$\begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow"] & \\ {} \arrow[rr, bend left, "I_{FA}"] \arrow[rr, "FI_A"'] \arrow[rr, phantom, "\phi_A"] & & {} \end{tikzcd} \quad = \quad \begin{tikzcd} & \cdot \arrow[d, phantom, "\Downarrow F\iota_A"] & \\ {} \arrow[rr, "FI_A"'] & & {} \end{tikzcd}$$

and there is no non-trivial naturality to check.

Finally we check that the axioms for a lax functor hold (see Appendix, Lemma C.2.6). So (F, ϕ) is indeed a lax functor, and we set $\zeta(F) = (F, \phi)$.

It is clear that the above construction of ζ is functorial, so we have defined a functor

$$\zeta : \textbf{Opic-2-Cat}_b \longrightarrow \textbf{Bicat};$$

it remains to show that ζ is surjective, full and faithful.

- We show that ζ is surjective.

Given a bicategory $\mathcal{B}$, we construct an opetopic 2-category X such that $\zeta(X) = \mathcal{B}$. The idea is

i) The 0-cells of X are the 0-cells of $\mathcal{B}$.

ii) The 1-cells of X are the 1-cells of $\mathcal{B}$.

iii) The 1-ary 2-cells of X are the 2-cells of $\mathcal{B}$.

iv) For $m \neq 1$, certain m-ary universals are fixed according to m-fold composites in $\mathcal{B}$; the remaining cells are then generated to ensure that these do indeed satisfy universality.

v) The 3-cells of X are determined from 2-cell composition in $\mathcal{B}$.

Put $X(0) = \mathrm{ob}(\mathcal{B})$ and set $X(1)$ to be the set of 1-cells of $\mathcal{B}$; the function $f_1 : X(1) \longrightarrow S(1)$ is defined so that the preimage of the frame $A \xrightarrow{?} B$ is the set of objects of the category $\mathcal{B}(A, B)$.

We now construct X(2) bearing in mind the comments in Section 5.2.3. Write $X(2)_m \subset X(2)$ for the set of m-ary 2-cells. First we define the set $X(2)_1$ of 1-ary 2-cells to be the set of 2-cells of $\mathcal{B}$.

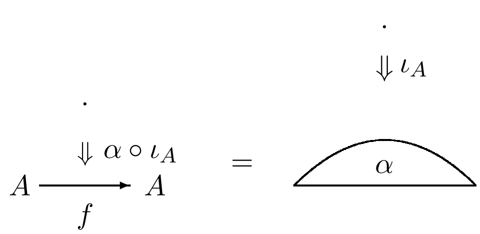
For 0-ary 2-cells, we first define for each $A \in X(0)$ a 2-cell

$$\begin{array}{c} \cdot \\ \Downarrow \iota_A \\ A \xrightarrow[I_A]{} A \end{array} .$$

We then define the set of occupants of the same niche to be

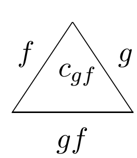
$$\{\alpha \circ \iota_A : \alpha \in X(2)_1, s(\alpha) = I_A\}$$

that is, cells of the form



where we put $1 \circ \iota = \iota$.

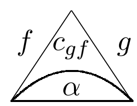
Similarly for $X(2)_2$ we first define for each composable pair of 1-cells f, g a 2-cell



where $g \circ f$ is the composite in $\mathcal{B}$. We then define the set of occupants of this niche to be

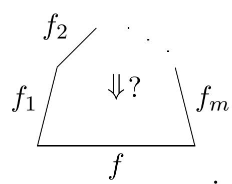
$$\{\alpha \circ c_{gf} : \alpha \in X(2)_1, s(\alpha) = g \circ f\}$$

that is, cells of the form

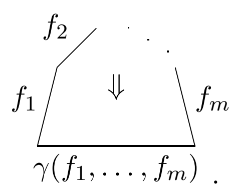


where we put $1 \circ c = c$.

For $X(2)_m$, $m > 2$, consider a 2-niche of the form



We have no preferred m-fold composite in $\mathcal{B}$; instead, for each composite $\gamma(f_1, \ldots, f_m)$ we define a 2-cell u_γ which is to be universal:



Now, suppose we have composites $\gamma(f_1, \ldots, f_m)$ and $\gamma'(f_1, \ldots, f_m)$. Then we have a unique invertible

$$a_{\gamma\gamma'} : \gamma(f_1, \ldots, f_m) \Longrightarrow \gamma'(f_1, \ldots, f_m)$$

given by composing components of the associativity constraint a. (Uniqueness follows from coherence for a bicategory.)

We then generate occupants of this niche as

$$\{\alpha \circ u_\gamma : \alpha \in X(2)_1, s(\alpha) = \gamma(f_1, \ldots, f_m)\} \Big/ \sim$$

where $\sim$ is the equivalence relation generated by

i) $\alpha \circ u_\gamma = \beta \circ u_{\gamma'} \iff \beta \circ a_{\gamma\gamma'} = \alpha \ \in \mathcal{B}$

ii) $1 \circ u_\gamma = u_\gamma$.

Note in particular that since $1 \circ a_{\gamma\gamma'} = a_{\gamma\gamma'}$ we have

$$a_{\gamma\gamma'} \circ u_\gamma = u_{\gamma'}.$$

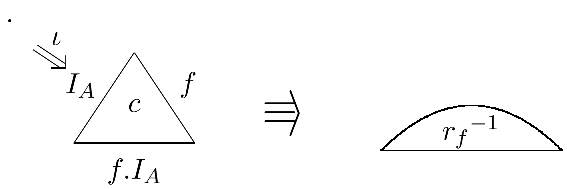
So, given any γ, every occupant of the niche is uniquely expressible as $\alpha \circ u_\gamma$, with $\alpha \in X(2)_1$. This shows that u_γ is indeed universal, and completes the definition of $X(2)$.

Note that the universality of the u_γ follows from coherence for classical bicategories, as it depends on the fact that any two composites of given 1-cells are uniquely isomorphic.

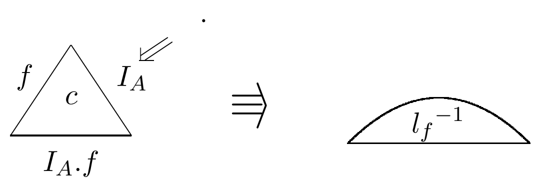
We now construct $X(3)$. We must specify a unique 3-cell for any 3-niche, that is, a unique composite 2-cell for any formal composite of 2-cells.

1) First, composites of 1-ary 2-cells are determined by 2-cell composition in $\mathcal{B}$.

2) Next we consider any composite of the form $c \circ \iota$. We define the composites by

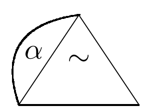


and similarly

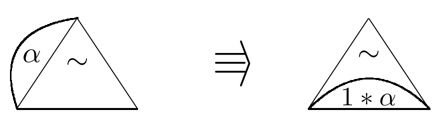


.

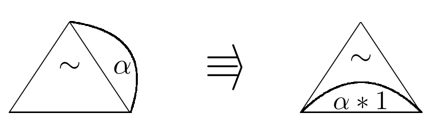
3) Now consider a composite of the form



where α is any 1-ary 2-cell. We put

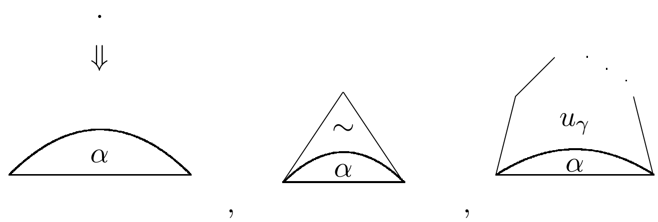


and similarly



4) Now consider a formal composite of chosen 2-ary 2-cells c_{gf}. Such a diagram uniquely determines a composite γ in $\mathcal{B}$ of its boundary 1-cells. So we set the composite 2-cell in X to be u_γ. Conversely, any 2-cell u_γ thus arises as the composite of some 2-cells c.

5) Finally, since we require that 2-cell composition be strictly associative, we have determined all 3-cells in X. For, using the above cases, any nullary, 2-ary or m-ary composite can be written in the form



respectively, where α is a composite of 1-ary 2-cells which we can then compose in $\mathcal{B}$.

This completes the definition of the opetopic set X; it remains to check that X is 2-coherent. Certainly, every 3-niche has a unique occupant by construction. A 2-cell $\alpha \circ \iota$, $\alpha \circ c$ or $\alpha \circ u_\gamma$ is universal if and only if α is universal, that is, if and only if α is invertible in $\mathcal{B}$. So every 2-niche has a universal occupant and composites of universal 2-cells are universal.

We can check that a 1-cell in X is universal if and only if it is an (internal) equivalence in $\mathcal{B}$; this follows by an analogous argument to the 'Yoneda' proof of Proposition 5.2.2. So every 1-niche has a universal occupant I_A, and composites of universal 1-cells are universal.

So X is a biased opetopic 2-category, with chosen universal 2-cells ι and c, and it is clear that $\zeta(X) = \mathcal{B}$. So ζ is surjective.

- We show that ζ is full.

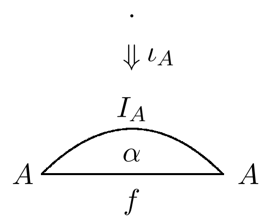
Let X and X' be biased opetopic 2-categories, and suppose we have a morphism of bicategories

$$(G, \phi) : \mathcal{B}_X \longrightarrow \mathcal{B}_{X'}.$$

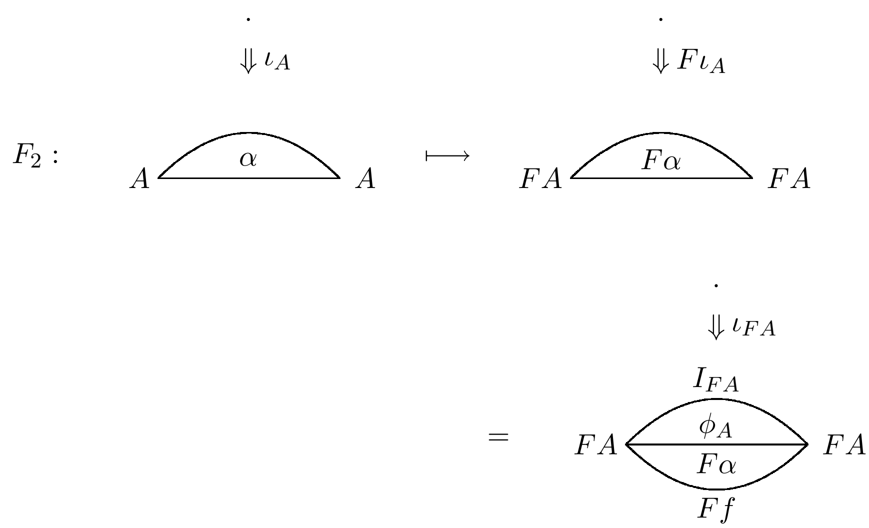
We define a morphism $F : X \longrightarrow X'$ as follows. For $k = 0$ and $k = 1$ the functions

$$F_k : X(k) \longrightarrow X'(k)$$

are given by the action of G on objects and 1-cells respectively. We construct F_2 as follows. The action of F_2 on 1-ary 2-cells is the action of G on 2-cells of $\mathcal{B}_X$. For 0-ary 2-cells, we observe that any such is expressible uniquely as

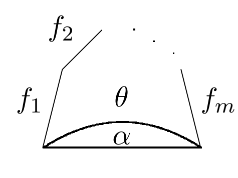


where ι_A is the chosen universal for$X$. Then we define

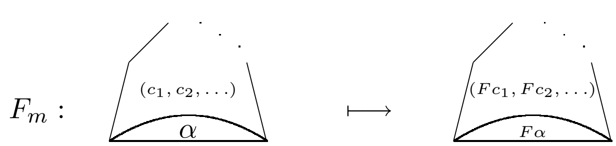


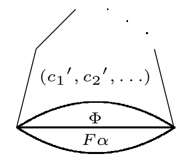
where ι_{FA} is the appropriate chosen universal for X'; this assignation is well-defined by uniqueness.

For $m \geq 2$, any m-ary 2-cell is expressible in the form



Here θ is the composite of some configuration of chosen universals c, determining a 1-cell composite $\gamma(f_1, \ldots, f_m)$ in $\mathcal{B}$, and $\alpha : \gamma \Longrightarrow g$. Then we define



$=$ 

where Φ is the appropriate composite of components of the constraint ϕ. This assignation is well-defined by uniqueness and the axioms for a morphism of bicategories.

It is clear from the construction that this is a morphism of biased opetopic 2-categories, and that

$$\zeta(F) = (G, \phi).$$

So ζ is full.

- We show that ζ is faithful.

Consider morphisms F, F' of unbiased opetopic 2-categories, such that $\zeta(F) = \zeta(F')$. Write $\zeta(F) = (G, \phi)$ and $\zeta(F') = (G', \phi')$.

Certainly since $G = G'$ on objects and 1-cells we have $F_0 = F_0'$ and $F_1 = F_1'$. Similarly, $G = G'$ on (bicategorical) 2-cells gives $F_2 = F_2'$ on (opetopic) 1-ary 2-cells. For m-ary 2-cells with $m \neq 1$ consider again the above presentation of 2-cells. Then $\phi = \phi'$ gives $F_2 = F_2'$ on all opetopic 2-cells. So ζ is faithful.

So finally we may conclude that ζ exhibits an equivalence

$$\textbf{Opic-2-Cat}_b \xrightarrow{\sim} \textbf{Bicat}$$

as required. $\square$

Proof of Theorem 5.2.4. By Proposition 5.2.7 we have

$$\textbf{Opic-2-Cat}_b \xrightarrow{\sim} \textbf{Bicat}$$

and by Proposition 5.2.6 we have

$$\textbf{Opic-2-Cat}_b \xrightarrow{\sim} \textbf{Opic-2-Cat}$$

so we have an equivalence

$$\textbf{Opic-2-Cat} \simeq \textbf{Bicat}$$

as required. $\square$

Remarks 5.2.8.

1) Note that the final equivalence is not surjective in *either* direction. Left-to-right involves a choice of universal 2-cells; right-to-left involves generating sets of 3-cells and k-ary 2-cells (for $k \neq 1$) which are only defined up to isomorphism. Observe that a different choice of universal 2-cells yields a bicategory non-trivially isomorphic but with the same cells.

2) The term 'biased opetopic 2-category' is used in the spirit of Leinster's work on biased and unbiased bicategories ([Lei5]). Rather than pick universal m-ary 2-cells for just $m = 0, 2$, we might pick universals for all $m \geq 0$. Again with no further stipulations on morphisms, this yields an equivalent category of 'unbiased opetopic 2-categories'. By a straightforward modification of the above proof, we may see that this corresponds to the theory of unbiased bicategories; Leinster has shown directly that the biased and unbiased theories are equivalent.

3) In fact, we may choose any number of universal m-ary 2-cells for each m and define a category obviously equivalent to **Opic-**2**-Cat**, by making no stipulation on morphisms. We might then ask: when does this yield a theory of bicategories? In order to modify the above construction as required, we need enough chosen universals to give a complete presentation of the 2-cells of X. From the observations in Section 5.2.3 we see that this is possible provided we have chosen at least one 0-ary universal, and at least one m-ary universal for some $m > 1$ (for each appropriate niche). This idea is discussed in [Lei5] (Appendix A); in the opetopic setting it is immediate that each resulting category of 'bicategories' is equivalent.

4) Like Leinster, we might observe that the equivalence of *categories*

$$\textbf{Opic-}2\textbf{-Cat} \simeq \textbf{Bicat}$$

is two levels 'better' than we might have asked; we have a comparison at the 1-dimensional level without having to invoke 3- or even 2-dimensional structures. So the theory might already be seen as fruitful despite the lack of an $(n+1)$-category of n-categories.

In summary, we have the following equivalences, surjective in the directions shown:

$$\textbf{Opic-}2\textbf{-Cat} \xleftarrow{\sim} \textbf{Opic-}2\textbf{-Cat}_b \xrightarrow{\sim} \textbf{Bicat}.$$

5.2.5 Strictness

In this section we discuss (informally) various possible notions of strictness in the opetopic setting, and compare these with the classical biased and unbiased settings.

In the classical theory of bicategories, 'strictness' (of bicategories or their morphisms) is determined by the 'strictness' of the constraints; in general 'lax' for plain morphisms, 'weak' for isomorphisms and 'strict' for identities.

In the opetopic theory we cannot make such definitions, since we do not have those constraints unless we have chosen universal 2-cells. Even then the constraints are not explicitly given. So we must define strictness by some other means; we may define stricter and weaker notions in terms of universals.

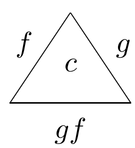
We first turn our attention to morphisms. Recall that the original Baez-Dolan definition demanded that a morphism preserve universality; this is stronger than the general morphisms we use in our definition of **Opic-2-Cat**.

Proposition 5.2.9. *Recall (Proposition 5.2.7) that we have an equivalence*

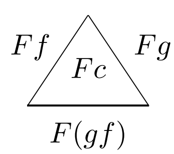
$$\zeta : \mathbf{Opic\text{-}2\text{-}Cat}_b \xrightarrow{\sim} \mathbf{Bicat}.$$

Let F be a morphism of opetopic 2-categories. Then F preserves universals iff $\zeta(F)$ is a weak functor (homomorphism) of bicategories.

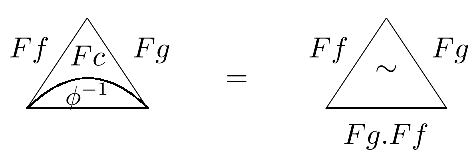
Proof. Suppose $F : X \longrightarrow X'$ preserves universals. Then the chosen universal in X



becomes, under the action of F, a universal in X'

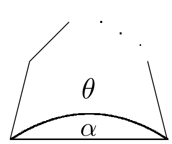


inducing

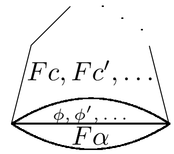


so ϕ_{ABC} is an isomorphism.

Conversely suppose ϕ_{gf} and ϕ_A are invertible for all f, g, A. First note that 1-ary universal 2-cells are always preserved (clear from the case $n = 1$). Now, any universal can be expressed as



where θ is some composite of chosen universals and α is universal. Now applying F we have



which is universal since $F\alpha$ is universal.

The result for 1-cells follows (with some effort). $\square$

Definition 5.2.10. *We write* **Opic-2-Cat(weak)**, **Opic-2-Cat$_b$(weak)** *and* **Bicat(weak)** *for the lluf subcategories with only weak morphisms.*

Proposition 5.2.11. *The equivalences given in the proofs of Propositions 5.2.6 and 5.2.7 restrict to equivalences*

$$\textbf{Opic-2-Cat(weak)} \xleftarrow{\sim} \textbf{Opic-2-Cat}_b\textbf{(weak)} \xrightarrow{\sim} \textbf{Bicat(weak)}$$

surjective in the directions shown.

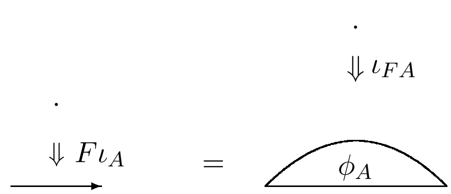
Proof. The first equivalence is clear from the definitions and the second follows from Proposition 5.2.9. Since these are lluf subcategories the functors are clearly still surjective. $\square$

Since we have still made no stipulation about the action of morphisms on chosen universals, it is clear that we will still have a result of the form 'all theories are equivalent' (cf [Lei4]). That is, regardless of the number of universals chosen, the category-with-weak-morphisms will remain equivalent to the category **Opic-2-Cat(weak)**. This ceases to be so in the strict case.

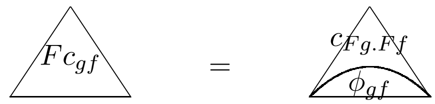
There is no obvious way of further strengthening the conditions imposed on morphisms in **Opic-2-Cat(weak)**, but if we consider **Opic-2-Cat$_b$(weak)**, we can further demand that chosen universals be preserved.

Proposition 5.2.12. *Let F be a weak morphism of biased opetopic 2-categories. Then F preserves chosen universals iff $\zeta(F)$ is strict.*

Proof. '$\Rightarrow$' is clear from the definition of ζ. Now for *any* morphism (F, ϕ) of opetopic 2-categories we have



and



so clearly if (F, ϕ) is strict then F preserves chosen universals. $\square$

Definition 5.2.13. *We call a weak morphism of biased opetopic 2-categories* strict *if it preserves chosen universal 2-cells.*

Write **Opic-**2**-Cat**$_b$**(str)** and **Bicat(str)** for the lluf subcategories with only strict morphisms.

Proposition 5.2.14. *The previously defined equivalence restricts to an equivalence*

$$\textbf{Opic-}2\textbf{-Cat}_b\textbf{(str)} \xrightarrow{\sim} \textbf{Bicat(str)}$$

surjective in the direction shown.

Proof. Follows immediately from Proposition 5.2.12 $\square$

We now consider the possibility of altering the structures of the 2-categories themselves. Considering the structures used so far as 'weak', we might try to find either lax or strict opetopic n-categories.

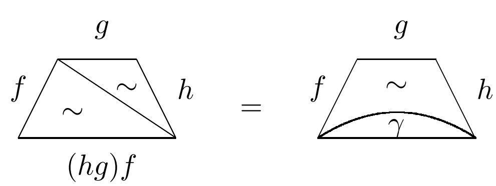
In the lax direction we might consider removing the condition that universals compose to universals. Observe that in the case $n = 1$ we do not use this condition to prove

$$\textbf{Opic-}1\textbf{-Cat} \simeq \textbf{Cat}$$

so a 'lax opetopic 1-category' would be just the same as a weak one, as we would hope.

However, for $n = 2$ it is not clear that this 'laxification' produces a useful structure for the general or biased theories. Consider instead the case in which m-ary universal 2-cells have been chosen for every $m \geq 0$. That is, we define an 'unbiased opetopic 2-category' to be one in which *every* 2-niche has a chosen universal occupant.

If we now remove the condition that composites of universals be universals, we have certain 2-cell 'constraints' induced by the chosen universals. For example we have



and thus an induced 2-cell

$$\gamma : hgf \Rightarrow (hg)f.$$

This produces a structure something like a 'lax unbiased bicategory' in the sense of Leinster ([Lei5]) except that the constraints γ are acting in the opposite direction.

For strictness there is likewise no obvious way of imposing stronger conditions on an opetopic 2-category. Once we have chosen universals, we might demand that the chosen universals compose to chosen universals, but this will certainly not be possible unless we have chosen m-ary universals for all $m \geq 0$. So once again we find ourselves in the unbiased theory.

If we have one chosen universal for each 2-niche, the above condition forces strict associativity and left and right unit action. So we have a 2-category; this is to be expected since Leinster has already observed that unbiased 2-categories are in one-to-one correspondence with 2-categories. (There is a possibility of more interesting structure if a niche has more than one chosen universal.)

From this informal discussion we see that the theory of opetopic 2-categories neither laxifies nor strictifies particularly naturally. In the lax direction, this is perhaps consistent with the fact that there is no very satisfactory lax version of classical bicategories. In the strict direction, this demonstrates why we have found it hard to state a coherence theorem of the form 'every bicategory is biequivalent to a 2-category'; we simply do not know what a 'strict opetopic 2-category' is. (Note however that statements of the form 'all diagrams commute' are much less problematic.)

We have already observed that there are (at least) two possible ways of removing the bias in a bicategory: we may choose m-ary composites for no m, or all m. It appears that, although the former philosophy may be viewed as being more egalitarian towards all universal cells, the latter provides more footholds for exploring the theory.

Chapter 6

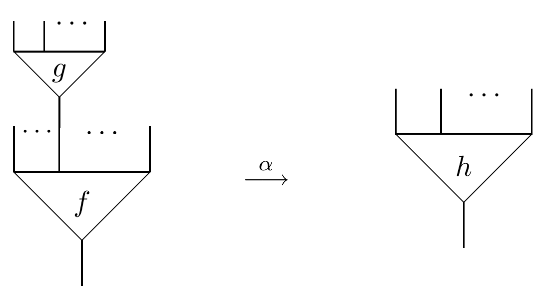
An alternative notion of universality

In this chapter we discuss an alternative characterisation of universal cells in opetopic n-categories. While the theory of opetopes and opetopic sets deals with the underlying data for k-cells in the opetopic theory of n-categories, it is universality that deals with composition and coherence. However, there are many possible ways of characterising universal cells, just as there are many ways of characterising, say, isomorphisms in a category. We now propose an alternative characterisation to the one given in Section 5.1.1.

Terminology and Notation

In this chapter we will avoid any detailed discussion of the language of multicategories and construction of opetopic sets since this has been discussed in the previous chapters of this work. We will adopt the (more practical) method of Hermida, Makkai and Power ([HMP1]), picking one ordering of source elements in order to represent a symmetry class. For a general k-cell we write its source as $\underline{a}$, say, to indicate a formal composite whose constituent $(k-1)$-cells may be placed in some order.

Furthermore, we may adopt the following convention for 2-ary cells. A 2-ary k-cell α has the form



where f, g, and h are $(k-1)$-cells (and necessarily $k \geq 2$). We write this k-cell as

$$\alpha : (f, g) \longrightarrow h$$

employing this ordering of the source elements to indicate that f and g are pasted at the target of g; we also write $s_1(\alpha) = f$ and $s_2(\alpha) = g$.

6.1 Preliminaries

We begin by examining the motivating example in categories. Let $\mathcal{C}$ be a category and $f : A \longrightarrow B$ a morphism in $\mathcal{C}$. Then we have a natural transformation

$$H^f : \mathcal{C}(B, \_\,) \longrightarrow \mathcal{C}(A, \_\,)$$

with components

$$\_ \circ f : \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C)$$

for each $C \in \mathcal{C}$. Then

$$\begin{array}{rcl} f \text{ is an isomorphism} & \Longleftrightarrow & H^f \text{ is an isomorphism} \\ & \Longleftrightarrow & \forall\, C \in \mathcal{C},\ \_ \circ f \text{ is an isomorphism} \\ & \Longleftrightarrow & \text{“composition with } f \text{ is an isomorphism”} \end{array}$$

Here “composition with f” is a function on homsets.

Now let X be an opetopic n-category and $f : \underline{a} \longrightarrow b$ a k-cell in X. Then given any $(k-1)$-cell c we have $(n-k)$-categories $X(b, c)$ and $X(\underline{a}, c)$ whose 0-cells are k-cells of X with the appropriate source and target, and whose j-cells are $(k+j)$-cells.

Since composition in an opetopic n-category is not uniquely defined, we cannot expect $\_ \circ f$ to be a well-defined operation $X(b, c) \longrightarrow X(\underline{a}, c)$. Instead, we will have a span of $(n-k)$-categories

$$\begin{tikzcd} & C_f \arrow[dl, "\sigma_f"'] \arrow[dr, "\tau_f"] & \\ X(b,c) & & X(\underline{a},c) \end{tikzcd}$$

where C_f gives all possible ways of composing with f. Here σ_f and τ_f are $(n-k)$-functors i.e. morphisms of the underlying opetopic sets. (σ has more properties that we will not discuss here.)

We then have the following definition.

Definition 6.1.1. *A k-cell f is* universal *iff*

1) $k > n$ and f is unique in its niche, or

2) $k \leq n$ and τ_f is an $(n-k)$-equivalence of $(n-k)$-categories.

Definition 6.1.2. *An m-functor is an m-*equivalence of m-categories *iff*

1) it is an $(m-1)$-equivalence on hom-$(m-1)$-categories

2) it is “essentially surjective on 0-cells” i.e. surjective up to universal 1-cells

We observe (without giving details) that since σ_f will be an $(n-k)$-equivalence of $(n-k)$-categories, the above condition for universality will also result in $X(b,c)$ and $X(\underline{a},c)$ being $(n-k)$-equivalent.

Furthermore it will follow from the construction of the composition span that in an n-category the above definition is equivalent to demanding "on the nose" surjectivity. i.e. f is universal iff $\forall \underline{x}, y \in C_f$

$$\tau : C_f(\underline{x}, y) \longrightarrow X(\underline{\tau x}, \tau y)$$

is surjective on objects. This is a consequence of the fact that composites of universals are universal in an opetopic n-category.

In the next section we construct the composition span itself.

6.2 Construction of composition span

In this section we give the construction of a composition span; in the next section we give some explicit examples at low dimensions.

Composition of k-cells is given by universal $(k+1)$-cells, so in order to construct a composition span for a k-cell f, we must assume that for all $m > k$ the universal m-cells have been defined.

We seek to construct a span of opetopic sets

$$\begin{tikzcd} & C_f \arrow[dl, "\sigma_f"'] \arrow[dr, "\tau_f"] & \\ X(b,c) & & X(\underline{a},c). \end{tikzcd}$$

For convenience we write $C_f = C$, $\sigma_f = \sigma$ and $\tau_f = \tau$. Also put $X(b,c) = X_1$ and $X(\underline{a},c) = X_2$. Recall that a morphism $F : A \longrightarrow B$ of opetopic sets has for each $j \geq 0$ a function

$$F_j : A(j) \longrightarrow B(j)$$

such that, for each $j \geq 1$ a certain square

$$\begin{tikzcd} A(j) \arrow[r, "F_j"] \arrow[d] & B(j) \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

commutes, ensuring that "underlying shapes are preserved". So we seek for each $j \geq 0$ functions

$$
\begin{tikzcd}
& C(j) \arrow[dl, "\sigma_j"'] \arrow[dr, "\tau_j"] & \\
X_1(j) & & X_2(j)
\end{tikzcd}
$$

such that for each $j \geq 1$ a certain diagram

$$
\begin{tikzcd}
& C(j) \arrow[dl] \arrow[dr] \arrow[d, "p_j"] & \\
X_1(j) \arrow[dd] & \cdot \arrow[dl] \arrow[dr] & X_2(j) \arrow[dd] \\
& & \\
\cdot & & \cdot
\end{tikzcd}
$$

commutes. Then a j-cell $\theta \in C(j)$ exhibits $\tau_j(\theta) \in X_2(j)$ as a composite of f with $\sigma_j(\theta) \in X_1(j)$. p_j gives the frame of each j-cell in C.

- $j = 0$

Put

$$C(0) = \{u \in \mathcal{U}(k+1) \mid s_2(u) = f\}$$

where $\mathcal{U}(m)$ is the set of 2-ary universal m-cells. Put $\sigma_0 = s_1$ and $\tau_0 = t$.

- $j = 1$

A 1-frame in C has the form $u_1 \longrightarrow u_2$. We form the set of occupants of this frame as follows. Write

$$
\begin{aligned}
\mathcal{U}_1 &= \{u \in \mathcal{U}(k+2) \mid s_1(u) = u_2\} \\
\mathcal{U}_2 &= \{u \in \mathcal{U}(k+2) \mid s_2(u) = u_1\}
\end{aligned}
$$

and form the pullback

$$
\begin{tikzcd}
& & \cdot \arrow[dl] \arrow[dr] & & \\
& \mathcal{U}_1 \arrow[dl, "s_2"'] \arrow[dr, "t"] & & \mathcal{U}_2 \arrow[dl, "t"'] \arrow[dr, "s_1"] & \\
X_1(1) & & \cdot & & X_2(1)
\end{tikzcd}
\quad .
$$

- $j > 1$

For higher values of j we construct for each j a pullback over 2^j subsets of $\mathcal{U}(k+j+1)$ as follows.

Let θ be a $(j-1)$-frame in C with target α. α is a $(j-1)$-cell of C so is a string of 2^{j-1} universal $(k+j)$-cells $u_1, \ldots, u_{j-1}$, say. Now write $\mathcal{U} = \mathcal{U}(k+j+1)$ and for each $1 \le i \le 2^{j-1}$

$$\mathcal{U}_i = \{u \in \mathcal{U}(k+j+1) \mid s_1(u) = u_i\} \ .$$

For the set of occupants of the frame θ we form a pullback over 2^j sets as follows:

$$\begin{tikzcd}[column sep=tiny]
& \mathcal{U}_1 \arrow[dl, "s_2"'] \arrow[dr, "t"] & & \mathcal{U} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & \mathcal{U} \arrow[dl, "s_1"'] \arrow[dr, "t"] & & \mathcal{U}_2 \arrow[dl, "t"'] \arrow[dr, "s_2"] & & \mathcal{U}_3 \arrow[dl, "s_2"'] \arrow[dr, "t"] & & \mathcal{U} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & \mathcal{U} \arrow[dl, "s_1"'] \arrow[dr, "t"] & & \mathcal{U}_4 \arrow[dl, "t"'] \arrow[dr, "s_2"] & & \mathcal{U}_5 \arrow[dl, "s_2"'] \arrow[dr, "t"] & & \mathcal{U} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & \cdots & & \mathcal{U}_{2^{j-1}} \arrow[dl, "t"'] \arrow[dr, "s_2"] & \\
{} & & {} & & {} & & {} & & {} & & {} & & {} & & {} & & {} & & {} & & {} & & {} & & {}
\end{tikzcd} .$$

This completes the definition of C_f.

6.3 Some examples at low dimensions

In this section we give some examples of elements of the composition span $C = C_f$ for a 1-cell f.

- $j = 0$

$C(0)$ is the set of universal 2-cells of the form

$$\begin{tikzcd}
& \cdot \arrow[dr, "g"] & \\
\cdot \arrow[ur, "f"] \arrow[rr, "\bar{g}"'] & \arrow[u, phantom, "\Downarrow"] & \cdot
\end{tikzcd}$$

exhibiting $\bar{g}$ as a composite of f and g.

- $j = 1$

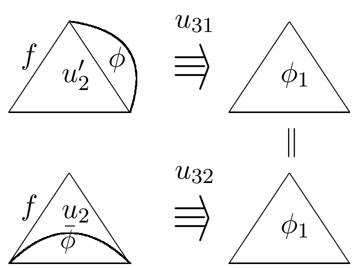
We form a pullback over

$$\begin{tikzcd}
& \mathcal{U}_1 \arrow[dl, "s_2"'] \arrow[dr, "t"] & & \mathcal{U}_2 \arrow[dl, "t"'] \arrow[dr, "s_1"] & \\
X_1(1) & & \cdot & & X_2(1)
\end{tikzcd} .$$

So a typical element is of the form (u_{31}, u_{32}) with projections as shown below

$$\begin{tikzcd}
& u_{31} \arrow[dl, "s_2"'] \arrow[dr, "t"] & & u_{32} \arrow[dl, "t"'] \arrow[dr, "s_1"] & \\
\phi & & \phi_1 & & \bar{\phi}
\end{tikzcd}.$$

For example, the following two universal 3-cells exhibit $\bar{\phi}$ as a composite of f with ϕ; this element of $C(2)$ is in the frame $u_2 \longrightarrow u_2'$.



- $j = 2$

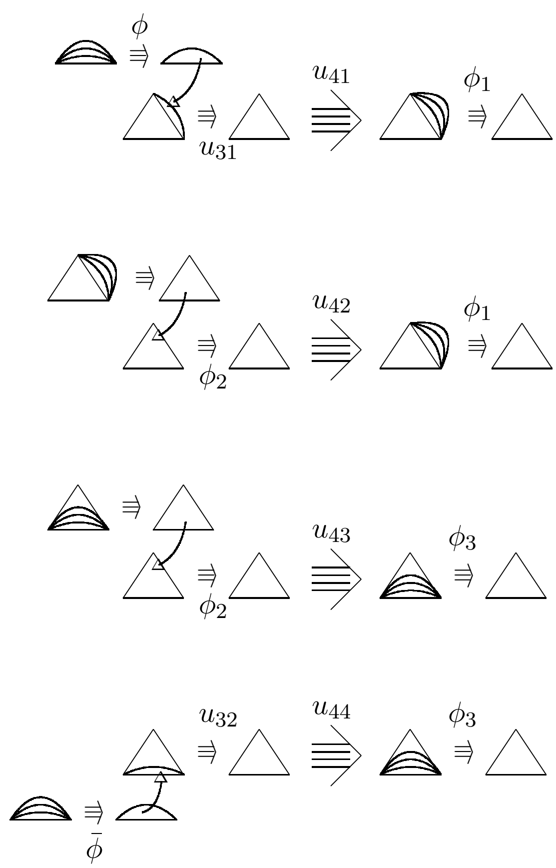
We form a pullback over

$$\begin{tikzcd}
& \mathcal{U}_1 \arrow[dl, "s_2"'] \arrow[dr, "t"] & & \mathcal{U} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & \mathcal{U} \arrow[dl, "s_1"'] \arrow[dr, "t"] & & \mathcal{U}_2 \arrow[dl, "t"'] \arrow[dr, "s_2"] & \\
X_1(2) & & \cdot & & \cdot & & \cdot & & X_2(2)
\end{tikzcd}.$$

A typical element is of the form $(u_{41}, u_{42}, u_{43}, u_{44})$ with projections as shown below

$$
\begin{tikzcd}[column sep=small]
 & u_{41} \arrow[dl, "s_2"'] \arrow[dr, "t"] & & u_{42} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & u_{43} \arrow[dl, "s_1"'] \arrow[dr, "t"] & & u_{44} \arrow[dl, "t"'] \arrow[dr, "s_2"] & \\
\phi & & \phi_1 & & \phi_2 & & \phi_3 & & \bar{\phi}
\end{tikzcd}
$$

exhibiting $\bar{\phi}$ as a composite of f with ϕ. For example, the following element of $C(2)$ is in a frame with target (u_{31}, u_{32}).

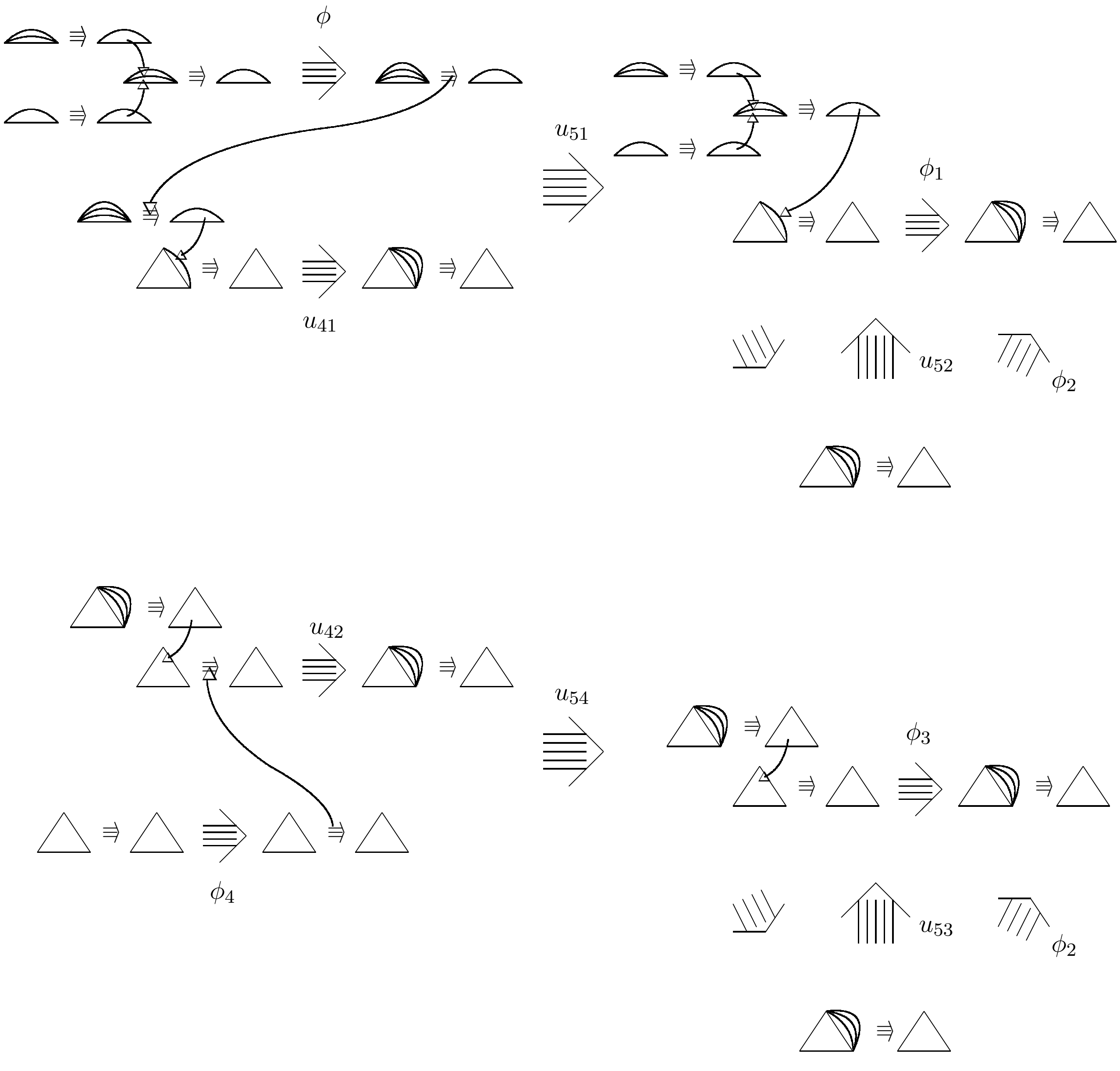


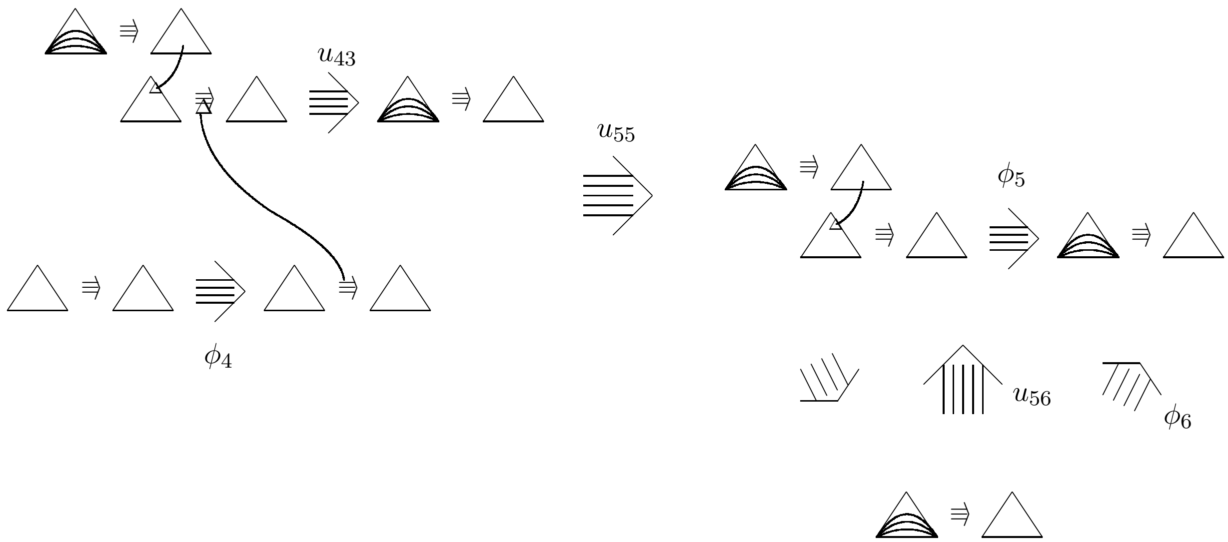
- $j = 3$

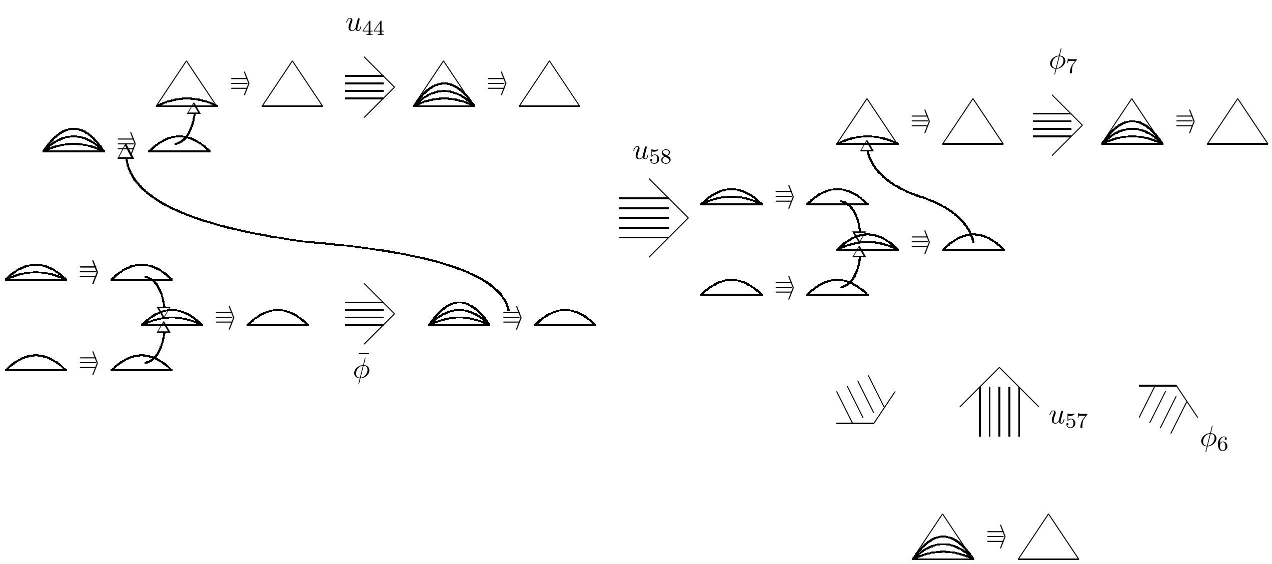
Similarly, in $C(3)$ we have a typical element $(u_{51}, \ldots, u_{58})$

$$
\begin{tikzcd}[column sep=tiny]
 & u_{51} \arrow[dl, "s_2"'] \arrow[dr, "t"] & & u_{52} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & u_{53} \arrow[dl, "s_1"'] \arrow[dr, "t"] & & u_{54} \arrow[dl, "t"'] \arrow[dr, "s_2"] & & u_{55} \arrow[dl, "s_2"'] \arrow[dr, "t"] & & u_{56} \arrow[dl, "t"'] \arrow[dr, "s_1"] & & u_{57} \arrow[dl, "s_1"'] \arrow[dr, "t"] & & u_{58} \arrow[dl, "t"'] \arrow[dr, "s_2"] & \\
\phi & & \phi_1 & & \phi_2 & & \phi_3 & & \phi_4 & & \phi_5 & & \phi_6 & & \phi_7 & & \bar{\phi}
\end{tikzcd}
.
$$

For example the following element of $C(3)$ (running over two pages) has target $(u_{41}, u_{42}, u_{43}, u_{44})$:




u_{43}
u_{55}
ϕ_5
ϕ_4
u_{56}
ϕ_6


u_{44}
ϕ_7
u_{58}
$\bar{\phi}$
u_{57}
ϕ_6

6.4 Conclusions

We conclude that the outline of the basic syntax of opetopes seems secure, notwithstanding the more abstract notions of symmetric multicategory that still require further work. However, universality is less well understood, and we remain unsure of the ideal from in which it should be defined. The alternative approach described in this chapter seems right in 'spirit', but in the end the mathematics that emerges is not as 'slick' as might be hoped. It is therefore not yet clear what this alternative approach has achieved, but rather, there is much scope for further work in this area.

Appendix A

Proof of Proposition 2.2.4

We now give the proof of Proposition 2.2.4 deferred from Section 2.2.2.

Proposition 2.2.4. Let Q be a tidy symmetric multicategory. Then

$$\zeta(Q)' \cong \zeta(Q^+)$$

that is

$$(\mathcal{E}_Q{}', T_Q{}') \cong (\mathcal{E}_{Q^+}, T_{Q^+})$$

in the category **CartMonad**.

Proof. First we show that $\mathcal{E}_Q{}' \cong \mathcal{E}_{Q^+}$. Now $\mathcal{E}_{Q^+} = \mathbf{Set}/S_{Q^+}$ where $S_{Q^+} \cong o(Q^+) = \mathrm{elt}\, Q$, and $\mathcal{E}_Q{}' = \mathbf{Set}/S_Q{}'$ where

$$\begin{pmatrix} S_Q{}' \\ \downarrow \\ S_Q \end{pmatrix} = T_Q \begin{pmatrix} S_Q \\ \downarrow 1 \\ S_Q \end{pmatrix}.$$

So $S_Q{}'$ is equivalent to the pullback

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \mathcal{F}S_Q{}^{\mathrm{op}} \arrow[d, "1"] \\ \mathrm{elt}\, Q \arrow[r] & \mathcal{F}S_Q{}^{\mathrm{op}} \end{tikzcd}$$

so $S_Q{}' \simeq \mathrm{elt}\, Q$, giving $S_Q{}' \cong S_{Q^+}$. So we have $\mathcal{E}_Q{}' \cong \mathcal{E}_{Q^+}$. By abuse of notation, we write elements of both these categories as sets over S', since confusion is unlikely.

Consider $(A, f) = (A \xrightarrow{f} S') \in \mathcal{E}_Q{}' \cong \mathcal{E}_{Q^+}$. Write $T_Q{}'(A, f) = (A_1, f_1)$ and $T_{Q^+}(A, f) = (A_2, f_2)$. We show $(A_1, f_1) \cong (A_2, f_2)$. To construct A_2, first form the pullback

$$\begin{tikzcd}
\cdot \arrow[rr] \arrow[d] & & \mathcal{F}A^{\mathrm{op}} \arrow[d, "\mathcal{F}f^{\mathrm{op}}"] \\
\mathrm{elt}\,Q^{+} \arrow[r, "s"'] & \mathcal{F}(\mathrm{elt}\,Q)^{\mathrm{op}} \arrow[r, "\sim"'] & \mathcal{F}S'^{\mathrm{op}}
\end{tikzcd}.$$

Then $A_2 \simeq \mathrm{elt}\,Q^{+} \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}$, and f_2 is given by the composite

$$A_2 \simeq \mathrm{elt}\,Q^{+} \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}} \longrightarrow \mathrm{elt}\,Q^{+} \xrightarrow{t_{Q^{+}}} \mathrm{elt}\,Q \xrightarrow{\sim} S'$$

where $t_{Q^{+}}$ is the target map of Q^{+}.

Informally, since we are here considering $S' \simeq o(Q^{+}) = \mathrm{elt}\,(Q)$, the object $(A \xrightarrow{f} S')$ may be thought of as a set of labels for arrows of Q. Then A_2 is the set of all possible source-labelled arrows of Q^{+}. Since an arrow of Q^{+} is given by a tree with nodes corresponding to arrows of Q, an element of A_2 may be thought of as such a tree, with nodes labelled by compatible elements of A. Alternatively, it may be thought of as a configuration for composing labelled arrows of Q via object-isomorphisms, where composition is according to the underlying arrows only. f_2 acts by composing the underlying arrows of Q and then taking isomorphism classes.

We now turn our attention to the action of $T_Q{}'$. (For full details of the free multicategory construction we refer the reader to [Lei3].) For convenience we write $T_Q = T$ and $S_Q = S$, so we need to form

$$(T, \mathbf{Set}/S)' = (T', S').$$

To construct A_1, we form the free multicategory on the following graph:

$$\begin{tikzcd}
 & \begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \arrow[dl, "f"'] \arrow[dr, "!"] & \\
T\begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix}
\end{tikzcd}.$$

Recall we have

$$T\begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} S' \\ \downarrow \\ S \end{pmatrix}$$

and the map $A \longrightarrow S$ is the composite $A \xrightarrow{f} S' \longrightarrow S$. The graph underlying the free operad is then

$$\begin{tikzcd} & \begin{pmatrix} A_1 \\ \downarrow \\ S \end{pmatrix} \arrow[dl, "f'"'] \arrow[dr] & \\ T \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} \end{tikzcd}.$$

The construction gives a sequence of graphs

$$\begin{tikzcd} & \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \arrow[dl, "d_k"'] \arrow[dr, "c_k"] & \\ T \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} & & \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} \end{tikzcd}$$

where $C^{(0)} = S$, $d_0 = \eta_T$ and

$$\begin{pmatrix} C^{(k+1)} \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} S \\ \downarrow 1 \\ S \end{pmatrix} + \begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \circ \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix}.$$

Here $\circ$ is composition in the bicategory of spans, so the composite

$$\begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \circ \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix}$$

is given by the pullback

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \arrow[d, "f"] & \\ T \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \arrow[r, "Tc_k"'] & T \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} S' \\ \downarrow \\ S \end{pmatrix} \end{tikzcd}$$

and d_{k+1} is given by the composite

$$\begin{pmatrix} A \\ \downarrow \\ S \end{pmatrix} \circ \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \longrightarrow T \begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} \xrightarrow{Td_k} TT \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix} \xrightarrow{\mu_T} T \begin{pmatrix} S \\ \downarrow \\ S \end{pmatrix}.$$

This construction gives a nested sequence $(C^{(k)}, f^{(k)}) \in \mathbf{Set}/S$ with $(C^{(0)}, f^{(0)}) = (S, 1)$ and

$$C^{(k+1)} = S \amalg T(C^{(k)}) \times_{S'} A$$

where (by further abuse of notation) we write

$$T\begin{pmatrix} C^{(k)} \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} T(C^{(k)}) \\ \downarrow \\ S \end{pmatrix}.$$

$f^{(k+1)}$ is given by $1 \amalg (T(C^{(k)}) \times_{S'} A \xrightarrow{d_{k+1}} S' \longrightarrow S)$ and $\begin{pmatrix} A_1 \\ \downarrow \\ S \end{pmatrix}$ is then the colimit of this nested sequence.

Informally, the sets $C^{(k)}$ may be thought of as k-fold formal composites (or composites of 'depth' at most k). The formula for $C^{(k)}$ says that a composite is either null or is a generating arrow composed with other composites. We aim to show that these formal composites correspond to the formal composites given by the source-labelled arrows of Q^+.

We show that $A_1 \cong A_2 \simeq \mathrm{elt}\, Q^+ \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}$ as follows. For each k we exhibit an embedding

$$g_k : C^{(k)} \hookrightarrow A_2$$

which makes the following diagram commute

$$\begin{tikzcd} C^{(k)} \arrow[rr, "g_k"] \arrow[dr, "d_k"'] & & A_2 \arrow[dl, "f_2"] \\ & S' & \end{tikzcd}.$$

Then the colimit induces the map required.

We proceed by induction. Define $g_0 : S \longrightarrow \mathrm{elt}\, Q^+ \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}$ as follows. Let $[x] \in S$ denote the isomorphism class of $x \in o(Q)$. Given any $[x] \in S$, we have a nullary arrow $\alpha_x \in Q^+(\cdot\ ; 1_x)$. Recall that an arrow of Q^+ may be regarded as a tree with nodes corresponding to the source elements (which are themselves arrows of Q) and edges labelled by object-morphisms of Q. Then $\alpha_x \in Q^+(\cdot\ ; 1_x)$ is given by a tree with no nodes, that is, a single edge labelled by 1_x as shown below.

$$\big\downarrow 1_x$$

The source of α is empty, so we can define g_0 by

$$g_0([x]) = [(\alpha_x, \cdot)]$$

where $(\alpha_x, \cdot) \in \mathrm{elt}\, Q^+ \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}$, and observe immediately that

$$x \cong x' \in o(Q) \iff 1_x \cong 1_{x'} \in \mathrm{elt}\, Q.$$

Furthermore we have

$$d_0[x] = \mu_T[x] = [1_x] = f_2 g_0[x]$$

as required.

For the induction step, suppose we have constructed g_k satisfying the commuting condition; we seek to construct

$$g_{k+1} : C^{(k+1)} \hookrightarrow A_2$$

satisfying the condition. Consider

$$y \in C^{(k+1)} = S \amalg T(C^{(k)}) \times_{S'} A.$$

If $y \in S$ then put $g_{k+1}(y) = g_0(y)$. Otherwise, we have

$$y = (\alpha, a) \in T(C^{(k)}) \times_{S'} A.$$

Here the map $T(C^{(k)}) \longrightarrow S'$ is given by $Tf^{(k)}$. Recall that by definition of T, $T(C^{(k)})$ is equivalent to the pullback

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \mathcal{F}(C^{(k)})^{\mathrm{op}} \arrow[d, "\mathcal{F}(f^{(k)})^{\mathrm{op}}"] \\
\mathrm{elt}\, Q \arrow[r] & \mathcal{F}S^{\mathrm{op}}
\end{tikzcd}$$

So, an element of $T(C^{(k)})$ is an isomorphism class of arrows of Q source-labelled by compatible elements of $C^{(k)}$. We write the pullback as $\mathbb{C}^{(k)}$. Then $Tf^{(k)}$ is the map given by the composite

$$T(C^{(k)}) \xrightarrow{\sim} \mathbb{C}^{(k)} \longrightarrow \mathrm{elt}\, Q \xrightarrow{\sim} S'.$$

Informally, $Tf^{(k)}$ removes the labels, leaving only the (isomorphism class of the) underlying arrow of Q.

Now we in fact exhibit a full and faithful functor

$$\mathbb{C}^{(k)} \times_{S'} A \longrightarrow \mathrm{elt}\, Q^+ \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}.$$

Let $((\beta, \underline{b}), a) \in \mathbb{C}^{(k)} \times_{S'} A$. So $\beta \in \mathrm{elt}\, Q$, $\underline{b} = b_1, \ldots, b_n \in \mathcal{F}(C^{(k)})^{\mathrm{op}}$ and $a \in A$ such that $[s_Q(\beta)] = (f^{(k)}(b_1), \ldots, f^{(k)}(b_n))$ and $f(a) = [\beta]$.

Informally, we have an arrow β of Q, source-labelled by the $b_i \in C^{(k)}$, and a compatible label $a \in A$. We seek a formal composite of labelled arrows, of depth up to $k+1$. By induction, we already have for each element of $C^{(k)}$ a formal composite of labelled arrows, of depth up to k. So we aim to form a formal composite of these together with β labelled by a.

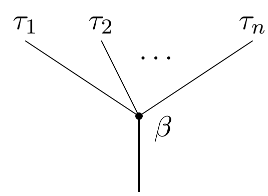
By induction we have for each $1 \leq i \leq n$

$$g_k(b_i) = (\pi_i, p_i) \in \mathrm{elt}\, Q^+ \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}.$$

The commuting condition implies that for each i

$$[s_Q(\beta)_i] = [t_Q t_{Q^+}(\pi_i)].$$

This gives us a way of constructing a new element of $\mathrm{elt}\, Q^+$ from the data given, since each π_i can be composed with β at the ith place, via the appropriate object-isomorphism. That is, we form a tree by induction, as shown in the following diagram



where τ_i is the tree for π_i. Each π_i has its nodes (that is, source elements) labelled by elements of A; to complete the definition it remains only to 'label' the node corresponding to β. But we have $f(a) = [\beta]$, that is, a is a compatible label for β. So we let a be the label for β.

So we have defined a full and faithful functor

$$\mathbb{C}^{(k)} \times_{S'} A \longrightarrow \mathrm{elt}\, Q^+ \times_{\mathcal{F}S'^{\mathrm{op}}} \mathcal{F}A^{\mathrm{op}}$$

inducing, on isomorphism classes, an embedding

$$g_{k+1} : C^{(k)} \hookrightarrow A_2$$

as required. We now check the commuting condition. Informally, d_k acts by ignoring the labels and composing the underlying arrows of Q, as does μ. Since μ is induced from composition in Q, and t_{Q^+} is constructed from composition of a formal composite of arrows of Q, we have $f_2 \circ g_{k+1} = d_{k+1}$ as required.

So we have for each $k \geq 0$ an embedding g_k as required. The g_k then induce a map $A_1 \longrightarrow A_2$. It is straightforward to check that this is surjective; by construction it makes the following diagram commute

$$\begin{tikzcd}
A_1 \arrow[rr] \arrow[dr] & & A_2 \arrow[dl] \\
& S' &
\end{tikzcd}$$

so we have an isomorphism

$$(A_1, f_1) \cong (A_2, f_2)$$

as required.

Finally we check that the naturality condition for a monad opfunctor is satisfied. Given a morphism $(A, f) \longrightarrow (B, g) \in \mathbf{Set}/S'$ it is clear from the constructions that the following diagram commutes in $\mathbf{Set}/S'$

$$\begin{tikzcd}
A_1 \arrow[rr, "\simeq"] \arrow[dd] \arrow[dr] & & A_2 \arrow[dd] \arrow[dl] \\
& S' & \\
B_1 \arrow[rr, "\simeq"] \arrow[ur] & & B_2 \arrow[ul]
\end{tikzcd}$$

and the other axioms for a monad opfunctor are easily checked. So we have

$$(\mathcal{E}_{Q^+}, T_{Q^+}) \cong (\mathcal{E}_Q{}', T_Q{}')$$

as required. $\square$

Appendix B

Opetopes via Kelly-Mac Lane Graphs

In this appendix we show how opetopes can be constructed using Kelly-Mac Lane graphs. This arises from the fact that a tree can be expressed as a Kelly-Mac Lane graph, and thus the slice construction can also be expressed in terms of such graphs.

In [KM], Kelly and Mac Lane introduce a notion of graph to study coherence for symmetric monoidal closed categories. In Section B.2, we study the trees used in the slice construction for symmetric multicategories and show how to express such trees as Kelly-Mac Lane graphs; we will use the formal description of trees as given in Section 3.1.1.

Then in Section B.3 we use this characterisation of trees to restate the definition of opetopes, and prove that this construction does indeed give equivalent categories of k-opetopes to the ones constructed in Chapter 2.

Blute ([Blu]) has established a relationship between Kelly-Mac Lane graphs and the proof nets of Linear Logic, so the material in this appendix should in turn give a relationship between opetopes and proof nets. However, we do not pursue this matter here.

We begin by giving a minimal account of the theory of Kelly-Mac Lane graphs, including no more than what is required for the purposes of this work. We refer the reader to [KM] for the full details.

B.1 Background on Kelly-Mac Lane Graphs

In this section we give a brief account of the theory of Kelly-Mac Lane graphs. In [KM], Kelly and Mac Lane study coherence for symmetric monoidal closed categories. In brief, a symmetric monoidal closed category is a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, a, b, c)$ equipped, in addition, with a functor

$$[\ ,\ ] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$$

and natural transformations

$$d = d_{AB} : A \longrightarrow [B, A \otimes B]$$

$$e = e_{AB} : [A, B] \otimes A \longrightarrow B$$

satisfying certain axioms. (Here a, b and c are the natural isomorphisms for associativity, unit and symmetric action respectively.) In particular we have a natural isomorphism

$$\pi : \mathcal{C}(A \otimes B, C) \longrightarrow \mathcal{C}(A, [B, C]).$$

Kelly and Mac Lane refer to such categories simply as *closed categories* and we do the same.

Kelly and Mac Lane introduce a notion of graph which enables a partial solution to the question: when does a diagram in a closed category commute? In fact we are not concerned with the coherence question here, so we only give the construction of the graphs and state one theorem from [KM] which will later be useful.

Kelly and Mac Lane define a category G whose objects are *shapes* and whose morphisms are *graphs*; this is seen to be a closed category. They then define a subcategory whose morphisms are the *allowable morphisms*. These are defined as precisely those morphisms of G demanded by the symmetric monoidal closed structure.

We do not need to use the notion of 'free symmetric monoidal closed category' although this notion should give a more abstract treatment of the material; the graphs we use should be morphisms in such a category. However, this is somewhat beyond the scope of this thesis.

B.1.1 Shapes

We define *shapes* by the following inductive rules:

1) I is a shape

2) 1 is a shape

3) if S and T are shapes then so is $S \otimes T$

4) if S and T are shapes then so is $[S, T]$

Thus shapes are formal objects built from 1, I, $\otimes$ and $[\ ,\ ]$.

We assign to each shape T a *variable set* $v(T)$ which may be considered as a list of $+$'s and $-$'s, defined inductively as follows:

1) $v(I) = \emptyset$

2) $v(1) = \{+\}$

3) $v(T \otimes S) = v(T) \coprod v(S)$

4) $v([T, S]) = v(T)^{\text{op}} \coprod v(S)$

Here $\coprod$ is the concatenation of lists and $v(T)^{\text{op}}$ is $v(T)$ with all signs reversed. Kelly and Mac Lane write

$$v(T) \coprod v(S) = v(T) \hat{+} v(S)$$

$$v(T)^{\mathrm{op}} \coprod v(S) = v(T) \tilde{+} v(S)$$

and call these the *ordered sum* and *twisted sum* respectively. The sign of each variable is called its *variance.*

In fact we only need the strict monoidal version of this theory. That is, we put

$$(T \otimes S) \otimes R = T \otimes (S \otimes R)$$

and

$$T \otimes I = T.$$

For example,

$$[\ [1,1] \otimes 1 \otimes 1\ ,\ I\ ] \otimes 1$$

is a shape with

$$v(T) = \{+,-,-,-,+\}.$$

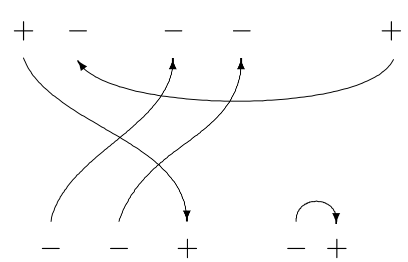
B.1.2 Graphs

A *graph* $T \longrightarrow S$ is defined to be a fixed point free pairing of the variables in T and S such that paired elements have opposite variances in $v(T)^{\mathrm{op}} \coprod v(S)$. (Kelly and Mac Lane refer to such paired elements as "mates".) Equivalently, this is a bijection between the $+$'s and the $-$'s in $v(T)^{\mathrm{op}} \coprod v(S)$.

For example, the following is a graph

$$[\ [1,1] \otimes 1 \otimes 1\ ,\ I\ ]\ \otimes\ 1 \longrightarrow [\ 1 \otimes 1\ ,\ 1 \otimes [1,1]\ ]$$

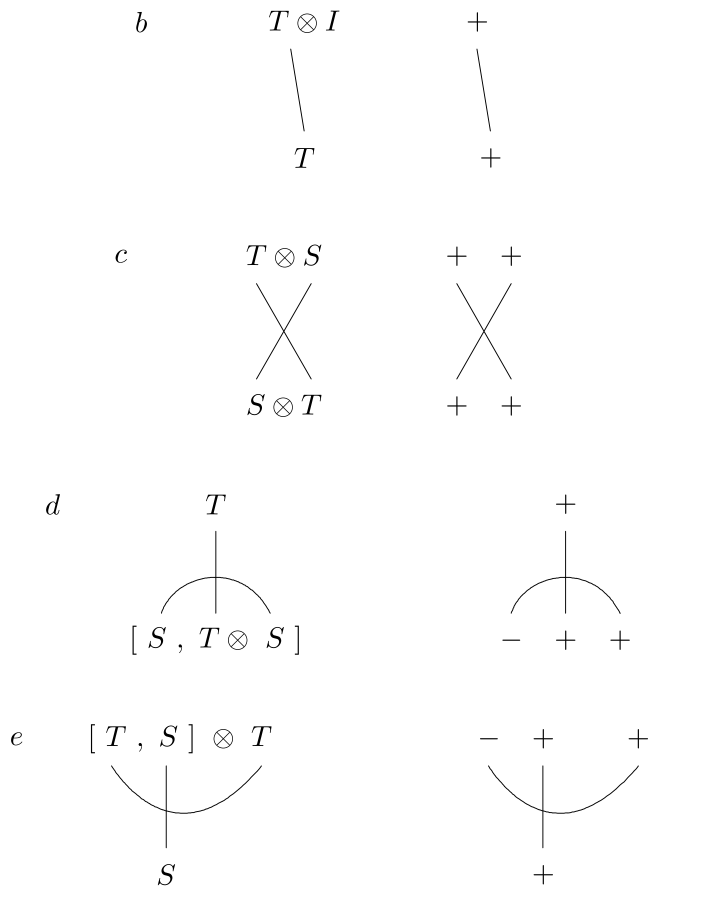
showing variances:



.

Graphs are composed in the obvious way, so that shapes and graphs form a category G. Moreover, G has the structure of a closed category as follows. $\otimes$ and $[\ ,\ ]$ are defined on graphs in the obvious way, and the constraints are given by the following graphs:

a



The diagrams on the right give variances, showing that these are indeed graphs; note that in the twisted sum the variances of the domain are reversed. For the strict monoidal version we have $a = 1$ and $b = 1$.

Observe that we realise Kelly-Mac Lane graphs as pictorial graphs by joining paired objects with an edge. In the diagrams above, the objects are in fact shapes, so the drawn edges in fact represent multiple edges as necessary.

We will later introduce the notion of graphs labelled in a category $\mathbb{C}$ (Section B.3.1); these are the morphisms of a category which we will call $K\mathbb{C}$. We will then see that the graphs above may be considered as graphs labelled in the category $\mathbf{1}$. So for consistency we write $G = K\mathbf{1}$.

B.1.3 Allowable morphisms

The *allowable morphisms* are then defined to be the smallest class of morphisms of $K\mathbf{1}$ satisfying the following conditions:

1) For any T, S, R each of the following morphisms is in the class:

$$\begin{array}{rcl} 1 & : & T \longrightarrow T \\ a & : & (T \otimes S) \otimes R \longrightarrow T \otimes (S \otimes R) \\ a^{-1} & : & T \otimes (S \otimes R) \longrightarrow (T \otimes S) \otimes R \\ b & : & T \otimes I \longrightarrow T \\ b^{-1} & : & T \longrightarrow T \otimes I \\ c & : & T \otimes S \longrightarrow S \otimes T. \end{array}$$

2) For any T, S each of the following morphisms is in the class:

$$\begin{array}{rcl} d & : & T \longrightarrow [S, T \otimes S] \\ e & : & [T, S] \otimes T \longrightarrow S. \end{array}$$

3) If $f : T \longrightarrow T'$ and $g : S \longrightarrow S'$ are in the class so is

$$f \otimes g : T \otimes S \longrightarrow T' \otimes S'.$$

4) If $f : T \longrightarrow T'$ and $g : S \longrightarrow S'$ are in the class then so is

$$[f, g] : [T', S] \longrightarrow [T, S'].$$

5) If $f : T \longrightarrow S$ and $g : S \longrightarrow R$ are in the class then so is $gf : T \longrightarrow R$.

We write $A\mathbf{1}$ for the category of shapes and allowable morphisms.
The main theorem of [KM] that we use is as follows:

Theorem B.1.1. *If $f : T \longrightarrow S$ and $g : S \longrightarrow R \ \in G$ are allowable then they are compatible, that is, composing them gives no closed loops.*

For the proof, see [KM].

B.1.4 Duality

Since $K\mathbf{1}$ is closed, given any graph

$$\xi : S \otimes T \longrightarrow U \ \in K\mathbf{1}$$

there is a unique dual

$$\bar{\xi} : S \longrightarrow [T, U]$$

so in particular, given a graph

$$\alpha : S \longrightarrow T$$

there is a unique dual

$$\bar{\alpha} : I \longrightarrow [S, T].$$

We will eventually be concerned with graphs of the form

$$\alpha : A_1 \otimes \cdots \otimes A_k \longrightarrow B;$$

it is sometimes convenient or indeed necessary to use the dual

$$\bar{\alpha} : I \longrightarrow [\ A_1 \otimes \cdots \otimes A_k\ ,\ B\ ]$$

and we may refer to either of these graphs as α when the exact form is not relevant.

B.2 Trees

In this section we show how a tree may be expressed as an allowable graph, that is, as a morphism in the closed category $A\mathbf{1}$.

The trees in question are those used in the slice construction as defined in Section 2.1.1. We are thus able to restate the slice construction using the language of closed categories, which then enables us to give another construction of opetopes.

We begin by recalling the trees in question, and the more formal definition of such trees as given in Section 3.1.1. This formalisation enables us to express such trees as graphs in $K\mathbf{1}$ of a certain shape, not *a priori* allowable. We then show that in fact any graph arising in this way is allowable, and that, conversely, all such allowable graphs arise in this way.

B.2.1 Background on trees

We will first consider unlabelled, 'combed' trees, with ordered nodes, as in Section 3.1

That is, a tree $T = (T, \rho, \tau)$ consists of

i) A planar tree T

ii) A permutation $\rho \in \mathbf{S}_l$ where l = number of leaves of T

iii) A bijection $\tau : \{\text{nodes of } T\} \longrightarrow \{1, 2, \ldots, k\}$ where k = number of nodes of T; equivalently an ordering on the nodes of T.

Suppose we have nodes $N_1, \ldots, N_k$, where N_i has inputs $\{x_{i1}, \ldots, x_{im_i}\}$ and output x_i. Also, let N be a node with inputs $\{z_1, \ldots, z_l\}$ and output z, with $l = (\sum_{i=1}^{k} m_i) - k + 1$. A tree with nodes N_i is given by a bijection

$$\alpha \ : \ \coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}$$

such that no closed loop arises; a closed loop arises precisely when there is a non-empty sequence of indices

$$\{t_1, \ldots, t_n\} \subseteq \{1, \ldots, k\}$$

such that for each $2 \leq j \leq n$

$$\alpha(x_{t_j b_j}) = x_{t_{j-1}}$$

for some $1 \leq b_j \leq m_j$, and

$$\alpha(x_{t_1 b_1}) = x_{t_n}$$

for some $1 \leq b_1 \leq m_1$.

B.2.2 Trees as morphisms in $K\mathbf{1}$

We now show how trees may be expressed as graphs. Here we consider unlabelled trees; the labelled version follows easily.

Let $\mathbf{1}$ be the category with just one object and one (identity) morphism. We write the single object of $\mathbf{1}$ as 1. Then we express a node of a tree as the following object in $K\mathbf{1}$

$$X_m = [1 \otimes \ldots \otimes 1, 1] = [1^{\otimes m}, 1]$$

where m is the number of inputs of the node.

Now consider a tree T with (ordered) nodes $N_1, \ldots N_k$ where N_i has m_i inputs. We show that this tree may be represented as a morphism

$$X_{m_1} \otimes \ldots \otimes X_{m_k} \xrightarrow{\xi_T} X_l \;\in\; K\mathbf{1}$$

using the formal description of trees as in Section 3.1.1.

Lemma B.2.1. *Let T be a tree with $N_1, \ldots, N_k$ be nodes where N_i has inputs $\{x_{i1}, \ldots, x_{im_i}\}$ and output x_i. Then T is given by a morphism*

$$\xi_T : X_{m_1} \otimes \ldots \otimes X_{m_k} \longrightarrow X_l \;\in\; K\mathbf{1}$$

where $l = (\sum_{i=1}^{k} m_i) - k + 1$. Note that if $k = 0$ then the left hand side of the above expression becomes I.

Proof. Recall that a graph ξ_T as above is precisely a bijection from the $-$'s to the $+$'s in the twisted sum

$$v(X_{m_1} \otimes \ldots \otimes X_{m_k}) \tilde{+} v(X_l).$$

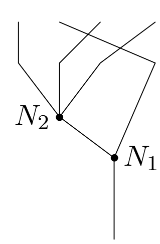
By Lemma 3.1.1, T is given by a bijection

$$\coprod_i \{x_{i1}, \ldots, x_{im_i}\} \coprod \{z\} \longrightarrow \coprod_i \{x_i\} \coprod \{z_1, \ldots, z_l\}.$$

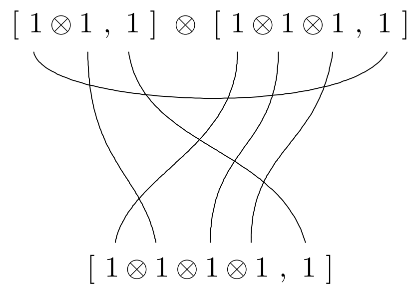
Observe that the elements of the left hand side of this expression are precisely the $-$'s in the twisted sum above, and those of the right hand side are precisely the $+$'s. $\square$

As in Section 3.1.1, the idea is that a tree is constructed by identifying each node output with the node input to which it is joined, unless it is the root; similarly each input is identified with a node output unless it is a leaf. This identification gives the mates in the graph ξ_T, where the codomain X_l is representing the leaves and the root of the tree T.

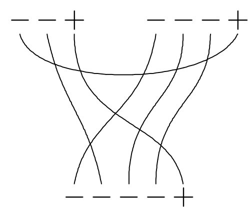
For example the following tree as described in Section 3.1.1



is be expressed as the following morphism in $K\mathbf{1}$



and the following representation giving variances shows that this is indeed a graph:



Formally, the graph for a tree T as above is given as follows. We write

$$X_{m_i} = [\ A_{i1} \otimes \ldots \otimes A_{im_i}\ ,\ A_i\ ]$$

$$X_l = [\ B_1, \otimes \ldots \otimes B_l,\ ,\ B\ ]$$

where each $A_{ij}, A_i, B_i, B = 1$ and in the twisted sum we have variances

$$\begin{array}{rl} v(A_{ij}) = +, & v(A_i) = - \\ v(B_p) = -, & v(B) = +. \end{array}$$

Then the graph ξ_T is given as follows.

- considering node inputs

For each i, j, either

i) the jth input of N_i is joined to the output of N_r, say, in which case A_{ij} is the mate of A_r, or

ii) the jth input of N_i is the pth leaf of the tree T, in which case A_{ij} is the mate of B_p in ξ_T.

- considering node outputs

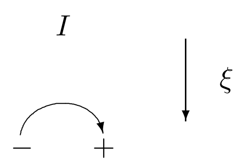
For each r, either

i) the output of N_r is the root of the tree, in which case B_r is the mate of B, or

ii) the output of N_r is joined to the jth input of N_i, say, in which case A_r is the mate of A_{ij}.

Note that the null tree

is a graph $I \xrightarrow{\xi} X_1$ as follows:



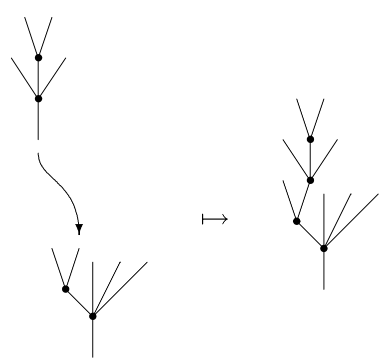
.

So we have shown that every tree is given by a graph in $K\mathbf{1}$; in Section B.2.4 we show that any such graph is allowable. The proof is by induction, and the following section enables us to makes the induction step.

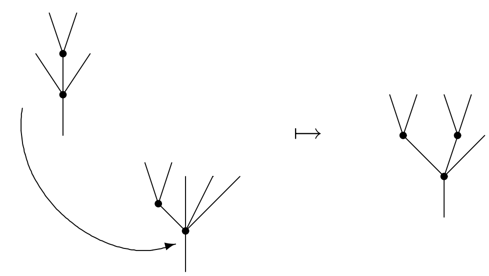
B.2.3 Composition of trees

We now discuss two ways of composing trees:

1) *leaf-root composition* in which a leaf of one tree is attached to the root of another, for example



2) *node-replacement composition* in which a node of one tree is replaced by another tree, for example



In the first case the inputs of the tree are considered to be the leaves, and the output the root; note that an issue of node-ordering arises, so that this 'composition' is not associative. However, it facilitates the induction argument in Section B.2.4, which is why we discuss it here.

In the second case the inputs are the nodes, and the output a node with one input edge for each leaf of the tree. This form of composition is used in Section B.4 in the slice construction.

We show how each of these forms of composition arises for trees represented as graphs as in Section B.2.2

Recall that a tree is expressed as a morphism

$$X_{m_1} \otimes \cdots \otimes X_{m_k} \longrightarrow X_l \ \in K\mathbf{1}.$$

Now in general, given any morphisms in $K\mathbf{1}$

$$B_1 \otimes \cdots \otimes B_n \xrightarrow{f} A_p$$

$$A_1 \otimes \cdots \otimes A_m \xrightarrow{g} A$$

for some $1 \leq p \leq m$, we may form the composite

$$f \circ (1 \otimes \cdots \otimes 1 \otimes g \otimes 1 \otimes \cdots \otimes 1)$$

which we write as

$$g \circ_p f : A_1 \otimes \cdots \otimes A_{j-1} \otimes B_1 \otimes \cdots \otimes B_n \otimes A_{p+1} \otimes \cdots \otimes A_m \longrightarrow A.$$

Note that if p is evident from the context we simply write $g \circ f$.

This composition gives node-replacement composition of trees. Consider trees S, T with graphs

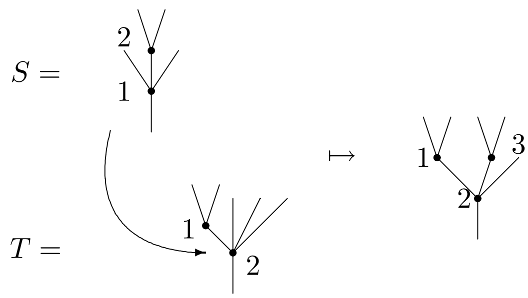
$$\xi_S : X_{s_1} \otimes \cdots \otimes X_{s_n} \longrightarrow X_l$$

$$\xi_T : X_{t_1} \otimes \cdots \otimes X_{t_m} \longrightarrow X_k \ .$$

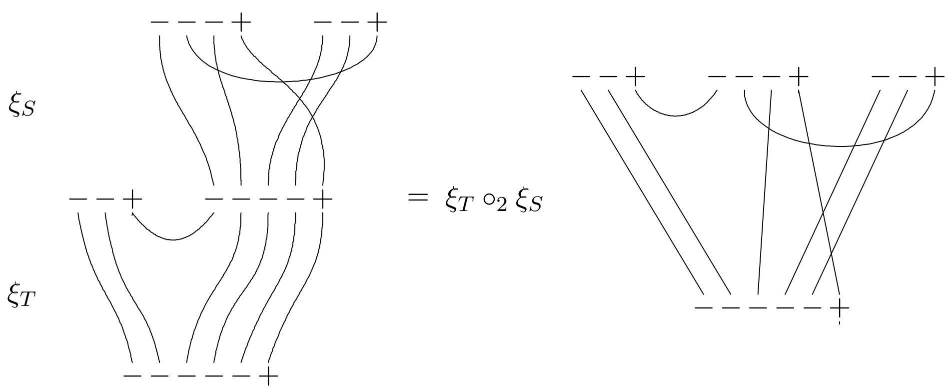
Then S may be composed at the pth node of T if the number of leaves of S equals the number of inputs of the pth node, that is, if $X_l = X_{t_p}$. Then the graph for the composite tree is given by

$$\xi_S \circ_p \xi_T.$$

For example as above, suppose we have $p = 2$ and



then we express this with graphs as follows



In fact, considering the dual forms $\bar{\xi}_S$ and $\bar{\xi}_T$, we see that this composite may also be expressed by means of a 'composition graph' ξ as follows. We have

$$\bar{\xi}_S : I \longrightarrow [X_{s_1} \otimes \cdots \otimes X_{s_n}\ ,\ X_l]$$
$$\bar{\xi}_T : I \longrightarrow [X_{t_1} \otimes \cdots \otimes X_{t_m}\ ,\ X_k].$$

Then ξ is a graph

$$\begin{array}{c}[X_{t_1} \otimes \cdots \otimes X_{t_m}\ ,\ X_k] \otimes [X_{s_1} \otimes \cdots \otimes X_{s_n}\ ,\ X_l] \\ \downarrow \\ [X_{t_1} \otimes \cdots \otimes X_{t_{p-1}} \otimes X_{s_1} \otimes \cdots \otimes X_{s_n} \otimes X_{t_{p+1}} \otimes \cdots \otimes X_{t_m}\ ,\ X_k]\end{array}$$

where X_l is joined to X_{t_p} in the domain, and for all other j, X_j in the domain is joined to X_j in the codomain.

We now consider leaf-root composition. Consider trees S, T as above. We seek to attach the root of S to the qth leaf of T, and we adopt the convention that the nodes of S are then listed before those of T in the final tree.

This is achieved in $K\mathbf{1}$ by placing the graphs ξ_S and ξ_T side by side, that is, taking their tensor product, and composing the result with a 'composition graph' that joins up the correct leaf and root as required. We write

$$X_l = [A_1 \otimes \cdots \otimes A_l, A]$$

$$X_k = [B_1 \otimes \cdots \otimes B_k, B]$$

$$X_{l+k-1} = [C_1 \otimes \cdots \otimes C_{l+k-1}, C]$$

and the 'composition graph' as

$$\xi : X_l \otimes X_k \longrightarrow X_{l+k-1}.$$

The idea is that the leaves of S are inserted into the list of leaves of T at the qth place to give

$$[B_1 \otimes \cdots \otimes B_{q-1} \otimes A_1 \otimes \cdots \otimes A_l \otimes B_{q+1} \otimes \cdots \otimes B_k \ , \ B]$$

so the composition graph ξ is given as follows:

i) the mate of A is B_q

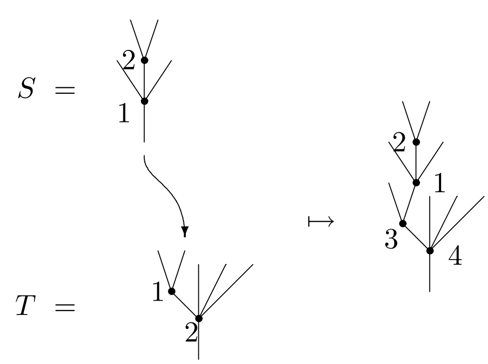
ii) the mate of B is C

iii) for $1 \leq i \leq l$ the mate of A_i is C_{q+i-1}

iv) for $1 \leq i \leq q-1$ the mate of B_i is C_i

v) for $q+1 \leq i \leq k$ the mate of B_i is C_{l+i-1}.

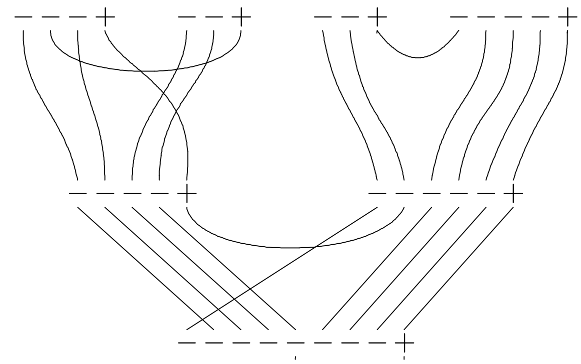
For example, suppose we have $q = 2$ with



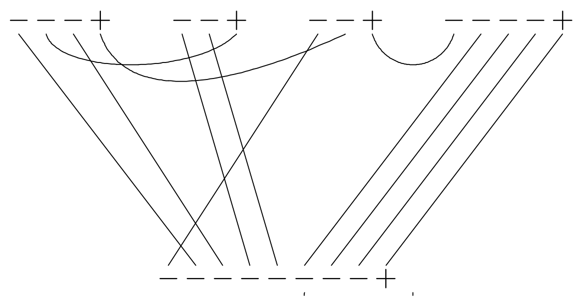
then this is represented by the following graph in $K\mathbf{1}$:

$\xi_S \otimes \xi_T$

ξ



$=$



Note that we could adopt a different convention for ordering the nodes of the composite tree, using $\xi_T \otimes \xi_S$. Of course, neither convention yields an associative composition, but since we are not at this time trying to form a category (or multicategory) of such trees, we do not pursue this matter here.

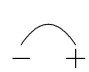
B.2.4 The graph of a tree is allowable

We have shown how any tree is represented by a graph. We now show that any such graph is *allowable.*

Proposition B.2.2. *Given a tree T as above, the graph ξ_T is allowable.*

Proof. By induction on the height of trees. Here the height of a tree is the maximum number of nodes on any path from a leaf to the root. A tree of height 0 is the null tree

represented by the graph

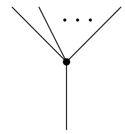


which is the morphism

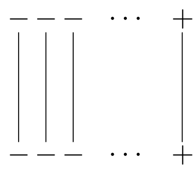
$$I \xrightarrow{d_{I1}} [1, I \otimes 1] = [1, 1]$$

which is allowable.

A tree of height 1 is just a node

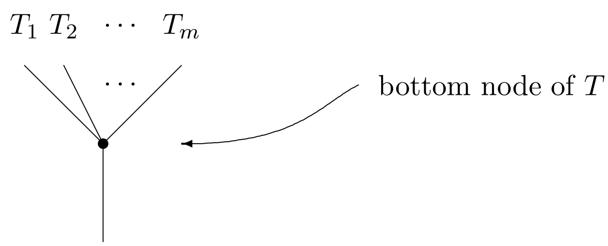


which is represented by an identity graph



which is allowable.

A tree of height $h \geq 1$ may be considered as a composite



where the T_i are subtrees of T; by construction they have height $\leq h$. So by induction each of these is represented by an allowable graph.

It is therefore sufficient to show that leaf-root composition of allowable graphs gives an allowable graph. Note that leaf-root composition as defined in Section B.2.3 will not necessarily give the correct node ordering on the final tree; however, this can be achieved by composing with symmetries as necessary. This will not affect the allowability of the graph since symmetries are allowable graphs, and composites of allowable graphs are allowable.

Furthermore, since tensors and composites of allowable graphs are allowable, it is sufficient to show that all 'composition graphs' ξ as defined in Section B.2.3 are allowable.

Since any permutation may be written as a composite of transpositions, and is therefore allowable, we may assume without loss of generality that $q = 1$ in the composition. So it is sufficient to show that any graph ξ of the following form is allowable. Writing

$$X_{m_1} = [A_1 \otimes \cdots \otimes A_{m_1}, A]$$

$$X_{m_2} = [B_1 \otimes \cdots \otimes B_{m_2}, B]$$

$$X_{m_1+m_2-1} = [C_1 \otimes \cdots \otimes C_{m_1+m_2-1}, C]$$

then

$$\xi : X_{m_1} \otimes X_{m_2} \longrightarrow X_{m_1+m_2-1}$$

is given as follows.

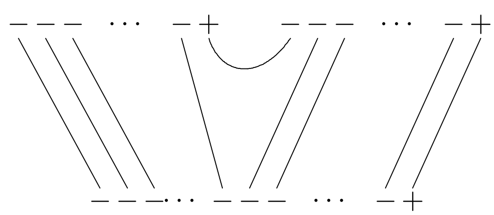
i) the mate of A is B_1

ii) the mate of B is C

iii) for all $1 \leq i \leq m_1$ the mate of A_i is C_i

iv) for $2 \leq i \leq m_2$ the mate of B_i is C_{m_1+i-1}.

So ξ has the form

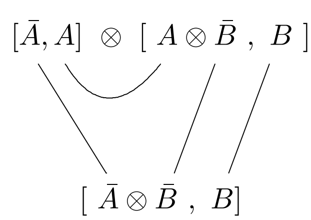


Writing

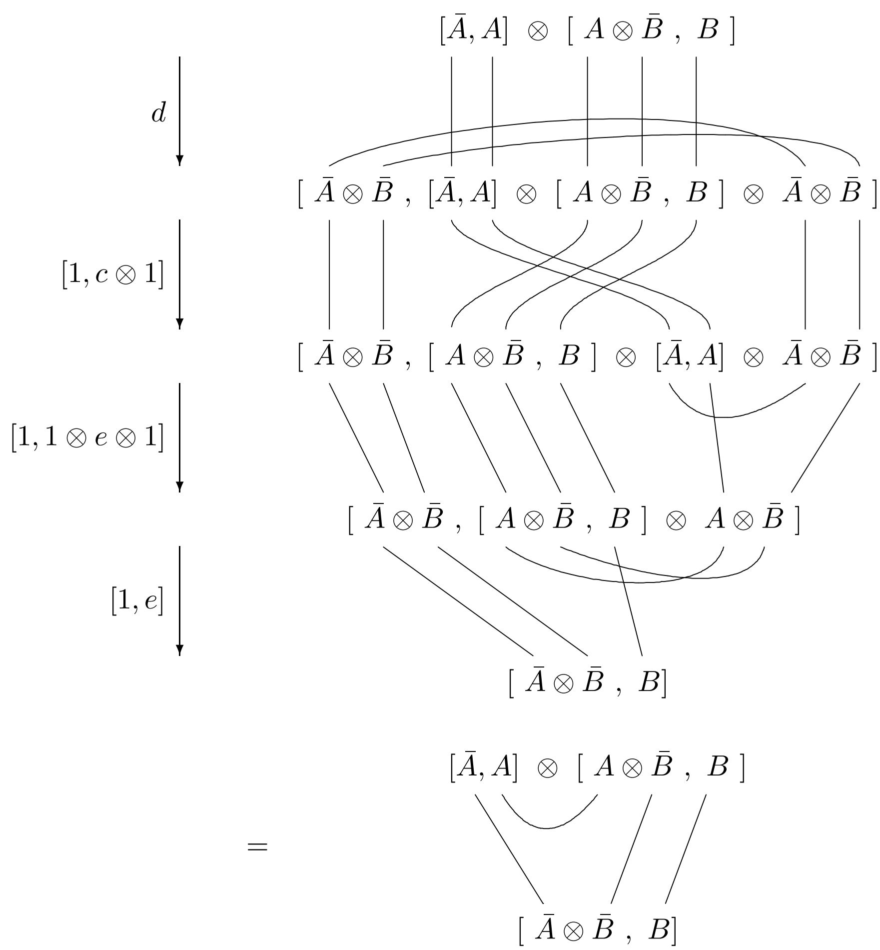
$$A_1 \otimes \cdots \otimes A_{m_1} = \bar{A}$$

$$B_2 \otimes \cdots \otimes B_{m_2} = \bar{B}$$

we may abbreviate this as



which may be written as the following composite of allowable graphs:



so ξ is allowable as required. □

B.2.5 Every allowable graph is a tree

We have seen that every tree is represented by a unique graph, and that this graph is allowable. In this section we prove the converse, that every allowable graph of the correct shape represents a unique tree.

We now use the characterisation of trees as in Section 3.1.1. As in that section, for the converse we see that every morphism

$$X_{m_1} \otimes \cdots \otimes X_{m_k} \longrightarrow X_l \ \in K\mathbf{1}$$

gives a graph but that it is not necessarily a tree; we need to ensure that the resulting graph has no closed loops. We copy Lemmas 3.1.2 and 3.1.3, "translating" them into the language of closed categories. Note that the word 'graph' is used in the ordinary sense; for clarity we refer to Kelly-Mac Lane graphs as 'morphisms in $K\mathbf{1}$'.

Lemma B.2.3. *Let $N_1, \ldots, N_k$ be nodes where N_i has inputs*

$$\{A_{i1}, \ldots, A_{im_i}\}$$

and output x_i. *Let* ξ *be a morphism*

$$\xi : X_{m_1} \otimes \ldots \otimes X_{m_k} \longrightarrow X_l \ \in \ K\mathbf{1}$$

where $l = (\sum_{i=1}^{k} m_i) - k + 1$. *Then* ξ *defines a graph with nodes* $N_1, \ldots, N_k$.

Lemma B.2.4. *Let* ξ *be a graph as above. Then* ξ *has a closed loop if and only if there is a non-empty set of indices*

$$\{t_1, \ldots, t_n\} \subseteq \{1, \ldots, k\}$$

such that for each $2 \leq j \leq n$ *the mate of* $A_{t_{j-1}}$ *under* ξ *is* $A_{t_j b_j}$ *and the mate of* A_{t_n} *is* $A_{t_1 b_1}$ *for some* $1 \leq b_j \leq m_j$.

Proposition B.2.5. *If there is a set of indices* $\{t_1, \ldots t_n\}$ *as above then* ξ *is not allowable.*

Corollary B.2.6. *Let* ξ *be a morphism as above. Then* ξ *is a tree if and only if it is allowable.*

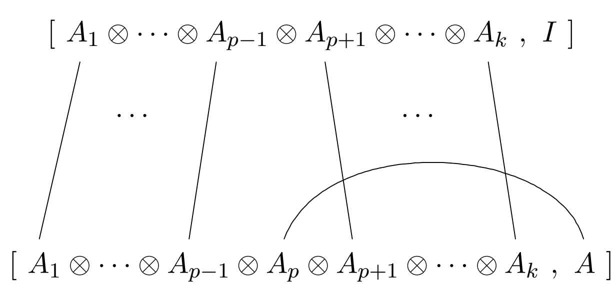
To prove this we will use Theorem B.1.1 (Theorem 2.2 of [KM]) which states that if two composable morphisms are allowable then they are compatible, that is, composing them does not result in any closed loops. So to show that ξ as above is not allowable, we aim to construct an allowable morphism η such that η and ξ are not compatible. The following lemma provides us with such a morphism.

Lemma B.2.7. *Write* $X_k = [A_1 \otimes \cdots \otimes A_k, A]$ *with* $A_i, A = 1$ *and let* $1 \leq p \leq k$.

Then there is an allowable morphism

$$\theta_p : [A_1 \otimes \cdots \otimes A_{p-1} \otimes A_{p+1} \otimes \cdots \otimes A_k, I] \longrightarrow X_k$$

with graph

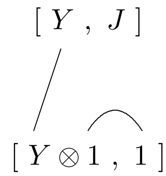


.

Proof. Write $Y = A_1 \otimes \cdots \otimes A_{p-1} \otimes A_{p+1} \otimes \cdots \otimes A_k$. Since symmetries are allowable, it is sufficient to exhibit an allowable morphism

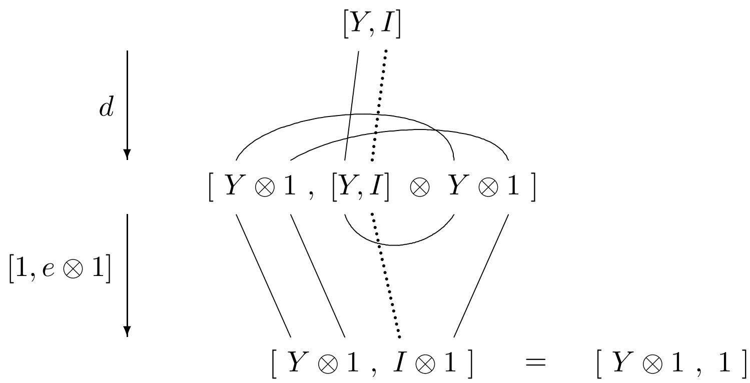
$$[Y, I] \longrightarrow [Y \otimes 1, 1]$$

with underlying graph



.

We have the following composite of allowable morphisms:



which has the underlying graph as required; since composites of allowable morphisms are allowable, the composite is allowable. □

Proof of Proposition B.2.5. To show that

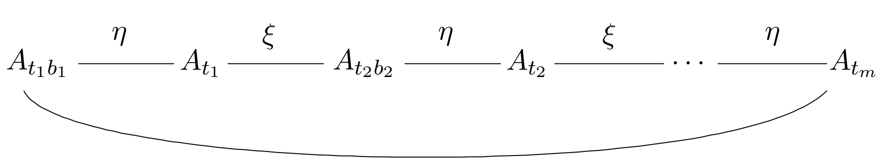
$$\xi : X_{m_1} \otimes \cdots \otimes X_{m_k} \longrightarrow X_l$$

is not allowable we construct an allowable morphism

$$\eta : T \longrightarrow X_{m_1} \otimes \cdots \otimes X_{m_k}$$

such that η and ξ are not compatible, that is, composing them produces a closed loop.

We aim to construct η in such a way that for each $1 \leq j \leq n$ the mate of A_{t_j} is $A_{t_j b_j}$ so that in the composite graph we have the following closed loop:



.

We use morphisms of the form θ_p as given in Lemma B.2.7.

Put $T = Y_1 \otimes \cdots \otimes Y_k$ where

$$Y_i = [A_{t_j 1}, \otimes \cdots \otimes A_{t_j (b_j - 1)} \otimes A_{t_j (b_j + 1)} \otimes \cdots \otimes A_{t_j m_i} \ , \ I]$$

if $i = t_j$ for some $1 \leq j \leq n$, and

$$Y_i = X_{m_i}$$

We define η as a tensor product

$$f_1 \otimes \cdots \otimes f_k \;:\; Y_1 \otimes \cdots \otimes Y_k \;\longrightarrow\; X_{m_1} \otimes \cdots \otimes X_{m_k}$$

where

$$f_i = \begin{cases} \theta_{b_j} & \text{if } i = t_j \text{ for some } 1 \leq j \leq n \\ 1 & \text{otherwise} \end{cases}$$

By Lemma B.2.7 each f_i is allowable, so η is allowable.

Since the mate of A_{t_j} under θ_{b_j} is $A_{t_j b_j}$ we have a closed loop as above, so η and ξ are not compatible. Since η is allowable, it follows from Theorem B.1.1 that ξ is not allowable. □

Finally we sum up the results of this section in the following proposition.

Proposition B.2.8. *A tree is a unique morphism of the form*

$$X_{m_1} \otimes \cdots \otimes X_{m_k} \longrightarrow X_l \in K\mathbf{1}$$

and this morphism is allowable. Conversely, any such allowable morphism represents a unique tree.

Corollary B.2.9. *A tree is a unique allowable morphism of the form*

$$I \longrightarrow [X_{m_1} \otimes \cdots \otimes X_{m_1}, X_l] \in K\mathbf{1}.$$

Conversely, any such allowable morphism represents a unique tree.

Proof. Follows from the closed structure of $K\mathbf{1}$. □

In order to make Proposition B.2.8 and Corollary B.2.9 more precise, we seek an equivalence between a 'category of trees' and a 'category of allowable morphisms'. In fact, trees of this form arise naturally by considering configurations for composing arrows of a symmetric multicategory. That is, they arise from the 'slicing' process as defined in [BD2] and 2.1.1; the trees then appear as arrows of the multicategory I^{2+}, and so as objects of I^{3+}, forming a category $\mathbb{C}_3$.

So we proceed by considering the slice construction using the representation in closed categories. In considering this for constructing trees, we in fact deal with all the machinery used in constructing k-opetopes for all $k \geq 0$, since these are formed by iterating the construction. This is the subject of the next section.

B.3 Opetopes

In this section we use the results of the previous section to construct opetopes. However we first need to introduce the notion of *labelled* Kelly-Mac Lane graphs.

B.3.1 Preliminaries

For the construction of opetopes we require the 'labelled' version of the theory presented in Sections B.1 and B.2: labelled shapes, labelled graphs and labelled trees.

Given a category $\mathbb{C}$ we can form *labelled shapes* (in $\mathbb{C}$), that is, shapes labelled by the objects of $\mathbb{C}$. A labelled shape is a shape T with each 1 'labelled' by an object A_i of $\mathbb{C}$. We write this as

$$|T|(A_1, \ldots, A_k).$$

The variable set is then defined to be the variable set of the underlying shape.

For example given

$$T = [\ [1,1] \otimes 1 \otimes 1\ ,\ I\ ] \otimes 1$$

we have a labelled shape

$$\alpha = |T|(A_1, \ldots, A_5) = [\ [A_1, A_2] \otimes A_3 \otimes A_4\ ,\ I\ ] \otimes A_5$$

with underlying shape T, and

$$v(\alpha) = v(T) = \{+, -, -, -, +\}.$$

Given a category $\mathbb{C}$ we can form labelled graphs, that is, graphs whose edges are labelled by morphisms of $\mathbb{C}$ as follows. Consider labelled shapes α and β with underlying shapes T and S respectively. A *labelled graph*

$$\alpha \longrightarrow \beta$$

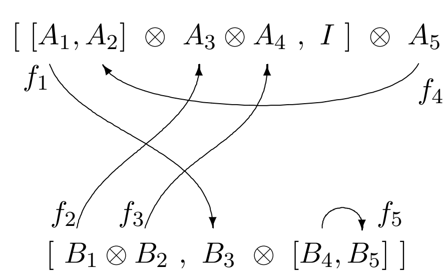
is a graph

$$\xi : T \longrightarrow S$$

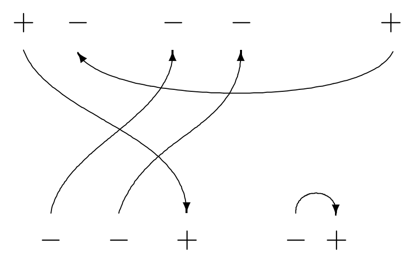
together with a morphism $x \longrightarrow y$ for each pair of labels x, y whose underlying variables are mates under ξ, with $v(x) = -$ and $v(y) = +$ in the twisted sum. That is, the morphism is in the direction

$$- \longrightarrow +.$$

For example, the following is a labelled graph



with underlying graph and variances as below



.

Observe that, since the variances of the domain are reversed in the twisted sum, the direction of morphisms is also reversed in the domain.

We write $K\mathbb{C}$ for the category of labelled shapes and labelled graphs in $\mathbb{C}$; thus $G = K\mathbf{1}$ as mentioned in Section B.1.2.

A labelled graph is called allowable if and only if its underlying graph is allowable. We write $A\mathbb{C}$ for the category of labelled shapes and allowable labelled morphisms. We observe immediately that the correspondence between trees and graphs exhibited in Section B.2 generalises to a correspondence between labelled graphs and labelled trees.

Lemma B.3.1. *A labelled tree in $\mathbb{C}$ is precisely an allowable morphism*

$$\alpha_1, \otimes \cdots \otimes \alpha_k \;\longrightarrow\; \alpha \;\in A\mathbb{C}$$

with underlying shape

$$X_{m_1} \otimes \cdots \otimes X_{m_k} \longrightarrow X_{(\sum_i m_i)-k+1}.$$

Recall (Section 2.1.1) that a labelled tree gives a 'configuration for composing' arrows of a symmetric multicategory via object-morphisms, as used in the slice construction. By the above correspondence, a labelled graph as above may also be considered to give such a configuration; thus in Section B.4.1 we will use such graphs to give an alternative description of the slice construction. We will need the following construction.

Given categories $\mathbb{C}$ and $\mathbb{D}$ and a functor

$$F : \mathbb{C} \longrightarrow K\mathbb{D}$$

we define a functor

$$KF : K\mathbb{C} \longrightarrow K\mathbb{D}$$

as follows.

- on objects

An object in $K\mathbb{C}$ is a labelled shape

$$\alpha = |T|(x_1, \ldots, x_n);$$

put

$$KF(\alpha) = |T|(Fx_1, \ldots Fx_n) \in K\mathbb{D}.$$

- on morphisms

Given a morphism

$$|T|(x_1,\ldots,x_n) \xrightarrow{f} |S|(x_{n+1},\ldots,x_m) \in K\mathbb{C}$$

we define KFf as follows. Suppose f has underlying graph ξ, say. Consider a pair of mates a and b in ξ, with the edge joining them 'labelled' with morphism

$$g : a \longrightarrow b \in \mathbb{C}.$$

This gives a morphism

$$Fg : Fa \longrightarrow Fb \in K\mathbb{D}.$$

So Fg is a graph labelled in $\mathbb{D}$. Then KFf consists of all such graphs given by mates in ξ, positioned according to the positions in ξ.

Furthermore, if $F : \mathbb{C} \longrightarrow \mathbb{D}$ then we get

$$AF : A\mathbb{C} \longrightarrow A\mathbb{D}$$

by restricting the functor KF.

B.3.2 The construction of opetopes

We seek to define, for each $k \geq 0$, a category $\mathbf{Ope}_k$ of k-opetopes. A k-opetope θ is to have a list of input $(k-1)$-opetopes $\alpha_1,\ldots,\alpha_m$, say, and an output $(k-1)$-opetope α, say. This data is to be expressed as an object

$$[\ \alpha_1 \otimes \cdots \otimes \alpha_m\ ,\ \alpha\ ] \in A\mathbf{Ope}_{k-1}$$

called the *frame* of θ (see [BD2]). Each frame has shape $X_m = [1^{\otimes m}, 1]$ for some $m \geq 0$. So, for each k we will have a functor

$$\phi_k : \mathbf{Ope}_k \longrightarrow A\mathbf{Ope}_{k-1}$$

and thus

$$A\phi_k : A\mathbf{Ope}_k \longrightarrow A\mathbf{Ope}_{k-1}.$$

$\mathbf{Ope}_k$ is defined inductively; for $k \geq 2$ it is a certain full subcategory of the comma category

$$(I \downarrow A\phi_{k-1})$$

with the following motivation. A k-opetope θ with frame

$$[\ \alpha_1 \otimes \cdots \otimes \alpha_m\ ,\ \alpha\ ]$$

is a configuration for composing $\alpha_1,\ldots,\alpha_m$ to result in α. That is, it is an allowable morphism

$$I \xrightarrow{\theta} [\phi_{k-1}\alpha_1 \otimes \cdots \otimes \phi_{k-1}\alpha_m\ ,\ \phi_{k-1}\alpha\ ] \in A\mathbf{Ope}_{k-2}$$

such that the composition does result in α. Such a θ is clearly an object of $(I \downarrow A\phi_{k-1})$; so we take the full subcategory whose objects are all those of the correct form.

In fact we begin with a more general construction for building up dimensions.

Definition B.3.2. *A* ladder *is given by*

- *for each* $k \geq 0$ *a category* $\mathbb{D}_k$
- *for each* $k \geq 1$ *a functor* $F_k : \mathbb{D}_k \longrightarrow A\mathbb{D}_{k-1}$

such that for each $k \geq 2$*,* F_k *is of the form*

$$\mathbb{D}_k \longrightarrow (I \downarrow AF_{k-1}) \longrightarrow A\mathbb{D}_{k-1}$$

where the second morphism is the forgetful functor.

Note that given $F_k : \mathbb{D}_k \longrightarrow A\mathbb{D}_{k-1}$ we have a functor

$$AF_k : A\mathbb{D}_k \longrightarrow A\mathbb{D}_{k-1}$$

and the comma category $(I \downarrow AF_{k-1})$ has as its objects pairs (θ, z) where $z \in A\mathbb{D}_{k-1}$ and θ is an allowable morphism

$$I \xrightarrow{\theta} AF_{k-1}(z) \ \in A\mathbb{D}_{k-1}.$$

Definition B.3.3. *The* opetope ladder *is given as follows.*

- $\mathbb{D}_0 = 1 = \{x\}$, say
- $\mathbb{D} = 1 = \{u\}$, say, with

$$\begin{array}{rccc} \phi_1 : & \mathbb{D}_1 & \longrightarrow & A\mathbb{D}_0 \\ & u & \longmapsto & [x, x] \end{array}$$

- For $k \geq 2$, $\mathbb{D}_k$ is a full subcategory of $(I \downarrow A\phi_{k-1})$. This comma category has objects (θ, z) where $z \in A\mathbb{D}_{k-1}$ and

$$I \xrightarrow{\theta} A\phi_{k-1}(z)$$

  is an allowable morphism in $A\mathbb{D}_{k-2}$. Then the subcategory $\mathbb{D}_k$ by the following two conditions:

  A. The objects of $\mathbb{D}_k$ are all (θ, z) such that z has shape X_m for some $m \geq 0$. So $z = [\alpha_1 \otimes \cdots \otimes \alpha_m, \alpha]$ for some $\alpha_i, \alpha \in \mathbb{D}_{k-1}$.

  B. For $k \geq 3$ we require in addition that

$$A\phi_{k-2}\bar{\theta} \circ (\alpha_1 \otimes \cdots \otimes \alpha_m) = \alpha$$

  as morphisms in $A\mathbb{D}_{k-3}$.

- For $k \geq 2$ the functor $\phi_k : \mathbb{D}_k \longrightarrow A\mathbb{D}_{k-1}$ is the following composite

$$\mathbb{D}_k \hookrightarrow (I \downarrow A\phi_{k-1}) \longrightarrow A\mathbb{D}_{k-1}$$

  where the functors shown are the inclusion followed by the forgetful functor.

Note that the composition in condition B is possible: each α_i is an object of $\mathbb{D}_{k-1}$, so is by definition a morphism

$$I \longrightarrow A\phi_{k-2}(\phi_{k-1}\alpha_i) \in A\mathbb{D}_{k-3}.$$

Now θ is a morphism

$$I \longrightarrow [\phi_{k-1}\alpha_1 \otimes \cdots \otimes \phi_{k-1}\alpha_m \ , \ \phi_{k-1}\alpha\ ]$$

so

$$\bar{\theta} \ : \ \phi_{k-1}\alpha_1 \otimes \cdots \otimes \phi_{k-1}\alpha_m \ \longrightarrow \ \phi_{k-1}\alpha$$

so the domain of $A\phi_{k-2}\bar{\theta}$ is indeed the codomain of $(\alpha_1 \otimes \cdots \otimes \alpha_m)$ and the composite in $A\mathbb{D}_{k-3}$ may be formed.

Definition B.3.4. *For each $k \geq 0$ the category $\mathbb{D}_k$ defined above is the category of k-*opetopes. *We write $\mathbb{D}_k = \mathbf{Ope}_k$. If the frame of a k-opetope has shape X_m we say θ is an m-*ary opetope*.*

Remarks B.3.5.

1) In general (that is for $k \geq 3$) the objects of $\mathbb{D}_k$ are those of $(I \downarrow A\phi_{k-1})$ satisfying the conditions A and B. Condition A restricts our scope only to those objects having the correct shape; condition B ensures that the 'output' of the opetope is indeed the composite given. For $k = 2$ condition B does not apply; any configuration of composing identity maps gives the identity.

2) A morphism $(\theta, z) \xrightarrow{f} (\theta', z')$ in $(I \downarrow A\phi_{k-1})$ is a morphism

$$f : z \longrightarrow z' \in A\mathbb{D}_{k-1}$$

such that the following diagram commutes:

$$\begin{tikzcd} & A\phi_{k-1}(z) \arrow[dd, "A\phi_{k-1}(f)"] \\ I \arrow[ur, "\theta"] \arrow[dr, "\theta'"'] & \\ & A\phi_{k-1}(z') \end{tikzcd}$$

so a morphism $\theta \xrightarrow{f} \theta'$ in $\mathbb{D}_k$ is given as follows. Writing

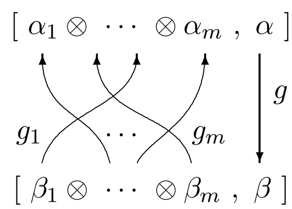
$$\phi_k\theta = [\alpha_1 \otimes \cdots \otimes \alpha_m, \alpha]$$

$$\phi_k\theta' = [\beta_1 \otimes \cdots \otimes \beta_j, \beta]$$

f must be a morphism

$$[\alpha_1 \otimes \cdots \otimes \alpha_m, \alpha] \longrightarrow [\beta_1 \otimes \cdots \otimes \beta_j, \beta] \in A\mathbb{D}_{k-1}.$$

So we must have $m = j$ and f has the form

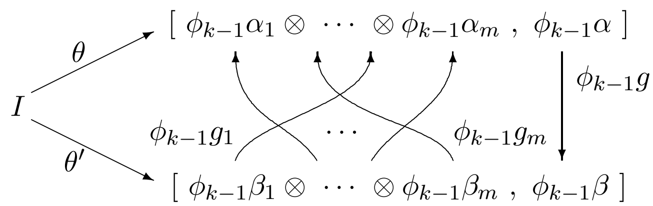


that is, a permutation $\sigma \in \mathbf{S}_m$ and morphisms

$$g_i : \beta_i \longrightarrow \alpha_{\sigma(i)}, \quad \text{for each } 1 \leq i \leq m$$

$$g : \alpha \longrightarrow \beta$$

in $\mathbb{D}_{k-1}$, such that the following diagram commutes



B.3.3 Examples

We now give the first few stages of the construction explicitly, together with some examples.

- $k = 0$

$\mathbf{Ope}_0 = \mathbf{1}$, that is, there is only one 0-opetope. We may think of this as an object $\cdot$; we write x for convenience.

- $k = 1$

$\mathbf{Ope}_1 = \mathbf{1}$, that is, there is only one 1-opetope, u, say. We have

$$\phi_1(u) = [x, x] \in A\mathbf{Ope}_0$$

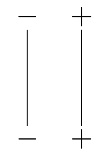
that is, the unique 1-opetope u has one input 0-opetope and one output 0-opetope. We may think of this as

$$\longrightarrow$$

or, showing variances

$$- \quad +$$

and then we have

$\phi_1(1_u) \quad =$ 

,

an allowable morphism in $A\mathbf{Ope}_0$. (We do not show arrowheads since all arrows in $\mathbf{Ope}_0$ are identity arrows.)

- $k = 2$

We seek to construct the category $\mathbf{Ope}_2$. First we consider an object $\alpha \in \mathbf{Ope}_2$. α has frame

$$\phi_2\alpha \in A\mathbf{Ope}_1$$

where $\phi_2\alpha$ has shape X_m for some $m \geq 0$. So we have

$$\phi_2\alpha = [u^{\otimes m}, u] = [u \otimes \cdots \otimes u, u].$$

Now α is an allowable morphism

$$I \xrightarrow{\alpha} [\ \phi_1 u \otimes \cdots \otimes \phi_1 u\ ,\ \phi_1 u\ ] \in A\mathbf{Ope}_0 = A\mathbf{1}$$

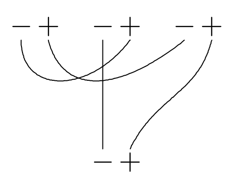
that is

$$I \xrightarrow{\alpha} [\ [x,x] \otimes \cdots \otimes [x,x]\ ,\ [x,x]\ ]$$

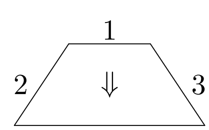
or equivalently a morphism

$$[x,x] \otimes \cdots \otimes [x,x] \longrightarrow [x,x] \in A\mathbf{1}.$$

For example, for $m = 3$ we may have a graph



which we will later see corresponds to the following



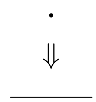
where the numbers show the order in which the input 1-opetopes are listed.

For the nullary case $m = 0$ we seek an allowable morphism

$$I \longrightarrow [x,x].$$

There is precisely one such, given by the following graph

and we will later see that this corresponds to the nullary 2-opetope



.

We now consider a morphism

$$\alpha \xrightarrow{f} \alpha' \in \mathbf{Ope}_2.$$

We must have

$$\phi_2\alpha = \phi_2\alpha' = [u^{\otimes m}, u],$$

say. Then f is a morphism

$$[u^{\otimes m}, u] \longrightarrow [u^{\otimes m}, u] \in A\mathbf{Ope}_1 = A\mathbf{1}.$$

So f must be a permutation $\sigma \in \mathbf{S}_m$, an isomorphism. So we have

$$\mathbf{Ope}_2(\alpha, \alpha') = \begin{cases} \mathbf{S}_m & \text{if } \alpha \text{ and } \alpha' \text{ are both } m\text{-ary} \\ \emptyset & \text{otherwise} \end{cases}$$

and $\mathbf{Ope}_2$ is equivalent to a discrete category whose objects are the natural numbers.

Note that the action of ϕ_2 on morphisms is given as follows. Given a morphism f as above, the morphism

$$\phi_2 f : \phi_2\alpha \longrightarrow \phi_2\alpha' \in A\mathbf{Ope}_1$$

is given by the forgetful functor

$$(I \downarrow A\phi_1) \longrightarrow A\mathbf{Ope}_1$$

so is simply the graph given by the permutation σ.

- $k = 3$

We now seek to construct the category $\mathbf{Ope}_3$. We first consider an m-ary opetope $\theta \in \mathbf{Ope}_3$ with frame

$$[\alpha_1 \otimes \cdots \otimes \alpha_m, \alpha] \in A\mathbf{Ope}_2$$

such that

$$\phi_2\alpha_i = [u^{\otimes n_i}, u] \quad \text{for each } 1 \le i \le m$$
$$\phi_2\alpha = [u^{\otimes n}, u].$$

So θ is an allowable morphism

$$I \xrightarrow{\theta} [\ [u^{\otimes n_1}, u] \otimes \cdots \otimes [u^{\otimes n_m}, u], [u^{\otimes n}, u]\ ]$$

or equivalently

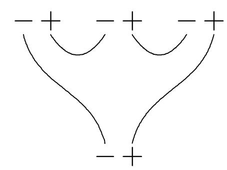
$$[u^{\otimes n_1}, u] \otimes \cdots \otimes [u^{\otimes n_m}, u] \ \xrightarrow{\bar{\theta}}\ [u^{\otimes n}, u] \in A\mathbf{Ope}_1,$$

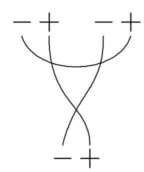
such that

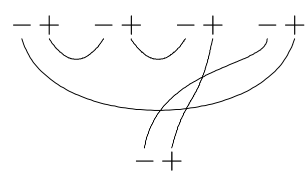
$$(A\phi_1)\bar{\theta} \circ (\alpha_1 \otimes \cdots \otimes \alpha_m) = \alpha$$

as morphisms in $A\mathbf{Ope}_0$.

For example for $m = 2$ consider

$\bar{\alpha}_1 \ =$ 

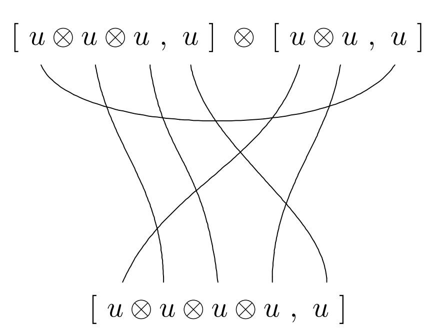
$\bar{\alpha}_2 \ =$ 

$\bar{\alpha} \ =$ 

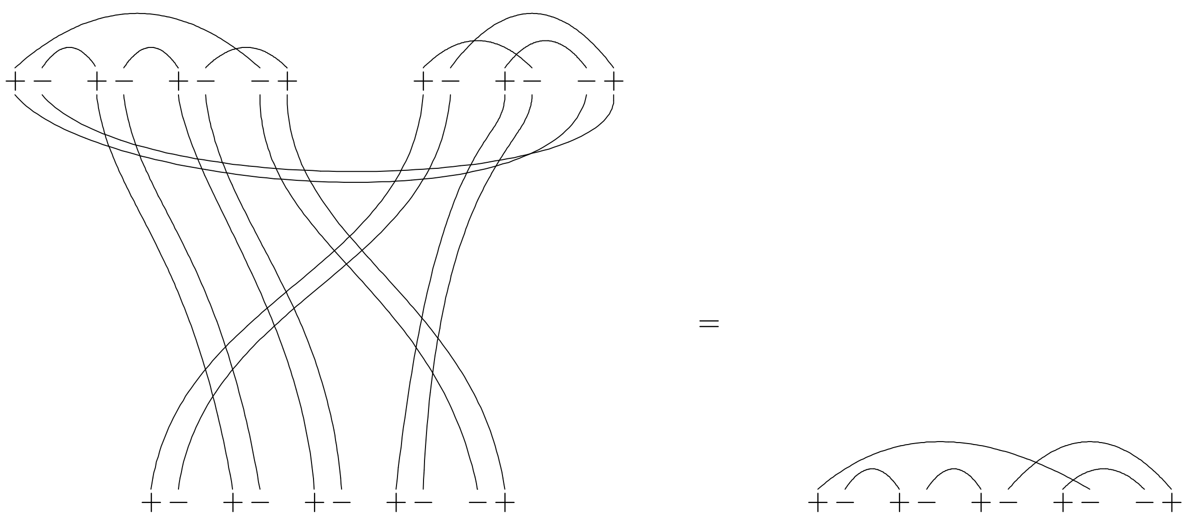
so

$$\phi_2\alpha_1 = [u \otimes u \otimes u\ ,\ u]$$
$$\phi_2\alpha_2 = [u \otimes u\ ,\ u]$$
$$\phi_2\alpha = [u \otimes u \otimes u \otimes u\ ,\ u]$$

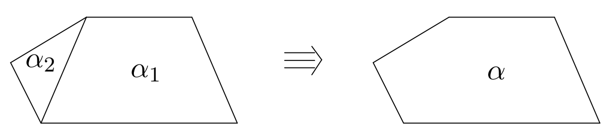
Then $\bar{\theta}$ may have the following graph in $A\mathbf{Ope}_1$



The condition B is seen to be satisfied by the following diagram; we apply ϕ_1 to each component, and compose with $\alpha_1 \otimes \alpha_2$:



This corresponds to a 3-opetope of the form



.

Note that we still do not need to label the edges of the graph since $\mathbf{Ope}_1$ also only has identity arrows.

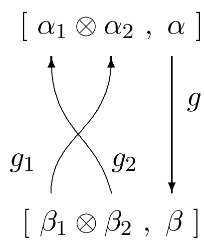
A morphism

$$\theta \xrightarrow{f} \theta' \in \mathbf{Ope}_3$$

then has one of the following two forms

$$\begin{array}{ccc} [\ \alpha_1 \otimes \alpha_2 & , & \alpha\ ] \\ g_1 \Big\uparrow \ \ g_2 \Big\uparrow & & \Big\downarrow g \\ [\ \beta_1 \otimes \beta_2 & , & \beta\ ] \end{array}$$

or



where g_1, g_2, g are morphisms in $\mathbf{Ope}_2$. Since all morphisms in $\mathbf{Ope}_2$ are isomorphisms, it follows that all morphisms in $\mathbf{Ope}_3$ are isomorphisms. In fact, since $\mathbf{Ope}_2$ is equivalent to a discrete category, $\mathbf{Ope}_3$ is also, and similarly $\mathbf{Ope}_k$ for all $k \geq 0$; this is proved in Section B.4.

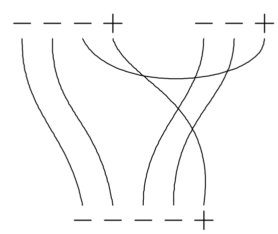
- $k = 4$

Finally we give an example of a 4-opetope $\gamma \in \mathbf{Ope}_4$, with

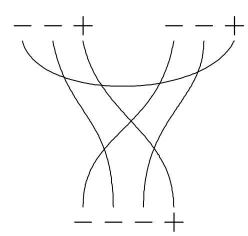
$$\phi_4\gamma = [\theta_1 \otimes \theta_2\ ,\ \theta]$$

where

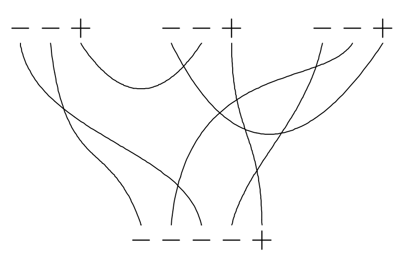
$\bar{\theta}_1 =$



$\bar{\theta}_2 =$



$\bar{\theta} =$

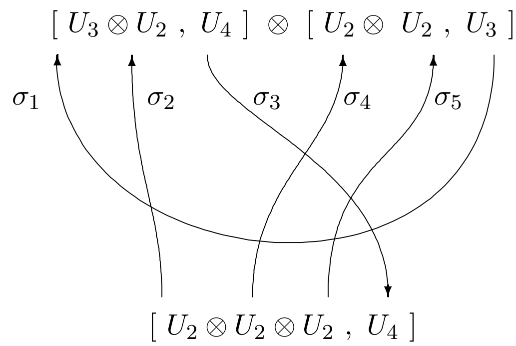


and we have

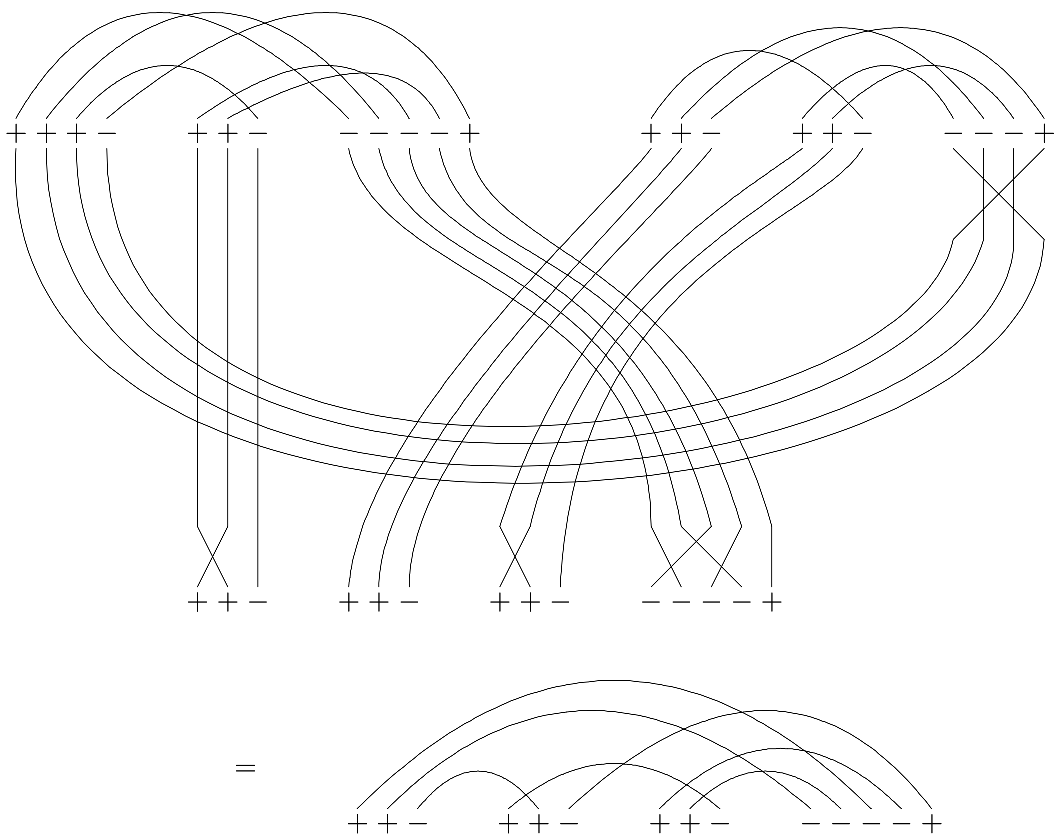
$$\phi_3\theta_1 = [\ [u^{\otimes 3}, u] \otimes [u^{\otimes 2}, u]\ ,\ [u^{\otimes 4}, u]\ ] = [U_3 \otimes U_2\ ,\ U_4],\ \text{say}$$

$$\phi_3\theta_2 = [\ [u^{\otimes 2}, u] \otimes [u^{\otimes 2}, u]\ ,\ [u^{\otimes 3}, u]\ ] = [U_2 \otimes U_2\ ,\ U_3]$$
$$\phi_3\theta = [U_2 \otimes U_2 \otimes U_2\ ,\ U_4].$$

Then $\bar{\gamma}$ may be given by the following graph in $A\mathbf{Ope}_2$

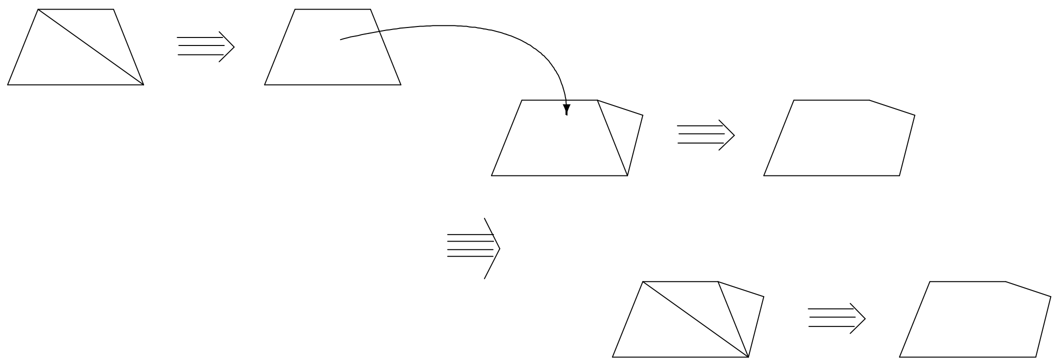


where each σ_i is a morphism in $\mathbf{Ope}_2$, that is, a permutation. We then check condition B by the following diagram:



giving the composite θ as required. Note that the permutations σ_i appear as permutations of the appropriate edges in the above diagram.

This corresponds to an opetope of the following form:



B.4 Comparison with the multicategory approach

In [BD2], opetopes are constructing using symmetric multicategories. Dimensions are built up using the slicing process. We compare this process with the use of closed categories as above.

B.4.1 The slice construction

Recall the slice construction for a symmetric multicategory. Let Q be a symmetric multicategory. Then the slice multicategory Q^+ is given by

- Objects: $o(Q^+) = \mathrm{elt}\, Q$
- Arrows: $Q^+(f_1, \ldots, f_n; f)$ is given by the set of 'configurations' for composing $f_1, \ldots, f_n$ as arrows of Q, to yield f.

Recall further that such a configuration for composing is given by a labelled tree (T, ρ, τ) where the nodes give the positions for composing the f_i. So by Corollary B.2.9 we may restate this using allowable morphisms in $K\mathbb{C}$, where $\mathbb{C} = o(Q)$.

Let Q be a symmetric multicategory with category of objects $\mathbb{C}$. Given an arrow $f \in Q(x_1, \ldots, x_m; x)$ we write

$$\phi f = [x_1 \otimes \cdots \otimes x_m, x] \in A\mathbb{C}.$$

Then the slice multicategory Q^+ is given as follows.

- objects $o(Q^+) = \mathrm{elt}\, Q$
- an arrow $\theta \in Q^+(f_1, \ldots, f_j; f)$ is an arrow

  $$\theta \in A\mathbb{C}(\ I\ ,\ [\phi f_1 \otimes \cdots \otimes \phi f_j\ ,\ \phi f]\ )$$

  such that composing the f_i in this configuration gives f.

Lemma B.4.1. *ϕ extends to a functor*

$$\phi : \mathrm{elt}\, Q \longrightarrow A\mathbb{C}.$$

Proof. Let

$$f \in Q(x_1, \ldots, x_m; x)$$
$$g \in Q(y_1, \ldots, y_j; y).$$

Then $\mathrm{elt}\, Q(f, g) = \emptyset$ unless $m = j$. If $m = j$ then a morphism $f \xrightarrow{\gamma} g$ is given by a permutation $\sigma \in \mathbf{S}_m$ together with morphisms

$$t_i : y_i \longrightarrow x_{\sigma(i)}$$

$$t : x \longrightarrow y$$

satisfying certain conditions. This specifies a unique allowable morphism

$$[x_1 \otimes \cdots \otimes x_m, x] \longrightarrow [y_1 \otimes \cdots \otimes y_m, y] \in A\mathbb{C}$$

and we define $\phi\gamma$ to be this morphism. This makes ϕ into a functor. □

We call ϕ the *frame functor* for Q. We now show how the slicing process corresponds to moving one rung up the 'ladder'.

Lemma B.4.2. *Let Q be a symmetric multicategory with category of objects $\mathbb{C}$. Then the category $\mathrm{elt}\, Q^+$ is isomorphic to a full subcategory of the comma category $(I \downarrow A\phi)$ and the frame functor for Q^+ is given by*

$$\mathrm{elt}\, Q^+ \hookrightarrow (I \downarrow A\phi) \longrightarrow A(\mathrm{elt}\, Q)$$

where the functors shown are the inclusion followed by the forgetful functor.

Proof. Write $\mathbb{C}_1 = \mathrm{elt}\, Q = o(Q^+)$.

An object of $\mathrm{elt}\, Q^+$ is (θ, p) where $p \in \mathcal{F}\mathbb{C}_1{}^{\mathrm{op}} \times \mathbb{C}_1$ and $\theta \in Q^+(p)$.

Write

$$p = (f_1, \ldots, f_m; f).$$

Then θ is an allowable morphism

$$I \xrightarrow{\theta} A\phi[f_1 \otimes \cdots \otimes f_m, f]$$

that is, an object

$$(\ \theta\ ,\ [f_1 \otimes \cdots \otimes f_m, f]\ ) \in (I \downarrow A\phi)$$

such that composing the f_i according to θ results in f.

A morphism $(\theta, p) \longrightarrow (\theta', p')$ in $\mathrm{elt}\, Q^+$ is a morphism $p \longrightarrow p'$ in $\mathcal{F}\mathbb{C}_1{}^{\mathrm{op}} \times \mathbb{C}_1$ such that a certain commuting condition holds. Such a morphism is precisely an allowable morphism

$$[f_1 \otimes \cdots \otimes f_m, f] \longrightarrow [f'_1 \otimes \cdots \otimes f'_m, f'] \in A\mathbb{C}_1$$

and the commuting condition is precisely that ensuring that this is a morphism $\theta \longrightarrow \theta'$ in $(I \downarrow A\phi)$.

It is then clear that the frame functor is given by the inclusion followed by the forgetful functor as asserted. □

Corollary B.4.3. *The category* $\operatorname{elt} Q^+$ *is the full subcategory of* $(I \downarrow A\phi)$ *whose objects are all* (θ, p) *satisfying the following two conditions:*

i) p *has shape* X_m *for some* $m \geq 0$ *so* $p = [f_1 \otimes \cdots \otimes f_m, f]$

ii) the result of composing the f_i *according to* θ *is* f.

If Q is itself a slice multicategory then we can state the condition (ii) in the language of closed categories as well, since each f_i is itself an allowable graph.

So we now consider forming Q^{++}, that is, the slice of a slice multicategory. Let Q be a symmetric multicategory with category of objects $\mathbb{C}_0$. We write

$$\mathbb{C}_1 = o(Q^+)$$

with frame functor

$$\begin{array}{rccc} \phi_1 : & \mathbb{C}_1 & \longrightarrow & A\mathbb{C}_0 \\ & f \in Q(x_1, \ldots, x_m; x) & \longmapsto & [x_1 \otimes \cdots \otimes x_m, x] \end{array}$$

Also, we write

$$\mathbb{C}_2 = \operatorname{elt} Q^+$$

with frame functor

$$\begin{array}{rccc} \phi_2 : & \mathbb{C}_2 & \longrightarrow & A\mathbb{C}_1 \\ & \alpha \in Q^+(f_1, \ldots, f_m; f) & \longmapsto & [f_1 \otimes \cdots \otimes f_m, f] \end{array}$$

Lemma B.4.4. *Let* θ *be a configuration for composing* $\alpha_1, \ldots, \alpha_j \in \operatorname{elt} Q^+ = \mathbb{C}_2$ *expressed as an allowable morphism*

$$I \xrightarrow{\theta} [\ \phi_2\alpha_1 \otimes \cdots \otimes \phi_2\alpha_j\ ,\ \phi_2\alpha] \in A\mathbb{C}_1.$$

Then the result of composing the α_i *in this configuration is*

$$(A\phi_1)\bar{\theta}\ \circ\ (\alpha_1 \otimes \cdots \otimes \alpha_j)$$

composed as morphisms of $A\mathbb{C}_0$.

Proof. By definition, each α_i is a morphism in $A\mathbb{C}_0$ of shape

$$I \longrightarrow [\ X_{im_1} \otimes \cdots \otimes X_{im_i}\ ,\ X\ ],$$

so is a tree labelled in $\mathbb{C}_0$. These trees are composed by node-replacement composition (see Section B.2.3) and the "composition graph" is given by $\bar{\theta}$. $\square$

Corollary B.4.5. *An arrow* $\theta \in Q^{++}(\alpha_1, \otimes \cdots \otimes, \alpha_j; \alpha)$ *is precisely a morphism*

$$\theta \in A\mathbb{C}_1(\ I\ ,\ [\phi_2\alpha_1 \otimes \cdots \otimes \phi_2\alpha_j\ ,\ \phi_2\alpha]\ )$$

such that

$$(A\phi_1)\bar{\theta}\ \circ\ (\alpha_1 \otimes \cdots \otimes \alpha_j) = \alpha\ \in A\mathbb{C}_0$$

Corollary B.4.6. elt Q^{++} *is the full subcategory of* $(I \downarrow A\phi_2)$ *whose objects are all* (θ, p) *satisfying the following two conditions:*

1) p *has shape* X_m *for some* $m \geq 0$, *so* $p = [\alpha_1 \otimes \cdots \otimes \alpha_m; \alpha] \in A\mathbb{C}_2$

2) $(A\phi_1)\bar{\theta} \circ (\alpha_1 \otimes \cdots \otimes \alpha_j) = \alpha$.

Finally we are ready to show that the opetopes constructed using symmetric multicategories correspond to those constructed in closed categories.

Corollary B.4.7. *Let* Q *be the symmetric multicategory with just one object and one (identity) morphism. Then for all* $k \geq 0$

$$o(Q^{k+}) \cong \mathbf{Ope}_k$$

where $Q^{0+} = Q$.

Proof. For $k \leq 1$ the result is immediate by Definition B.3.3. For $k = 2$ we use Corollary B.4.3 on Q^+; the result follows since condition (ii) is trivially satisfied. For $k \geq 3$ we use Corollary B.4.6 on $Q^{(k-3)+}$; the result follow since the ϕ_2 in the Corollary is ϕ_{k-2} in the case in question. $\square$

B.5 The category of opetopes

Recall that in Chapter 3 we defined the category $\mathcal{O}$ of opetopes. It is possible to restate this definition in the framework of Kelly-Mac Lane graphs described in this Appendix; we copy the definition exactly, using the fact that the bijection giving the formal definition of a tree gives the mates in the corresponding Kelly-Mac Lane graph.

Although we do not give the construction explicitly here, we give some examples of low-dimensional face maps. We use the example of a 3-opetope as given in Section B.3.3.

For the 2-opetopes we have face maps

$$\begin{tikzcd}
\alpha_1 \arrow[dr, "s_1"] & \\
\alpha_2 \arrow[r, "s_2"] & \theta \\
\alpha \arrow[ur, "t"'] &
\end{tikzcd}$$

together with the isomorphic cases.

For 1-opetopes we then have

$$s_1, s_2, s_3, t \ : \ u \longrightarrow \alpha_1$$

$$s_1, s_2, t \ : \ u \longrightarrow \alpha_2$$

$$s_1, s_2, s_3, s_4, t \ : \ u \longrightarrow \alpha$$

but by considering the generating relations, here given by mates in the graph θ, we have

$$\begin{array}{rcl} s_1 s_1 & = & s_2 t \\ s_1 s_2 & = & t s_2 \\ s_1 s_3 & = & t s_3 \\ s_1 t & = & tt \\ s_2 s_1 & = & t s_1 \\ s_2 s_2 & = & t s_4; \end{array}$$

note that $s_i s_j$ give the jth source of the ith source of θ.

For 0-opetopes we have in addition face maps

$$x \longrightarrow u$$

and the relations on composites

$$x \longrightarrow \theta$$

are generated by relations on composites

$$x \longrightarrow \alpha_i$$

as well as by those on composites

$$u \longrightarrow \theta.$$

For the former relations we are considering mates under graphs $\alpha_i \in A\mathbf{Ope}_0$, and for the latter, mates under the graph $(A\phi_1)\bar{\theta} \in A\mathbf{Ope}_0$. So in fact we are considering, in total, all objects connected in the composite graph

$$(A\phi_1)\bar{\theta} \circ (\alpha \otimes \cdots \otimes \alpha_m) \in A\mathbf{Ope}_1.$$

So we have

$$\begin{array}{ccccccccccc} ts_1 s & = & s_2 s_1 s & = & s_2 s_2 t & = & ts_4 t & & & & \\ ts_1 t & = & s_2 s_1 t & = & s_2 tt & = & s_1 s_1 t & = & s_1 s_2 s & = & ts_2 s \\ ts_2 t & = & s_1 s_2 t & = & s_1 s_3 s & = & ts_3 s & & & & \\ ts_3 t & = & s_1 s_3 t & = & s_1 tt & = & ttt & & & & \\ ts_4 s & = & s_2 s_2 s & = & s_2 ts & = & s_1 s_1 s & = & s_1 ts & = & tts \end{array}$$

Note that since

$$(A\phi_1)\bar{\phi} \circ (\alpha_1 \otimes \cdots \otimes \alpha_m) = \alpha$$

the 0-cell face maps for θ are precisely those of the form tf where f is a 0-cell face map of $\alpha = t(\theta)$. This reflects the fact that, when 2-opetopes are composed along 1-opetopes, the composite is formed by 'deleting' the boundary 1-opetopes, but no 0-cells are deleted. This result generalises to k-opetopes, but we do not prove this here.

Appendix C

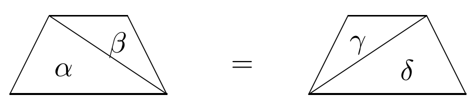
Calculations for Section 5.2.4

In this appendix we perform the calculations deferred from Section 5.2.4. However, we first introduce some shorthand to deal with some of the more unwieldy parts of the algebra.

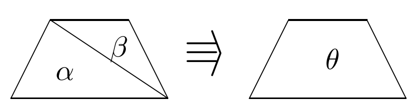
C.1 Shorthand for calculations

The following shorthand is used for calculations in an opetopic 2-category.

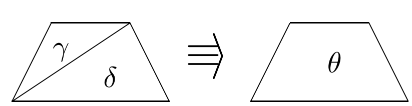
i) Since 3-niche occupants are unique, we may omit the target of a 3-cell without ambiguity. We then write an equality to indicate that the 3-cells have the same target. For example we might write



meaning

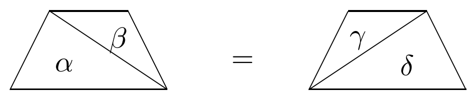


and

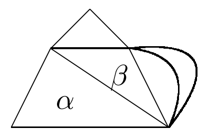


.

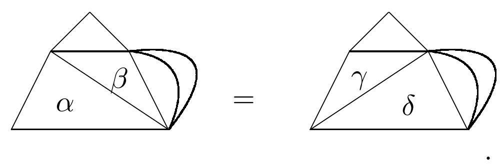
ii) Recall that, by uniqueness of 3-niche occupants, we have associativity of 2-cell composition. So we may substitute 'equal' (in the above sense) 2-cell composites in part of the domain of another 3-cell. For example, given



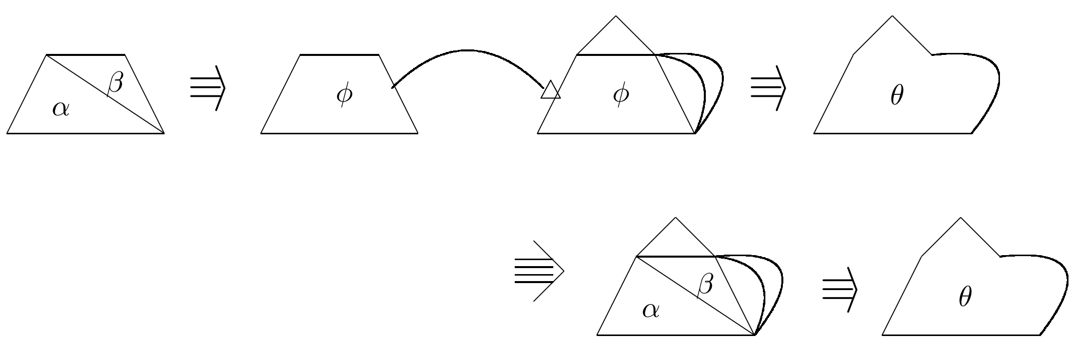
and a 3-cell



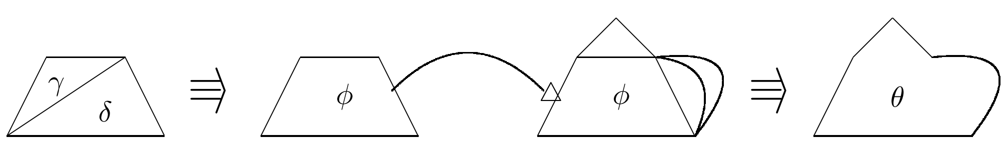
we have

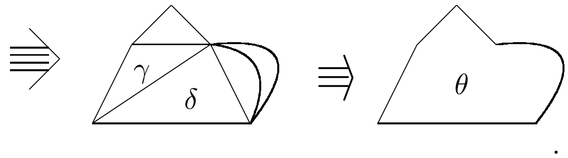


This is shorthand for the following

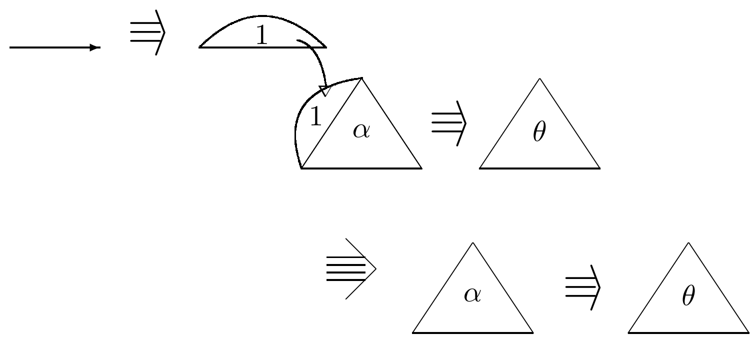


and

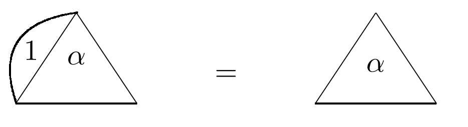




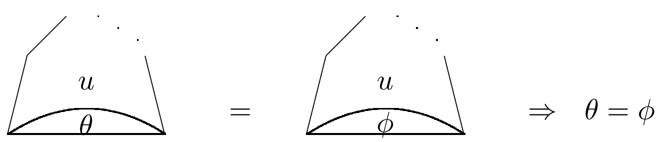
iii) Recall that 2-cell identities act as identities on k-ary 2-cells for all k (not only 1-ary 2-cells), for example



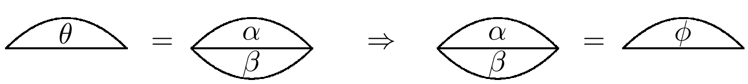
so we have $\alpha = \theta$, that is



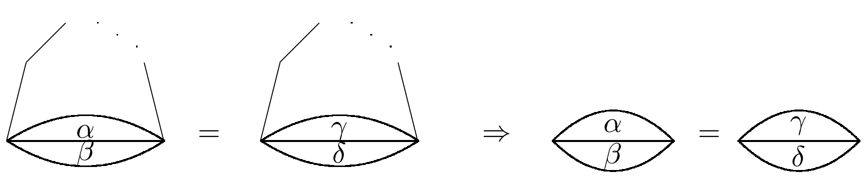
iv) Note that if u is any universal 2-cell, we have



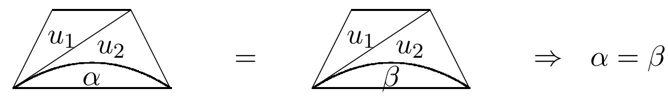
by definition of universality. This also holds if θ and ϕ are 2-cell composites, for example



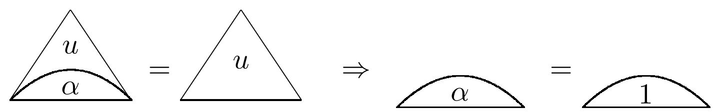
and



Furthermore, this holds if u is a composite of universals, since a composite of universals is universal, for example if u_1 and u_2 are universal then



and in particular



C.2 Calculations

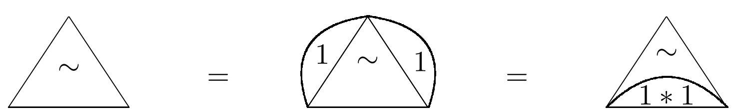
Throughout this section, we use the notation and constructions exactly as given in Section 5.2.4.

Lemma C.2.1. *i)* $1_g * 1_f = 1_{gf}$

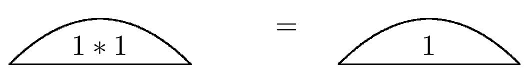
ii) $(\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1) = (\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1)$ *(middle 4 interchange)*

Proof.

i) We have

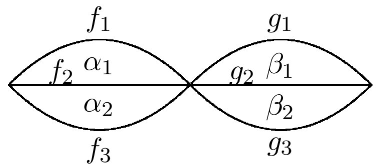


by the action of 1 and definition of $*$, so

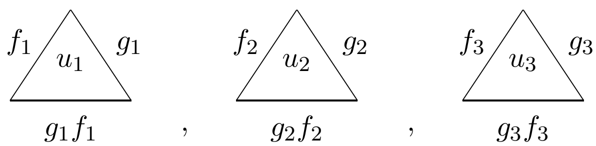


as required.

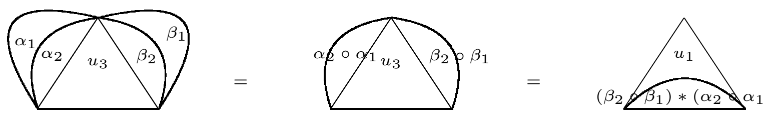
ii) Given



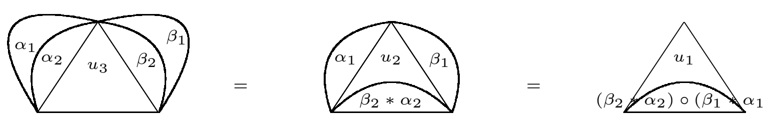
we write



for the chosen universal 2-cells as shown. Then we have



by definition, but also

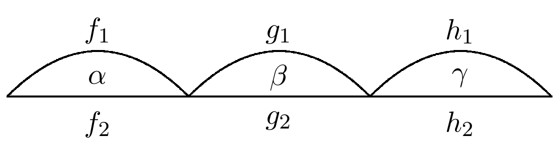


by definition, hence the result.

□

Lemma C.2.2. *a is natural*

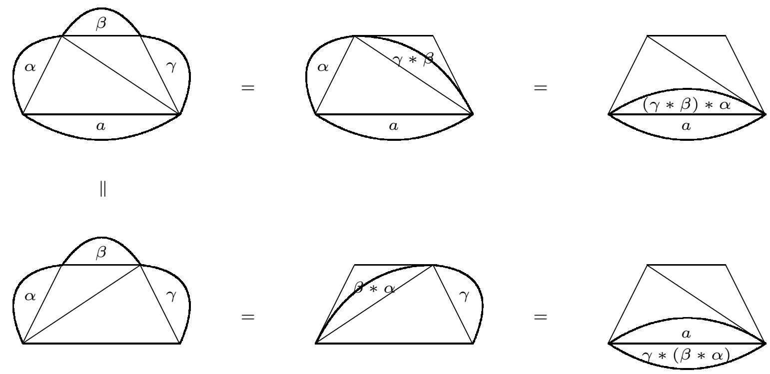
Proof. Given 2-cells



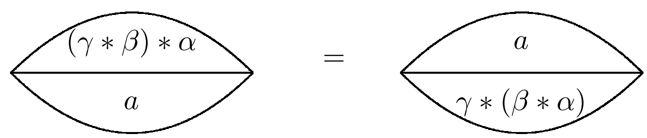
we need to show that the following naturality square commutes

$$\begin{tikzcd}
(h_1 g_1) f_1 \arrow[r, "a"] \arrow[d, "(\gamma * \beta) * \alpha"'] & h_1 (g_1 f_1) \arrow[d, "\gamma * (\beta * \alpha)"] \\
(h_2 g_2) f_1 \arrow[r, "a"'] & h_2 (g_2 f_2)
\end{tikzcd} .$$

We have



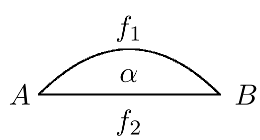
so by uniqueness we have



as required. ☐

Lemma C.2.3. *r is natural*

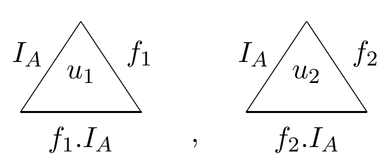
Proof. Given a 2-cell



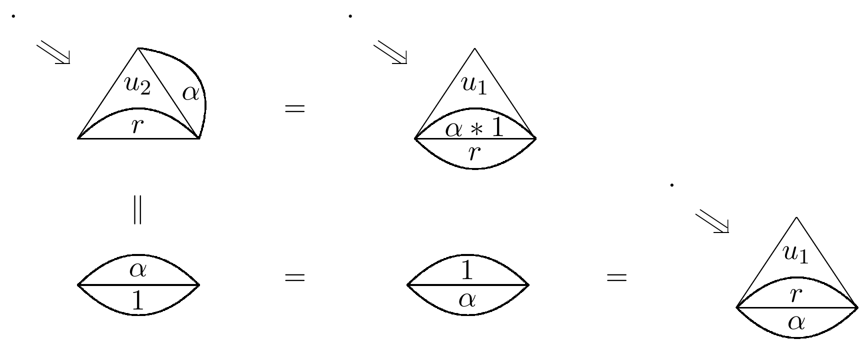
we need to show that the following naturality square commutes

$$\begin{tikzcd} f_1 \circ I_A \arrow[r, "r"] \arrow[d, "\alpha * 1"'] & f_1 \arrow[d, "\alpha"] \\ f_2 \circ I_A \arrow[r, "r"'] & f_2 \end{tikzcd} .$$

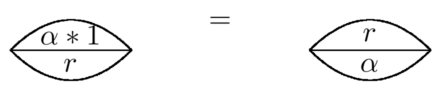
Writing chosen composites as



we have



so by uniqueness

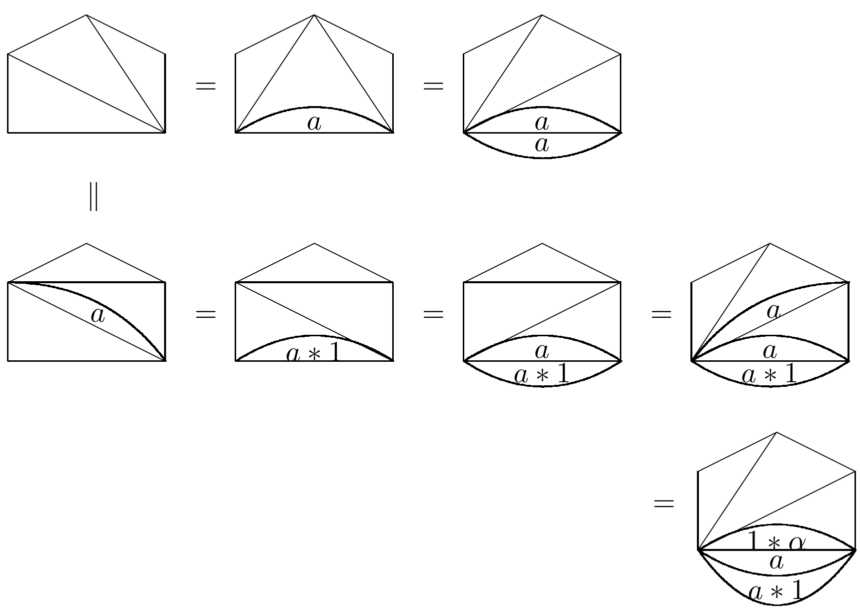


as required. □

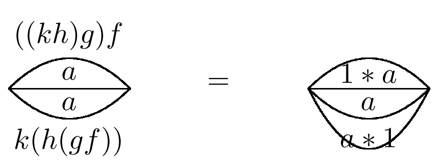
Lemma C.2.4. *a, l and r satisfy the axioms for a bicategory.*

Proof.

i) associativity pentagon

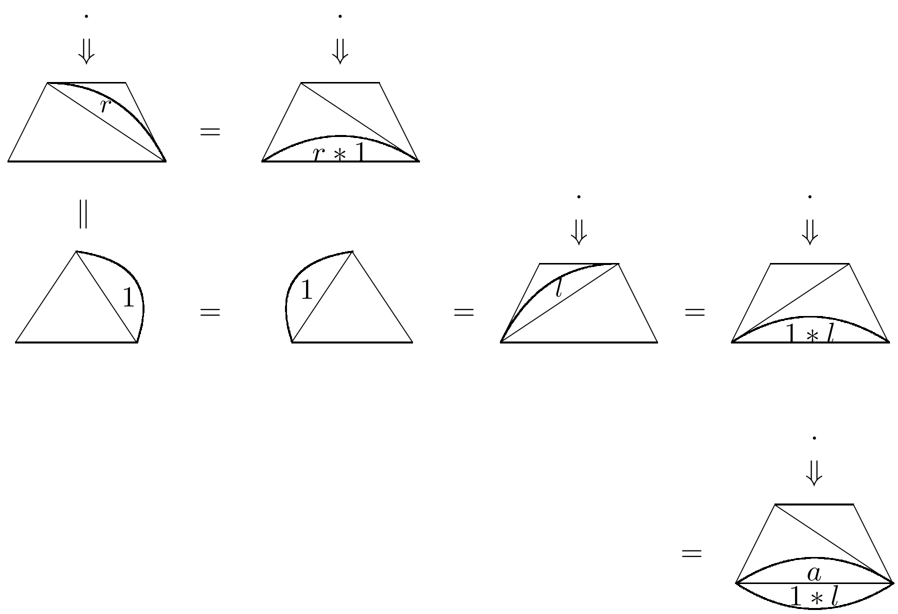


so

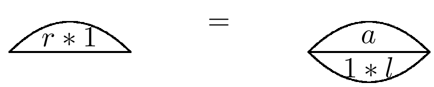


as required.

ii) unit triangle



so

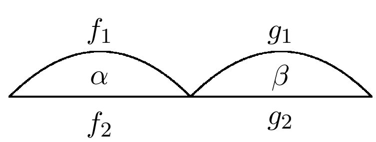


as required.

∎

Lemma C.2.5. *ϕ is natural.*

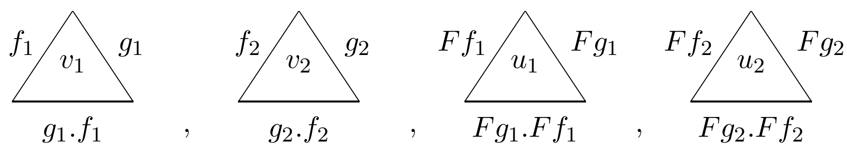
Proof. Given 2-cells



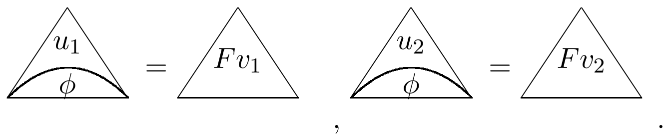
we need to show that the following diagram commutes

$$\begin{tikzcd}
Fg_1 \circ Ff_1 \arrow[r, "\phi_{g_1 f_1}"] \arrow[d, "F\beta * F\alpha"'] & F(g_1 \circ f_1) \arrow[d, "F(\beta * \alpha)"] \\
Fg_2 \circ Ff_2 \arrow[r, "\phi_{g_2 f_2}"'] & F(g_2 \circ f_2)
\end{tikzcd} \quad .$$

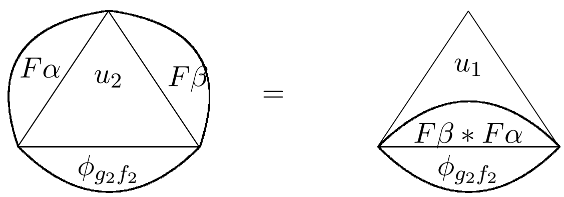
We write the chosen universal 2-cells as



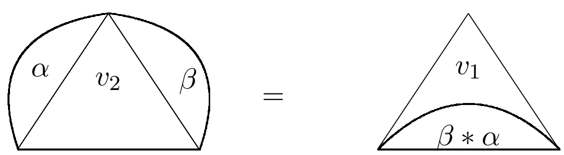
so



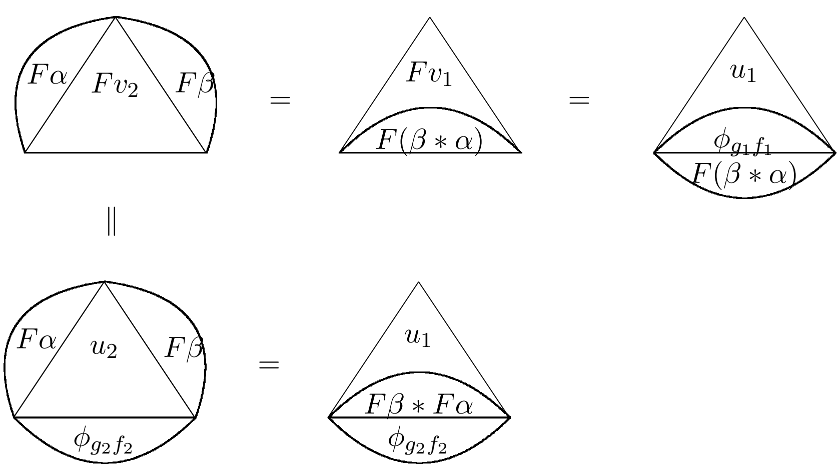
We have



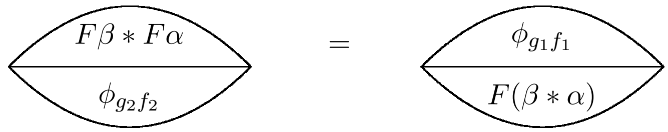
in X', and in X we have



so applying F, we have, since F is strictly functorial on 2-cells,



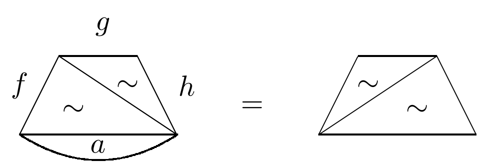
so by uniqueness we have



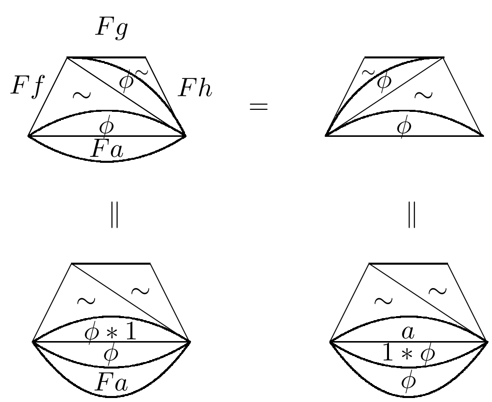
as required. □

Lemma C.2.6. *(F, ϕ) satisfies the axioms for a morphism of bicategories.*

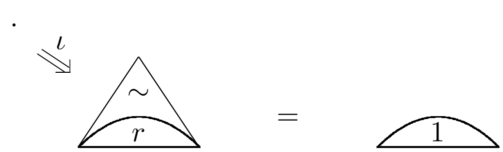
Proof. We have in X



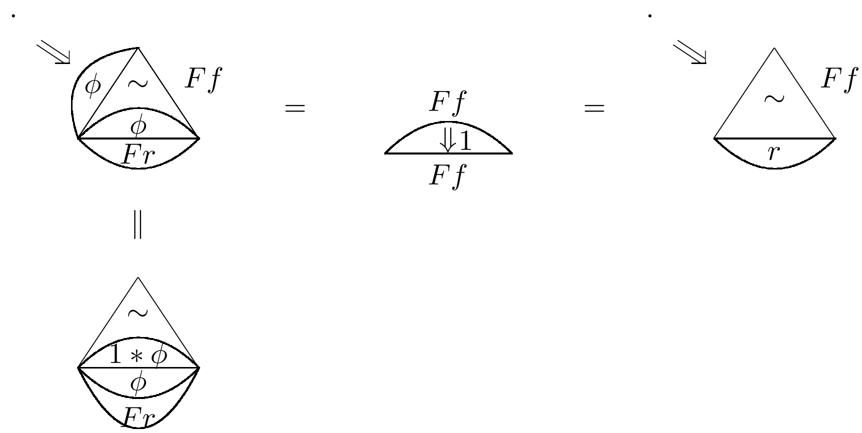
so applying F, we get in X'



as required. For r we have in X



so applying F, we get in X'



as required. The axiom for l holds similarly. $\square$

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