General Equilibrium Theory: Lecture Note

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1 Introduction to General Equilibrium Theory

Debreu's book is an excellent introduction to general equilibrium theory.

1.1 Purpose

- Gather consumers and producers in a unifying framework and analyze how the *price mechanism* will lead to an *equilibrium.*
- Emphasize the analysis of the interaction among different markets.

1.2 Methodology

- Start from the description of the fundamentals (such as endowments, preference relations, and production possibilities) of the economy.
- Assume the price-taking behavior and anonymous markets.
- Avoid the detailed descriptions of allocation mechanisms.
- Take a somewhat abstract approach to the analysis of equilibria.

2 Economy, Efficiency, and Equilibrium

The following materials can also be found in Section 16.C of MWG.

2.1 An Economy

We assume that there are L commodities. We study an economy consisting of I *consumers* and J *firms.* Each consumer $i = 1, \dots, I$, is characterized by his *consumption set* $X_i \subset \boldsymbol{R}^L$ and a *preference relation* $\succsim_i$ defined on X_i. Each firm $j = 1, \dots, J$ is characterized by its *production set* $Y_j \subset \boldsymbol{R}^L$. The (aggregate) *endowments* in the economy for the L commodities are denoted by $\overline{\omega} \in \boldsymbol{R}^L$.

A *feasible allocation* of this economy is a vector $(x_1, \dots, x_I, y_1, \dots, y_J)$ in $X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J$ such that

$$\sum_{i=1}^{I} x_i = \overline{\omega} + \sum_{j=1}^{J} y_j.$$

What we shall call a "Walrasian allocation" is a feasible allocation of this economy led to by the "(competitive) price mechanism". Other trading mechanisms would lead to other feasible allocations.

2.2 Pareto Efficiency

We shall judge the desirability of an allocation, and hence of the price and other mechanisms that lead to them, with respect to the following criterion on efficiency.

2.1 Definition A feasible allocation $(x_1, \ldots, x_I, y_1, \ldots, y_J)$ is *Pareto efficient* if there is no feasible allocation $(x'_1, \ldots, x'_I, y'_1, \ldots, y'_J)$ such that $x'_i \succsim_i x_i$ for every i and $x'_i \succ_i x_i$ for some i.

2.3 Price Equilibrium

2.2 Definition A feasible allocation $(x^*_1, \ldots, x^*_I, y^*_1, \ldots, y^*_J)$ and a price vector $p \in \boldsymbol{R}^L$ constitute a *price equilibrium* (*with transfers*) if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y^*_j$.
2. For every i and every $x_i \in X_i$, if $p \cdot x_i \leq p \cdot x^*_i$, then $x^*_i \succsim_i x_i$.

It is easy to check that this definition is equivalent to Definition 16.B.4 of MWG. The above definition is closer to the definition of an equilibrium in Chapter 6 of "Theory of Value".

3 Two Fundamental Theorems of Welfare Economics

The materials of this sections can also be found in Section 16.D of MWG.

3.1 First Fundamental Theorem of Welfare Economics

3.1 Definition The pair $(X_i, \succsim_i)$ of the consumption set X_i and the preference relation $\succsim_i$ is *locally non-satiated* if, for every $x_i \in X_i$ and every $\varepsilon > 0$, there exists an $x'_i \in X_i$ such that $\|x'_i - x_i\| < \varepsilon$ and $x'_i \succ_i x_i$. We may also say, more simply, that the preference relation $\succsim_i$ is locally non-satiated.

Exercise 1 Suppose that $\succsim_i$ is complete, transitive, and locally non-satiated. Let $x^*_i \in X_i$ and $p \in \boldsymbol{R}^L$ be such that for every $x_i \in X_i$, if $p \cdot x_i \leq p \cdot x^*_i$, then $x^*_i \succsim x_i$. Prove then that for every $x_i \in X_i$, if $x_i \succsim x^*_i$, then $p \cdot x_i \geq p \cdot x^*_i$.

3.2 Theorem (First Fundamental Theorem of Welfare Economics) *Suppose that the preference relations are complete, transitive, and locally non-satiated. If a feasible allocation $(x^*_1, \ldots, x^*_I, y^*_1, \ldots, y^*_J)$ and some price vector constitute a price equilibrium, then $(x^*_1, \ldots, x^*_I, y^*_1, \ldots, y^*_J)$ is Pareto efficient.*

3.2 Second Fundamental Theorem of Welfare Economics

The following definition is a weaker notion of an equilibrium, termed "quasi-equilibrium". Although it plays only a technical role in the second welfare theorem, it is worth presenting because it will also appear in the existence problem of a Walrasian equilibrium.

3.3 Definition A feasible allocation $(x^*_1, \ldots, x^*_I, y^*_1, \ldots, y^*_J)$ and a price vector $p \in \boldsymbol{R}^L \setminus \{0\}$ constitute a *price quasi-equilibrium* if:

1. For every j and every $y_j \in Y_j$, we have $p \cdot y_j \leq p \cdot y^*_j$.
2. For every i and every $x_i \in X_i$, if $p \cdot x_i < p \cdot x^*_i$, then $x^*_i \succsim_i x_i$.

A price equilibrium is a price quasi-equilibrium. In general, the converse does not hold. Roughly speaking, however, if every consumer i can "survive" under p with some wealth level lower than $p \cdot x_i^*$, then the quasi-equilibrium is also an equilibrium. Note that if $p \cdot x_i^*$ is equal to the minimum wealth necessary for survival, then Condition 2 in Definition 3.3 is trivially met.

3.4 Theorem (Second Fundamental Theorem of Welfare Economics) *Suppose that $\succsim_i$ is complete, transitive, convex, and locally non-satiated for every i and that Y_j is convex for every j. If a feasible allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ is Pareto efficient, then there exists a price vector p such that $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and p constitute a price quasi-equilibrium.*

This is a consequence of the following theorem.

3.5 Theorem (Separating Hyperplane Theorem) *If A and B are mutually disjoint convex subsets of $\boldsymbol{R}^L$, then there exist a $p \in \boldsymbol{R}^L \setminus \{0\}$ and a $c \in \boldsymbol{R}$ such that $p \cdot a \geq c \geq p \cdot b$ for every $a \in A$ and $b \in B$.*

4 Ownership Structures

The materials of this sections can also be found in Section 16.B of MWG.

4.1 A Private Ownership Economy

A private ownership economy is nothing but an economy with a list of specifications regarding who owns what. More specifically, a *private ownership economy* is defined, in addition to the economy as defined in Section 2.1, by the consumers' endowments $\omega_i \in \boldsymbol{R}^L$ for $i = 1, \ldots, I$ and shareholdings $\theta_{ij} \geq 0$ in the J firms for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. We assume that $\sum_{i=1}^I \omega_i = \overline{\omega}$ and $\sum_{i=1}^I \theta_{ij} = 1$ for every j. This list of specifications of the private ownership determines to whom various profits and revenues are paid.

4.2 Walrasian Equilibrium

4.1 Definition A feasible allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ and a price vector $p \in \boldsymbol{R}^L$ constitute a *Walrasian equilibrium* if they constitute a price equilibrium and $p \cdot x_i^* \leq p \cdot \omega_i + \sum_{j=1}^J \theta_{ij} p \cdot y_j^*$ for every i.

An Walrasian equilibrium allocation $(x_1^*, \ldots, x_I^*, y_1^*, \ldots, y_J^*)$ is a feasible allocation led to by the price "mechanism", though the definition does not give any concrete idea on what the mechanism is like. The price vector p determines the "exchange rate" between any two commodities. If the "value" of a commodity is defined to be what it can buy in terms of other commodities, the value of a commodity is nothing but its price.

5 Examples of Private Ownership Economies

The materials in this section can also be found in Sections 15.B and 15.C of MWG.

5.1 An Edgeworth Box Economy

An *Edgeworth box economy* is an economy such that $L = 2$, $I = 2$, $X_1 = X_2 = \boldsymbol{R}_+^2$, $Y_1 = \cdots = Y_J = \{0\}$, and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2) \in \boldsymbol{R}_{++}^2$. This is an *exchange economy* with two consumers and two goods. Since the feasibility condition for an allocation is reduced to $x_2 = \overline{\omega} - x_1$, the set of feasible allocations can be identified with a rectangle with length $\overline{\omega}_1$ and height $\overline{\omega}_2$. This economy is a simplest possible framework in which we can see how the prices coordinate different consumers' demands to arrive at a feasible allocation. Exercises 15.B.1 and 15.B.2 of MWG are routine but recommended.

Exercise 2

1. Give a diagrammatical example of an Edgeworth box economy to show that the first fundamental theorem of welfare economics would not hold without the local non-satiation assumption.
2. Give a diagrammatical example of an Edgeworth box economy to show that the second fundamental theorem of welfare economics would not hold without the convexity assumption.
3. Give a diagrammatical example of an Edgeworth box economy to show that without adding any assumption to those of the second fundamental theorem of welfare economics, a Pareto-efficient allocation need not constitute a Radner equilibrium. Then state what kind of additional assumptions would guarantee that a Pareto-efficient allocation constitutes a Radner equilibrium.

5.2 A Robinson Crusoe Economy

A *Robison Crusoe economy* is an economy such that $L = 2$, $I = 1$, $J = 1$, $X_1 = \boldsymbol{R}^2_+$, and $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2) \in \boldsymbol{R}^2_+$. It is often considered that one of the two commodities is an input, whose endowment is positive, and the other is an output, whose endowment is zero. If we let the first commodity is the input and the second the output, then $\overline{\omega}$ lies on the positive part of the horizontal axis and Y_1 is included in the left half space of $\boldsymbol{R}^2$. The feasibility condition is reduced to $x_1 \in Y_1 + \{\overline{\omega}\}$. This economy is a simplest possible framework in which we can describe how the production and consumption decision are made separately and yet the price mechanism leads to a feasible allocation.

Exercise 3

1. Give a diagrammatical example of an Robinson Crusoe economy to show that the second fundamental theorem of welfare economics would not hold if the preference relation is not locally non-satiated.
2. Give a diagrammatical example of an Edgeworth box economy to show that the second fundamental theorem of welfare economics would not hold if the production set is not convex.

6 Excess Demand Function

The materials of this section can also be found in Section 17.B of MWG.

In the rest of this lecture note, we consider an exchange economy ($Y_1 = \cdots = Y_J = \{0\}$) and assume that $X_i = \boldsymbol{R}^L_+$ and $\omega_i \in \boldsymbol{R}^L_+$ for every i, and $\overline{\omega} = \sum_{i=1}^I \omega_i \in \boldsymbol{R}^L_{++}$. We also assume that the preference relations $\succsim_i$ are complete, transitive, continuous, strictly convex, and strongly monotone.

The *excess demand function* of this exchange economy is a mapping $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ defined by

$$z(p) = \sum_{i=1}^{I} \left(x_i(p, p \cdot \omega_i) - \omega_i \right),$$

where each x_i is the demand function of consumer i. A price vector p is a Walrasian equilibrium price vector if and only if $z(p) = 0$.

6.1 Proposition *The excess demand function z has the following properties.*

Continuity *z is continuous.*

Homogeneity *z is homogeneous of degree zero.*

Walras' Law $p \cdot z(p) = 0$ *for every* $p \in \boldsymbol{R}^L_{++}$.

Boundedness from Below *z is bounded from below.*

Boundary Behavior *If a sequence $p^1, p^2, \dots$ in $\boldsymbol{R}^L_{++}$ converges to a price vector $p = (p_1, p_2, \dots, p_L) \in \boldsymbol{R}^L_+$ with $p_\ell > 0$ for some ℓ and $p_\ell = 0$ for some other ℓ, then*

$$\max\{z_1(p^n), \dots, z_L(p^n)\} \to \infty$$

as $n \to \infty$, where $z_\ell(p^n)$ is the ℓ-th coordinate of $z(p^n)$.

Exercise 4 Suppose that a sequence $p^1, p^2, \dots$ in $\boldsymbol{R}^L_{++}$ converges to a price vector $p = (p_1, p_2, \dots, p_L) \in \boldsymbol{R}^L_+$ with $p_\ell > 0$ for some ℓ and $p_\ell = 0$ for some other ℓ. Suppose also that $p \cdot \omega_i > 0$. Show then that

$$\max\{x_{1i}(p, p \cdot \omega_i), \dots, x_{Li}(p, p \cdot \omega_i)\} \to \infty$$

as $n \to \infty$, where $x_{\ell i}(p, p \cdot \omega_i)$ is the ℓ-th coordinate of $x_i(p, p \cdot \omega_i)$.

7 Existence of a Walrasian Equilibrium

The materials of this sections can also be found Section 17.C of MWG.

7.1 Theorem *Under the assumptions stated in Section 6, there exists a Walrasian equilibrium.*

This theorem can be derived from the following lemma.

7.2 Lemma *If $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ satisfies the five properties of Proposition 6.1, then there exists a $p^* \in \boldsymbol{R}^L_{++}$ such that $z(p^*) = 0$.*

This lemma is, in turn, a consequence of the following fixed point theorem.

7.3 Theorem (Kakutani's Fixed Point Theorem) *Let P be a non-empty, convex, and compact subset of $\boldsymbol{R}^L$ and $f : P \to P$ be a correspondence having the following two properties:*

1. *For every $p \in P$, $f(p)$ is a non-empty and convex subset of P.*
2. *The graph of f, $\{(p, q) \in P \times P \mid q \in f(p)\}$, is a closed subset of $P \times P$.*

Then there exists a $p^* \in P$ such that $p^* \in f(p^*)$.

Proof of Lemma 7.2 Let $P = \{p \in \boldsymbol{R}^L_+ \mid \|p\|_1 = 1\}$, where $\|\cdot\|_1$ is the $\mathscr{L}^1$-norm in $\boldsymbol{R}^L$, and define a correspondence $f : P \to P$ by

$$f(p) = \begin{cases} \{q \in P \mid \text{for every } \ell, \text{ if } z_\ell(p) < \max\{z_1(p), \dots, z_L(p)\}, \text{ then } q_\ell = 0\}, & \text{if } p \in \boldsymbol{R}^L_{++} \\ \{q \in P \mid \text{for every } \ell, \text{ if } p_\ell > 0, \text{ then } q_\ell = 0\}, & \text{otherwise.} \end{cases}$$

It can be shown that P and f satisfy the two properties of Theorem 7.3, and that if $p^* \in f(p^*)$, then $p^* \in \boldsymbol{R}^L_{++}$ and $z(p^*) = 0$. ///

8 Sonnenschein-Mantel-Debreu Theorem

The materials of this sections can also be found Section 17.E of MWG.

The SMD theorem asserts that if there are as many consumers as commodities, then the continuity, homogeneity, and Walras' law in Propositions 6.1 exhaust all the implications of consumers' utility maximization behavior on the excess demand function of an exchange economy over any compact subset of $\boldsymbol{R}^L_{++}$.

To see why the number of consumers matters, let's assume that the x_i are continuously differentiable. Then

$$Dz_i(p) = S_i(p, p \cdot \omega_i) - D_w x_i(p, p \cdot \omega_i) z_i(p)^\top \in \boldsymbol{R}^{L \times L},$$

where $S_i(p, p \cdot \omega_i) \in \boldsymbol{R}^{L \times L}$ is the Slutsky substitution matrix and $z_i(p) \in \boldsymbol{R}^L$ (a column vector) is the excess demand of consumer i. (Note that this notation is different from that of Proposition 6.1.) Then $Dz_i(p)$ is negative semi-definite on the linear subspace

$$\left\{ v \in \boldsymbol{R}^L \mid p \cdot v = z_i(p, p \cdot \omega_i) \cdot v = 0 \right\}.$$

Hence $Dz(p) = \sum_{i=1}^I Dz_i(p)$ is negative semi-definite on the linear subspace

$$\begin{aligned} & \bigcap_{i=1}^{I} \left\{ v \in \boldsymbol{R}^L \mid p \cdot v = z_i(p, p \cdot \omega_i) \cdot v = 0 \right\} \\ = \quad & \left\{ v \in \boldsymbol{R}^L \mid p \cdot v = z_1(p, p \cdot \omega_i) \cdot v = \cdots = z_I(p, p \cdot \omega_i) \cdot v = 0 \right\}. \end{aligned}$$

If p is an equilibrium price vector, then the dimension of this linear subspace may be $L - I$ but not higher. Hence, if there are fewer consumers than commodities, then there is at least one direction of price variations along which $Dz(p)$ is negative semi-definite.

Exercise 5 For every i, every $p \in \boldsymbol{R}^L_{++}$, and every $q \in \boldsymbol{R}^L_{++}$, if $q \cdot z_i(p) \leq 0$, then $p \cdot z_i(q) \geq 0$.

Exercise 6

1. For every $p \in \boldsymbol{R}^L_{++}$ and every $q \in \boldsymbol{R}^L_{++}$, if $q \cdot z_i(p) \leq 0$ for every i, then $p \cdot z(q) \geq 0$.
2. If $L \geq I+2$, then for every $p \in \boldsymbol{R}^L_{++}$, there exists a $q \in \boldsymbol{R}^L_{++}$ such that $q \neq p$, $p \cdot (p-q) = 0$, and $p \cdot z(q) \geq 0$.
3. If $L \geq I+1$, then for every $p \in \boldsymbol{R}^L_{++}$ with $z(p) = 0$, there exists a $q \in \boldsymbol{R}^L_{++}$ such that $q \neq p$, $p \cdot (p - q) = 0$, and $p \cdot z(q) \geq 0$.

8.1 Theorem (Sonnenschein-Mantel-Debreu) *Let $z : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ be an arbitrary function that satisfies the continuity, homogeneity, and Walras law of Proposition 6.1. Let C be a compact subset of $\boldsymbol{R}^L_{++}$. Then there is an exchange economy consisting of L consumers whose excess demand function coincides with z on C.*

9 Generic Determinacy of Walrasian Equilibria

The materials of this and next sections can also be found Section 17.D of MWG.

9.1 Regular Equilibrium

In the rest of this lecture note, we assume that the consumers' demand functions are continuously differentiable.

9.1 Definition A Walrasian equilibrium price vector p is *regular* if $\text{rank} Dz(p) = L - 1$. An exchange economy is *regular* if all of its Walrasian equilibrium price vectors are regular.

Exercise 7 Show that the regularity is equivalent to each one of the following two conditions:

1. The column space of $Dz(p)$ is equal to the hyperplane normal to p going through the origin.
2. Define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \to \boldsymbol{R}^{L-1}$ by

$$\widehat{z}(\widehat{p}) = (z_1(\widehat{p}, 1), \ldots, z_{L-1}(\widehat{p}, 1)),$$

for every $\widehat{p} \in \boldsymbol{R}^{L-1}_{++}$, where $z_\ell(\widehat{p}, 1)$ is the ℓ-th coordinate of $z(\widehat{p}, 1)$. Then the $(L-1) \times (L-1)$ matrix $D\widehat{z}(p_1/p_L, \ldots, p_{L-1}/p_L)$ is invertible.

9.2 Proposition (Local Uniqueness of a Regular Equilibrium) *For every regular Walrasian equilibrium price vector p there exists an $\varepsilon > 0$ such that if a price vector p' is not proportional to p and satisfies $\|p' - p\| < \varepsilon$, then p' is not a Walrasian equilibrium price vector.*

9.2 Genericity Analysis

We now consider a class of exchange economies parameterized by $q \in Q$, where Q is an open subset of $\boldsymbol{R}^S$ and S is a positive integer. Denote by $z(\cdot, q) : \boldsymbol{R}^L_{++} \to \boldsymbol{R}^L$ the excess demand function of the exchange economy of parameter $q \in Q$. This defines the *parameterized excess demand function* $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$. Note that the domain of z has been expanded to include the *parameter space* Q.

9.3 Example We assume that the preference relations $\succsim_i$ $(i = 1, \ldots, I)$ of all consumers and the endowments ω_i $(i = 2, \ldots, I)$ of all consumers but the first one are prespecified. The economy is parameterized by the (strictly positive) endowments $\omega_1 \in \boldsymbol{R}^L_{++}$ of the first consumer. The parameter space Q is thus equal to $\boldsymbol{R}^L_{++}$.

9.4 Example We assume that the preference relations $\succsim_i$ $(i = 2, \ldots, I)$ of all consumers but the first one and the endowments ω_i $(i = 1, \ldots, I)$ of all consumers are prespecified. The preference relation $\succsim_1$ of the first consumer is represented by the Cobb-Douglas utility function $u_1(x_1) = x_{11}^a x_{21}^{1-a}$, where $x_1 = (x_{11}, x_{21})$ and $a \in (0, 1)$. The parameter space Q is thus equal to the open unit interval $(0, 1)$.

9.5 Definition The parametrization by Q is *regular* if the following condition is satisfied: the parameterized excess demand function $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$ is continuously differentiable and, for every $(p, q) \in \boldsymbol{R}^L_{++} \times Q$, if p is a Walrasian equilibrium price vector of parameter q then $\text{rank} Dz(p, q) = L - 1$.

Since $Dz(p, q) = [D_p z(p, q)\ D_q z(p, q)]$, the regularity of the parameter space Q is a weaker requirement than the regularity of the exchange economy with every parameter $q \in Q$. It can be shown that Examples 9.3 and 9.4 are both regular parameterizations.

Given a parameter space Q, we say that a property holds for almost every exchange economy in Q if there exists an open and full-measure subset Q' of Q such that the property holds for every exchange economy in Q'.

9.6 Theorem *If the parametrization by Q is regular, then almost every economy in Q is regular.*

This theorem is a consequence of the following theorem.

9.7 Theorem (Transversality Theorem) *Let P be an open subset of $\boldsymbol{R}^M$, Q be an open subset of $\boldsymbol{R}^N$, and $f : P \times Q \to \boldsymbol{R}^L$ be infinitely many times differentiable. If* $\operatorname{rank} Df(p,q) = L$ *for every $(p,q) \in P \times Q$ with $f(p,q) = 0$, then there exists a subset C of Q having Lebesgue measure zero such that* $\operatorname{rank} D_p f(p,q) = L$ *for every $(p,q) \in P \times (Q \setminus C)$ with $f(p,q) = 0$.*

9.3 Comparative Statics Analysis

Given a parameter space Q and a parameterized excess demand function $z : \boldsymbol{R}^L_{++} \times Q \to \boldsymbol{R}^L$, define $\widehat{z} : \boldsymbol{R}^{L-1}_{++} \times Q \to \boldsymbol{R}^{L-1}$ by

$$\widehat{z}\,(\widehat{p}, q) = \left(z_1\left(\left(\widehat{p}, 1\right), q\right), \ldots, z_{L-1}\left(\left(\widehat{p}, 1\right), q\right)\right),$$

for every $(\widehat{p}, q) \in \boldsymbol{R}^{L-1}_{++} \times Q$. If $(\widehat{p}^*, 1)$ is a regular Walrasian equilibrium price vector of q^*, then $\operatorname{rank} D_{\widehat{p}}\widehat{z}\,(\widehat{p}^*, q^*) = L - 1$ and hence the implicit function theorem implies that there exist an open subset V of $\boldsymbol{R}^{L-1}_{++}$, an open subset Q' of Q, and a continuously differentiable mapping $p : V \to Q'$ such that $(\widehat{p}^*, q^*) \in V \times Q'$ and, for every $(\widehat{p}, q) \in V \times Q'$, $(\widehat{p}, 1)$ is a regular Walrasian equilibrium price vector of q if and only if $p(q) = \widehat{p}$. The implicit function theorem also implies that

$$Dp(q^*) = -D_{\widehat{p}}\widehat{z}\,(\widehat{p}^*, q^*)^{-1}\, D_q\widehat{z}\,(\widehat{p}^*, q^*)\,.$$

10 Economy under Uncertainty

The framework to be presented in this and the next two sections can be found in Chapter 7 of Debreu's book, Magill and Quinzii's book, and Chapter 19 of MWG. Also, Section 4 of D. Duffie and H. Sonnenschein's "Arrow and general equilibrium theory" in *Journal of Economic Literature* vol. 27 (1989), pp. 565–598, will be very helpful.

10.1 Setup

There are two *periods*. There is no uncertainty in the first period, but no economic agent then knows which one of the S conceivably possible *states of the world* $1, \ldots, S$, is to be realized in the second period. The first period is often designated as *state* 0.

There are L types of physically and spatially distinguished perishable goods. Hence there are $L(1+S)$ types of *contingent commodities*. A consumption vector for consumer i is now $x_i = (x_{0i}, x_{1i}, \ldots, x_{Si}) \in \boldsymbol{R}^{L(1+S)}_+$, where $x_{si} \in \boldsymbol{R}^L_+$ for every $s = 0, 1, \ldots, S$.

There is no firm. The economy under consideration is thus an exchange economy.

An economy and a private ownership economy are defined as before, except that the commodity space $\boldsymbol{R}^L$ is now replaced by $\boldsymbol{R}^{L(1+S)}$ and there is no firm or shareholding.

10.2 Pareto Efficiency and Arrow-Debreu Equilibrium

A price vector is now $p = (p_0, p_1, \ldots, p_S) \in \boldsymbol{R}^{L(1+S)}$, where $p_s \in \boldsymbol{R}^L$ for every $s = 0, 1, \ldots, S$. Thus, $p \cdot x_i = \sum_{s=0}^{S} p_s \cdot x_{si}$.

A Walrasian equilibrium in this context is often referred to as an *Arrow-Debreu equilibrium.*

10.1 Remark The $L(1+S)$ prices in p represent the present values in the first period and reflect the time discount rate and betting rates. Also, the budget constraint $p \cdot x_i \leq p \cdot \omega_i$ may accommodate borrowing, lending, insurance, and gambles.

10.2 Remark Arrow-Debreu equilibria inherit all properties of Walrasian equilibria, such as the existence, efficiency, and generic local uniqueness.

11 Asset Markets and Sequential Trades

11.1 Setup

There are J *assets*, $j = 1, \ldots, J$, each characterized by the contingent payoffs $a_j = (a_{1j}, \ldots, a_{Sj}) \in \boldsymbol{R}^{LS}$, where an owner of one unit of asset j is entitled to the vector $a_{sj} \in \boldsymbol{R}^L$ of the L goods in state s. Assets of this type are often called *real* assets, as opposed to *nominal* assets, whose payoffs are specified in monetary terms.

In the first period, consumers trade the L goods and the J assets without knowing what the true state of the world is. In the second period, they come to know what it is, receive and/or deliver the payments of the assets they have traded, and trade the L goods.

A consumption vector $x_i = (x_{0i}, x_{1i}, \ldots, x_{Si}) \in X_i$ now consists of a current consumption vector x_{0i} and a list of *plans* x_{si} of consumptions in each state $s = 1, \ldots, S$.

A price vector $p = (p_0, p_1, \ldots, p_S) \in \boldsymbol{R}^{L(1+S)}$ now consists of a current price vector p_0 and a list of price vectors p_s *expected* to prevail in the spot markets of each state $s = 1, \ldots, S$.

A *portfolio* that consumer i takes in the first periods is denoted by $z_i = (z_{1i}, \ldots, z_{Ji}) \in \boldsymbol{R}^J$. An *asset price vector* is denoted by $q = (q_1, \ldots, q_J) \in \boldsymbol{R}^J$. The cost of the portfolio z_i under q is equal to $q \cdot z_i = \sum_{j=1}^J q_j z_{ji}$.

11.2 Radner Equilibrium

A consumption vector $x_i = (x_{0i}, x_{1i}, \ldots, x_{Si}) \in X_i$ is *budget-feasible under price vectors* (p, q) if there is a portfolio $z_i = (z_{1i}, \ldots, z_{Ji}) \in \boldsymbol{R}^J$ such that

$$
\begin{aligned}
&p_0 \cdot x_{0i} + q \cdot z_i = p_0 \cdot \omega_{0i}, \\
&\qquad p_s \cdot x_{si} = p_s \cdot \omega_{si} + \sum_{j=1}^{J} (p_s \cdot a_{sj}) z_{ji} \text{ for every } s = 1, \ldots, S.
\end{aligned}
$$

Then, we also say that (x_i, z_i) is *budget-feasible under* (p, q).

11.1 Definition (Radner Equilibrium) A pair of commodity and asset price vectors, (p, q), and a list of budget feasible consumption vectors and portfolios, $((x_1^*, z_1^*), \ldots, (x_I^*, z_I^*))$, constitute a *Radner equilibrium* if x_i^* is preference-maximizing among all budget-feasible consumption vectors under (p, q) for every i, $\sum_{i=1}^I x_i^* = \sum_{i=1}^I \omega_i$ (that is, $(x_1^*, \ldots, x_I^*)$ is feasible), and $\sum_{i=1}^I z_i^* = 0$.

11.2 Remark Since $p_s \cdot e_{si} + \sum_{j=1}^J (p_s \cdot a_{sj}) z_{ji} = p_s \cdot \left(e_{si} + \sum_{j=1}^J z_{ji} a_{sj} \right)$, there are $(1 + S)$ degrees of freedom in prices. In particular, these prices reflect neither discount rates nor betting rates; they only reflect relative prices among the L goods in each state. The discount rates and betting rates are in fact reflected by (p_0 and) q. Also, Walras's law can be applied $(1 + S)$ times.

12 Complete and Incomplete Asset Markets

12.1 Definition

If (x_i, z_i) is budget-feasible under (p, q), then

$$\begin{bmatrix} p_1 \cdot (x_{1i} - \omega_{1i}) \\ \vdots \\ p_S \cdot (x_{Si} - \omega_{Si}) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{J} (p_1 \cdot a_{1j}) z_{ji} \\ \vdots \\ \sum_{j=1}^{J} (p_S \cdot a_{Sj}) z_{ji} \end{bmatrix} = \begin{bmatrix} p_1 \cdot a_{11} & \cdots & p_1 \cdot a_{1J} \\ \vdots & \ddots & \vdots \\ p_S \cdot a_{S1} & \cdots & p_S \cdot a_{SJ} \end{bmatrix} \begin{bmatrix} z_{1i} \\ \vdots \\ z_{Ji} \end{bmatrix}.$$

Denote the $S \times J$ matrix on the right hand side by $V(p)$ or $V(p_1, \dots, p_S)$.

12.1 Definition The asset markets are *complete* under p (or $(p_1, \dots, p_S)$) if the rank of $V(p)$ (or $V(p_1, \dots, p_S)$) equals S. Otherwise, they are *incomplete*.

A necessary but not sufficient condition for complete asset markets is that $S \geq J$.

Exercise 8 Suppose that $L = S = J = 2$. The first asset pays one unit of the first good (but none of the second) in both states, and the second asset pays one unit of the second good (but none of the first) in both states. Give an equivalent condition in terms of spot prices (which you may assume are all strictly positive) for the asset markets to be complete.

Exercise 9 Show that the rank of $V(p)$ does not depend on p if, at each state, all assets pay off in the same good, that is, for every state $s = 1, \dots, S$, there exists a good $\ell = 1, \dots, L$ (which may depend on the choice of s) such that $a_{ksj} = 0$ for every security $j = 1, \dots, J$ and every other good $k = 1, \dots, L$ with $k \neq \ell$, where a_{ksj} is the k-th coordinate of $a_{sj} \in \boldsymbol{R}^L$.

12.2 Equivalence between Arrow-Debreu and Radner Equilibria

12.2 Theorem *1. If p and $(x_1^*, \dots, x_I^*)$ constitute an Arrow-Debreu equilibrium and if the rank of $V(p)$ equals S, then $(x_1^*, \dots, x_I^*)$ is a Radner equilibrium consumption allocation.*

2. If (p, q) and $((x_1^*, z_1^*), \dots, (x_I^*, z_I^*))$ constitute a Radner equilibrium and if the rank of $V(p)$ equals S (that is, the asset markets are complete under p), then $(x_1^*, \dots, x_I^*)$ is an Arrow-Debreu equilibrium consumption allocation.

Exercise 10 Suppose, contrary to the above definitions, that the assets payoffs are nominal (in terms of monetary amounts) rather than real (in terms of consumption bundles).

1. Give the definition of a Radner equilibrium.
2. Explain, using the payoffs of the nominal assets, how to determine whether asset markets are complete.
3. Discuss whether Theorem 12.2 remains to hold.
4. Discuss whether the Radner equilibria are (generically) determinate, first for the case of complete markets, and then for the case of incomplete markets.

13 Constrained Efficiency in Incomplete Asset Markets

While it is clear that Radner equilibrium allocations with incomplete asset markets will in general not be fully Pareto efficient (first-best), it is far less clear what the "right" notion of *constrained efficiency* is and whether equilibrium allocations are constrained efficient. In this section, we present two approaches, which differ in whether to take *pecuniary externalities* into consideration.

This section contains advanced materials. The best introduction to them is probably "Incomplete Markets" by Michael Magill and Wayne Shafer, Chapter 30 of the *Handbook of Mathematical Economics* vol. 4, edited by Werner Hildenbrand and Hugo Sonnenschein.

13.1 Constrained Efficiency without Pecuniary Externalities

Define $M(p)$ to be the set of all $y_i = (y_{0i}, y_{1i}, \dots, y_{Si}) \in \boldsymbol{R}^{L(1+S)}$ for which there exists a $z_i \in \boldsymbol{R}^J$ such that $p_s \cdot y_{si} = \sum_{j=1}^{J} (p_s \cdot a_{sj}) z_{ji}$ for every $s = 1, \dots, S$. The set $M(p)$ is a linear subspace of $\boldsymbol{R}^{L(1+S)}$.

13.1 Remark If a price vector p and a feasible allocation $(x_1^*, \dots, x_I^*)$ constitute a Radner equilibrium, then $x_i^* - \omega_i \in M(p)$ for every i.

13.2 Proposition *Suppose that $\succsim_i$ is locally non-satiated for every i. If a price vector p and the consumption allocation $(x_1^*, \dots, x_I^*)$ constitute a Radner equilibrium, then there is no feasible allocation $(x_1, \dots, x_I)$ that Pareto-improves $(x_1^*, \dots, x_I^*)$ and satisfies $x_i - \omega_i \in M(p)$ for every i.*

Exercise 11 Prove Proposition 13.2.

Exercise 12 Show that if $L = 1$, then $M(p)$ coincides with the Cartesian product of $\boldsymbol{R}$ and the column space of

$$\begin{bmatrix} a_{111} & \cdots & a_{11J} \\ \vdots & \ddots & \vdots \\ a_{1S1} & \dots & a_{1SJ} \end{bmatrix}$$

(included in $\boldsymbol{R}^S$), and, in particular, does not depend on p. Deduce from this result that every equilibrium allocation is efficient among those which can be achieved by reallocating assets.

The notion of efficiency in Exercise 12 is the standard notion of *constrained efficiency for the one-good case.*

13.2 Constrained Inefficiency with Pecuniary Externalities

If $L \geq 2$, then it is conceivable that any reallocation of assets would induce spot prices to change. The notion of constrained efficiency in this case should probably take this *pecuniary externality* into consideration.

Throughout this section, we assume that the utility functions u_i can be written in the separable form:

$$u_i(x_i) = u_{0i}(x_{0i}) + u_{1i}(x_{1i}) + \cdots + u_{Si}(x_{Si}).$$

13.3 Definition A feasible allocation $(x_1, \dots, x_I)$ is *achievable through spot markets* if there exist a price vector p and a list of portfolios $(z_1, \dots, z_I)$ with $\sum_{i=1}^{I} z_i = 0$ such that for every $s = 1, \dots, S$, the spot price vector p_s and the allocation $(x_{s1}, \dots, x_{sI})$ constitute a Walrasian equilibrium of the L-good economy in which consumer i has the utility function u_{si} and initial endowment vector $\omega_{si} + \sum_{j=1}^{J} z_{ji} a_{sj}$.

13.4 Remark 1. Every Radner equilibrium allocation is achievable through spot markets.

2. In the case of $L = 1$, a feasible allocation $(x_1, \dots, x_I)$ is achievable through spot markets if and only if $x_i - \omega_i \in M(p)$ for every i.

3. Assume that a feasible allocation $(x_1, \dots, x_I)$ is achievable through spot markets under a price vector p. Then $x_i - \omega_i \in M(p)$. But this does *not* imply that p is a Radner equilibrium price vector (for the overall, two-period economy), because some x_i need not satisfy the (overall) utility maximization condition under p (and any asset price vector q).

4. Even if p is a Radner equilibrium price vector and a feasible allocation $(x_1, \dots, x_I)$ satisfies $x_i - \omega_i \in M(p)$ for every i, it need not be achievable through spot markets. A possible reason for this is that for some $s \geq 1$, the allocation $(x_{si}, \dots, x_{sI})$ is not an efficient allocation within state s.

The last two points of Remark 13.4 show that if $L \geq 2$, then, in general, either the set of the feasible allocations that are achievable through spot markets or the set of the feasible allocations $(x_1, \dots, x_I)$ for which there exists a Radner equilibrium price vector p such that $x_i - \omega_i \in M(p)$ for every i need not include the other.

13.5 Definition (Constrained Efficiency) A feasible allocation is *constrained efficient* if there is no feasible allocation that Pareto-improves it and is achievable through spot markets.

Exercise 13 Extend Definition 13.3 to the case of non-separable utility functions, and then discuss whether every Radner equilibrium allocation remains to be achievable through spot markets.

13.3 Generic Constrained Inefficiency with Pecuniary Externalities

By imposing some additional assumptions and using the techniques introduced in Section 9.2, we can show that if there are more than one types of goods, then, generically, every Radner equilibrium allocation is constrained inefficient.

The first set of assumptions is concerned with assets payoffs.

13.6 Assumption 1. For every j and s, write $a_{sj} = (a_{sj1}, \dots, a_{sjL}) \in \boldsymbol{R}^L$, then $a_{sj\ell} = 0$ for every s, j, and $\ell \geq 2$. In the following, we denote by A the $S \times J$ matrix

$$\begin{bmatrix} a_{111} & \cdots & a_{11J} \\ \vdots & \ddots & \vdots \\ a_{1S1} & \dots & a_{1SJ} \end{bmatrix}.$$

2. A is in general position, that is, all of its square submatrices are invertible.

3. There exists a $z_i \in \boldsymbol{R}^J$ such that $Az_i \in \boldsymbol{R}^S_{++}$.

The first assumption says that the assets are *real numeraire assets*, in the sense that their payoffs are all in terms of the same single good, considered as the numeraire. Although we are assuming that the first good is the numeraire in every state, we can allow the numeraire to depend on the realization of states, as in Exercise 9, without losing the validity of Theorem

13.8. The second assumption is not satisfied if the assets are the so-called *Arrow securities*, that is, $J \leq S$, and $a_{sj1} = 1$ whenever $s = j$, and $a_{sj1} = 0$ otherwise.

The second set of assumptions is concerned with the numbers of goods, consumers, and assets.

13.7 Assumption 1. $L \geq 2$.

2. $2(L-1) \leq I \leq S(L-1) - 1$.

3. $2 \leq J \leq S - 1$.

Note that the first two assumptions implies that $I \geq 2$. The upper bound $S(L-1) - 1$ on the number of consumers (or, more precisely, the number of *types* of consumers) may be considered as inconsistent with the hypothesis of perfect competition.

We shall not rigorously specify the space of parameters defining preference relations (utility functions) and endowments, but we can state the generic constrained inefficiency result roughly as follows

13.8 Theorem (Generic Constrained Inefficiency) *Under Assumptions 13.6 and 13.7, for almost all every economy, none of its Radner equilibrium allocations is constrained efficient in the sense of Definition 13.5.*