Relativity Fundamentals for Time Scales and Astrometry

Abridged from *Guide to Geotemporal Sciences: Issues, Principles, and References*

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1. Introduction

The precision now available from satellite and terrestrial timekeeping systems requires the application of the special and general theories of relativity. Satellites with on–board time–registration capabilities and earth stations that disseminate time–based data must incorporate these relativistic effects in order to gather data or perform calculations self–consistently. It is the purpose of this paper to provide a theoretical background and identify the principal effects that must be considered.

The Global Positioning System is an example of an engineering system in which relativity is not merely of theoretical scientific interest but rather is essential to its performance. GPS satellites revolve around the earth with a velocity of 3.874 km/s at an altitude of 20,184 km. The satellites carry atomic clocks that maintain time to a precision of a few parts in 10^{14} over a day, or about 10 nanoseconds. In the GPS, there are three relativistic effects that are taken into account. First, the combined effects of time dilation and gravitational red shift phenomena cause the clocks to appear to run fast by 38 microseconds per day. This rate difference is compensated by a clock adjustment relative to the nominal frequency of 10.23 MHz prior to launch so that the satellite clocks appear to have the same average rate as earth clocks. Second, although the orbits are nominally circular, the inevitable residual eccentricity causes a sinusoidal variation over one revolution between the time registered by a satellite clock and the time registerd by a similar clock on the ground. For an eccentricity of 0.02, the peak–to–peak amplitude is 92 nanoseconds. Finally, because of the universality of the speed of light in every inertial frame of reference, there is a correction, called the "Sagnac effect," which is due to the change in position of the receiver during the time of flight of the signal. The maximum value of this effect is 133 nanoseconds. The eccentricity and Sagnac corrections are performed in the user's receiver.

Corresponding effects for other satellite orbits may be even larger. For example, for a satellite in a 12–hour highly elliptical orbit with eccentricity 0.722, the peak–to–peak amplitude of the difference in time between a satellite clock and an earth station clock is 3.3 microseconds. For the propagation of a signal from a satellite in geostationary orbit to an earth station on the equator, the Sagnac effect is 215 nanoseconds. These effects are considerable. At the subnanosecond level, there are a variety of other effects, such as the red shift due to the contribution to the earth's gravitational potential from oblateness, and the effect of gravitation on the speed of signal propagation itself.

These examples alone should be sufficient to alert the system designer for the need to incorporate relativity into the measurements of time. Moreover, because time and position are interconnected by the speed of light through the approximate correspondence one nanosecond = 30 centimeters, an error in time determination will result in an error in position and vice versa. The following discussion will summarize the principal ideas to give some appreciation for the nature of the theory and the resulting effects.

2. Gravitation

The birth of modern physics and astronomy may be identified with the work of the Danish astronomer Tycho Brahe during the latter half of the sixteenth century. Tycho became famous in Europe from his writings on the supernova of 1572. To encourage him to continue his research in Denmark, King Fredrick II provided him with land on the island of Hven in the Danish sound near Copenhagen and funds to build an observatory there. Between 1576 and 1597, Tycho carried out extensive measurements of the planetary positions with unprecedented precision. Before the age of the telescope, his observations were distinguished by the use of large instruments on firm foundations and by a continuous measurement routine extending over a long span of time.

Following the death of his sponsor, Tycho moved his records to Prague, where he hired Johannes Kepler as his assistant. When Tycho died in 1601, Kepler acquired the data and from these observations, especially those of Mars, he discovered empirically three laws of planetary motion. Kepler's first law states that the orbits of the planets are ellipses with the sun at one focus. His second law states that the line joining the sun to a planet sweeps out equal areas in equal times. His third law states that the square of the period of revolution is proportional to the cube of the semimajor axis. The first and second laws were published in *Astronomia Nova* in 1609 and the third was published in *Harmonice Mundi* in 1619.

At this time in Italy, Galileo established observational evidence for the heliocentric model of the solar system. With the newly invented telescope, Galileo observed the phases of Venus. The existence of a gibbous phase proved that during some portion of its orbit, Venus is on the other side of the sun with respect to the earth, a geometrical condition that was impossible in the geocentric model. He also discovered the four inner satellites of Jupiter, demonstrating the existence of bodies that revolve around a center other than that of the earth. In addition, Galileo investigated terrestrial motion and showed that all bodies fall with equal acceleration, provided that their masses are sufficiently great that air resistance is negligible. He also established the law of inertia, which states that in the absence of an unbalanced external force, a body moves in a straight line at constant velocity or remains at rest.

The fundamental laws of mechanics and of gravitation were discovered by Isaac Newton while studying at his home in Woolsthorpe, outside of London, during the plague years of 1665 – 1666 and were enunciated in his famous treatise *Philosophiae Naturalis Principia Mathematica*, published in 1687. Newton's first law of motion was the law of inertia stated by Galileo. His second law of motion states that the force $\mathbf{F}$ acting on a body is equal to the product of its mass m and its acceleration $\mathbf{a}$, or

$$\mathbf{F} = m\,\mathbf{a}$$

The force was assumed to be a function of position and velocity, but not of acceleration. Thus, according to Newton, the essential kinematical quantity is the acceleration, from which the velocity and position as functions of time can be determined once the force is specified. In particular, the gravitational force between two point masses is given by the inverse square law

$$\mathbf{F} = -\frac{GMm}{r^2}\,\hat{r}$$

where M and m are the masses of the bodies, r is the distance between them, and G is the universal gravitational constant. The force is directed along the line connecting the bodies and is attractive, as indicated by the unit vector $\hat{r} = \mathbf{r}/r$ along the direction of $\mathbf{r}$ and by the minus sign. The mathematical form of this force law is an "action at a distance" law, according to which bodies interact directly and instantaneously without the intervention of a medium between them.

By these laws Newton unified terrestrial physics with celestial mechanics. He concluded that the same gravitational interaction responsible for the fall of a body near the earth's surface was responsible for holding the moon in its orbit around the earth. Thus the inverse square law of force extended all the way to the moon and beyond, and the period of revolution of the moon could be derived from the known acceleration of a freely falling body near the surface of the earth.

Newton understood the distinction between gravitational and inertial mass. The gravitational mass is the measure of the weight of a body, as indicated by the force required to support it against gravity, while the inertial mass is the measure of the inertia, as indicated by the force required to change its state of motion. Mathematically, the inertial mass is the quantity that appears in Newton's second law of motion, while the gravitational mass is the quantity that appears in his law of gravitation. Nevertheless, these two quantities are equal, which accounts for why we speak simply of the "mass." The weight will increase as the gravitational mass increases, but the force required to produce a given acceleration will increase as the inertial mass increases in the same proportion. Thus, as shown by Galileo, all bodies subject only to gravitation will undergo the same acceleration. Newton himself performed experiments using pendulums, in which he demonstrated that for the same pendulum length, the period of oscillation is independent of the mass or composition of the suspended body.

Newton was also able to demonstrate that, for two interacting bodies, the second law of motion together with the law of gravitation implied the three laws of planetary motion discovered empirically by Kepler. However, he showed that the orbit can be any curve in the family of conic sections, including a parabola or an hyperbola, not just an ellipse. Kepler's second law can be shown to be equivalent to the law of conservation of angular momentum, which applies to any central force, not just gravitation. Kepler's third law has the mathematical form

$$T^2 = \frac{4\,\pi^2}{GM}\,a^3$$

where T is the orbital period and a is the semimajor axis. However, when the orbiting body has a mass m that is not negligible in comparison with the mass M of the central body, this equation is not strictly correct because the central body does not remain fixed in space. Newton's amended form of Kepler's third law is

$$T^2 = \frac{4\,\pi^2}{G(M+m)}\,a^3$$

The two bodies revolve around a common center of mass. For example, the moon has a mass roughly 1/81 that of the earth and the two bodies revolve around a point that is 1/81 of the distance between their centers, or about 4700 km from the center of the earth.

When more than two bodies interact, the net force is given by the vector sum of all the individual forces. The equation of motion of the ith body, due to its interaction with $n-1$ other bodies, is

$$m_i\,\mathbf{a}_i = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{Gm_i\,m_j}{r_{ij}^2}\,\hat{r}_{ij}$$

or, explicitly in terms of Cartesian coordinates,

$$m_i\,\ddot{x}_i = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{Gm_i\,m_j(x_i - x_j)}{r_{ij}^3}$$

$$m_i\,\ddot{y}_i = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{Gm_i\,m_j(y_i - y_j)}{r_{ij}^3}$$

$$m_i\,\ddot{z}_i = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{Gm_i\,m_j(z_i - z_j)}{r_{ij}^3}$$

where a dot indicates differentiation with respect to time and the distance between bodies i and j is

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

During the eighteenth and nineteenth centuries, the implications of this equation were studied extensively by mathematicians and astronomers, such as Laplace, Poisson, Euler, and Gauss. They were delighted to find that all of the fine details of motion in the solar system, including the observed departures from elliptical motion, could be understood as arising from the mutual perturbations of the planets in addition to the gravitational attraction of the sun. Indeed, when the planet Uranus, discovered by Herschel in 1781, did not follow its predicted path, astronomers were confident that the discrepancy was due, not to a fault in the law, but rather to the existence of a previously unseen planet beyond Uranus. Subsequently, Leverrier and Adams independently calculated the orbit of the new planet from its perturbations on Uranus. Leverrier sent his calculations to the astronomer Galle at the Berlin Observatory, and on September 23, 1846 Neptune was discovered at its predicted position.

3. Electromagnetism

The success of Newton's law of gravitation led scientists to develop analogous laws for electrical phenomena. The quantitative study of electricity and magnetism began with the scientific research of the French physicist Charles Augustin Coulomb. In 1787 Coulomb proposed a law of force for charges that, like Newton's law of gravitation, varied inversely as the square of the distance. Using a sensitive torsion balance, he demonstrated its validity experimentally for forces of both repulsion and attraction. Like the law of gravitation, Coulomb's law was based on the notion of "action at a distance," wherein bodies can interact instantaneously and directly with one another without the intervention of any intermediary.

At the beginning of the nineteenth century, the electrochemical cell was invented by Alessandro Volta, professor of natural philosophy at the University of Pavia in Italy. The cell created an electromotive force, which made the production of continuous currents possible. Then in 1820 at the University of Copenhagen, Hans Christian Oersted made the momentous discovery that an electric current in a wire could deflect a magnetic needle. News of this discovery was communicated to the French Academy of Sciences two months later. The laws of force between current bearing wires were at once investigated by Andre-Marie Ampere and by Jean-Baptiste Biot and Felix Savart. Within six years the theory of steady currents was complete. These laws were also "action at a distance" laws, that is, expressed directly in terms of the distances between the current elements.

Subsequently, in 1831, the British scientist Michael Faraday demonstrated the reciprocal effect, in which a moving magnet in the vicinity of a coil of wire produced an electric current. This phenomenon, together with Oersted's experiment with the magnetic needle, led Faraday to conceive the notion of a magnetic field. A field produced by a current in a wire interacted with a magnet. Also, according to his law of induction, a time varying magnetic field incident on a wire would induce a voltage, thereby creating a current. Electric forces could similarly be expressed in terms of an electric field created by the presence of a charge.

Faraday's field concept implied that charges and currents interacted directly and locally with the electromagnetic field, which although produced by charges and currents, had an identity of its own. This view was in contrast to the concept of "action at a distance," which assumed bodies interacted directly with one another. Faraday, however, was a self-taught experimentalist and did not formulate his laws mathematically.

It was left to the Scottish physicist James Clerk Maxwell to establish the mathematical theory of electromagnetism based on the physical concepts of Faraday. In a series of papers published between 1856 and 1865, Maxwell restated the laws of Coulomb, Ampere, and Faraday in terms of Faraday's electric and magnetic fields. Maxwell thus unified the theories of electricity and magnetism, in the same sense that two hundred years earlier Newton had unified terrestrial and celestial mechanics through his theory of universal gravitation.

As is typical of abstract mathematical reasoning, Maxwell saw in his equations a certain symmetry that suggested the need for an additional term, involving the time rate of change of the electric field. With this generalization, Maxwell's equations also became consistent with the principle of conservation of charge. In modern mathematical notation, Maxwell's equations for free space are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \, \epsilon_0 \, \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

where μ_0 and ϵ_0 are the permeability and permittivity of free space. Furthermore, Maxwell made the profound observation that his set of equations, thus modified, predicted the existence of electromagnetic waves, since from these equations it follows that the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ satisfy the wave equation

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \, \frac{\partial^2 \mathbf{E}}{\partial t^2} \; ; \qquad \nabla^2 \mathbf{B} = \frac{1}{c^2} \, \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Thus the electromagnetic field propagates through space with the unique speed c. Using the values of known experimental constants obtained solely from measurements of charges and currents (equivalent to ϵ_0 and μ_0), Maxwell deduced that the speed of propagation was equal to speed of light. This quantity had been measured astronomically by Olaf Romer in 1676 from the eclipses of Jupiter's satellites and determined experimentally from terrestrial measurements by H.L. Fizeau in 1849. He then asserted that light itself was an electromagnetic wave, thereby unifying optics with electromagnetism as well.

Maxwell was aided by his superior knowledge of dimensional analysis and units of measure. He was a member of the British Association committee formed in 1861 that eventually established the centimeter-gram-second (CGS) system of absolute electrical units.

Maxwell's theory was not accepted by scientists immediately, in part because it had been derived from a bewildering collection of mechanical analogies and difficult mathematical concepts. In their original form, they were thus obscure and not readily accepted. The equations we know today as Maxwell's equations were actually first stated by Hertz, who simplified them and eliminated unnecessary auxiliary assumptions. Hertz's analysis established a new paradigm for scientific reasoning. Once having arrived at Maxwell's equations in final form, by whatever method, he emphasized that they should be regarded independently of how they were derived and that their validity should rest on the physical phenomena which they encompass. "To the question, 'What is Maxwell's theory?'," Hertz wrote, "I know of no shorter or more definite answer than the following: – Maxwell's theory is Maxwell's system of equations."

Hertz's interest in Maxwell's theory was occasioned by a prize offered by the Berlin Academy of Sciences in 1879 for research on the relation between polarization in insulators and electromagnetic induction. By means of his experiments, Hertz discovered how to generate high frequency electrical oscillations. He was surprised to find that these oscillations could be detected at large distances from the apparatus. Up to that time, it had been generally assumed that electrical forces decreased rapidly with distance according to the Newtonian law. He therefore sought to test Maxwell's prediction of the existence of electromagnetic waves.

In 1888, Hertz set up standing electromagnetic waves using an oscillator and spark detector of his own design and made independent measurements of their wavelength and frequency. He found that their product was indeed the speed of light. He also verified that these waves behaved according to all the laws of reflection, refraction, and polarization that applied to visible light, thus demonstrating that they differed from light only in wavelength and frequency. "Certainly it is a fascinating idea," Hertz wrote, "that the processes in air that we have been investigating represent to us on a million-fold larger scale the same processes which go on in the neighborhood of a Fresnel mirror or between the glass plates used in exhibiting Newton's rings."

It was not long until the discovery of electromagnetic waves was transformed from pure physics to engineering. After learning of Hertz's experiments through a magazine article, the young Italian engineer Guglielmo Marconi constructed the first transmitter for wireless telegraphy in 1895. Within two years he used this new invention to communicate with ships at sea. Marconi's transmission system was improved by Karl F. Braun, who increased the power, and hence the range, by coupling the transmitter to the antenna through a transformer instead of having the antenna in the power circuit directly. Transmission over long distances was made possible by the reflection of radio waves by the ionosphere. For their contributions to wireless telegraphy, Marconi and Braun were awarded the Nobel Prize in physics in 1909.

Marconi created the American Marconi Wireless Telegraphy Company in 1899, which competed directly with the transatlantic undersea cable operators. On the early morning of April 15, 1912, a 21-year old Marconi telegrapher in New York City by the name of David Sarnoff received a wireless message from the Marconi station in Newfoundland, which had picked up faint SOS distress signals from the steamship *Titanic*. Sarnoff relayed the report of the ship's sinking to the world. This singular event dramatized the importance of the new means of communication. The American Marconi Wireless Telegraphy Company eventually became the Radio Corporation of America (RCA) with Sarnoff as its Director.

4. Special relativity

A fundamental question was the nature of the frame of reference in which the laws of gravitation and electromagnetism applied. Newton assumed that his laws applied in a frame of reference that was at absolute rest, called an inertial frame of reference. Physically, this frame was imagined to coincide with the frame of reference of the fixed stars. However, Newton's second law of motion is expressed in terms of the acceleration and the gravitational force is dependent only on the separation of the interacting bodies. Thus the equation of motion is invariant under a Galilean transformation of coordinates between a frame of reference at rest and one moving with constant velocity in a straight line, given by

$$
\begin{aligned}
x &= x' + V\,t' \\
y &= y' \\
z &= z' \\
t &= t'
\end{aligned}
$$

where x, y, and z are the Cartesian spatial coordinates in the rest system, x', y', and z' are the Cartesian spatial coordinates in the moving system, and V is the velocity of the moving system with respect to the rest system (where for simplicity it is assumed that the motion is along the x–axis). The transformation of velocity has a simple and intuitive form:

$$v = v' + V$$

where v and v' are the velocities as measured in the stationary and moving frames, respectively.

For Newtonian mechanics, time is universal and so the coordinates t' and t are assumed to be the same in both systems. In the moving frame Newton's second law thus has the form

$$\mathbf{F}' = m\,\mathbf{a}'$$

Consequently, as far as gravitational phenomena were concerned, any frame of reference moving with constant velocity in a straight line with respect to an inertial frame of reference is also an inertial frame of reference. This result is called the "Galilean principle of relativity."

The same idea could not be applied to electromagnetic phenomena. Maxwell's equations predict a unique speed of propagation of electromagnetic waves c. Thus they are not invariant under a Galilean transformation. It was assumed, therefore, that Maxwell's equations applied in some fundamental inertial system and that the speed of propagation in a moving frame must be $c \pm V$. Furthermore, since it could not be imagined that electromagnetic waves could propagate in the absence of a medium, it was assumed that some kind of ephemeral substance, called the "ether," pervaded all of interplanetary space.

Near the close of the nineteenth century, Hertz attempted to generalize Maxwell's equations and derive the form that they take in a moving frame of reference. Meanwhile, in 1887, Michelson and Morely performed their famous experiment in which they attempted to observe the effect of the earth's motion through the ether using an interferometer. Their null result led to speculations by Lorentz and Fitzgerald as to how bodies must contract in the direction of motion so as to cancel out the anticipated effect.

A different line of thought was investigated by Albert Einstein purely on philosophical grounds. Einstein was generally aware of the Michelson–Morely experiment (he made reference to "unsuccessful attempts to discover any motion of the earth relatively to the 'light medium' "), but the consensus of historians is that he was not greatly influenced by it. Rather, he saw in the reciprocity of the behavior of moving magnets and electric currents evidence that the principle of relativity should apply to electromagnetic phenomena as well as to gravitation. He called this postulate the "special principle of relativity." Einstein also introduced another postulate: that the speed of light should have the definite velocity c in *every* inertial frame and should be independent of the motion of the source. If these postulates were true, then the Galilean transformation could not be correct and must be modified.

Einstein then sought the transformation of coordinates under which the form of Maxwell's equations would remain invariant with respect to two inertial frames moving at a constant velocity V relative to one another. He found that this transformation is the same as one found previously by Lorentz, who had designed it specifically to account for the null Michelson–Morely measurement. For motion along the x–axis, the Lorentz transformation is

$$\begin{aligned} x &= \gamma\,(x' + V\,t') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + V\,x'/c^2) \end{aligned}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}}$$

When the velocity V is negligible in comparison to c, the Lorentz transformation reduces to the Galilean transformation and is therefore consistent with observations known previously. Einstein published his findings in 1905 in a paper entitled "On the Electrodynamics of Moving Bodies."

The Lorentz transformation implies that spatial dimensions are rendered differently in the stationary and moving reference frames. The length of a moving rod transforms as

$$\Delta x = \Delta x'\,\sqrt{1 - V^2/c^2} = \frac{1}{\gamma}\,\Delta x'$$

This property, called length contraction, implies that the measured length of the rod is less as seen from the stationary frame than when it is when measured in the moving frame in which it is at rest. However, it does not imply that the visual appearance of a moving three–dimensional object is foreshortened; rather, the object appears to be rotated.

The Lorentz transformation also implies that the time is not the same in both frames of reference. Thus two events that are simultaneous in one frame may not be simultaneous in another frame moving with respect to it. This property is called the "relativity of simultaneity." Moreover, the time interval between two events in a moving frame of reference appears to be greater with respect to the stationary frame than as measured in the moving frame. The time interval between two events transforms as

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - V^2/c^2}} = \gamma\,\Delta t'$$

This "time dilation property" has been extensively verified using atomic clocks.

In addition, the velocities v' and v of an object as seen from the moving and stationary frames are related by

$$v = \frac{v' + V}{1 + v' V/c^2}$$

This is the Einstein velocity addition theorem. It is thus never possible for the velocity of an object with respect to any reference frame to exceed the velocity of light. Even if one produced a beam of light with speed $v' = c$ from a reference frame that itself had a velocity close to the speed of light, $V = c$, the apparent speed from the stationary frame would still be $v = c$.

The formulation of special relativity was given a powerful abstract generalization by Hermann Minkowski in 1908. Minkowski reasoned that if light propagates with the same speed c in all inertial frames of reference, the equation of an expanding spherical wavefront can be expressed in two different inertial frames as

$$c^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2$$

This equation can be rewritten as

$$0 = dx^2 + dy^2 + dz^2 - c^2\,dt^2 = dx'^{\,2} + dy'^{\,2} + dz'^{\,2} - c^2\,dt'^{\,2}$$

Minkowski then made the mental leap of assuming that in special relativity the spacetime interval

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2$$

is an invariant, such that in the special case of a propagating light wavefront, $ds = 0$. Also, the time recorded by a clock, its "proper time" τ, is given by $ds^2 = -c^2\,d\tau^2$ and is not equal to the coordinate time t when the clock is in motion.

The transformation of coordinates that preserves the invariance of ds^2 is the Lorentz transformation. Thus Minkowski was able to reformulate the properties of space and time according to special relativity in terms of the properties of a geometrical invariant. The spacetime interval resembles the Pythagorean theorem in a four–dimensional Riemannian manifold, except that the coefficient of the coordinate ct is negative instead of positive. (Originally, the equation was given the exact form of a four-dimensional Pythogorean theorem with the contrivance of defining ict as a fourth coordinate, but this convention is no longer generally used.) It is possible to introduce a coordinate system by a transformation, such as a spatial rotation, in which the line element does not have this simple form. However, the geometrical properties of the spacetime are not altered by such a transformation. In terms of the Riemannian theory of geometry, this spacetime is said to be flat.

5. General relativity

Following his success with the special theory of relativity, Einstein sought to devise a relativistic theory of gravitation. Just as Einstein had built the theory of special relativity on a fundamental philosophical principle, the universality of the speed of light, he founded his theory of general relativity on another fundamental principle: the universality of free fall.

A freely falling frame of reference is locally equivalent to an inertial frame without gravitation. This equivalence, as it applies to mechanics and gravitation, is due to the equivalence of inertial and gravitational masses, as first demonstrated by Galileo and Newton. It has been confirmed by many high precision experiments, for example those of Eötvos and Dicke, and is responsible for the phenomenon of weightlessness in an orbiting spacecraft. Einstein elevated this proposition to the status of a principle: the principle of equivalence. It is a philosophical cornerstone of general relativity and, as postulated by Einstein, applies to *all* physical phenomena. Thus, in the neighborhood of the origin of coordinates of a freely falling frame of reference, all the laws of physics assume the form they take according to special relativity, *i.e.* in the absence of gravitation.

Conversely, an accelerated frame of reference should be indistinguishable locally from one that is in a gravitational field. That is, an accelerated reference frame and a reference frame in a gravitational field cannot be distinguished locally, even by an optical experiment. (Over a finite region of space a gravitational field could be identified by tidal forces.) Since a light ray that appears straight from an inertial reference frame would appear to be curved in an accelerated frame, Einstein concluded that the same would be true in the presence of a gravitational field. He thus predicted that light from a distant star should be deflected by the gravitational field of the sun.

Special relativity was based on the form invariance ("covariance") of physical laws under a particular group of linear transformations: the Lorentz transformations. However, to include accelerated reference frames, nonlinear transformations of the four coordinates must be admitted. The principle of equivalence thus implies that the laws of nature must be covariant with respect to *all* continuous transformations of coordinates. This is the fundamental tenet of general relativity. The mathematical machinery in which this tenet is embodied is the theory of tensor analysis. Physical quantities are represented by tensors, which are defined by their transformation properties under a transformation of coordinates.

The next step was to generalize the spacetime line element, so that it has the mathematical form

$$ds^2 = -c^2\, d\tau^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = g_{00}\, c^2 dt^2 + 2\, g_{0j}\, cdt\, dx^j + g_{ij}\, dx^i\, dx^j$$

where the Greek indices α and β assume the range $0 - 3$, the Latin indices i and j assume the range $1 - 3$, and by the Einstein summation convention a repeated index is summed over its range. The coordinates x^α are defined as $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$ (the superscripts are raised indices, not exponents). The quantities $g_{\alpha\beta}$ are called the components of the metric tensor and play the role of the gravitational potentials. They are functions of position and time and are determined from the distribution of matter by the field equations. The metric tensor is symmetric ($g_{\beta\alpha} = g_{\alpha\beta}$) and thus in four–dimensional spacetime has ten independent components.

The spacetime is said to be curved. The curvature properties are described by the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$, which is a function of the metric components and their first and second derivatives. In the particular case of flat spacetime, the metric is the Minkowski metric: $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$, all others zero. It is possible to find a transformation of coordinates that reduces a given metric to the Minkowski metric if and only if $R^{\alpha}{}_{\beta\gamma\delta} = 0$.

When the spacetime is not flat, a system of coordinates exists in which the first derivatives of the metric vanish at a selected point. This property is consistent with the principle of equivalence, which implies that the metric of a freely falling frame is locally given by the Minkowski metric. Einstein thus required that the field equation must contain derivatives of the metric up to the second but no higher. Mathematically, this implies that the law of gravitation necessarily must involve the Riemann tensor; the only geometric tensors that do not contain derivatives of $g_{\mu\nu}$ beyond the second are functions of $g_{\mu\nu}$ and $R^{\alpha}{}_{\beta\gamma\delta}$.

Between 1910 and 1915, Einstein tried various alternatives for the field equations from which the metric tensor is determined. At last, with the help of his colleague Marcel Grossman, Einstein arrived at the final form of his field equation,

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \frac{8\pi G}{c^4}\, T_{\mu\nu}$$

where $E_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor (which is a function of the metric coefficients and is obtained from the Riemann tensor by summing on the first and third indices, *i.e.* $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$), R is the scalar curvature (which is obtained from the Ricci tensor by summing on its two indices), and $T_{\mu\,\nu}$ is the matter tensor (which is determined by the density, pressure, and velocity of matter). The field equation is nonlinear so that exact solutions are difficult to obtain and the principle of superposition is not valid. In free space the field equation has the deceptively simple form

$$R_{\mu\nu} = 0$$

However, this equation does not imply the stronger condition $R^{\alpha}{}_{\beta\gamma\delta} = 0$, which implies the spacetime is flat. The final form of the theory was published in 1916 in a paper entitled "The Foundation of the General Theory of Relativity."

The Einstein tensor is symmetric and satisfies four identities. Thus the field equation effectively comprises $10 - 4 = 6$ component equations for six unknowns. Since the metric has ten independent components, there are four degrees of freedom that correspond to the choice of an arbitary system of coordinates. The four identities also imply conservation of momentum. Thus the equation of motion is contained within the field equation and, unlike in Newtonian mechanics, does not have to be postulated separately. For a point particle, the equation of motion reduces to the only possible invariant path: the equation of a geodesic in four–dimensional spacetime.

In the first approximation, Einstein's field equation implies Poisson's equation,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 4\pi G\,\rho$$

where ϕ is the Newtonian gravitational potential and ρ is the density of matter. Further, the equation for a geodesic becomes in this approximation

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{\partial\phi}{\partial x}\\ \frac{d^2y}{dt^2} &= -\frac{\partial\phi}{\partial y}\\ \frac{d^2z}{dt^2} &= -\frac{\partial\phi}{\partial z}\end{aligned}$$

Thus to lowest order, Einstein's theory reduces to Newton's theory. Nevertheless, even in this approximation, coordinate time and proper time are distinct. The components of the spacetime metric $g_{\alpha\beta}$ are given by

$$\begin{aligned}-g_{00} &= 1 + 2\,\phi/c^2\\ g_{0j} &= 0\\ g_{ij} &= \delta_{ij}\ .\end{aligned}$$

The Kronecker delta is defined as $\delta_{ij} \equiv 1$ if $i = j$; 0 if $i \neq j$. To this order, the spacetime interval is

$$ds^2 = -c^2\,d\tau^2 = -(1 + 2\,\phi/c^2)\,c^2\,dt^2 + \delta_{ij}\,dx^i\,dx^j$$

For a moving clock this becomes

$$d\tau^2 = (1 + 2\,\phi/c^2 - V^2/c^2)\,dt^2$$

where $V \equiv \sqrt{\delta_{ij}\,(dx^i/dt)(dx^j/dt)}$ is the velocity of the clock. This implies that the elapsed coordinate time is

$$\Delta t = \int \frac{d\tau}{\sqrt{(1 + 2\,\phi/c^2 - V^2/c^2)}} \approx \int \left(1 - \frac{1}{c^2}\,\phi + \frac{1}{2}\,\frac{1}{c^2}\,V^2\right)\,d\tau$$

using the binomial expansion $(1 + x)^n \approx 1 + nx$ for $x << 1$ with $n = -1/2$. In the reference frame described by the coordinates (ct, x^i), the coordinate time t of the clock experiences both a "redshift" due to the potential ϕ and a "time dilation" due to the velocity V compared to the proper time τ as measured in the rest frame of the clock.

6. Clock transport

For comparison of clocks by means of a transported intermediate clock, the coordinate time interval is given to the required approximation as,

$$\Delta t = \int\limits_{\text{path}} \left(1 + 2\,\frac{1}{c^2}\,\phi - \frac{1}{c^2}\,\delta_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}\right)^{-1/2} d\tau \approx \int\limits_{\text{path}} \left(1 - \frac{1}{c^2}\,\phi + \frac{1}{2}\,\frac{1}{c^2}\,v^2\right) d\tau$$

where v^i is the clock velocity as measured in the inertial frame. Therefore, when transfering time from point P to point Q by means of a clock, the coordinate time elapsed during the motion of the clock is

$$\Delta t = \int_P^Q \left[1 - \frac{1}{c^2}\,\phi(\mathbf{r}) + \frac{1}{2}\,\frac{1}{c^2}\,v^2\right] d\tau$$

where $\phi(\mathbf{r})$ is the gravitational potential and v is the velocity relative to an inertial frame of reference. In the vicinity of the earth, the gravitational potential may be approximated by

$$\phi(r, \lambda) = -\,\frac{GM}{r}\left[1 - \frac{1}{2}\,J_2\left(\frac{R}{r}\right)^2 (3\,\sin^2\lambda - 1)\right]$$

where r is the distance from the center of the earth, λ is the latitude, R is the equatorial radius of the earth, and J_2 is the lowest order oblateness coefficient.

6.1 Clock on the geoid

For a standard clock at rest on the rotating geoid, the velocity relative to an inertial frame is $v = \omega\, r\,\cos\lambda$. Thus

$$\Delta t = \int_P^Q \left[1 - \frac{1}{c^2}\,\phi(\mathbf{r}) + \frac{1}{2}\,\frac{1}{c^2}\,\omega^2\,r^2\,\cos^2\lambda\right] d\tau = (1 - U_0/c^2)\,\Delta\tau$$

where U_0 is the geopotential evaluated on the geoid. It is convenient to define a new scale of coordinate time given by

$$\Delta t' \equiv (1 + U_0/c^2)\,\Delta t$$

Then for a standard clock at rest on the geoid, $\Delta t' = \Delta\tau$ and the standard clock becomes a coordinate clock.

6.2 Clock on a satellite

For a clock at an arbitrary position $\mathbf{r}$ with velocity $\mathbf{v}$ in an inertial frame of reference,

$$\Delta t = \int_P^Q \left[1 - \frac{1}{c^2}\,\phi(\mathbf{r}) + \frac{1}{2}\,\frac{1}{c^2}\,v^2 \right] d\tau$$

In particular, for a clock onboard a satellite following a Keplerian orbit, the potential is (neglecting oblateness perturbations)

$$\phi = -\frac{GM}{r}$$

and by conservation of energy the velocity is given by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

where a is the orbital semimajor axis. Therefore,

$$\Delta t = \int_{\text{orbit}} \left[1 + \frac{1}{c^2}\,\frac{GM}{r} + \frac{1}{2}\,\frac{1}{c^2}\,GM\left(\frac{2}{r} - \frac{1}{a}\right)\right] d\tau$$

$$= \int_{\text{orbit}} \left(1 - \frac{1}{2}\,\frac{1}{c^2}\,\frac{GM}{a} + 2\,\frac{1}{c^2}\,\frac{GM}{r} \right) d\tau$$

$$= \left(1 - \frac{1}{2}\,\frac{1}{c^2}\,\frac{GM}{a} \right) \Delta\tau + 2\,\frac{1}{c^2}\,GM \int_0^\tau \frac{1}{r}\,d\tau$$

The distance r can be expressed in terms of the eccentric anomaly E as

$$r = a\,(1 - e\,\cos E)$$

where e is the orbital eccentricity. The mean anomaly M and the eccentric anomaly E are related by Kepler's equation

$$M \equiv n\,(t - t_p) = E - e\,\sin E$$

where $n = 2\pi/T = \sqrt{GM/a^3}$ is the mean motion and t_p is the time of perigee. Thus assuming $d\tau \approx dt$,

$$\int_0^\tau \frac{1}{r}\,d\tau \approx \int_0^E \left[\frac{1}{a\,(1 - e\cos E)} \right] \left[\frac{1}{n}\,(1 - e\cos E)\,dE \right] = \frac{E}{na} \approx \frac{1}{na}\,(n\,\Delta\tau + e\,\sin E)$$

$$= \frac{1}{a}\,\Delta\tau + \sqrt{\frac{a}{GM}}\;e\,\sin E$$

Therefore,

$$\Delta t = \left(1 - \frac{1}{2}\frac{1}{c^2}\frac{GM}{a}\right)\Delta\tau + 2\,\frac{1}{c^2}\,GM\,\left(\frac{1}{a}\,\Delta\tau + \sqrt{\frac{a}{GM}}\;e\,\sin E\right)$$

$$= \left(1 + \frac{3}{2}\frac{1}{c^2}\frac{GM}{a}\right) + 2\,\frac{1}{c^2}\sqrt{GM\,a}\,e\,\sin E$$

It is convenient to apply the change of coordinate time scale

$$\Delta t' \equiv (1 + U_0/c^2)\,\Delta t$$

so that the coordinate time is given by the reading of a standard clock on the geoid. Then

$$\Delta t' = \left[1 + \frac{3}{2}\frac{1}{c^2}\frac{GM}{a} - \frac{1}{c^2}\frac{GM}{R}\left(1 + \frac{1}{2}\,J_2\right) - \frac{1}{2}\frac{1}{c^2}\,\omega^2\,R^2\right]\Delta\tau + 2\,\frac{1}{c^2}\sqrt{GM\,a}\,e\,\sin E$$

$$\equiv (1 + k)\,\Delta\tau - \Delta t_r$$

In the WGS–84 model of the earth, $GM = 398\,600.5\ \mathrm{km}^3/\mathrm{s}^2$, $R = 6378.137$ km, $\omega = 7.292\,115 \times 10^{-5}$ rad/s, and $J_2 = 0.001\,082\,63$. The first term has a constant factor that represents an apparent rate offset between the satellite clock and a clock on the geoid. By adjusting the rate of proper time on the satellite clock according to

$$\Delta\tau' = (1 - k)\,\Delta\tau$$

the satellite clock can be made to appear to have the same average rate as a clock on the geoid. The second term is a periodic relativistic correction given by

$$\Delta t_r = -\,2\,\frac{1}{c^2}\sqrt{GMa}\;e\sin E = F\,\sqrt{a}\,e\,\sin E$$

where

$$F \equiv -\,\frac{2}{c^2}\sqrt{GM}$$

Consequently, the coordinate time is

$$\Delta t' = \Delta\tau' - \Delta t_r$$

Therefore, apart from a small periodic correction, the satellite clock becomes a coordinate clock.

The nominal clock rate on a GPS satellite is 10.23 MHz. A GPS satellite has an orbital semimajor axis $a = 26\,561.8$ km. Thus a satellite clock has a net rate offset of 4.4647×10^{-10} and runs fast by 38 μs per day with respect to a standard clock on the geoid, of which 45 μs per day is fast due to difference in gravitational potential and 7 μs per day is slow due to difference in velocity. This effect is enormous compared to the available precision of clocks at the nanosecond and subnanosecond level. To compensate for this rate difference, the satellite clock is given a fractional rate offset prior to launch of $\Delta f/f = -k = -4.4647 \times 10^{-10}$. The change in the is thus $\Delta f = -0.004\,5674$ Hz. The resulting frequency is equal to 10.229 999 995 43 MHz, so that, as it appears to an observer on the ground, the frequency remains unchanged. For the GPS satellite, $F = -4.442\,807\,633 \times 10^{-10}$ s/m$^{1/2}$. For a maximum orbital eccentricity of 0.02, the term Δt_r can have an amplitude of 46 nanoseconds and a peak–to–peak variation of 92 nanoseconds at the orbital period of 11.967 hours. The relativistic correction Δt_r is applied in the user receiver. For a 12–hour highly elliptical orbit with semimajor axis 26 561.8 km and eccentricity 0.722, the average rate offset is the same but the peak–to–peak variation is 3.3 microseconds.

7. Conclusion

We have in the Global Positioning System (GPS) an example of an actual engineering system in which the special and general theories of relativity are not merely of scientific interest, but rather are essential to its operation. The GPS serves as a kind of laboratory for the demonstration of relativistic effects on satellite clocks and as a model for the appropriate application of relativity algorithms in other systems.

Typically, the effects of relativity become important for individual time measurements made to a precision of one microsecond or better. For any satellite system within the DoD in which there are on-board clocks and on-board time-tagging of events, it is possible that relativistic effects may be even more dramatic than in the case of GPS, such as a satellite in a highly elliptical orbit cited above. It is therefore important that these effects should be appropriately identified and taken into account. To facilitate the dissemination of information on this subject, I have prepared a technical manual entitled *Guide to Geotemporal Science: Issues, Principles, and References*. This report may be obtained by contacting me at 301-657-9641.

Appendix
Relativity Corrections for a Flying Atomic Clock

The coordinate time t is the time recorded by a clock at rest on the rotating geoid. By definition, t is equal to the coordinate time with respect to the underlying inertial frame of reference. The proper time τ is the time recorded by the transported clock in its own frame of reference. The coordinate time accumulated by the transported clock in moving from A to B is given by

$$\Delta t = \frac{2\omega A}{c^2} + \int_A^B (1 - \frac{1}{c^2} g\, h + \frac{1}{2}\frac{1}{c^2} v^2)\; d\tau$$

where g is the acceleration of gravity (9.8 m/s^2), h is the height above the geoid, and v is the velocity relative to the rotating geoid. The term gh/c^2 under the integral is the correction for the gravitational potential (redshift/blueshift). The term $v^2 / 2\, c^2$ is the correction for velocity (time dilation).

The first term is the Sagnac effect, which is given by

$$\frac{2\omega A}{c^2} = \frac{1}{c^2}\int_A^B (\vec{\omega}\times\vec{r}) \bullet \vec{v}\; d\tau = \frac{1}{c^2}\int_A^B (\vec{\omega}\times\vec{r}) \bullet dr = \frac{1}{c^2}\int_A^B \vec{\omega} \bullet (\vec{r}\times d\vec{r}) = 2\frac{1}{c^2}\int_A^B \vec{\omega} \bullet d\vec{A}$$

where $\vec{r}$ is the position vector relative to the center of the Earth, ω is the angular velocity of rotation of the Earth (7.292×10^{-5} rad/s), and A is the area swept out by the position vector with respect to the center of the Earth projected onto the equatorial plane. The Sagnac effect can be written

$$\frac{2\omega A}{c^2} = \frac{1}{c^2}\int_A^B (\vec{\omega}\times\vec{r}) \bullet \vec{v}\;\; d\tau \;=\; \frac{1}{c^2}\int_A^B \omega\, r^2 \cos^2\lambda\, \frac{d\Lambda}{dt}\;\; d\tau \approx \frac{1}{c^2}\int_A^B \omega\; r^2 \cos^2\lambda\; d\Lambda$$

where λ is the latitude and Λ is the longitude. This term is positive for a clock traveling east and is negative for a clock traveling west; it is zero for a clock traveling north or south. By Stokes' theorem, for a closed path bounding an area S on the Earth's surface,

$$\oint_{path} (\vec{\omega}\times\vec{r}) \bullet \vec{v}\; d\tau = \oint_{path} (\vec{\omega}\times\vec{r}) \bullet d\vec{r} = \iint_S curl(\vec{\omega}\;\times\vec{r}) \bullet d\vec{S} = \iint_S 2\,\vec{\omega} \bullet d\vec{S} \;= 2\omega \iint_S \sin\lambda\, dS = 2\omega\, A$$

where $d\vec{S}$ is oriented perpendicular to the surface and is positive (pointing outward) for a counter-clockwise circuit and where A is the total area projected onto the equatorial plane. For a small area, $A \approx S \sin\bar{\lambda}$, where $\bar{\lambda}$ is the mean latitude evaluated at the center of S. Note that by a vector identity,

$$curl(\vec{\omega}\times\vec{r}) = \vec{\omega}\;\, div\,\vec{r} - \vec{r}\;\, div\vec{\omega} + (\vec{r} \bullet \nabla)\,\vec{\omega} - (\vec{\omega} \bullet \nabla)\,\vec{r} = 3\vec{\omega} - 0 + 0 - \vec{\omega}\; = 2\,\vec{\omega}$$

The gravitational term can be written

$$g\;\, h\; = \Delta U \;=\; U - U_0 \;=\; \phi \;-\; \frac{1}{2}\,(\vec{\omega}\;\times\vec{r})^2 - \;U_0 = \Delta\phi - \frac{1}{2}\,(\vec{\omega}\;\times\vec{r})^2$$

where ΔU is the difference in geopotential, $U \equiv \phi - (\vec{\omega}\times\vec{r})^2 / 2$ is the geopotential at height h, ϕ is the Newtonian gravitational potential at height h, $-(\vec{\omega}\times\vec{r})^2 / 2$ is the centrifugal potential due to the Earth's rotation and height h, and $\Delta\phi \;\equiv\; \phi \;-\; U_0$ is the difference between the Newtonian gravitational potential ϕ at height h and the geopotential at the Earth's surface U_0 .

By substitution of these expressions, the elapsed coordinate time as the clock is transported from A to B may be expressed

$$\Delta t = \int_A^B \{1 - \frac{1}{c^2}\Delta\phi + \frac{1}{2}\frac{1}{c^2}[\, v^2 + 2\,(\vec{\omega}\times\vec{r})\bullet\vec{v} + (\vec{\omega}\times\vec{r})^2\ ]\}\, d\tau = \int_A^B (1 - \frac{1}{c^2}\Delta\phi + \frac{1}{2}\frac{1}{c^2}V^2)\, d\tau$$

where the velocity relative to an inertial frame of reference is

$$\vec{V} = \vec{v} + \vec{\omega}\times\vec{r}$$

If measurements are performed at time intervals $\delta\tau_i$, the elapsed coordinate time may be expressed approximately as

$$\Delta t = \Delta\tau + \frac{1}{c^2}\,\omega \sum_{i=1}^{N} r_i^{\,2}\,\cos^2\lambda_i\ \delta\Lambda_i - \frac{1}{c^2}\,g \sum_{i=1}^{N} h_i\ \delta\tau_i + \frac{1}{2}\frac{1}{c^2}\sum_{i=1}^{N} v_i^{\,2}\ \delta\tau_i$$

where $r = R + h$ and where R is the radius of the Earth (6378 137 m).

Example 1. An aircraft flies over a triangular path starting from Edwards AFB, California (latitude 35° N) with a velocity of 300 kt (154 m/s) at an altitude of 35 000 ft (11 000 m) [1 kt = 1 nmi/h = 1.852 km/h; 1 ft = 0.3048 m]. The aircraft flies due west for 1.0 h, then flies due south for 1.0 h, and returns on a direct path for 1.4 h. The aircraft carries a cesium atomic clock, which reads proper time τ, and a GPS receiver, which reads coordinate time t.

(a) First segment (due west). The distance from the center of the Earth is

$$r = 6378\,000 \text{ m} + 11\,000 \text{ m} = 6389\,000 \text{ m}$$

The distance traveled is (westward)

$$d = -\,(154 \text{ m/s})(3600 \text{ s}) = -\,554\,000 \text{ m}$$

The Sagnac correction is

$$\Delta t = (1/c^2)\,\omega\, r^2 \cos^2\lambda\,\Delta\Lambda = (1/c^2)\,\omega\, r\cos\lambda\,[\,(r\cos\lambda)\,\Delta\Lambda\,] = (1/c^2)\,\omega\, r\, d\cos\lambda$$

$$= (7.292\times 10^{-5} \text{ rad/s})(6389\,000 \text{ m})(-\,554\,000 \text{ m})(\cos 35°)\,/\,(3\times 10^8 \text{ m/s})^2$$

$$= -\,2.3\times 10^{-9} \text{ s} = -\,2.3 \text{ ns}$$

The gravitational correction is

$$\Delta t = -\,g\,h\,\Delta\tau\,/\,c^2 = -\,(9.8 \text{ m/s}^2)(11\,000 \text{ m})(3600 \text{ s})\,/\,(3\times 10^8 \text{ m/s})^2 = -\,4.3\times 10^{-9} \text{ s} = -\,4.3 \text{ ns}$$

The velocity correction is

$$\Delta t = +\,v^2\,\Delta\tau\,/\,(2\,c^2) = +\,(1/2)(154 \text{ m/s})^2\,(3600 \text{ s})\,/\,(3\times 10^8 \text{ m/s})^2 = +\,0.5\times 10^{-9} \text{ s} = +\,0.5 \text{ ns}$$

The total correction after the first segment is

$$\Delta t_1 = -\,2.3 \text{ ns} - 4.3 \text{ ns} + 0.5 \text{ ns} = -\,6.1 \text{ ns}$$

Since

coordinate time elapsed (GPS receiver) = proper time elapsed (cesium clock) + correction

this means that 6.1 ns must be subtracted from the reading of the cesium clock to obtain the reading of the GPS receiver, *i.e.*, the cesium clock is ahead of the GPS receiver by 6.1 ns after the end of the first segment.

(b) Second segment (due south). There is no correction due to the Sagnac effect, since the aircraft's velocity is perpendicular to the Earth's rotational velocity. The gravitational and velocity corrections are the same as for the first segment. The total correction over the second segment is

$$\Delta t_2 = 0.0 \text{ ns} - 4.3 \text{ ns} + 0.5 \text{ ns} = -3.8 \text{ ns}$$

Thus at the end of the second segment, the cesium clock is ahead of the GPS receiver by 6.1 ns + 3.8 ns = 9.9 ns.
(c) Third segment (direct return). During the second segment, the latitude of the aircraft changes south by (554 000 m / 40 000 000 m)(360°) = 5°. Thus the average latitude on the third segment is 32.5°. The Sagnac correction is + $(\cos 32.5°/\cos 35°)^2$ (2.3 ns) = + 2.4 ns. The return time is 5040 s, so the gravitational correction is (5040/3600)(– 4.3 ns) = – 6.0 ns and the velocity correction is (5040/3600)(+ 0.5 ns) = + 0.7 ns. The total correction over the third segment is

$$\Delta t_3 = +2.4 \text{ ns} - 6.0 \text{ ns} + 0.7 \text{ ns} = -2.9 \text{ ns}$$

Thus at the end of the third segment, the cesium clock is ahead of the GPS receiver and the ground clock by 9.9 ns + 2.9 ns = 12.8 ns. The net Sagnac correction is = + 0.1 ns; the net gravitational correction is – 14.6 ns; and the net velocity correction is + 1.7 ns. Therefore, the net total correction is – 12.8 ns. Note that for the closed path, the bounded area on the Earth's surface is approximately $S = 0.5 \times (554\,000 \text{ m})^2 = 1.53 \times 10^{11} \text{ m}^2$, so that the net Sagnac correction may be expressed

$$\Delta t = 2\,\omega S \sin \lambda / c^2 = 2\,(7.292 \times 10^{-5} \text{ rad/s})(1.53 \times 10^{11} \text{ m}^2) \sin 32.5° / (3 \times 10^8 \text{ m/s})^2 = +0.1 \times 10^{-9} \text{ s} = +0.1 \text{ ns}$$

in agreement with the sum of the contributions over the three segments.

Example 2. Two aircraft at the equator synchronize their atomic clocks in time and frequency. They simultaneously take off in opposite directions, one going east and the other going west. They both fly at the same altitude and the same speed. After traveling in opposite directions for a distance of 100 nautical miles (1852 km) each, they both turn north and travel along their respective meridians until they meet at the north pole. They then compare their clocks.

The gravitational (redshift/blueshift) and velocity (time dilation) corrections for both aircraft are the same.

The area swept out by the path projected on the equatorial plane is

$$A = (\,d / 2\,\pi\, r\,)(\,\pi\, r^2\,) = d\, r / 2$$

The Sagnac correction for the aircraft flying east is (clock runs slow)

$$\Delta t = 2\,\omega A / c^2 = \omega\, d\, r / c^2 = (7.292 \times 10^{-5} \text{ rad/s})(1852 \text{ km})(6378 \text{ km}) / (3 \times 10^5 \text{ km/s})^2 = 1.0 \text{ ns}$$

The Sagnac correction for the aircraft flying west is similarly – 1.0 ns (clock runs fast). There is no correction for either aircraft along a meridian. When the aircraft meet at the north pole, the westward flying clock will be ahead of the eastward flying clock by 2.0 ns.

Example 3. An aircraft flies completely around the equator in the eastward direction and returns to its starting point.

The Sagnac correction is

$$\Delta t = 2\,\omega A / c^2 = (2)\,(7.2921 \times 10^{-5} \text{ rad/s})[\pi\;(6378\,137 \text{ m})^2] / (299\,792\,458 \text{ m/s})^2 = 2.074 \times 10^{-7} \text{ s} = 207.4 \text{ ns}$$