Lecturas Matemáticas
Volumen **27** *(2006), páginas 21–25*

Even from Gregory-Leibniz series π could be computed: an example of how convergence of series can be accelerated*

Vito Lampret
University of Ljubljana, Ljubljana, Slovenia

Abstract. Gregory-Leibniz series $\Sigma_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j-1} = \frac{\pi}{4}$, is discussed in literature as an example of beautiful, interesting and simple analytic expression for π. Unfortunately, this series is considered by most authors as unsuitable for computation of π. It is shown that, using a simple transformation of the series and applying the Euler–Maclaurin summation formula, slow convergence can be accelerated to the point where numerical computation of π can be performed accurately to several decimal places. It is also suggested that the Euler–Maclaurin formula, of some low order, should be included in the undergraduate mathematics curriculum.

Key words and phrases. acceleration, alternating, convergence, estimate, Euler-Maclaurin, Gregory-Leibniz, remainder, series.

2000 AMS Mathematics Subject Classification. 40A05, 40A25, 65B10, 65B15, 97D30, 97D40.

Resumen. La serie de Gregory-Leibniz $\Sigma_{j=1}^{\infty}\frac{(-1)^{j+1}}{2j-1} = \frac{\pi}{4}$, se conoce en la literatura como un ejemplo de una hermosa, interesante y sencilla expresión analítica de π. Desafortunadamente, la mayoría de los autores la consideran inapropiada para el cálculo de π. En esta nota se usa una simple transformación de la serie y una aplicación de la fórmula sumación de Euler-Maclaurin para mostrar cómo esta convergencia lenta puede acelerarse hasta el punto de que el cálculo numérico de π nos produzca un valor bastante preciso con varias cifras decimales. También se sugiere que la fórmula de Euler-Maclaurin debiera incluirse en los cursos de matemáticas del pregrado.

* Since next year is the third-centenary of Euler's birth, April 15, 1707, we dedicate this contribution to him, the main creator of the famous Euler-Maclaurin summation formula.

It seems that the Gregory-Leibniz series, figuring in the identities

$$\frac{\pi}{4} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2j-1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}, \tag{1}$$

is not suitable for computation of π. Indeed, from the well known Leibniz estimate[†] [6, p. 216-217]

$$0 \leq \; a_{n+1} - a_{n+2} \; \leq \; (-1)^n r_n \; \leq \; a_{n+1}$$

of remainder $r_n := \sum_{k=n+1}^{\infty} a_k$ of alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$, where a_n monotonically decreasingly converges to zero, we obtain, for the remainder in series (1), the estimate

$$\frac{2}{(2n+1)(2n+3)} \leq (-1)^n r_n \leq \frac{1}{2n+1}, \tag{1a}$$

valid for every positive integer n. Unfortunately, this estimate indicates slow convergence of series (1). To compute π from (1), using (1a), to six decimal places for example, we need to sum up much more than five millions terms of series (1) due to rounding the terms of series during the calculations. Figure 1 illustrates relation (1a) by showing the graphs of sequences $n \mapsto (-1)^n r_n \equiv \frac{\pi}{4} - s_n$ and $n \mapsto 1/(2n+1)$ for series (1).

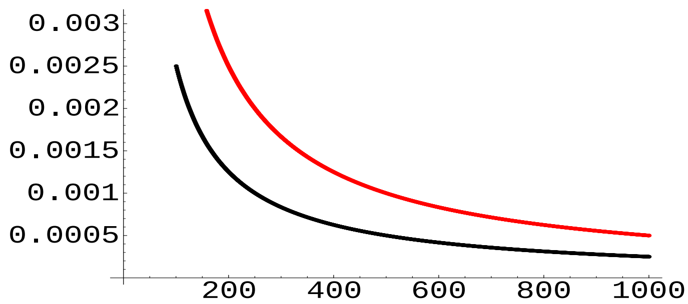

Figure 1: Graphs of sequences $n \mapsto (-1)^n r_n$ and $n \mapsto 1/(2n+1)$.

In order to accelerate convergence in (1) we first substitute alternating conditionally convergent series (1) by absolutely convergent one. Indeed, in partial sums of (1) with even indexes, we collect pairwise the terms with odd and even

[†]This estimate is sharp as it could be seen from the example of geometric series.

indexes

$$\begin{aligned}\sum_{k=1}^{2n}\frac{(-1)^k}{2k+1} &= \sum_{i=1}^{n}\frac{1}{4i+1}-\sum_{i=1}^{n}\frac{1}{4i-1}\\ &= \sum_{i=1}^{n}\frac{-2}{16i^2-1}\,.\end{aligned}$$

This way, using (1), we get the expression

$$\frac{\pi}{4} = 1-2\sum_{i=1}^{\infty}\frac{1}{16i^2-1}\,. \tag{2}$$

The obtained series converges faster than the original one. To accelerate also convergence in (2) we use the Euler-Maclaurin summation formula of order four [3, p. 320, Theorem] or [4, p. 119, Theorem 2]. It says that for any $m \in \mathbb{N}$ and any function $f \in C^4[1,\infty)$, such that $f^{(4)}(x)$ does not change sign and $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} f'(n) = \lim_{n\to\infty} f^{(3)}(n) = 0$ and $\int_1^\infty \left|f^{(4)}(x)\right| dx$ converges, there exists a number $\omega(m)$, depending on m, such that

$$\sum_{k=1}^{\infty} f(k) = \sum_{k=1}^{m-1} f(k) + \int_m^\infty f(x)\,dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \omega(m)\cdot f^{(3)}(m) \tag{3}$$

and

$$0 \le \omega(m) \le \frac{1}{384}\,. \tag{3a}$$

The infinite series in (2) originates from function $\varphi : [1,\infty) \to \mathbb{R}$, such that

$$\varphi(x) \equiv \frac{1}{16x^2-1} \tag{4}$$

$$\varphi'(x) \equiv -\frac{32x}{\left(16x^2-1\right)^2} < 0 \tag{4a}$$

$$\varphi^{(2)}(x) \equiv \frac{32\left(48x^2+1\right)}{\left(16x^2-1\right)^3} > 0 \tag{4b}$$

$$\varphi^{(3)}(x) \equiv -\frac{6144\left(16x^3+x\right)}{\left(16x^2-1\right)^4} < 0 \tag{4c}$$

$$\varphi^{(4)}(x) \equiv \frac{6144\left(1280x^4+160x^2+1\right)}{\left(16x^2-1\right)^5} > 0. \tag{4d}$$

Because

$$\int_m^\infty \varphi(x)\,dx = \frac{1}{8}\ln\left(\frac{4m+1}{4m-1}\right),$$

we obtain from equations (3)–(3a) and (4)–(4c), for any $m \in \mathbb{N}$, the following expression

$$\sum_{i=1}^{\infty} \frac{1}{16i^2-1} = \sum_{i=1}^{m-1} \frac{1}{16i^2-1} + \frac{1}{8}\ln\left(\frac{4m+1}{4m-1}\right) + \frac{1}{2\,(16m^2-1)} + \frac{8m}{3\,(16m^2-1)^2} - \omega(m)\cdot\frac{6144\,(16m^3+m)}{(16m^2-1)^4}\,.$$

Hence, from (2) we conclude

$$\pi = \pi_m + \rho_m\,, \tag{5}$$

where

$$\pi_m = 4 - \left[8\sum_{i=1}^{m-1}\frac{1}{16i^2-1} + \ln\left(\frac{4m+1}{4m-1}\right) + \frac{4}{16m^2-1} + \frac{64m}{3\,(16m^2-1)^2}\right] \tag{5a}$$

and

(5b)
$$0 \le \rho_m \le 8\omega(m)\cdot\frac{6144\,(16m^3+m)}{(16m^2-1)^4} < \frac{32}{(4m+1)\,(4m-1)^4} < \frac{1}{(2m-1)^5}$$

for every $m \in \mathbb{N}$. Figure 2 shows graphs of sequences $m \mapsto \rho_m$ and $m \mapsto (2m-1)^{-5}$.

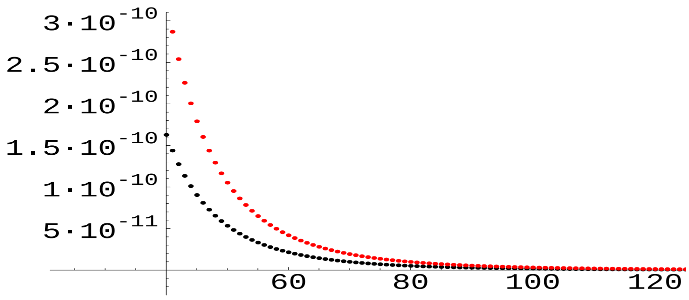

Figure 2: Graphs of sequences $m \mapsto \rho_m$ and $m \mapsto (2m-1)^{-5}$.

Using [7] we find from (5a) that $\pi_{10} = 3.1415924875\ldots$ and from (5b) that $0 \le \rho_{10} < 4.04 \times 10^{-7}$. Hence $3.141592 \le \pi < 3.141592488 + 4.04 \times 10^{-7} = 3.141592892$. This way we determined π to six decimal places $\pi = 3.141592\ldots$. We point out that during this computation we have to sum only 13 terms. To obtain more decimal places of π we should increase parameter m. However, sequence $m \mapsto \pi_m$ converges relatively slowly and therefore is not suitable for the computation of large number of digits of π. For example, we have $\pi - \pi_{1000} > 1.6 \times 10^{-17}$ and $\rho_{1000} < 3.2 \times 10^{-17}$. Thus to obtain 17 decimal places we should sum much more than 1000 terms, due to their rounding during the computation. Faster convergent sequence, originating in series (2), could be obtained using the Euler-Maclaurin formula of higher order than four [4].

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(Recibido en septiembre de 2005. Aceptado para publicación en febrero de 2006)

Vito Lampret
Faculty of Civil and Geodetic Engineering
University of Ljubljana
Ljubljana 1000, Slovenia 386
e-mail: `vlampret@fgg.uni-lj.si, vito.lampret@fgg.uni-lj.si`