

Misconceptions about the Golden Ratio

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George Markowsky received his B.A. in mathematics from Columbia University and his M.A. and Ph.D in mathematics from Harvard University. From 1974 to 1984 he worked at IBM's Thomas J. Watson Research Center and since then he has been a member of the Computer Science Department at the University of Maine. His primary mathematical interests are lattice theory, discrete mathematics and the applications of computers to mathematical problems. He became interested in the Golden Ratio as a result of preparing a talk on the subject for the Classics Club at the University.

The golden ratio, also called by different authors the golden section [Cox], golden number [Fi4], golden mean [Lin], divine proportion [Hun], and division in extreme and mean ratios [Smi], has captured the popular imagination and is discussed in many books and articles. Generally, its mathematical properties are correctly stated, but much of what is presented about it in art, architecture, literature, and esthetics is false or seriously misleading. Unfortunately, these statements about the golden ratio have achieved the status of common knowledge and are widely repeated. Even current high school geometry textbooks such as [Ser] make many incorrect statements about the golden ratio.

It would take a large book to document all the misinformation about the golden ratio, much of which is simply the repetition of the same errors by different authors. This paper discusses some of the most commonly repeated misconceptions.

Some Mathematical Properties of the Golden Ratio

The *golden ratio* arises from dividing a line segment so that the ratio of the whole segment to the larger piece is equal to the ratio of the larger piece to the smaller piece. This was called *division in extreme and mean ratio* by Euclid (see [Smi; Vol. II, p. 291] and [Her]).



Figure 1
Dividing a line segment according to the golden ratio

Figure 1 shows a line segment of length 1 divided into two pieces. This division produces the golden ratio if $(1/X) = X/(1 - X)$ or $X^2 + X - 1 = 0$. The positive root of this equation is $X = (-1 + \sqrt{5})/2 \approx 0.61803398875\dots$, so the ratio $1/X = (1 + \sqrt{5})/2 \approx 1.61803398875\dots$. Note that $1/X$ satisfies the equation $Y^2 - Y - 1 = 0$.

Commonly, the Greek letters Φ [Hil; p. 78] or τ [Bur; p. 128] are used to represent the golden ratio 1.61803... . I will use Φ throughout this paper to represent 1.61803... . (Some authors use Φ to represent 0.618... .)

The golden ratio appears in many geometrical constructions. For example, it appears as the ratio of a side to the base in the $72^\circ, 72^\circ, 36^\circ$ isosceles triangle. Figure 2 shows an isosceles triangle ABC with two sides of length 1 and a base of length X . Bisecting angle A creates an isosceles triangle ADC similar to the first. Triangle ADB is also isosceles so that the lengths of BD , AD and AC are all equal to X . CD has length $1 - X$, and since ABC and CAD are similar, $1/X = X/(1 - X)$ and $X = 1/\Phi$. From $1/\Phi$ it is easy to construct the $72^\circ, 72^\circ, 36^\circ$ triangle and using this triangle it is easy to construct pentagons, pentagrams and decagons. For additional geometric constructions see [Odo] and [Rig; p. 29].

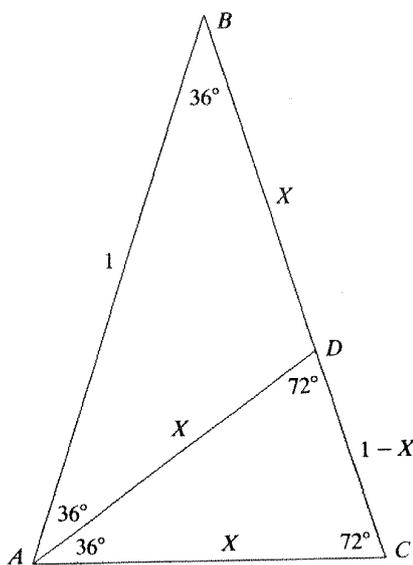


Figure 2
Deriving the proportions of a $72^\circ, 72^\circ, 36^\circ$ triangle

The self-reproducing capability of the golden rectangle is also commonly cited. R is a *golden rectangle* with dimensions $a > b$ if $a/b = \Phi$. Removing a square with dimensions $b \times b$ from one end leaves a golden rectangle with dimensions b and $(a - b)$. Figure 3(a) shows how constructing a sequence of smaller golden rectangles yields a spiral.

Spirals can be constructed using rectangles of any ratio other than 1 : 1. Figure 3(b) shows a sequence of smaller rectangles based on the ratio $\sqrt{2} : 1$. The smaller rectangles are created from the larger rectangles by dividing the larger rectangle exactly in half. Unlike Figure 3(a), every rectangle appearing in Figure 3(b) has exactly the same proportion. For a general discussion of such constructions and how they appear in living creatures see [Tho; pp. 181-187].

For additional information on the golden ratio and its connection with the Fibonacci numbers see [Cov], [Cox], [Fo2], [Ga2], [Gru], [Knu] and [Rig]. Browsing

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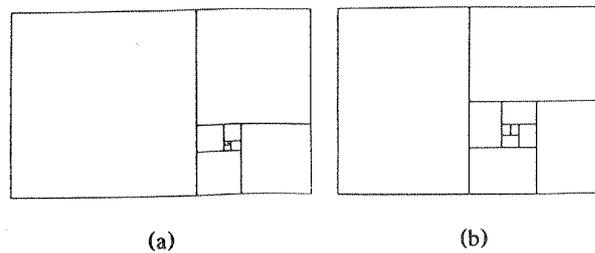


Figure 3
Spirals of rectangles

through the *Fibonacci Quarterly* will also turn up much information about the golden ratio and the Fibonacci numbers.

Misconception: The Name “Golden Ratio” Was Used in Antiquity

Many people assume that the names “golden ratio” and “golden section” are very old. For example, François Lasserre states [Las; p. 76]

The proportion, famous throughout antiquity, has been known since Leonardo da Vinci’s time as the *golden section*.

However, the use of the adjective “golden” in connection with Φ is a relatively modern one. Even the term “divine proportion” goes back only to the Renaissance. David Eugene Smith [Smi; Vol. II, p. 291] states:

The solution (of the problem of drawing 36° and 72° angles) is related to that of the division of a line in extreme and mean ratio.² This was referred to by Proclus when he said that Eudoxus (c. 370 B.C.) ‘greatly added to the number of the theorems which Plato originated regarding the *section*.’ This is the first trace that we have of this name for such a cutting of the line.

In comparatively modern times the section appears first as ‘divine proportion,’³ and then, in the 19th century,⁴ as the ‘golden section.’

In the above passage footnote 2 refers to Euclid, footnote 3 refers to Pacioli’s book *De Divina Proportione* and footnote 4 refers to an 1844 article in the *Archiv der Math. und Physik* (IV, 15–22).

D. H. Fowler [Fol; p. 146] gives the following history.

It may surprise some people to find that the name ‘golden section,’ or more precisely, *goldener Schnitt*, for the division of a line AB at a point C such that $AB \cdot CB = AC^2$, seems to appear in print for the first time in 1835 in the book *Die reine Elementar-Mathematik* by Martin Ohm, the younger brother of the physicist Georg Simon Ohm. By 1849, it had reached the title of a book: *Der allgemeine goldene Schnitt und sein Zusammenhang mit der harmonischen Theilung* by A. Wiegand. The first use in English appears to have been in the ninth edition of the *Encyclopaedia Britannica* (1875), in an article on Aesthetics by James Sully, The first English use in a purely mathematical context appears to be in G. Chrystal’s *Introduction to Algebra* (1898).

The term "golden mean" was used in classical times to denote "the avoidance of excess in either direction" [Oxf]. Some authors (for example [Lin]) use the term "golden mean" to denote the golden ratio. The confusion of names might have led some people to conclude that "golden mean" was used in classical times to denote the golden ratio. For a detailed history of the golden ratio up until 1800 see [Her].

How to Find the Golden Ratio

Throughout this paper you will see passages from different works that assert the presence of the golden ratio in some work of art or architecture. In some cases, authors will draw golden rectangles that conveniently ignore parts of the object under consideration. In the absence of any clear criteria or standard methodology it is not surprising that they are able to detect the golden ratio.

Following Martin Gardner's lead I will call such unsystematic searching for Φ the *Pyramidology Fallacy*. Pyramidologists use such numerical juggling to justify all sorts of claims concerning the dimensions of the Great Pyramid. Martin Gardner [Ga1; pp. 177-8] describes this methodology.

It is not difficult to understand how Smyth achieved these astonishing scientific and historical correspondences. If you set about measuring a complicated structure like the Pyramid, you will quickly have on hand a great abundance of lengths to play with. If you have sufficient patience to juggle them about in various ways, you are certain to come out with many figures which coincide with important historical dates or figures in the sciences. Since you are bound by no rules, it would be odd indeed if this search for Pyramid 'truths' failed to meet with considerable success.

This process of juggling is rendered infinitely easier by two significant facts. (1) Measurements of various Pyramid lengths are far from established....(2) The figures which represent scientific truths are equally vague. The distance to the sun...varies considerably because the earth's path is not a circle but an ellipse. In such cases you have a wide choice of figures. You can use the earth's shortest distance to the sun, or the longest, or the mean.

Martin Gardner proceeds to illustrate this principle in action by deriving "amazing numerical properties" of the Washington Monument based on statistics taken from an almanac. [Ga1] is well worth reading.

Another point overlooked by many golden ratio enthusiasts is the fact that measurements of real objects can only be approximations. Surfaces of real objects are not perfectly flat. Furthermore, it is necessary to specify the precision of any measurements and to realize that inaccuracies in measurements lead to greater inaccuracies in ratios. For example, a $\pm 1\%$ variation in the measurement of two lengths can lead to a roughly $\pm 2\%$ variation ($0.99/1.01 \approx 0.98$ to $1.01/0.99 \approx 1.02$) in the ratio that is computed. Thus, someone eager to find the golden ratio somewhere can alter two numbers by $\pm 1\%$ and alter their ratio by roughly $\pm 2\%$.

It is unfortunate that many writers on mathematical subjects treat measurements of real objects as if they were exact numbers. To discuss the claims about Φ intelligently it is necessary to create some guidelines for dealing with measurements and ratios.

I propose the following guidelines. If measurements are given without an error range I will assume that they are accurate to within $\pm 1\%$. In practice, error ranges

can be substantially better than this. For example, codes of practice for structural engineers call for tolerances of 0.2% (see [Ame; pp. 6–235, 6–236]). For measurements done with a ruler, a $\pm 1\%$ error range represents roughly a $1/16''$ error in a 6 inch object.

As a consequence of this assumption, I will consider a claim for the presence of Φ to be at least reasonable if the computed ratio is within about 2% of Φ . To be more generous, I will expand these bounds a little and use the range 1.58 to 1.66. For convenience I refer to the range [1.58, 1.66] as the *acceptance range*. If a ratio falls outside the acceptance range I will not consider it reasonable to claim that it is Φ .

Even if a ratio falls within the acceptance range, this will not constitute automatic proof that Φ is present. This simply means that a claim has passed the first test and is worth investigating further. Since the acceptance range includes infinitely many numbers near Φ it is necessary to justify the claim that Φ is the preferred number. Some other ratio coincidentally near Φ might be the important one.

I will compute ratios to at most 3 significant figures since we have an error range of about $\pm 2\%$.

Misconception: The Great Pyramid Was Designed to Conform to Φ

A variety of people have looked for Φ in the dimensions of the Great Pyramid of Khufu (Cheops), which was built before 2500 B.C. According to [Tas; p. 12] the lengths of the sides of the base of the Great Pyramid range from 755.43 feet to 756.08 feet, so it is not a perfect square. The average length is 755.79 feet. The height of the Great Pyramid is given as 481.4 feet. Every source that I have checked for the dimensions gives values within 1% of these (e.g. [Gil; p. 185]). Throughout this section I will use 755.79 feet as the length of the base and 481.4 feet as the height.

Some authors claim that the Great Pyramid was designed so that the ratio of the slant height of the pyramid to half the length of the base would be Φ . In Figure 4, h represents the height, b half the base, and s the slant height of the Great Pyramid. From the Pythagorean theorem $s \approx 612.01$ feet. This gives a ratio of $612.01/377.90 \approx 1.62$ which differs from Φ by only 0.1%. Thus, we must examine the claims put forward for the presence of the golden ratio in the dimensions of the Great Pyramid.

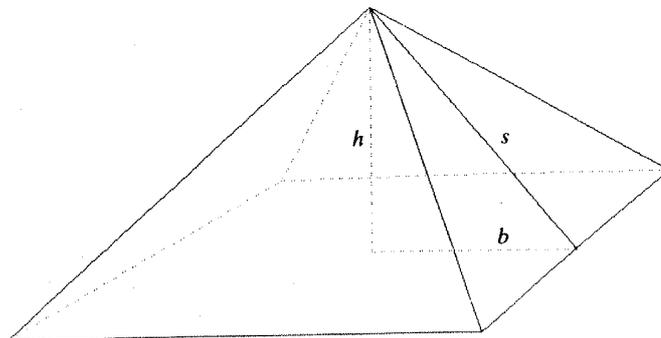


Figure 4
A square pyramid

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Quite a few books repeat the claim that Φ is present in the Great Pyramid by design. For example, Martin Gardner [Ga1; p. 178], Herbert Westren Turnbull [Tur; p. 80] and David Burton repeat essentially the same story:

Herodotus related in one passage that the Egyptian priests told him that the dimensions of the Great Pyramid were so chosen that the area of a square whose side was the height of the great pyramid equaled the area of a face triangle. [Bur; p. 62]

This passage implies that the ratio of the slant height of a face to half the length of the base is the golden ratio. If the area of a face (Figure 4) is equal to the area of a square whose side is equal h we get the equation $h^2 = sb$. The Pythagorean theorem yields $h^2 + b^2 = s^2$. Let $r = s/b$. Dividing both equations by b^2 and expressing the results in terms of r yields $(h/b)^2 = r$ and $(h/b)^2 + 1 = r^2$. Combining these equations yields $0 = r^2 - r - 1$, which has the golden ratio as its only positive root.

Fischler [Fi2] and Gillings [Gil; pp. 238–239] have decided that this interpretation of Herodotus is bogus. Fischler traces it to the book *The Great Pyramid, Why Was It Built and Who Built It?* which was published in 1859 by the pyramidologist John Taylor.

Neither Gardner, Turnbull nor Burton specifies the location of this passage in Herodotus. I could find only one passage about the dimensions of the Great Pyramid in the translations of Herodotus's *History* (ca. 445–425 B.C.) by Rawlinson [He2] and Sélincourt [He1], and the commentaries of How and Wells [How]. Rawlinson [He2] translates this passage, paragraph 124 of Book II, as follows.

The Pyramid itself was twenty years in building. It is a square, eight hundred feet each way, and the height the same, built entirely of polished stone fitted together with the utmost care. The stones of which it is composed are none of them less than thirty feet in length.

The Sélincourt translation [He1; p. 179] is similar. Herbert Westren Turnbull [Tur; p. 80] admits that his interpretation depends on "the slightest literal emendation."

Figure 5 is the text from Herodotus [Hud; II.124, lines 16–20], and a translation. The text in parentheses gives the antecedents for the pronouns, while the text in braces lists alternative readings for the word. The text does not support the story repeated by [Bur], [Ga1] and [Tur].

τῇ δὲ πυραμίδι αὐτῇ χρόνον γενέσθαι εἴκοσι
ἔτεα ποιευμένη, τῆς ἐστι πανταχῇ μέτωπον ἕκαστον ὀκτώ
πλέθρα ζούσης τετραγώνου καὶ ὕψος ἴσον, λίθον δὲ ξεστοῦ
τε καὶ ἄρμοσμένον τὰ μάλιστα· οὐδεὶς τῶν λίθων τρήκοντα
ποδῶν ἐλάσσων.

(a) The original Greek.

of which (the pyramid) is in each direction (the face; the front) each one 8 {100 feet; 10,000 square feet} of being (the pyramid) (of four equal angles; square) and the height the same.

(b) A word by word translation.

Figure 5
Herodotus (*Herodoti Historiae*, p. 124, II, lines 16–20)

Furthermore, Herodotus's figures about the dimensions of the Great Pyramid are wildly off. The Great Pyramid neither is nor was (it has lost some height over the years) anywhere near 800 feet tall nor 800 feet square at the base. Finally, we should note that Herodotus wrote roughly two millennia after the Great Pyramid was constructed.

The distorted version of Herodotus's story makes little sense. Even the authors who quote it do not give a reason why the Egyptians would want to build a pyramid so that its height was the side of a square whose area is exactly the area of one of the faces. This idea sounds like something dreamt up to justify a coincidence rather than a realistic description of how the dimensions of the Great Pyramid were chosen. It does not appear that the Egyptians even knew of the existence of Φ much less incorporated it in their buildings (see [Gil; pp. 238–9]).

Some authors [Men; p. 64, 73] have noticed that the ratio of the circumference of the Great Pyramid to the height is approximately 2π . Using the figures given above yields $4 \times 755.79 / 481.4 \approx 6.28$ which is well within $\pm 2\%$ of $2\pi = 6.2831853\dots$. Now, of course, we must decide whether this is a coincidence or whether there is some reason why this ratio would be close to 2π .

A wide variety of theories have been advanced for the proportions of the Egyptian pyramids ranging from preserving a particular slope [Gil; pp. 185–187] to using rollers to measure horizontal distances and ropes to measure vertical distances [Men; pp. 64, 73]. [FiS] gives a survey of these theories.

Misconception: The Greeks Used Φ in the Parthenon

Many sources (for example, [Ber; pp. 94–95], [Bro; p. C11], [Hil; p. 79], [Hun; p. 63], [Man; p. 168], [Mit; p. 1445], [Pap; p. 102]) claim that the Parthenon embodies the golden ratio in some way. To support this claim authors often include a figure like Figure 6 where the large rectangle enclosing the end view of the Parthenon-like temple is a golden rectangle. None of these authors is bothered by the fact that parts of the Parthenon are outside the golden rectangle.

For example, Bergamini [Ber; pp. 94–95] presents a photograph and a line drawing to illustrate how the Parthenon fits snugly in a golden rectangle. The caption reads as follows.

The Parthenon at Athens fits into a golden rectangle almost precisely once its ruined triangular pediment is drawn in. Though it incorporates many geometric balances, its builders in the fifth century B.C. probably had no conscious knowledge of the golden ratio.

H. E. Huntley [Hun; p. 63] presents a figure that looks like Figure 6 with the caption,

The Parthenon at Athens, built in the fifth century, B.C., one of the world's most famous structures. While its triangular pediment was still intact, its dimensions could be fitted almost exactly into a Golden Rectangle, as shown above. It stands therefore as another example of the aesthetic value of this particular shape.

Similar statements can be found in the *Random House Encyclopedia* [Mit; p. 1445] and in this journal [Man; p. 168].

The dimensions of the Parthenon vary from source to source probably because different authors are measuring between different points. With so many numbers

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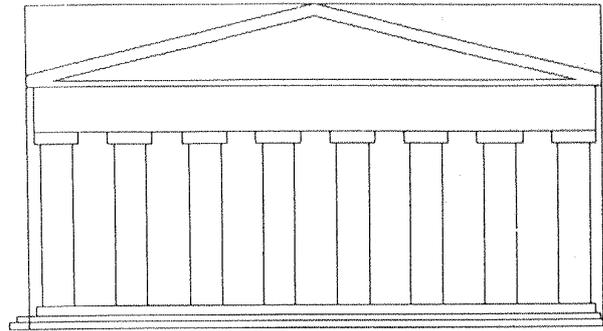


Figure 6
The Parthenon supposedly fitting into a golden rectangle

available a golden ratio enthusiast could choose whatever numbers gave the best result.

Marvin Trachtenburg and Isabelle Hyman [Tra; p. 90] give the dimensions of the Parthenon as: height = 45 feet 1 inch; width = 101 feet, 3.75 inches; length = 228 feet 1/8 inch. They do not specify the points between measurements. These numbers give the ratios width/height $\approx 2.25 = 9/4$ and length/width ≈ 2.25 which are well outside the acceptance range. The reader might be struck by the fact that the ratio 2.25 appears as the ratio of width/height and length/width. Stuart Rossiter [Ros; p. 88] gives the height of the apex above the stylobate as 59 feet. This gives a ratio of $101/59 \approx 1.71$ which also falls outside the acceptance range. According to Stuart Rossiter [Ros; p. 77]

Its (the Propylaea's) axis is alined to that of the Parthenon, its width would have equalled the length of the temple, and like the Parthenon, its proportions are worked out in the ratio of 4:9, thus affording the only certain example before Hellenistic times of designing one building in direct relationship to another.

More generally, Christine Flon [Flo; p. 131] dismisses much of the numerical mysticism about ancient structures with the following comment.

On the basis of a small number of ancient texts, an effort has been made to find (in buildings sufficiently well preserved) a coherent system of proportions based on the golden number, $\pi(\pi)$, or on the universal ratios of the Pythagoreans. Almost always, when all possible measurements have been taken, some system of geometric figures or some modular common denominator has come to light. However, the validity of this research remains uncertain: it is easy to overestimate the importance of an architectural speculation. It is not unlikely that some architects, in imitation of sculptures such as Polycleitos, should have wished to base their works on a strict system of ratios, but it would be wrong to generalize. In the conservative environment of ancient Greece, architectural activity was an empirical practice in which experience and intuition, that is to say 'mastery', played a large part.

conception: Many Painters, Including Leonardo da Vinci, Used Φ

Golden ratio enthusiasts (for example [Ber; pp. 94, 96], [Wil; p. 74]) also claim that Leonardo da Vinci used the golden ratio widely in his artistic works. In particular, Margaret F. Willerding [Wil; p. 74] states

THE GOLDEN SECTION. As we look about us, we see many geometric patterns in nature, art, architecture, and even in such mundane things as tables, chairs, and cups and saucers. A very special pattern that we find in leaves around the stems of plants, in seashells, and in the arrangement of sunflower seeds is called the *golden section*. Leonardo da Vinci, one of the greatest geniuses of all times, stated proportions for the ideal figure in terms of this geometric pattern.

Bergamini [Ber; p. 94] is more specific:

SYMMETRY IN A FACE. In Leonardo da Vinci's drawing of an old man, probably a self-portrait, the artist has overlaid the picture with a square subdivided into rectangles, some of which approximate Golden Rectangles.

The drawing Bergamini is describing is often reproduced (Figure 7). Since the rectangles in Figure 7 are very roughly drawn and do not have square corners it is difficult to see the significance of the claim that some rectangles "approximate" Golden rectangles.

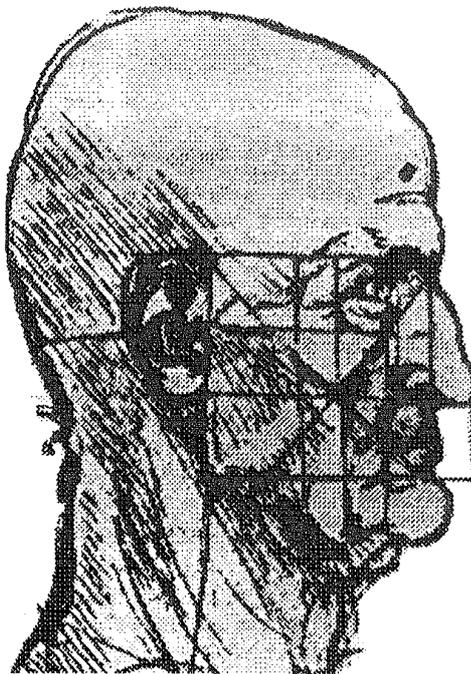


Figure 7
A drawing attributed to Leonardo da Vinci

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The claims that Leonardo da Vinci used the golden ratio seem to be based on the fact that he illustrated Luca Pacioli's book *De Divina Proportione* (1498, [Cla; p. 72]). The biographies of Leonardo da Vinci by Clark, Vallentin [Val], and Zammattio et al. [Zam] give no indication that he used the golden ratio in paintings or drawings not intended for Pacioli's book. Roger Fischler [Fi3; p. 31] claims that Pacioli "advocated a classical Vitruvian system, that is a system based on simple proportions," and did not advocate using the golden ratio for painting.

Another painting often used (for example [Ber; p. 96], [Pap; p. 33]) to support the claim that Leonardo da Vinci used the golden ratio extensively in his art is a painting of St. Jerome ([Cla; plate 18]). For example, Bergamini states

RECREATIONS of DA VINCI. St. Jerome, an unfinished canvas by Leonardo da Vinci painted about 1483, shows the great scholar with a lion lying at his feet. A Golden Rectangle (black overlay) fits so neatly around St. Jerome that some experts believe Leonardo purposely painted the figure to conform to those proportions. Such an approach would have been in keeping with the artist's ardent interest in mathematics. He took special delight in what he once described as 'geometrical recreations.'

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A glance at Figure 8 from [Ber; p. 96] is sufficient to show the flaws in the claims about this painting. The placement of the rectangle is somewhat arbitrary since the top does not touch the head. The rectangle is drawn using a very thick line. Its left side is tangent to a small fold of fabric and does not touch St. Jerome's body at any point. St. Jerome's right arm extends well past the left side of the superimposed rectangle. Finally, Leonardo da Vinci's acquaintance with the divine proportion dates from his meeting with Luca Pacioli, which occurred 13 years after he painted St. Jerome.

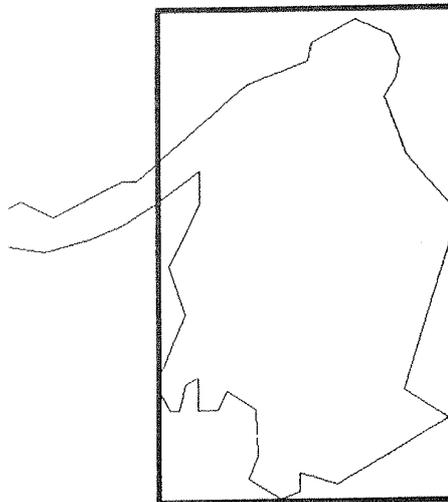


Figure 8
St. Jerome by Leonardo da Vinci

Bergamini [Ber; pp. 94-97] also claims that Mondrian and Seurat used the golden ratio in their paintings. Again no exact data are given, but rectangles are

superimposed over sketches and paintings with no justification being given for the particular lines being drawn. Roger Fischler's [Fi3; p. 31] detailed analysis of Seurat's writings, sketches and paintings shows that Seurat did not use the golden ratio as a basis for his paintings. Fischler also discusses the alleged use of the golden ratio by other painters, including Le Corbusier [Fi1].

Misconception: The UN Building Embodies the Ratio Φ

The UN Building is supposedly another example of the golden ratio. *The Random House Encyclopedia* [Mit; p. 1445] states

The Greeks saw beauty in number and shape and their excitement with the golden ratio [5] manifested itself in their art and architecture and has been echoed by later civilizations in such places as Notre Dame in Paris, in the architecture of Le Corbusier, and in the UN building in New York.

We can assume that the reference is to the Secretariat Building, the most prominent of the UN buildings in its New York complex. The UN gave these dimensions for the Secretariat Building over the phone: 505 feet high, 287 feet wide, and 72 feet thick. The only ratio even remotely close to Φ is height/width ≈ 1.76 which is outside the acceptance range 1.58 to 1.66. [Mob; p. 301] gives the height of the Secretariat building as 550 feet, while [Whi] gives the height as 544 feet. These values give height/width ratios of 1.92 and 1.90, which are even further away from Φ .

The explanation of the differences in height from the people in the UN's Architectural Planning Section was that the building rises 505 feet from the main entrance level, but it extends 41 feet below this level. Thus, the height depends on whether you measure it from the west side of the building and street level (41 feet up) or whether you measure it from the east side at river level. At any rate, the Secretariat building does not appear to be designed on the basis of the golden ratio.

To see how significant the golden ratio is in architecture I consulted several books on architecture. I could not find golden number, golden ratio, golden section or divine proportion in the indexes of [Mus] or [Tra]. An attack on the Φ cult in architecture is found in [Coo].

Misconception: A Golden Rectangle Is the Most Esthetically Pleasing Rectangle

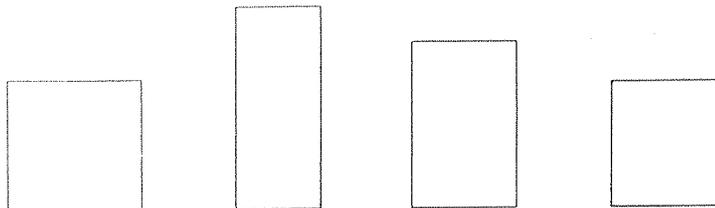
A common claim is that the golden rectangle is in some way the most esthetically pleasing of all rectangles. For example,

The golden rectangle was used by Greek architects in dimensions of their temples and other buildings. Psychologists have shown that most people will unconsciously select post cards, pictures, mirrors, and packages with these dimensions. For some reason, the golden rectangle holds the most artistic appeal. [Wil; p. 74]

The Golden Rectangle is said to be one of the most visually satisfying of all geometric forms; [Ber; p. 94]

Tastes may vary, but many people asked to select one of the shapes shown in Figure 13.5 for note-paper or for the frame of a picture would choose the third. It is not too square and not too elongated. [Lan; p. 222]

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Figure 9
A reconstruction of Figure 13.5 from Land's book

Figure 9 shows a computer reconstruction of Land's Figure 13.5. *The New Columbia Encyclopedia* [Har; p. 1103] in its article about the golden section states

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The Golden Rectangle, whose length and width are the segments of a line divided according to the Golden Section, occupies an important position in painting, sculpture, and architecture, because its proportions have long been considered the most attractive to the eye.

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Many of the claims about people's preference for the golden ratio seem to be based in large part on the experiments of Gustav Fechner performed in the 1860s. According to Leonard Zusne [Zus; p. 399],

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Fechner's procedure consisted in placing 10 rectangles before a subject and asking him to select the most pleasing rectangle. The rectangles varied in their height/length ratios from 1.00 (square) to .40... The modal rectangle had a height/length ratio of .62, i.e., the golden section, with 76% of all choices centering on three rectangles having the ratios .57, .62, and .67. While all other rectangles received less than 10% of the choices each, Fechner's results still indicated that many other rectangles besides the golden-section rectangle were considered the most pleasing by a fair number of subjects.

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The above statement can hardly be viewed as overwhelming evidence for the importance of the golden ratio in esthetics. Furthermore, Fechner's testing was rather limited since he offered only 10 choices. If the choices were presented ordered by increasing or decreasing proportion one could argue that people would tend to select the ones in the middle.

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H. R. Schiffman and D. J. Bobko [Sch; p. 102] state

Research on the golden section proportion as an empirically demonstrable preference has most often been applied to the rectangle where the results, on the whole, are negative.

Figures 10 and 11 can be used in your own tests to see whether people consistently select the golden ratio as the most pleasing ratio for a rectangle. Figure 10 shows 48 randomly arranged rectangles all having the same height but with their widths ranging from 0.4 times the height to 2.5 times the height.

In Figure 11 the same 48 rectangles are arranged by increasing length when read from left to right and bottom to top. Figures 10 and 11 each contain two rectangles that qualify as golden rectangles, one having ratio Φ and the other having ratio $1/\Phi$, and two rectangles that exhibit the ratio of the Parthenon: $9/4$ and $4/9$. See if you can identify them.

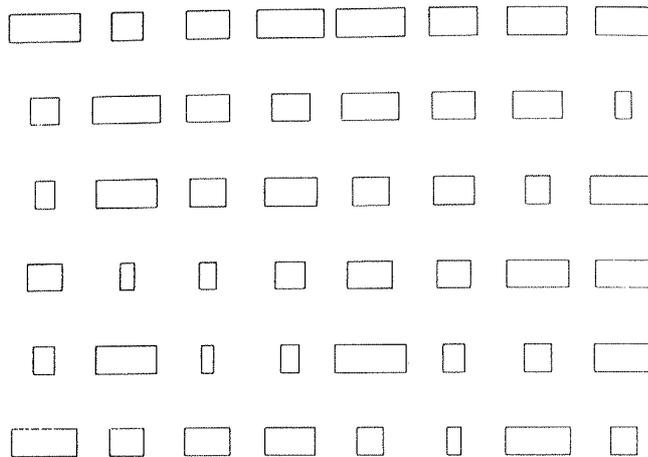


Figure 10
Rectangles with ratios ranging from 0.4 to 2.5 in random order

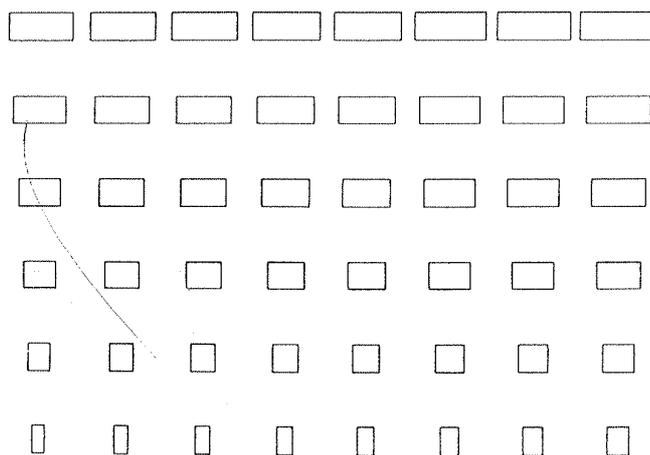


Figure 11
Rectangles with ratios ranging from 0.4 to 2.5 in linear order

My informal experiments asking people attending my lectures to select the “most pleasing rectangle” suggest that people cannot find the golden rectangle in Figure 10. Furthermore, they generally select slightly different rectangles as the most pleasing rectangles when shown both Figure 10 and Figure 11.

In the experiments I have conducted so far, the most commonly selected rectangle is one with a ratio of 1.83 (row 3, column 4 of Figure 10). In Figure 10, the numbers closest to Φ , $1/\Phi$, $9/4$ and $4/9$ are locations (4,5), (5,4), (3,8) and (6,6) respectively where the first coordinate gives the row and the second coordinate gives the column. In Figure 11, the corresponding locations are (4,4), (6,6), (1,2) and (6,2).

I also experimented with a collection of 48 rectangles with ratios ranging from 1.6 to 1.7, which has convinced me that most people cannot see any differences

among the rectangles whose ratios are so close together. This strongly suggests that even if people do prefer certain rectangles, the only reasonable claim would be that people prefer ratios in a certain range. The various claims made about the esthetic importance of the golden ratio seem to be without foundation.

Misconception: The Human Body Exhibits Φ

Some authors claim that the human body is designed according to the golden ratio. For example, Browne [Bro; p. C11] states

Penrose tiling has another characteristic that fascinates mathematicians and architects: it exhibits a feature known to the ancient Greeks as 'the golden mean,' a ratio that has been used in paintings, sculpture and architecture through the ages. The golden mean governs the proportions of the Parthenon and many other classical buildings. The ratio, as applied to artistic shapes and structures, is roughly equal to the ratio of lengths of the human body as divided at the navel, and is regarded as particularly pleasing to the eye.

This passage repeats many of the misconceptions we have already discussed and adds the claim that the ratio of a person's height to the height of his/her navel is roughly the golden ratio. We are not told why this is significant; the navel is a scar of no great importance in an adult human being. Much of the work relating the golden ratio to the human body suffers from the Pyramidology Fallacy.

While it might be entertaining to compute the ratio of many people's heights to the elevations of their navels, I did not spend much time on this effort. I did compute the ratios for the four members of my immediate family: 1.59, 1.63, 1.65 and 1.66. Their average is 1.63, which falls within our test interval for the golden ratio, although even in this small sample there is a significant amount of variation. However, there is some ambiguity about the precise location of the navel since it has a nontrivial length.

Boles and Newman [Bol; p. 47] find Φ in, among other things, two Greek statues, Doryphoros the Spearbearer by Polycleitos and Aphrodite of Cyrene.

The sketch of Doryphoros shows him divided into three zones: from the top of the head to the right nipple on the chest which is given the length 1; from the right nipple on the chest to the right knee which is given the length 1.61803; from the right knee to the big toe on the right foot which is given the length 1. It is unlikely that measurements were made accurately enough to justify a ratio with 5 decimal places. In particular, a knee covers a large area and should not be treated as a single point. A quick examination of their diagram shows that the left side of Doryphoros does not have the same proportions.

The sketch of Aphrodite of Cyrene is also divided into three zones, including one with a length of 1.61803. This time, since Aphrodite is missing her head, the first zone runs from the stump of the neck to the navel, the second zone runs from the navel to the right knee, and the third zone runs from the right knee to some indeterminate point on the right foot. Again, assigning 1.61803 as a length is nonsense and seems to imply that the sculptor of Aphrodite anticipated that she would lose her head.

Besides seeing the golden ratio in the statues just mentioned, Boles and Newman see it in many animal forms. [Bol; p. 59] shows a hawk, a dragonfly, a flying squirrel and a sunfish boxed by golden rectangles. The proportion of the golden rectangle is given as 1.61803 as before. Of course, wings, legs and fins can

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be moved over a wide range of positions and it is not surprising that the golden rectangle can often be produced. Like the arm of St. Jerome, the tail of the flying squirrel extends well past the boundaries of the bounding golden rectangle. Some of the wing feathers of the hawk also extend past the boundaries. On page 59 of [Bol] the left side of the bounding golden rectangle does not touch any part of the sunfish or dragonfly. Despite these difficulties, Boles and Newman express the proportions of the bounding rectangle using 5 significant figures.

Misconception: Virgils' Aeneid Exhibits Φ

George E. Duckworth wrote *Structural Patterns and Proportions in Vergil's Aeneid* [Duc] to prove that Virgil used Φ as a key element in designing the Aeneid. Duckworth arrives at this conclusion by computing the ratios of the lengths of different passages in the Aeneid. His work is criticized by Curchin and Fischler [Cur], Fischler [Fi4], and Bews [Bew]. Some of the points raised in [Fi4] are quite interesting to a mathematician.

Duckworth measures the number of lines in what he calls major (M) and minor (m) passages. If m/M is the reciprocal of the golden ratio $1/\Phi$, then $m/M = M/(m + M)$. On this basis Duckworth claims that he can use either measure and uses $M/(m + M)$ as being "slightly more accurate" [Duc, p. 43, Note 6]. On page 65 in Note 7, Duckworth observes that m/M shows a greater variation from the golden ratio than $M/(m + M)$. Unfortunately, he does not realize that he is fooling himself by using $M/(m + M)$. If m/M varies uniformly and randomly over $[0, 1]$, the ratio $M/(m + M)$ is restricted to the range $[0.5, 1]$ and is not uniformly distributed.

This point is illustrated in Figures 12 and 13. Figure 12 is a histogram for 1000 points chosen at random from the uniform distribution $[0, 1]$ for the value $r = m/M$.

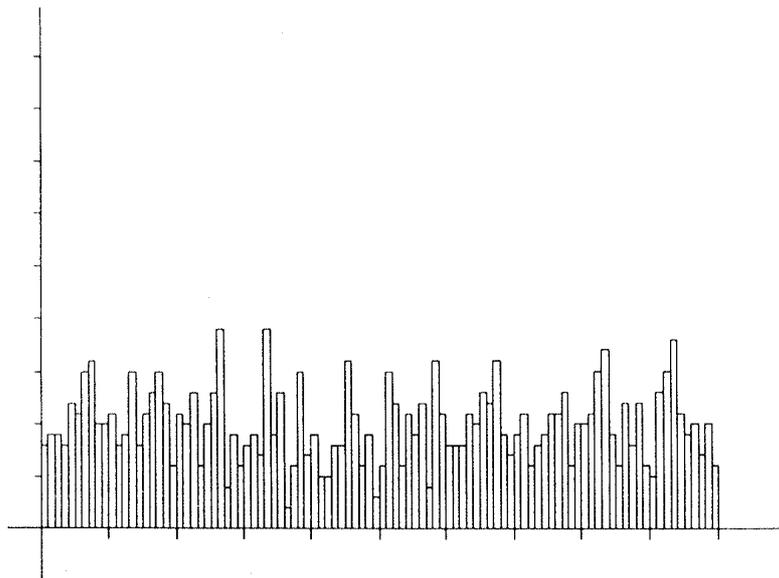


Figure 12
One thousand values for m/M chosen at random

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The points have been grouped into intervals of width 0.01. Figure 13 shows the 1000 points from Figure 12 replotted using $M/(m + M) = 1/(1 + r)$ instead of $m/M = r$.

The expected value of points chosen uniformly and randomly from the interval $[0, 1]$ is 0.5. On the other hand, choosing values from $[0, 1]$ at random and plotting them as $1/(1 + r)$ produces the following expected value:

$$\int_0^1 \frac{dx}{1+x} = \ln(1+x) \Big|_0^1 = \ln 2 - \ln 1 = \ln 2 \approx 0.69,$$

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which is not the golden ratio, but is nevertheless closer to 0.61803... than is 0.5. For additional mathematical analysis of Duckworth's approach see [Fi4].

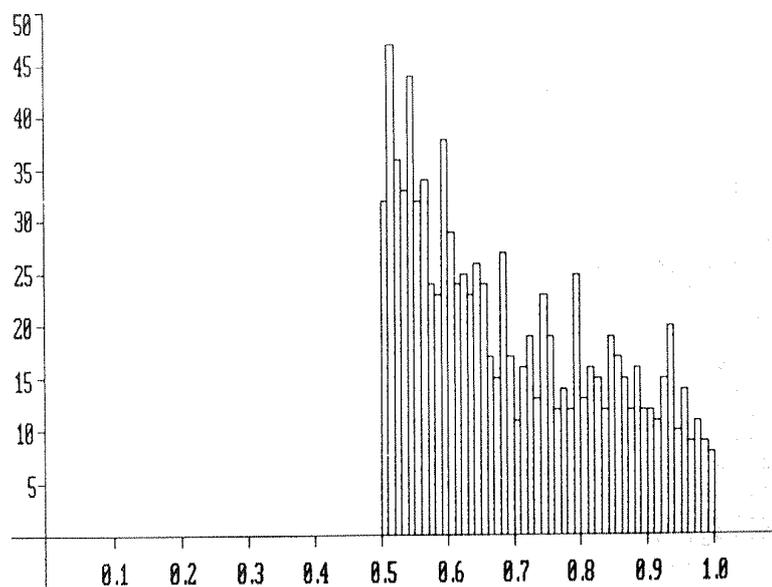


Figure 13
The values from Figure 12 replotted using $M/(m + M)$

Curchin and Fischler [Cur; p. 133] conclude that

An analysis, using the ratio m/M , has now been made with Duckworth's data and indicates that random scattering is indeed the case with Virgil."

Not only did Duckworth waste a lot of time on his misguided effort, but other people bandy his results about uncritically. An example can be found in [Nim; p. 317] in the chapter entitled "Golden Numbers." This chapter, besides repeating some of the errors discussed earlier in this paper, contains the following sentence on page 317:

And Vergil, Dante's guide, appears to have made an almost unbelievable use of the proportions of the golden section and the Fibonacci numbers (as they were later named).*

The * refers to a footnote referencing Duckworth's work.

Acknowledgments. I would like to thank Juliet Markowsky and Ed Lavery for helping to make this paper more readable. I am grateful to Jay Bregman and Tina Passman for translating the Greek passage from Herodotus. I would like to acknowledge helpful references from Pat Burnes, Tom Byther, Mohamed Elgaaly, Ed Ferguson, Martin Gardner, Bill Halteman, Paul Schroeder and Bill Soule. I would also like to thank my family for letting me measure their proportions and all the people who attended my lectures on the golden ratio and selected the most beautiful rectangles.

References

The following is not an exhaustive list of books and papers dealing with the golden ratio, but it should serve as a good starting point for your own investigations in this area. (Note that Roger Fischler and Roger Herz-Fischler are the same person.)

- [Ame] American Institute of Steel Construction, *Code of Standard Practice for Steel Buildings and Bridges*, AISC, Chicago, 1986.
- [Ber] David Bergamini and the Editors of LIFE, *Mathematics*, Time Incorporated, New York, 1963.
- [Bew] Jane B. Bews, Aeneid I and .618?, *Phoenix* 24, 2 (1970) 130–143.
- [Bol] Martha Boles and Rochelle Newman, *The Golden Relationship*, 2nd ed., Pythagorean Press, Bradford, MA, 1987.
- [Bro] Malcolm W. Browne, “Impossible” form of matter takes spotlight in study of solids, *New York Times*, September 5, 1989, pages C1 & C11.
- [Bur] David M. Burton, *The History of Mathematics: An Introduction*, Allyn and Bacon, Boston, 1985.
- [Cla] Kenneth Clark, *Leonardo da Vinci*, Penguin, Harmondsworth, England, 1976.
- [Coo] T. Cook, A new disease in architecture, *The Nineteenth Century & After* 91 (1922) 521–53.
- [Cov] Thomas M. Cover, Do longer games favor the stronger player?, *The American Statistician* 43, 4 (1989) 277–278.
- [Cox] H. S. M. Coxeter, The golden section, phyllotaxis, and Wythoff’s game, *Scripta Mathematica* 19 (1953) 135–143.
- [Cur] L. Curchin and R. Fischler, Hero of Alexandria’s numerical treatment of division in extreme and mean ratio and its implications, *Phoenix* 35, 2 (1981) 129–133.
- [Duc] George E. Duckworth, *Structural Patterns and Proportions in Vergil’s Aeneid: A Study in Mathematical Composition*, The University of Michigan Press, Ann Arbor, 1962.
- [Fi1] Roger Fischler, The early relationship of Le Corbusier to the “golden number,” *Environment and Planning B* 6 (1979) 95–103.
- [Fi2] Roger Fischler, What did Herodotus really say? or how to build (a theory of) the great pyramid, *Environment and Planning B* 6 (1979) 89–93.
- [Fi3] Roger Fischler, On the application of the golden ratio in the visual arts, *Leonardo* 14 (1981), 31–32.
- [Fi4] Roger Fischler, How to find the “golden number” without really trying, *Fibonacci Quarterly* 19 (1981) 406–410.
- [Fi5] Roger Fischler, Théories mathématiques de la grande pyramide, *Crux Mathematicorum* 4 (1978) 122–129.
- [Flo] Christine Flon, *The World Atlas of Architecture*, Mitchell Beazley Publishers, 1984. This is the English edition of *Le Grand Atlas de l’Architecture Mondiale*.
- [Fo1] D. H. Fowler, A generalization of the golden section, *Fibonacci Quarterly* 20 (1982) 146–158.
- [Fo2] D. H. Fowler, *The Mathematics of Plato’s Academy: A New Reconstruction*, Clarendon Press, Oxford, 1987.
- [Ga1] Martin Gardner, *Fads and Fallacies in the Name of Science*, Dover, New York, 1957.
- [Ga2] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers*, W. H. Freeman, New York, 1989.
- [Gil] Richard J. Gillings, *Mathematics in the Time of the Pharaohs*, MIT Press, Cambridge, 1972.
- [Gru] Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, New York, 1987.
- [Har] William H. Harris and Judith S. Levey, *The New Columbia Encyclopedia*, Columbia University Press, New York, 1975.
- [He1] Herodotus, *The Histories*, translated by Aubrey de Séincourt, Penguin Books, Harmondsworth, England, 1954 (1979 reprinting).
- [He2] Herodotus, *The History of Herodotus*, translated by George Rawlinson, reprinted in 1936 by J. M. Dent & Sons, London.

- [Her] Roger Herz-Fischler, *A Mathematical History of Division in Extreme and Mean Ratio*, Wilfrid Laurier University Press, Waterloo, Canada, 1987.
- [Hil] Francis S. Hill, Jr., *Computer Graphics*, Macmillan, New York, 1990.
- [How] W. W. How and J. Wells, *A Commentary on Herodotus*, Oxford University Press, 1912 (reprinted 1967).
- [Hud] Carolus Hude, ed., *Herodoti Historiae*, Oxonii, Londini, 1947.
- [Hun] H. E. Huntley, *The Divine Proportion*, Dover, New York, 1970.
- [Knu] Donald E. Knuth, *The Art of Computer Programming: Fundamental Algorithms*, Vol. 1, 2nd ed., Addison-Wesley, Reading, MA, 1973.
- [Lan] Frank Land, *The Language of Mathematics*, Doubleday, Garden City, NY, 1963.
- [Las] François Lasserre, *The Birth of Mathematics in the Age of Plato*, American Research Council, Larchmont, NY, 1964.
- [Lin] Charles F. Linn, *The Golden Mean*, Doubleday, Garden City, NY, 1974.
- [Man] George Manuel and Amalia Santiago, An unexpected appearance of the golden ratio, *The College Mathematics Journal* 19 (1988) 168-170.
- [Men] Kurt Mendelssohn, *The Riddle of the Pyramids*, Praeger Publishers, New York, 1974.
- [Mit] James Mitchell, ed., *The Random House Encyclopedia*, Random House, NY, 1977.
- [Mob] *Mobil Travel Guide (Northeastern States)*, Prentice-Hall, Englewood Cliffs, NJ, 1989.
- [Mus] John Musgrove, ed., *Sir Banister Fletcher's: A History of Architecture*, 19th ed., Butterworth's, London, 1987.
- [Nim] John F. Nims, *Western Wind: An Introduction to Poetry*, Random House, New York, 1983.
- [Odo] George Odom, Elementary problem E 3007, *American Mathematical Monthly* 90 (1983) 482.
- [Oxf] *The Compact Edition of the Oxford English Dictionary*, Oxford University Press, Oxford, 1971.
- [Pap] Theoni Pappas, *The Joy of Mathematics*, Wide World Publishing, San Carlos, CA, 1989.
- [Rig] J. F. Rigby, Equilateral triangles and the golden ratio, *The Mathematical Gazette* 72, 459, (1988) 27-30.
- [Ros] Stuart Rossiter, *Greece*, 3rd ed., (in the *Blue Guide Series*), Ernest Benn Ltd., London, 1977.
- [Sch] H. R. Schiffman and D. J. Bobko, Preference in linear partitioning: the golden section reexamined, *Perception & Psychophysics* 24 (1978) 102-103.
- [Ser] Michael Serra, *Discovering Geometry: An Inductive Approach*, Key Curriculum Press, Berkeley, CA, 1989.
- [Smi] David Eugene Smith, *History of Mathematics, Vol. II Special Topics of Elementary Mathematics*, Dover, New York, 1953.
- [Tas] Socrates G. Taseos, *Back in Time 3104 B.C. to the Great Pyramid*, SOC Publishers, Charlotte, NC, 1990.
- [Tho] D'arcy Thompson, *On Growth and Form*, abridged edition, edited by J. T. Bonner, Cambridge University Press, 1961.
- [Tra] Marvin Trachtenberg and Isabelle Hyman, *Architecture: From Prehistory to Post-Modernism/The Western Tradition*, Harry N. Abrams Inc., New York, 1986.
- [Tur] Herbert Westren Turnbull, The great mathematicians, in James R. Newman's *The World of Mathematics*, Simon and Schuster, New York, 1956, 75-168.
- [Val] Antonina Vallentin, *Leonardo da Vinci: The Tragic Pursuit of Perfection*, translated by E. W. Dickes, Viking Press, New York, 1938.
- [Whi] Norval White and Elliot Willensky, *ALA (American Institute of Architects) Guide to New York City*, Macmillan, New York, 1968.
- [Wil] Margaret F. Willerding, *Mathematical Concepts, A Historical Approach*, Prindle, Weber & Schmidt, Boston, 1967.
- [Zam] Carlo Zammattio, Augusto Marinoni, and Anna Maria Brizio, *Leonardo the Scientist*, McGraw-Hill, New York, 1980.
- [Zus] Leonard Zusne, *Visual Perception of Form*, Academic Press, New York, 1970.