The Shapley Value

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1. Introduction

To promote an understanding of the importance of Shapley's (1953) paper on the value, we shall start nine years earlier with the seminal book by von Neumann and Morgenstern that laid the foundations of cooperative game theory. Unlike non-cooperative game theory, cooperative game theory does not specify a game through a minute description of the strategic environment, including the order of moves, the set of actions at each move, and the payoff consequences relative to all possible plays, but, instead, it reduces this collection of data to the coalitional form. The cooperative game theorist must base his prediction strictly on the payoff opportunities available to each coalition, conveyed by a single real number: gone are the actions, the moves, and the individual payoffs. The chief advantage of this approach, at least in multiple-player environments, is its practical usefulness. A real-life situation fits more easily into a coalitional form game, whose structure has proved more tractable than that of a non-cooperative game, whether that be in normal or extensive form.

Prior to the appearance of the Shapley value, one solution concept alone ruled the kingdom of (cooperative) game theory: the von Neumann–Morgenstern solution. The core would not be defined until around the same time as the Shapley value. As set-valued solutions suggesting "reasonable" allocations of the resources of the grand coalition, both the von Neumann–Morgenstern solution and the core are based on the coalitional form game. However, no single-point solution concept existed as of yet to associate a single payoff vector to a coalitional form game. In fact the coalitional form game of those days had so little information in its black box that the creation of a single-point solution seemed untenable. It was in spite of these sharp limitations that Shapley came up with the solution. Using an axiomatic approach, Shapley constructed a solution remarkable not only for its attractive and intuitive definition but also for its unique characterization by a set of reasonable axioms. Section 2 of this chapter reviews Shapley's result, and in particular a version of it popularized in the wake of his original paper. Section 3 turns to a special case of the value for the class of voting games, the Shapley–Shubik index.

In addition to a model that attempts to predict the allocation of resources in multi-person interactions, Shapley also viewed the value as an index for measuring the power of players in a game. Like a price index or other market indices, the value uses averages (or weighted averages in some of its generalizations) to aggregate the power of players in their various cooperation opportunities. Alternatively, one can think of the Shapley value as a measure of the utility of players in a game. Alvin Roth took this interpretation a step further in formal terms by presenting the value as a von Neumann–Morgenstern utility function. Roth's result is discussed in Section 4.

Section 5 is devoted to alternative axiomatic characterizations of the Shapley value, with particular emphasis on Young's axiom of monotonicity, Hart and Mas-Colell's axiom of consistency, and the last named pair's notion of potential (arguably a single-axiom characterization).

To be able to apply the Shapley value to concrete problems such as voting situations, it is important to be able to characterize it on sub-classes of games. Section 6 discusses several approaches in that direction. Section 7 surveys several attempts to generalize the Shapley value to a framework in which the environment is described by some *a priori* cooperation structure other than the coalitional form game (typically a partition of the set of players). Aumann and Drèze (1974) and Owen (1977) pioneered examples of such generalizations.

While the Shapley value is a classic cooperative solution concept, it has been shown to be sustained by some interesting strategic (bargaining) games. Section 8 looks at some of these results. Section 9 closes the chapter with a discussion of the practical importance of the Shapley value in general, and of its role as an estimate of parliamentary and voter power and as a rule for cost allocation in particular.

Space forbids discussion of the vast literature inspired by Shapley's paper. Certain of these topics, such as the various extensions of the value to NTU games, and the values of non-atomic games to emerge from the seminal book by Aumann and Shapley (1974), are treated in other chapters of this Handbook.

2. The Framework

Recall that a *coalitional form game* (henceforth game) on a finite set of players N = {1,2,3 ..., n} is a function v from the set of all coalitions 2^N to the set of real numbers R with $v(\emptyset) = 0$. v(S) represents the total payoff or rent the coalition S can get in the game v.

A *value* is an operator ϕ that assigns to each game v a vector of payoffs $\phi(v) = (\phi_1, \phi_2, ..., \phi_n)$ in R^n. $\phi_i(v)$ stands for i's payoff in the game, or alternatively for the measure of i's power in the game.

Shapley presented the value as an operator that assigns an expected marginal contribution to each player in the game with respect to a uniform distribution over the set of all permutations on the set of players. Specifically, let π be a permutation (or an order) on the set of players, i.e., a one-to-one function from N onto N, and let us imagine the players appearing one by one to collect their payoff according to the order π. For each player i we can denote by $p_\pi^i = \{j: \pi(i) > \pi(j)\}$ the set of players preceding player i in the order π. The marginal contribution of player i with respect to that order π is $v(p_\pi^i \cup i) - v(p_\pi^i)$. Now, if permutations are randomly chosen from the set Π of all permutations, with equal probability for each one of the n! permutations, then the average marginal contribution of player i in the game v is

$$\phi_i(v) = 1/n! \sum_{\pi \in \Pi} [v(p_\pi^i \cup i) - v(p_\pi^i)], \quad (1)$$

which is Shapley's definition of the value.

While the intuitive definition of the value speaks for itself, Shapley supported it by an elegant axiomatic characterization. We now impose four axioms to be satisfied by a value:

The first axiom requires that players precisely distribute among themselves the resources available to the grand coalition. Namely,

Efficiency: $\sum_{i \in N} \phi_i(v) = v(N)$.

The second axiom requires the following notion of symmetry:
Players $i,j \in N$ are said to be symmetric with respect to game v if they make the same marginal contribution to any coalition, i.e., for each $S \subset N$ with $i,j \notin S$, $v(S \cup i) = v(S \cup j)$. The symmetry axiom requires symmetric players to be paid equal shares.

Symmetry: If players i and j are symmetric with respect to game v, then $\phi_i(v) = \phi_j(v)$.

The third axiom requires that zero payoffs be assigned to players whose marginal contribution is null with respect to every coalition:

Dummy : If i is a *dummy* player, i.e., $v(S \cup i)- v(S) = 0$ for every $S \subset N$, then $\phi_i(v) = 0$.

Finally, we require that the value be an additive operator on the space of all games, i.e.,

Additivity: $\phi(v+w) = \phi(v) +\phi(w)$, where the game v+w is defined by $(v+w)(S) = v(S) +w(S)$ for all S.

Shapley's amazing result consisted in the fact that the four simple axioms defined above characterize a value uniquely:

Theorem 1 (Shapley 1953): There exists a unique value satisfying the efficiency, symmetry, dummy, and additivity axioms: it is the Shapley value given in Equation (1).

The uniqueness result follows from the fact that the class of games with n players forms a 2^{n-1} -dimensional vector space in which the set of unanimity games constitutes a basis. A game u_R is said to be a unanimity game on the domain R if $u_R(S) = 1$, whenever $R \subset S$ and 0 otherwise. It is clear that the dummy and symmetry axioms together yield a value that is uniquely determined on unanimity games (each player in the domain should receive an equal share of 1 and the others zero.) Combined with the additivity axiom and the fact that the unanimity games constitute a basis for the vector space of games, this yields the uniqueness result.

Here it should be noted that Shapley's original formulation was somewhat different from the one described above. Shapley was concerned with the space of all games that can be played by some large set of potential players U, called the universe. For every game v, which assigns a real number to every finite subset of U, a *carrier* N is a subset of U such that $v(S) = v(S \cap N)$ for every $S \subset U$. Hence, the set of players who actually participate in the game must be contained in any carrier of the game. If for some carrier N a player i is not in N, then i must be a dummy player because he does not affect the payoff of any coalition that he joins. Shapley imposed the *carrier axiom* onto this framework, which requires that within any carrier N of the game the players in N share the total payoff of v(N) among themselves. Interestingly, this axiom bundles the efficiency axiom and the dummy axiom into one property.

3. Simple Games

Some of the most exciting applications of the Shapley value involve the measurement of political power. The reason why the value lends itself so well to this domain of problems is that in many of these applications it is easy to identify the real-life environment with a specific coalitional form game. In politics, indeed in all voting situations, the power of a coalition comes down to the question of whether it can impose a certain collective decision, or, in a typical application, whether it possesses the necessary majority to pass a bill. Such situations can be represented by a collection of coalitions W (a subset of 2^N), where W stands for the set of "winning" coalitions, i.e., coalitions with enough power to impose a decision collectively. We call these situations "simple games." While simple games can get rather complex, their coalitional function v assumes only two values: 1 for winning coalitions and 0 otherwise (see Chapter 36 in this Handbook). If we assume monotonicity, i.e., that a superset of a winning coalition is likewise winning, then the players' marginal contributions to coalitions in such games also assume the values 0 and 1. Specifically, player i's marginal contribution to coalition S is 1 if by joining S player i can turn the coalition from a non-winning (or losing) to a winning coalition. In such cases, we can say that player i is *pivotal* to coalition S. Recalling the definition of the Shapley value, it is easy to see that in such games the value assigns to each player the probability of being pivotal with respect to his predecessors, where orders are sampled randomly and with equal probability. Specifically,

$$\phi_i(v) = |\{\pi\in\Pi;\ p_\pi^i \cup i \in W \text{ and - } p_\pi^i \notin W\}|/n!$$

This special case of the Shapley value is known in the literature as the Shapley–Shubik index for simple games (Shapley and Shubik 1954).

A very interesting interpretation of the Shapley–Shubik index in the context of voting was proposed by Straffin (1977). Consider a simple (voting) game with a set of winning coalitions W representing the distribution of power within a committee, say a parliament. Suppose that on the agenda are several bills on which players take positions. Let us take an *ex ante* point of view (before knowing which bill will be discussed) by assuming that the probability of each player voting in favor of the bill is p (independent over i). Suppose that a player is asking himself what the probability is that his vote will affect the outcome. Put differently, what is the probability that the bill will pass if I support it? The answer to this question depends on p (as well as the distribution of power W). If p is 1 or 0, then I will have no effect unless I am a dictator. But because we do not know which bill is next on the agenda, it is reasonable to assume that p itself is a random variable. Specifically, let us assume that p is distributed uniformly on the interval [0,1]. Straffin points out that with this model for random bills the probability that a player is effective is equivalent to his Shapley–Shubik index in the corresponding game (see Chapter 32 in this Handbook). We shall demonstrate this with an example.

Example:

Let [3;2,1,1] be a weighted majority game[1], where the minimal winning coalitions are {1,2} and {1,3}. Player 2 is effective only if player 1 votes for and player 3 votes against.

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[1] In a weighted majority game $[q;w_1,...,w_n]$, a coalition S is winning if and only if $\sum_{i\in S} w_i \geq q$.

For a given probability p of acceptance, this occurs with probability p(1-p). Since 2 and 3 are symmetric, the same holds for player 3. Now player 1's vote is ineffective only if 2 and 3 both vote against, which happens with probability $(1-p)^2$. Thus player 1 is effective with probability $2p-p^2$. Integrating these functions between 0 and 1 yields $\phi_1 = 2/3$, $\phi_2 = \phi_3 = 1/6$.

It is interesting to note that with a different probability model for bills one can derive a different well-known power index, namely the Banzhaf index (see Chapter 32 in this Handbook). Specifically, if player k's probability of accepting the bill is p_k, and if $p_1, ..., p_n$ are chosen independently, each from a uniform distribution on [0,1], then player i's probability of being effective coincides with his Banzhaf index.

4. The Shapley Value as a von Neumann–Morgenstern Utility Function

If we interpret the Shapley value as a measure of the benefit of "playing" the game (as was indeed suggested by Shapley himself in his original paper), then it is reasonable to think of different positions in a game as objects for which individuals have preferences. Such an interpretation immediately gives rise to the following question: What properties should these preferences possess so that the cardinal utility function that represents them coincides with the Shapley value? This question is answered by Roth (1977).

Roth defined a position in a game as a pair (i,v), where i is a player and v is a game. He then assumed that individuals have preferences defined on the mixture set M that contains all positions and lotteries whose outcomes are positions. Using "~" to denote indifference and "≻" to denote strict preference, Roth imposed several properties on preferences. The first two properties are simple regularity conditions:

A1: Let v be a game in which i is a dummy. Then $(i,v) \sim (i,v_0)$, where v_0 is the null game in which every coalition earns zero. Furthermore, $(i,v_i) \succ (i,v_0)$, where v_i is the game in which i is a dictator, i.e., $v_i(S) = 1$ if $i \in S$ and $v_i(S) = 0$ otherwise.

The second property, which relates to Shapley's symmetry axiom, requires that individual preferences not depend on the names of positions, i.e.,

A2: For any game v and permutation π, $(i,v) \sim (\pi(i),\pi(v))$.[2]

The two remaining properties are more substantial and deal with players' attitudes towards risk. The first of these properties requires that the certainty equivalent of a lottery that yields the position i in either game v or game w (with probabilities p and 1-p) be the position i in the game given by the expected value of the coalitional function with respect to the same probabilities. Specifically, for two positions (i,v) and (i,w), we denote by [p(i,v);(1-p)(i,w)] the lottery where (i,v) occurs with probability p and (i,w) occurs with probability 1-p.

A3: Neutrality to Ordinary Risk: $(i,(pw + (1-p)v)) \sim [p(i,w);(1-p)(i,v)]$.

Note that a weaker version of this property requires that $(i,v) \sim [(1/c)(i,cv);(1-1/c)(i,v_0)]$ for $c > 1$. It can be shown that this property implies that the utility function u, which represents the preferences over positions in a game, must satisfy $u(cv,i) = cu(v,i)$.

The last property asserts that in a unanimity game with a carrier of r players the utility of a non-dummy player is 1/r of the utility of a dictator. Specifically, let v_R be defined by $v_R(S) = 1$ if $R \subset S$ and 0 otherwise.

A4: Neutrality to Strategic Risk: $(i,v_R) \sim (i,(1/r)v_i)$.

An elegant result now asserts that:

Theorem (Roth 1977): Let u be a von Neumann–Morgenstern utility function over positions in games, which represents preferences satisfying the four axioms. Suppose that

[2] $\pi(v)$ is the game with $\pi(v)(S) = v(\pi(S))$, where $\pi(S) = \{j;\ j=\pi(i) \text{ for some } i \in S\}$.

u is normalized so that $u(i,v_i) = 1$ and $u_i(i,v_0) = 0$. Then $u(i,v)$ is the Shapley value of player i in the game v.

Roth's result can be viewed as an alternative axiomatization of the Shapley value. I will now survey several other characterizations of the value, which, unlike Roth's utility function, employ the standard concept of a payoff function.

5. Alternative Axiomatizations of the Value

One of the most elegant aspects of the axiomatic approach in game theory is that the same solution concept can be characterized by very different sets of axioms, sometimes to the point of seeming unrelated. But just as a sculpture seen from different angles is understood in greater depth, so is a solution concept by means of different axiomatizations, and in this respect the Shapley value is no exception. This section examines several alternative axiomatic treatments of the value.

Perhaps the most appealing property to result from Definition (1) of the Shapley value is that a player's payoff is only a function of the vector of his marginal contributions to the various coalitions. This raises an interesting question: Without forgoing the above property, how far can we depart from the Shapley value? "Not much," according to Young (1985), whose finding also yields an alternative axiomatization of the value.

For a game v, a coalition S, and a player $i \notin S$, we denote by $D_i(v,S)$ player i's marginal contribution to the coalition S with respect to the game v, i.e., $D_i(v,S) = v(S \cup i) - v(S)$. Young introduced the following axiom:

Strong Monotonicity: Suppose that v and w are two games such that for some $i \in N$ $D_i(v,S) \geq D_i(w,S)$. Then $\phi_i(v) \geq \phi_i(w)$.

He then showed that this property plays a central role in the characterization of the Shapley value. Specifically,

Theorem (Young 1985): There exists a unique value ϕ satisfying strong monotonicity, symmetry, and efficiency, namely the Shapley value.

Note that Young's strong monotonicity axiom implies the marginality axiom, which asserts that the value of each player is only a function of that player's vector of marginal contributions. Young's axiomatic characterization of the value thus supports the claim that the Shapley value to some extent is a synonym for the principle of marginal contribution—a time-honored principle in economic theory. But we must be clear about what is meant by marginal contribution. In Young's framework, as in Shapley's definition of the value, players contribute by increasing the wealth of the coalition they join (or decreasing it if contributions are negative). This caused Hart and Mas-Colell (1989) to ask the following question: Can the Shapley value be derived by referring the players' marginal contributions to the wealth generated by the entire multilateral interaction, instead of tediously listing all the coalitions they can join? Offhand, the question sounds somewhat amorphous, for how is one to define an "entire interaction"? Absent a satisfactory definition, we shall proceed by way of supposition. Suppose each pair (N,v) is associated with a single real number P(N,v) that sums up the wealth generated by the entire interaction in the game. We are already within an ace of defining marginal contributions. Specifically, player i's marginal contribution with respect to (N,v) is simply:

$$D^i P(N,v) = P(N,v) - P(N\backslash i,v),$$

where (N\i,v) is the game v restricted to the set of players N\i. To be associated with a measure of power in the game, these marginal contributions need to add up to the total resources available to the grand coalitions, i.e., v(N). So we will say that P is a potential function if

$$\sum_{i \in N} D^i P(N,v) = v(N). \tag{2}$$

Moreover, we normalize P to satisfy $P(\varnothing,v) = 0$. Given the mild requirement on p, there seems to be enough flexibility for many potential functions to exist. The remarkable result of Hart and Mas-Colell (1989) is that there is in fact only one potential function, and it is closely related to the Shapley value. Specifically,

Theorem (Hart and Mas-Colell 1989): There exists a unique potential function P. Moreover, the corresponding vector of marginal contributions $(D^1P(N,v), ...D^nP(N,v))$ coincides with the Shapley value of the game (N,v).

Let us note that by rewriting Equation (2) we obtain the following recursive equation:

$$P(N,v) = (1/|N|)\,[v(N) + \sum_{i\in N} P(N\backslash i,v)]. \qquad (3)$$

Starting with $P(\emptyset,v) = 0$, Equation (3) determines p recursively. It is interesting to note that the potential is related to the notion of "dividends" used by Harsanyi (1963) to extend the Shapley value to games without transferable utility. Specifically, let $\sum_{T\subset N} a_T u_T$ be the (unique) representation of the game (N,v) as a linear combination of unanimity games. In any unanimity game u_T on the carrier T, the value of each player in T is the "dividend" $d_T = a_T/|T|$, and the Shapley value of player i in the game (N,v) is the sum of dividends that a player earns from all coalitions of which he is a member, i.e., $\sum_{\{T;\, i\in T\}} d_T$. Given the definition and uniqueness of the potential function, it follows that $P(N,v) = \sum_T d_T$, i.e., the total Harsanyi dividends across all coalitions in the game.

One can view Hart and Mas-Colell's result as a concise axiomatic characterization of the Shapley value—indeed, one that builds on a single axiom. In the same paper, Hart and Mas-Colell propose another axiomatization of the value by a different but related approach based on the notion of consistency.

Unlike in non-cooperative game theory, where the feasibility of a solution concept is judged according to strategic or optimal individual behavior, in cooperative game theory neither strategies nor individual payoffs are specified. A cooperative solution concept is considered attractive if it makes sense as an arbitration scheme for allocating costs or benefits. It comes as no surprise, then, that some of the popular properties used to support solution concepts in this field are normative in nature. The symmetry axiom is a case in point. It asserts that individuals indistinguishable in terms of their contributions to coalitions are to be treated equally. One of the most fundamental requirements of any legal system is that it be internally consistent. Consider some value (a single-point solution concept) ϕ, which we would like to use as a scheme for allocating payoffs in games. Suppose that we implement ϕ by first making payment to a group of players Z. Examining the environment subsequent to payment, we may realize that we are facing a different (reduced) game defined on the remaining players N\Z who are still waiting to be paid. The solution concept ϕ is said to be *consistent* if it yields the players in the reduced game exactly the same payoffs they are getting in the original game. Consistency properties play an important role in the literature of cooperative game theory. They were used in the context of the Shapley value by Hart and Mas-Colell (1989). The difference between Hart and Mas-Colell's notion of consistency and that of the related literature on other solution concepts lies in their definition of reduced game. For a given value ϕ, a game (N,v), and a coalition T, the reduced game (T,v_T) on the set of players T is defined as follows:

$$v_T(S) = v(S\cup T^c) - \sum_{i\in T^c} \phi_i(v|_{S\cup T^c}) \text{ for every } S \subset T,$$

where $v|_R$ denotes the game v restricted to the coalition R.

The worth of coalition S in the reduced game v_T is derived as follows. First, the players in S consider their options outside T, i.e., by playing the game only with the complementary coalition T^c. This restricts the game v to coalition $S\cup T^c$. In this game the total payoff of the members of T^c according to the solution ϕ is $\sum_{i\in T^c} \phi_i(v|_{S\cup T^c})$. Thus the resources left available for the members of S to allocate among themselves are precisely $v_T(S)$.

A value ϕ is now said to be *consistent* if for every game (N,v), every coalition T, and every $i\in T$, one has $\phi_i(T,v_T) = \phi_i(N,v)$.

Hart and Mas-Colell (1989) show that with two additional standard axioms the consistency property characterizes the Shapley value. Specifically,

Theorem (Hart and Mas-Colell 1989): There exists a unique value satisfying symmetry, efficiency and consistency, namely the Shapley value[3].

It is interesting to note that by replacing Hart and Mas-Colell's property with a different consistency property one obtains a characterization of a different solution concept, namely the pre-nucleolus. Sobolev's (1975) consistency property is based on the following definition of the reduced game:

$v_T^*(S) = \max_{Q \subset N \setminus T}[v(Q \cup S) - \sum_{i \in Q} \phi_i(v)]$ if $S \neq T, \emptyset$,

$v_T^*(S) = \sum_{i \in T} \phi_i(v)$ if $S = T$, and $v_T^*(S) = 0$ if $S = \emptyset$.

Note that in Sobolev's definition the members of S evaluate their power in the reduced game by considering their *most attractive* options outside T. Furthermore, the collaborators of S outside T are paid according to their share in the original game v (and not according to the restricted game as in Hart and Mas-Colell's property). It is surprising that while the definitions of the Shapley value and the pre-nucleolus differ completely, their axiomatic characterizations differ only in terms of the reduced game on which the consistency property is based. This nicely demonstrates the strength of the axiomatic approach in cooperative game theory, which not only sheds light on individual solution concepts, but at the same time exposes their underlying relationships.

Hart and Mas-Colell's consistency property is also related to the "balanced contributions" property due to Myerson (1977). Roughly speaking, this property requires that player i contribute to player j's payoff what player j contributes to player i's payoff. Formally, the property can be written as follows:

Balanced Contribution: For every two players i and j, $\phi_i(v) - \phi_i(v|_{N \setminus j}) = \phi_j(v) - \phi_j(v|_{N \setminus j})$.

Myerson (1977) introduced a value that associates a payoff vector with each game v and graph g on the set N (representing communication channels between players). His result

[3] Hart and Mas-Colell (1989) in fact showed that instead of the efficiency and symmetry axioms it is enough to require that the solution be "standard" for two-person games, i.e., that for such games $\phi_i(\{i,j\},v) = v(i) + (1/2)[v(\{i,j\}) - v(i) - v(j)]$.

implies that the balanced contributions property, the efficiency axiom, and the symmetry axiom characterize the Shapley value.

We will close this section by discussing another axiomatization of the value, proposed by Chun (1989). It employs an interesting property which generalizes the strategic equivalence property traceable to von Neumann and Morgenstern (1944). Chun's *coalitional strategic equivalence* property requires that if we change the coalitional form game by adding a constant to every coalition that contains some (fixed) coalition $T \subset N$, then the payoffs to players outside S will not change. This means that the extra benefit (or loss if the added constant is negative) accrues only to the members of T. Formally:

Coalitional Strategic Equivalence: For all $T \subset N$ and $a \in R$, let w_a^T be the game defined by $w_a^T(S) = a$ if $T \subset S$ and 0 otherwise. For all $T \subset N$ and $a \in R$, if $v = w + w_a^T$, then $\phi_i(v) = \phi_i(w)$ for all $i \in N \backslash T$.

Von Neumann and Morgenstern's strategic equivalence imposes the same requirement, but only for $|T| = 1$.

Another similar property proposed by Chun is *fair ranking*. It requires that if the underlying game changes in such a way that all coalitions but T maintain the same worth, then although the payoffs of members of T will vary, the ranking of the payoffs within T will be preserved. This directly reflects the idea that the ranking of players' payoffs within a coalition depends solely on the outside options of their members. Specifically:

Fair Ranking: Suppose that $v(S) = w(S)$ for every $S \neq T$; then for all $i,j \in T$, $\phi_i(v) > \phi_j(v)$ if and only if $\phi_i(w) > \phi_j(w)$.

To characterize the Shapley value an additional harmless axiom is needed.

Triviality: If v_0 is the constant zero game, i.e., $v_0(S) = 0$ for all $S \subset N$, then $\phi_i(v_0) = 0$ for all $i \in N$.

The following characterization of the Shapley value can now be obtained:

Theorem (Chun 1989): The Shapley value is the unique value satisfying efficiency, triviality, coalitional strategic equivalence, and fair ranking.

6. Sub-Classes of Games

Much of the Shapley value's attractiveness derives from its elegant axiomatic characterization. But while Shapley's axioms characterize the value uniquely on the class of all games, it is not clear whether they can be used to characterize the value on subspaces. It sounds puzzling, for what could go wrong? The fact that a value satisfies a certain set of axioms trivially implies that it satisfies those axioms on every subclass of games. However, the smaller the subclass, the less likely these axioms are to determine a *unique* value on it. To illustrate an extreme case of the problem, suppose that the subclass consists of all integer multiples of a single game v with no dummy players, and no two players are symmetric. On this subclass Shapley's symmetry and dummy axioms are vacuous: they impose no restriction on the payoff function. It is therefore easy to verify that any allocation of v(N) can be supported as the payoff vector for v with respect to a value on this subclass that satisfies all Shapley's axioms. Another problem that can arise when considering subclasses of games is that they may not be closed under operations that are required by some of the axioms. A classic example of this is the class of all simple games. It is clear that the additivity axiom cannot be used on such a class, because the sum of two simple games is typically not a simple game anymore. One can revise the additivity axiom by requiring that it apply only when the sum of the two games falls within the subclass considered. Indeed, Dubey (1975) shows that with this amendment to the additivity[4] axiom, Shapley's original proof of uniqueness still applies to some subclasses of games, including the class of *all* simple games. However, even in conjunction with the other standard axioms, this axiom does not yield uniqueness in the

[4] Specifically, one has to require the axiom only for games v_1, v_2 whose sum belongs to the underlying subclass.

class of all monotonic[5] simple games. To redress this problem, Dubey (1975) introduced an axiom that can replace additivity: For two simple games v and v', we define by min{v,v'} the simple game in which S is winning if and only if it is winning in both v and v'. Similarly, we define by max{v,v'} the game in which S is winning if and only if it is winning in at least one of the two games v and v'. Dubey imposed the property of

Modularity: $\phi(\min\{v,v'\}) + \phi(\max\{v,v'\}) = \phi(v) + \phi(v')$.

One can interpret this axiom within the framework of Roth's model in Section 3. Suppose that $\phi_i(v)$ stands for the utility of playing i's position in the game v, and player i is facing two lotteries. In the first lottery he will participate in either the game v or the game v' with equal probabilities. The other lottery involves two more "extreme" games: max{v.v'}, which makes winning easier than with v and v', and min{v,v'}, which makes winning harder. As before, each of these games will be realized with probability ½. Modularity imposes that each player be "risk neutral" in the sense that he be indifferent between these two lotteries.

Note that min{v,v'} and max{v,v'} are monotonic simple games whenever v and v' are too, so we do not need any conditioning in the formulation of the axiom. Dubey characterized the Shapley value on the class of monotonic simple games by means of the modularity axiom, showing that:

Theorem (Dubey 1975): There exists a unique value on the class of monotonic[6] simple games satisfying efficiency, symmetry, dummy, and modularity, and it is the Shapley–Shubik value.

When trying to apply a solution concept to a specific allocation problem (or game), one may find it hard to justify the Shapley value on the basis of its axiomatic characterization. After all, an axiom like additivity deals with how the value varies as a result of changes in the game, taking into account games which may be completely irrelevant to the

[5] Recall that a simple game v is said to be monotonic if $v(S) = 1$ and $S \subset T$ implies $v(T) = 1$.
[6] The same result was shown by Dubey (1975) to hold for the class of superadditive simple games.

underlying problem. The story is different insofar as Shapley's other axioms are concerned, because the three of them impose requirements only on the specific game under consideration. Unfortunately, one cannot fully characterize the Shapley value by axioms of the second type only[7] (save perhaps by imposing the value formula as an axiom). Indeed, if we were able to do so, it would mean that we could axiomatically characterize the value on subclasses as small as a single game. While this is impossible, Neyman (1989) showed that Shapley's original axioms characterize the value on the additive class (group) spanned by a single game. Specifically, for a game v let G(v) denote the class of all games obtained by a linear combination of restricted games of v, i.e., G(v) = {v' s.t. v' = $\sum_i k_i v|_{S_i}$, where k_i are integers and $v|_{S_i}$ is the game v restricted to the coalition S_i}.

Note that by definition the class G(v) is closed under summation of games, which makes the additivity axiom well defined on this class. Neyman shows that:

Theorem (Neyman 1989): For any v there exists a unique value on the subclass G(v) satisfying efficiency, symmetry, dummy, and additivity, namely the Shapley value.

It is worth noting that Hart and Mas-Colell's (1989) notion of Potential also characterizes the Shapley value on the subclass G(v), since the Potential is defined on restricted games only.

7. Cooperation Structures

One of Shapley's axioms which characterize the value is the symmetry axiom. It requires that a player's value in a game depend on nothing but his payoff opportunities as described by the coalitional form game, an in particular not on the his "name." Indeed, as we argued earlier, the possibility of constructing a unique measure of power axiomatically from very limited information about the interactive environment is doubtless one of the value's most appealing aspects. For some specific applications,

[7] A similar distinction can be made within the axiomatization of the Nash solution where the symmetry and efficiency axioms are "within games" while IIA and Invariance are "between games."

however, we might possess more information about the environment than just the coalitional form game. Should we ignore this additional information when attempting to measure the relative power of players in a game? One source of asymmetry in the environment can follow simply from the fact that players differ in their bargaining skills or in their property rights. This interpretation led to the concept of the *weighted Shapley value* by Shapley (1953) and Kalai and Samet (1985) (see Chapter 54 in this Handbook). But asymmetry can derive from a completely different source. It can be due to the fact that the interaction between players is not symmetric, as happens when some players are organized into groups or when the communication structure between players is incomplete, thus making it difficult if not impossible for some players to negotiate with others. This interpretation has led to an interesting field of research on the Shapley value, which is mainly concerned with generalizations.

The earliest result in this field is probably due to Aumann and Drèze (1974), who consider situations in which there exists an exogenous coalition structure in addition to the coalitional form game. A coalition structure $B = (S_1,...,S_m)$ is simply a partition of the set of players N, i.e., $\cup S_j = N$ and $S_i \cap S_j = \varnothing$ for $i \neq j$. In this context a value is an operator that assigns a vector of payoffs $\phi(B,v)$ to each pair (B,v), i.e., a coalition structure and a coalitional form game on N. Aumann and Drèze (1974) imposed the following axioms on such operators. First, the efficiency axiom is based on the idea that by forming a group, players can allocate to themselves only the resources available to their group. Specifically,

Relative Efficiency: For every coalition structure $B = (S_1,...,S_m)$ and $1 \leq k \leq m$, we have $\sum_{j \in Sk} \phi_j(B,v) = v(S_k)$.

The remaining three axioms are straightforward extensions of their counterparts in the benchmark model.

Symmetry: For every permutation π on N and every coalition structure B, we have $\phi(\pi B, \pi v) = \pi\phi(B,v)$, where πB is the coalition structure with $(\pi B)_i = \pi S_i$.

Dummy: If i is a dummy player in v, then $\phi_i(B,v) = 0$ for all B.

Additivity: $\phi(B,v+w) = \phi(B,v) + \phi(B,w)$ for all B and any pair of games v and w.

Aumann and Drèze showed that the above axioms characterize a value uniquely. This value assigns the Shapley value to each player in the game restricted to his group. Specifically, for each coalition S and game v, we define by v|S the game on S given by $(v|_S)(T) = v(T)$ for all $T \subset S$. We now have:

Theorem (Aumann and Drèze 1974): For a given set of players N and a coalition structure B, there exists a unique B-value satisfying the four above axioms, namely $\phi_i(B,v) = \phi_i(v|_{B(i)})$, where B(i) denotes the component of B that includes player i, and ϕ stands for the Shapley value.

Aumann and Drèze then defined the coalition structure of all the major solution concepts in cooperative game theory, in addition to the Shapley value. In so doing they exposed a startling property of consistency satisfied by all but one of them. The Shapley value is the exception. Hart and Mas-Colell (1989) would later show that the Shapley value satisfies a version of this property (discussed earlier in Section 5).

In Aumann and Drèze's framework, the coalition structure can be thought of as representing contractual relationships that affect players' property rights over resources, i.e., players in $S \in B$ "own" a total resource v(S), from which transfers to players outside S are forbidden. This is apparent from both the definition and the efficiency axiom that depends on the coalition structure. Roughly speaking, players from one component of the partition cannot share benefits with any player from another component. But in real life coalition formation often takes place merely for strategic reasons without affecting the fundamentals of the economy. A group may form in order to pressure another group, or to enhance the bargaining position of its members vis-à-vis other members without affecting the constraint that all players in N share the total pie v(N). It is thus reasonable to ask whether it is possible to extend the Shapley value to this context as well. Owen (1977), Hart and Kurz (1983), and Winter (1989, 1991, 1992) take up this question.

Owen proposed a value, like the Shapley value, in which each player receives his expected marginal contribution to coalitions with respect to a random order of players. But while the Shapley value assumes that all orders are of equal probability, the Owen value restricts orders according to the coalition structure. Specifically, for a given coalition structure B, let us consider a subset of the set of all orders, which includes only the orders in which players of the same component of B appear successively. Denote this set of orders $\Pi(B)$. The set of orders $\Pi(B)$ can be obtained by first ordering the components of B and then ordering players within each component of the partition. According to the Owen value, each player is assigned an expected marginal contribution to the coalition of preceding players with respect to a uniform distribution over the set of orders in $\Pi(B)$. More formally, let $B = (S_1,....,S_m)$ be a coalition structure. Set $\Pi(B) = \{\pi\in \Pi$; if $i,j\in S_k$ and $\pi(i) < \pi(r) < \pi(j)$, then $r\in S_k\}$, which is the set of all permutations consistent with the coalition structure B. The Owen value of player i in the game v with the coalition structure B is now given by

$$\phi_i(B,v) = (1/|\Pi(B)|) \Sigma_{\pi\in\Pi(B)}[v(p_\pi^{\ i} \cup i) - v(p_\pi^{\ i})].$$

Note that the Owen value satisfies efficiency with respect to the grand coalition, i.e., the total payoff across all players is v(N). This is due to the fact that the total marginal contribution of all players with respect to a fixed order sums up to precisely v(N).

Like the Shapley value, the Owen value can be characterized axiomatically in more than one way, including the way proposed by Owen himself and that of Hart and Kurz (1983). We will introduce here one version that was used in Winter (1989) for the characterization of level structure values, which generalize the Owen value.

First note that for a given coalition structure $B = (S_1, ...,S_m)$ and game v, we can define a "game between coalitions" in which each coalition S_i acts as a player. A coalition acting as a player will be denoted by $[S_i]$. Specifically, the worth of the coalition $\{[S_{i1}], ..., [S_{ik}]\}$ is $v(S_{i1}\cup S_{i2}, ..., \cup S_{ik})$. We will denote this game by v*.

We now impose two symmetry axioms:

Symmetry across Coalitions: If players $[S_k]$ and $[S_l]$ are symmetric in the game v*, then the total values for these coalitions are equal, i.e., $\Sigma_{Sl}\ \phi_j(B,v) = \Sigma_{Sk}\ \phi_j(B,v)$.

Symmetry within Coalitions: For any two players i and j who are symmetric in v and belong to the same coalition in B, i.e., $i,j \in S_k \in B$, we have $\phi_i(B,v) = \phi_j(B,v)$.

We recall that the Owen value satisfies:

Efficiency: $\Sigma_{i\in N}\ \phi_i(B,v) = v(N)$.

With the above axioms we now have:

Theorem (Owen 1977): For a given set of players N and a coalition structure B, the Owen value is the unique B-value satisfying symmetry across coalitions, symmetry within coalitions, dummy, additivity, and efficiency.

Hart and Kurz (1983) formulated a different axiom which can be used to characterize the Owen value requiring that:

Inessential Games among Coalitions: If $v(N) = \Sigma_k\ v(S_k)$, then for all k we have $\Sigma_{j\in Sk}\phi_j(B,v) = v(S_k)$.

Hart and Kurz (1983) show that this axiom together with symmetry, additivity, and a carrier axiom combining dummy and efficiency yield a characterization of the Owen value. Hart and Kurz also used the Owen value to model processes of coalition formation.

Other papers take the Owen value as a starting-point either to suggest alternative axiomatizations or to develop further generalizations. Winter (1992) used the consistency and the potential approach to characterize the Owen value. Calvo, Lasaga, and Winter (1996) used Myerson's approach of balanced contributions for another axiomatization of the Owen value. In Owen and Winter (1992), the multilinear extension approach (see

Owen (1972)) was used to propose an alternative interpretation of the Owen value based on a model of random coalitions. Finally, Winter (1991) proposed a general solution concept, special cases of which are the Harsanyi (1963) solution on standard NTU games and the Owen value for TU games with coalition structures (as well as other non-symmetric generalizations of the Shapley value).

As argued earlier, there is an intrinsic difference between Aumann and Drèze's interpretation of coalition structures and that of Owen. Thinking of coalition structures as unions or clubs, which define an asymmetric "closeness" relation between the various individuals, suggests several alternatives for introducing cooperation structures into the Shapley value framework. One such approach was proposed by Myerson (1977), who used graphs to represent cooperation structures between players. This important paper is discussed thoroughly in Monderer and Samet (see Chapter 54 in this Handbook). Another approach that is closer to Owen's (and indeed generalizes it) was analyzed by Winter (1989). Here cooperation is described by the notion of (hierarchical) level structures. Formally, a level structure is a finite sequence of partitions $\mathcal{B} = (B_1, B_2, ..., B_m)$ such that B_i is a refinement of B_{i+1}. That is, if $S \in B_i$, then $S \subset T$ for some $T \in B_{i+1}$.

The idea behind level structure is that individuals inherit connections to other individuals by association to groups with various levels of commitment. In the context of international trade, one can think of B_m (the coarsest partition) as representing free trade agreements between, say, Nafta, the European Union, Mercusor, etc. Each bloc in this coalition structure is partitioned into countries, each country into federal states or regions, and so on down to the elementary units of households.

In order to define the extension of the Owen value within this framework, we adopt the interpretation that the partition B_j represents a stronger connection between players than that of the coarser B_k where $k > j$. Let us think of a permutation as the order by which players appear to collect their payoffs. To make payoffs dependent on the cooperation structure, we restrict ourselves to orders in which no player i follows player j if there is another player, say k, who is "closer" to player i and who has still not appeared. Formally, we can construct this set of orders inductively as follows:

For a given level structure $\mathcal{B} = (B_1, ..., B_m)$, define

$\Pi_m = \{\pi\in\Pi$; for each $l,j \in S \in B_m$ and $i \in N$, $\pi(l) < \pi(i) < \pi(j)$ implies $i \in S\}$ and
$\Pi_r = \{\pi\in\Pi_{r+1}$; for each $l,j \in S \in B_r$ and $i \in N$, $\pi(l) < \pi(i) < \pi(j)$ implies $i \in S\}$.

The proposed value gives an expected marginal contribution to each player with respect to the uniform distribution over all the orders that are consistent with the level structure **B**, i.e., the orders in Π_1. Specifically,

$$\phi_i(\mathrm{B},v) = (1/|\Pi_1|)\ \Sigma_{\pi\in\Pi 1}[v(p_\pi^i \cup i) - v(p_\pi^i)]. \qquad (4)$$

Note that when the level structure consists of only one partition (i.e., m = 1), we are back in Owen's framework. Moreover, in contrast to the special case of Owen, in which there is only one game between coalitions, this framework gives rise to m games between coalitions, one for each hierarchy level (partition). We denote these games by $v^1, v^2, \ldots, v^m$. The following axiom is an extension of the axiom of symmetry across coalitions that we defined earlier in relation to the Owen value.

Coalitional Symmetry: Let $\mathrm{B} = (B_1,\ldots,B_m)$ be a level structure. For each level $1\leq i \leq m$, if $S, T \in B_i$ are symmetric as players ([S] and [T]) in the game v^i and if S and T are subsets of the same component in B_j for all $j > i$, then $\Sigma_{r\in S}\ \phi_r(\mathrm{B},v) = \Sigma_{r\in T}\ \phi_r(\mathrm{B},v)$.

In order to axiomatize the level structure value, we need another symmetry axiom that requires equal treatment for symmetric players within the same coalition:

Symmetry within Coalitions: If k and j are two symmetric players with respect to the game v, where for every level $1\leq i \leq m$, and any non-singleton coalition $S\in B_I$, then $k\in S$ iff $j\in S$, and $\phi_i(\mathrm{B},v) = \phi_j(\mathrm{B},v)$.

Using straightforward generalizations of the rest of the axioms of the Owen value, it can be shown that:

Theorem (Winter 1989): There exists a unique level structure value satisfying coalitional symmetry, symmetry within coalitions, dummy, additivity, and efficiency. This value is given by (4).

Several other approaches to cooperation structures have been proposed. We have already mentioned Myerson (1977), who uses a graph to represent bilateral connections (or communications) between individuals. An interesting application of Myerson's solution was proposed by Aumann and Myerson (1988). They considered an extensive form game in which players propose links to other players, sequentially. Using the Myerson value to represent the players' returns from each graph that forms, they analyze the endogenous graphs that form (given by the subgame perfect equilibria of the link formation game). Myerson (1980) discusses conference structures that are given formally by an arbitrary collection of coalitions representing a (possibly non-partitional) set of associations to which individuals belong. Derks and Peters (1993) consider a version of the Shapley value with restricted coalitions, representing a set of cooperation constraints. These are given by a mapping ρ: $2^N \to 2^N$, such that (1) $\rho(S) \subset S$, (2) $S \subset T$ implies $\rho(S) \subset \rho(T)$, and (3) $\rho(\rho(S)) = \rho(S)$. This mapping can be interpreted as institutional constraints on the communications between players, i.e., $\rho(S)$ represents the most comprehensive agreement that can be attained within the set of players S. Van den Brink and Gilles propose Permission Structures, based on the idea that some interactions take place in hierarchical organizations in which cooperation between two individuals requires the consent of their supervisors. Permission structures are thus given by a mapping p: $N \to 2^N$, where $j \in p(i)$ stands for "j supervises i." The function p imposes exogenous restrictions on cooperation and allows for an extension of the Shapley value.

I will conclude this section by briefly discussing another interesting (asymmetric) generalization of the value. Unlike the others, this one was proposed by Shapley himself (see also Chapter 32 in this Handbook). Shapley (1977) examines the power of indices in political games, where players' political positions affect the prospects of coalition formation. Shapley's aim was to embed players' positions in an m-dimensional Euclidean space. The point $x_i \in R^m$ of player i summarizes i's position (on a scale of support and opposition) on each of the relevant m "pure" issues. General issues faced by legislators typically involve a combination of several pure ones. For example, if the two pure issues are government spending (high or low) and national defense policy (aggressive or moderate), then the issue of whether or not to launch a new defense missile project is a combination of the two pure issues. Shapley's suggestion was to describe general issues as vectors of weights $w = (w_1, ..., w_m)$. Note that every vector w induces a natural order over the set of players. Specifically, j appears after i if $w \bullet x_i > w \bullet x_j$ (where $x \bullet y$ stands for the inner product of the vectors x and y). The main point to notice here is that different vectors induce different orders on the set of players. This is illustrated in Figure 1 for the case of 2 pure issues and 5 players. To measure the legislative power of each player in the game, one has to aggregate over all possible (general) issues. Let us therefore assume that issues occur randomly with respect to a uniform distribution over all issues (i.e., vectors w in R^m). For each order of players π, let $\theta(\pi)$ be the probability that the random issue generates the order π. Thus the players' profile of political positions $(x_1, x_2, ... , x_n)$ is mapped into a probability distribution over the set of all permutations. Shapley's political value yields an expected marginal contribution to each player, where the random orders are given by the probability distribution θ. Note that the political value is in the class of Weber's random order values (see Weber (1988)). A random order value is characterized by a probability distribution over the set of all permutations. According to this value, each player receives his expected marginal contribution to the players preceding him with respect to the underlying probability distribution on orders. The relation between the political value and the Owen value is also quite interesting. Suppose that the vector of positions is represented by m clusters of points in R^2, where the cluster k consists of the players in S_k, whose positions are very close to each other but further away than those of other players (in the extreme case we could think of the members of S_k as having identical positions). It is pretty clear that the payoff vector that will emerge from Shapley's political value in this case will be very close to the Owen value for the coalition structure $B = (S_1, ..., S_m)$.

8. Sustaining the Shapley Value via Non-Cooperative Games

If the Shapley value is interpreted merely as a measure for evaluating players' power in a cooperative game, then its axiomatic foundation is strong enough to fully justify it. But the Shapley value is often interpreted (and indeed sometime applied) as a scheme or a rule for allocating collective benefits or costs. The interpretation of the value in these situations implicitly assumes the existence of an outside authority—call it a planner—which determines individual payoffs based on the axioms that characterize value. However, situations of benefit (or cost) allocation are, by their very nature, situations in which individuals have conflicting interests. Players who feel underpaid are therefore likely to dispute the fairness of the scheme by challenging one or more of its axioms. It would therefore be nice if the Shapley value could be supported as an outcome of some decentralized mechanism in which individuals behave strategically in the absence of a planner whose objectives, though benevolent, may be disputable. This objective has been pursued by several authors as part of a broader agenda that deals with the interface between cooperative and non-cooperative game theory. The concern of this literature is the construction of non-cooperative bargaining games that sustain various cooperative solution concepts as their equilibrium outcomes. This approach, often referred to in the literature as "the Nash Program," is attributed to Nash's (1950) groundbreaking work on the bargaining problem, which, in addition to laying the axiomatic foundation of the solution, constructs a non-cooperative game to sustain it.

Of all the solution concepts in cooperative game theory, the Shapley value is arguably the most "cooperative," undoubtedly more so than such concepts as the core and the bargaining set whose definitions include strategic interpretations. Yet, perhaps more than any other solution concept in cooperative game theory, the Shapley value emerges as the outcome of a variety of non-cooperative games quite different in structure and interpretation.

Harsanyi (1985) is probably the first to address the relationship between the Shapley value and non-cooperative games. Harsanyi's "dividend game" makes use of the relation between the Shapley value and the decomposition of games into unanimity games. In the more recent literature which uses sequential bargaining games to sustain cooperative solution concepts, Gul (1989) makes a pioneering contribution. In Gul's model, players meet randomly to conduct bilateral trades. When two players meet, one of them is chosen randomly (with equal probabilities 1/2-1/2) to make a proposal. In a proposal by player i to player j at period t, player i offers to pay r_t to player j for purchasing j's resources in the game. If player j accepts the offer, he leaves the game with the proposed payoff, and the coalition {i,j} becomes a single player in the new coalitional form game, implying that i now owns the property rights of player j. If j rejects the proposal by i, both players return to the pool of potential traders who meet through random matching in a subsequent period. Each pair's probability of being selected for trading is $2/(n_t(n_t-1))$, where n_t is the number of players remaining at period t. The game ends when only a single player is left. For any given play path of the game, the payoff of player j is given by the current value of his stream of resources minus the payments he made to the other players. Thus, for a given strategy combination σ and a discount factor δ, we have

$$U^i(\sigma,\delta) = \Sigma_0^\infty (1-\delta)[V(M_i^t) - r_i^t]\delta^t,$$

where M_i^t is the set of players whose resources are controlled by player i at time t and δ is a discount factor. Gul confines himself to the stationary subgame perfect equilibria (SSPE) of the game, i.e., equilibria in which players' actions at period t depend only upon the allocation of resources at time t. He argues that SSPE outcomes may not be efficient in the sense of maximizing the aggregate equilibrium payoffs of all the players in the economy, but he goes on to show that in any no-delay equilibrium (i.e., an equilibrium in which all pairwise meetings end with agreements) players' payoffs converge to the Shapley value of the underlying game when the discount factor approaches 1. Specifically,

Theorem (Gul 1989): Let $\sigma(\delta_k)$ be a sequence of SSPEs with respect to the discount factors $\{\delta_k\}_0^\infty$ which converge to 1 as k goes to infinity. If $\sigma(\delta_k)$ are is a no-delay equilibrium for all k, then $U^i(\sigma(\delta_k),\delta_k)$ converges to i's Shapley value of V as k goes to infinity.

It should be argued that in general the SSPE outcomes of Gul's game do not converge to the Shapley value as the discount factor approaches 1. Indeed, if delay occurs along the equilibrium path, the outcome may not be close to the Shapley value even for δ close to 1. Gul's original formulation of the above theorem required that $\sigma(\delta_k)$ be efficient equilibria (in terms of expected payoffs). Gul argues that the condition of efficiency is a sufficient guarantee that along the equilibrium path every bilateral matching terminates in an agreement. However, Hart and Levy (1999) show in an example that efficiency does not imply immediate agreement. Nevertheless, in a rejoinder to Hart and Levy (1999), Gul (1999) points out that if the underlying coalitional game is strictly convex[8], then in his model efficiency indeed implies no delay.

A different bargaining model to sustain the Shapley value through its consistency property was proposed by Hart and Mas-Colell (1996) [9]. Unlike in Gul's model, which is based on bilateral agreements, in Hart and Mas-Colell's approach players submit proposals for payoff allocations to all the active players. Each round in the game is characterized by a set $S \subset N$ of "active players" and a player $i \in S$ who is designated to make a proposal after being randomly selected from the set S. A proposal is a feasible payoff vector x for the members in S, i.e., $\sum_{j \in S} x_j = v(S)$. Once the proposal is made, the players in S respond sequentially by either accepting or rejecting it. If all the members of S accept the proposal, the game ends and the players in S share payoffs according to the proposal. Inactive players receive a payoff of zero. If at least one player rejects the proposal, then the proposer i runs the risk of being dropped from the game. Specifically, the proposer leaves the game and joins the set of inactive players with a probability of 1-p, in which case the game continues into the next period with the set of active players being $S \setminus i$. Or the proposer remains active with probability p, and the game continues into the next period with the same set of active players. The game ends either when agreement is reached or when only one active player is left in the game. Hart and Mas-Colell analyzed the above (perfect information) game by means of its stationary subgame perfect equilibria, and concluded:

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[8] We recall that a game v is said to be strictly convex if $v(S \cup i) - V(S) > v(T \cup i) - v(T)$ whenever $T \subset S$ and $S \neq T$.

Theorem (Hart and Mas-Colell 1996): For every monotonic and non-negative[10] game v and for every $0 \leq p < 1$, the bargaining game described above has a unique stationary subgame perfect equilibrium (SSPE). Furthermore, if a_S is the SSPE payoff vector of a subgame starting with a period in which S is the active set of players, then $a_S = \phi(v|_S)$ (where ϕ stands for the Shapley value and $v|_S$ for the restricted game on S). In particular, the SSPE outcome of the whole game is the Shapley value of v.

A rough summary of the argument for this result runs as follows. Let $a_{s,i}$ denote the equilibrium proposal when the set of active players is S and the proposer is $i \in S$. In equilibrium, player $j \in S$ with $j \neq i$ should be paid precisely what he expects to receive if agreement fails to be reached at the current payoff. As the protocol specifies, with probability p we remain with the same set of players and the next period's expected proposal will be $a_S = 1/|S|\sum_{i\in S} a_{s,i}$. With the remaining probability 1-p, players i will be ejected so that the next period's proposal is expected to be $a_{S\backslash i}$. We thus obtain the following two equations for $a_{S,i}$:

(1) $\sum_j a^j_{S,i} = v(S)$ (feasibility condition) and

(2) $a^j_{S,i} = pa^j_S + (1-p)a^j_{S\backslash i}$ (equilibrium condition).

Rewriting the second condition we notice that the two of them correspond precisely to the two properties of Myerson (1977), which we discussed in Section 5 and which together with efficiency characterize the value uniquely.

In a recent paper, Perez-Castrillo and Wettstein (1999) suggested a game that modifies that of Hart and Mas-Colell (1996) so as to allow the (random) order of proposals to be endogenized. The game runs as follows: Prior to making a proposal there is a bidding phase in which each player i commits to pay a payoff v_i^j to player j. These bids are made simultaneously. The identity of the proposal is determined by the bids specifically, the proposer is chosen to be the player i for which the difference between the bids made by i and the bids made to i is maximized, i.e., $i = \text{argmax}_{k\in N}[\sum_j v_k^j - \sum_j v_j^k]$ (players' bids to themselves is always zero). If there is more than one player for which

[9] In the original Hart and Mas-Colell (1996) paper, the bargaining game was based on an underlying non-transferable utility game.

[10] $v(S) \geq 0$ for all $S \subset N$, and $v(T) \leq v(S)$ for $T \subset S$.

this maximum is attained, then the proposer is chosen from among these players with an equal probability for each candidate. Following player i's recognition to propose, the game proceeds according to Hart and Mas-Colell's protocol, so far as p = 1 is concerned. Namely, upon rejection, player i leaves the game with probability 1. Perez-Castrillo and Wettstein (1999) show that this game implements the Shapley value in a unique subgame perfect equilibrium (since the game is of a finite horizon, no stationarity requirement is needed).

Almost all the bargaining games that have been proposed in the literature on the implementation of cooperative solutions via non-cooperative equilibria are based on the exchange of proposals and responses. A different approach to multilateral bargaining was adopted in Winter (1994). Rather than a model in which players make full proposals concerning payoff allocations and respond to such proposals, a more descriptive feature of bargaining situations is sought by assuming that players submit only demands, i.e., players announce the share they request in return for cooperation. A coalition emerges when the underlying resources are sufficient to satisfy the demands of all members. I will describe here a simple version of the Winter (1994) model and some of the results that follow.

Consider the order in which players move according to their name, i.e., player 1 followed by 2, etc. Each player i in his turn publicly announces a demand d_i (which should be interpreted as a statement by player i of agreeing to be a member of any coalition provided that he is paid at least d_i). Before player i makes his demand, we check whether there is a compatible coalition among the i-1 players who already made their demands. A coalition S is said to be compatible (to the underlying game v) if S can satisfy the demands of all its members, i.e., $\sum_{j \in S} d_j \leq v(S)$. If compatible coalitions exist, then the largest one (in terms of membership) leaves the game and each of its members receives his demand. The game then proceeds with the set of remaining players. If no such coalition exists, then player i moves ahead and makes his demand. The game ends when all players have made their demands. Those players who are not part of a compatible coalition receive their individually rational payoff.

Consider now a game that starts with a chance move that randomly selects an order with a uniform probability distribution over all orders and then proceeds in accordance with the above protocol. We call this game the demand commitment game. Winter (1994) shows that the demand commitment game implements the Shapley value if the underlying game is strictly convex.

Theorem (Winter 1994): For strictly convex (cooperative) games, the demand commitment game has a unique subgame perfect equilibrium, and each player's equilibrium payoff equals his Shapley value.

Winter (1994) also considers a protocol that requires a second round of bidding in the event that the first round ends without a grand coalition that is compatible. It can be shown that with small delay costs the Shapley value emerges not only as the expected equilibrium outcome for each player, but that the actual demands made by the players in the first round coincide with the Shapley value of the underlying game.

Several other papers have followed the same approach in different contexts. Dasgupta and Chiu (1999) discuss a modified version of the Winter (1994) game, which allows for the implementation of the Shapley value in general games. Roughly, the idea is to allow outside transfers (or gifts) that will convexify a non-convex game. A balanced budget is guaranteed by a schedule of taxes dependent on the order of moves. Bag and Winter (1999) used a demand commitment-type mechanism to implement stable and efficient allocations in excludable public goods. Morelli (1999) modified Winter's (1994) model to describe legislative bargaining under various voting rules. Finally, Mutuswami and Winter (2000) used demand mechanisms of the same kind to study the formation of networks. They also noted that if the mechanism in Winter (1994) is amended to allow a compatible coalition to leave the game only when it is connected (i.e., only when it includes the last k players for some $1 \leq k \leq n$), then the resulting game implements the Shapley value not only in the case of convex games but in all games.

9. Practice

While game theory is thought of as "descriptive" in its attempt to explain social phenomena by means of formal modeling, cooperative game theory is primarily "prescriptive." It is not surprising that much of the literature on cooperative solution concepts finds its way not into economics journals but into journals of management science and operations research. Cooperative game theory does not set out to describe the way individuals behave. Rather, it recommends reasonable rules of allocation, or proposes indices to measure power. The prospect of using such a theory for practical applications is therefore quite attractive, the more so for its single-point solution and axiomatic foundation. In this section, I discuss two areas in which the Shapley value can be (and indeed has been) used as a practical tool: the measurement of voter power and cost allocation.

9.1 Measuring States Power in the U.S. Presidential Election

The procedure for electing a president in the United States consists of two stages. First, each state elects a group of representatives, or "Great Electors," who comprise the Electoral College. Second, the Electoral College elects the president by simple majority rule. It is assumed that each Great Elector votes for the candidate preferred by the majority of his/her state. Since the Electoral College of each state grows in proportion to its census count, a narrow majority in a densely populated state, like California, can affect an election's outcome more than wide majorities in several scarcely populated states. Mann and Shapley (1962) and Owen (1975) measured the voting power of voters from different states, using the Shapley value together with the interesting notion of compound simple games.

Let $M_1,M_2,...,M_n$ be a sequence of n disjoint sets of players. Let $w_1,w_2,...,w_n$ be a sequence of n simple games defined on the sets $M_1,...,M_n$ respectively. And let v be another simple game defined on the set $N = \{1,2,...,n\}$. We will refer to the players in N as districts. The compound game $u = v[w_1,...,w_n]$ is a simple game defined on the set $M = M_1 \cup M_2,..., \cup M_n$ by $u(S) = v(\{j| w_j(S \cap M_j) = 1\})$. In words: We say that S wins in district j if S's members in that district form a winning coalition, i.e., if $w_j(S \cap M_j) = 1$. S is said to be winning in the game u if and only if the set of districts in which S is winning is itself a winning coalition in v. In the context of the presidential race, M_j is the set of voters in state j, w_j is the simple majority game in state j, and v is the electoral college game. Specifically, the electoral college game can be written as the following weighted majority game $[270;p_1,...,p_{51}]$, where 51 stands for the number of states, and p_i is the number of electors nominated by a state i (e.g., 45 for California and 3 for the least populated states and the District of Columbia).

In general compound games, Owen has shown that the value of player i is the product of his value in the game within his district and the value of his district in the game v, i.e., $\phi_i(u) = \phi_j(v)\phi_i(w_j)$.

Since the districts' games are all symmetric simple majority games, the value of each player in the voting game in his state is simply 1 divided by the number of voters. To compute the value of the game v, Owen used the notion of a multilinear extension (see Owen (1972)). Overall, he found that the power of a voter in a more populated state is substantially greater than that of a voter in a less populated state. For example, California voters enjoy more than three times the power of their counterparts in Washington D.C.

Others have used the Shapley value (as well as other indices) to measure political power. Seidmann (1987) used it to compare the power of governments in Ireland following elections in the early and mid 80s. He argued that a government's durability greatly depends on the distribution of power across opposition parties, which can be estimated by means of the Shapley–Shubik index. Carreras, García-Jurado, and Pacios (1993) used the Shapley and the Owen value to evaluate the power of each of the parties in all the Spanish regional parliaments. Fedzhora (2000) uses the Shapley–Shubik index to study the voting power of the 27 regions (oblasts) in the Ukraine in the run-off stage of the presidential elections between 1994 and 1999. She also compares these indicators to the transfers that Ukrainian governments were making to the different regions. Another interesting application of the Shapley value to political science is due to Rapoport and Golan (1985). Immediately after the election of the tenth Israeli parliament, 21 students of political science, 24 Knesset members, and 7 parliamentary correspondents were invited to assess the political power ratios of the 10 parties represented in the Knesset. These assessments were then compared with various power indices, including the Shapley value. The value provided the best fit for 31% of the subjects, but the authors claimed the Banzhaf index performed better.

9.2 Allocation of Costs

The problem of allocating the cost of constructing or maintaining a public facility among its various users is of great practical importance. Young (1994) (see Chapter 34 in this Handbook) offers a comprehensive survey of the relation between game theory and cost allocation. An interesting allocation rule for such problems, which is closely related to the Shapley value, emerges from the "Airport Game" of Littlechild and Owen (1973). Specifically, consider n planes of different size for which a runway needs to be built. Suppose that there are m types of planes and that the construction of a runway sufficient to service a type j plane is c_j with $c_1 < c_2, \ldots, < c_m$. Let N_j be the number of planes of type j so that $\cup N_j = N$ is the set of all planes which need to be serviced. A runway servicing a subgroup of planes $S \subset N$ will have to be long enough to allow the servicing of the largest plane in S. This gives rise to the following natural cost-sharing game (in coalitional form) defined on the set of planes: $c(S) = c_{j(S)}$ and $c(\varnothing) = 0$, where $j(S) = \max\{j | S \cap N_j \neq \varnothing\}$. Littlechild and Owen's (1973) suggestion was to use the game c to determine the allocation of cost by applying the Shapley value on the game.

Note that the game v can be decomposed into m unanimity games. Specifically, define $R_k = R_k \cup R_{k+1}, \ldots, \cup R_m$ and consider the coalitional form games v_k with $v_k(S) = 0$ when $S \cap R_k = \varnothing$ and $v_k(S) = c_k - c_{k-1}$ otherwise (we set $c_0 = 0$). It is easy to verify that the sum of the games v_k is exactly the cost-sharing game, i.e., $v_1(S) + \ldots + v_m(S) = c(S)$ for every coalition of planes S. The additivity of the Shapley value implies that the value of the game c is $\phi(c) = \phi(v_1) +, \ldots, +\phi(v_m)$. But each v_k is a unanimity game with $\phi_i(v_k) = 0$ for $i \in N \backslash R_k$ and $\phi_i(v_k) = (c_k - c_{k-1})/|R_k|$ for $i \in R_k$. We therefore obtain that the Shapley value of the game c is given by $\phi_i(c) = (c_2-c_1)/|R_1| + (c_3-c_2)/|R_2| +, \ldots, + (c_j-c_{j-1})/|R_j|$ for a plane of type j.

The rule suggested by Littlechild and Owen (1973) has the following interesting interpretation: First, all players share equally the cost of a runway for type 1 planes. Then all players who need a larger facility share the marginal extra cost, i.e., c_2-c_1. Next, all players who need yet a larger runway share equally the cost of upgrading to a runway large enough to service type 3 planes. We now continue in this manner until all the residual costs are allocated, which will ultimately allow for the acquisition of a runway long enough to service all planes.

A version of the Airport game was studied by Fragnelli et al. (1999). Their work was part of a research project funded by the European Commission with the aim of determining cost allocation rules for the railway infrastructure in Europe. Fragnelli et al. realized that the original Airport game is ill-suited to their problem since the maintenance cost of a railway infrastructure depends on the number of users. They constructed a new game which distinguishes construction costs (which do not depend on the number of users) from maintenance costs, and derived a simple formula for the Shapley value of the game. They also used real data concerning the maintenance of railway infrastructures to estimate the allocation of cost among different users.

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