13. Math Before Calculus

13.1 Four Problems

There were four problems facing mathematicians in the sixteen hundreds. All four problems can thought of in terms of geometry, given the graph of a function. The first problem was that of finding the tangent line to a curve at a particular point, meaning a line that intersects the curve at a single point in the vicinity of the chosen point. The second problem involves finding the length of a curve from one point to another. The third problem was that of computing the area between the x-axis and the curve over a certain domain. The fourth problem involved finding the maximum and minimum values attained by a function for a specified domain.

Using some nice trickery, some clever mathematicians figured out ways to solve some of these problems in certain cases, such as finding the tangent line to the graph of $y = x^2$. In other cases the solution was thought not even to exist at all, such as the problem of finding the length of a general curve. It is one of the greatest achievements in the history of human thought that has connected these four problems to that of a single idea. Yet it is only through these first attempts at solutions that the real idea became evident. And for this I feel it is worthwhile to study these problems as they historically came to light.

13.2 Optimization of a Function

Finding the maximum or the minimum of a function is so inherent to everyday experience it is often looked over without second thought. We don't like to do two hours of work where one hour will get the job done right. We do our best not to pay $1.80 for a gallon of gas when we know of a station that will sell a gallon for $1.75. And certainly we do not do more homework problems than we are assigned. It is our nature to attempt to minimize workload and stress, while maximizing fun time.

Maximization and minimization also come up in explicit circumstances. At what angle should you throw a baseball to insure that it will stay in the air for the longest possible time? At what angle should you throw the ball to have it travel the maximum possible distance? Problems of these types involve maximizing quadratic polynomials, that is functions of the form $y = ax^2 + bx + c$.

Locating the minimum
value of a function
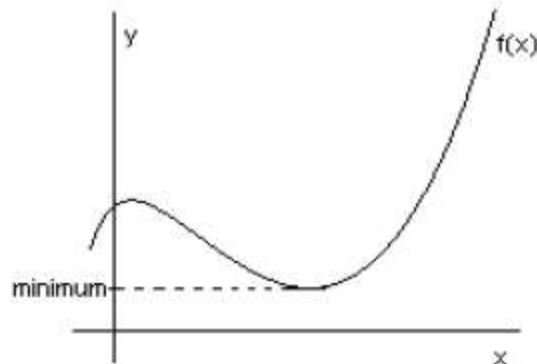
Example 13.2.1. Galileo found that if you throw a ball from the ground straight upinto the air with the intial velocity (speed) of v_0 feet per second, then the height in feet is given by

$$h(t)=-16t^2+v_0t.$$

Suppose we throw a ball straight up from the ground with initial velocity of 4 feet per second. What is the maximum height the ball will reach? We can solve this problem as before by expressing h as a translation of the function $h(t)=-16t^2$ and then locating the vertex (by completing the square). We solve to find the ball reaches a height of 1/4 feet, or 3 inches.

Of course, the success of the previous example relied crucially on our ability to express the problem in terms of a previously known problem, shifting the function $y=x^2$. In general we have no such way of doing so.

13.3 Tangent Lines

Intuitively you might say that a tangent line at $x=a$ is a line which touches the graph of a function at the one point $(a,f(a))$. This is misleading however, as the tangent line may come in contact with the function again at another point on its graph. So what we might say is that a tangent line touches the graph of f at one point in the vicinity of x = a: It should always be clear as to how big this vicinity should be. As in the figure below, we see that the tangent line touches the graph of the f at the single point $(a,f(a))$. Though it crosses the function again later on to the right, this has nothing to do with how the function is behaving near the point $x=a$:
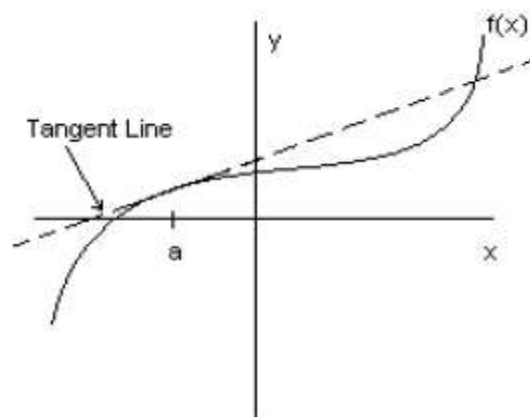
The question then becomes, given a function f, is it possible to find the tangent line at any point $x=a$? And if so, how do we find it? You might see why this problem was of interest to mathematicians in the 17th century, as well as today. For suppose the function represents distance traveled. Then we would find that the slope of the tangent line would give us the instantaneous velocity, meaning the exact velocity at the exact time in question. Or perhaps the function would represent earnings at time t. In this case the slope of the tangent line would represent the instantaneous rate of change of the earnings.

13.4 Descartes' Method

Descartes was successful in solving the tangent line problem for certain functions. In order to find the tangent line to f at the point a, his method was to find a circle tangent to the curve at the point a. The beauty of this method is that once the circle is found, we know the tangent line, because by nature the tangent line to a circle is perpendicular to the line passing through the radius of the circle!

Example 13.4.1. Find the slope of the tangent line to the function $y=x^2$ at $x=1$.
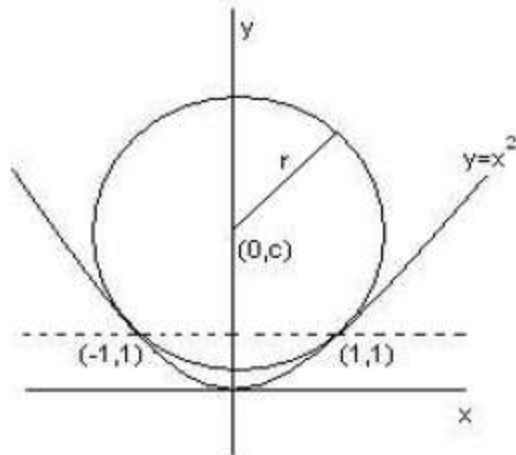
From the figure we can see that if we can find the value of c, we can find the slope of the line between $(0,c)$ and $(1,1)$. Since this line is perpendicular to the tangent line of the function $y=x^2$ at $x=1$ we can easily obtain the tangent line. We begin by writing the equation of the circle (from the figure)
$$x^2+(y-c)^2=r^2.$$

As the circle will pass the point $(1,1)$, we can set $x=1, y=1$ in the above equation to get a relation involving only r and c.
$$1+(1-c)^2=r^2 \quad \rightarrow \quad r^2=c^2-2c+2.$$

If we place this value of r^2 into the original equation for the circle above, we have
$$x^2+y^2-2cy+2c-2=0.$$

So far we haven't used any facts about the function $y=x^2$ (other than the fact that it passes through the point $(1,1)$ of course). If we substitute $y=x^2$ into the latest equation for the circle we will find that the y-values of the points lying both on the circle and on the function $y=x^2$. So, substituting in we then have
$$y+y^2-2cy+c2-2=0.$$

Notice that this is a quadratic equation in y
$$y^2+(1-2c)y+(2c-2)=0,$$

and that the quadratic equation will give at most two solutions, (see the figure below)

Intersection of the curve with a circle.

Curve intersects circle in two points

Curve intersects circle at a single point.

Of course, the choice we are looking for is the *single* intersection. This corresponds to the right-hand-side of the figure. When is there a single solution to the quadratic equation $\alpha x^2 + \beta x + \gamma = 0$? When $\beta^2 - 4\alpha\gamma = 0$. In this case, this means when

$$(1 - 2c)^2 - (4)(1)(2c - 2) = 0.$$

Solving this equation for c gives a value of 3/2. Now that we know c we can calculate the slope of the line passing through the radius of the circle as

$$\text{slope of radial line} = \frac{\frac{3}{2} - 1}{0 - 1} = -\frac{1}{2}.$$

As the slope of the tangent line is perpendicular to the slope of the radial line, we find the slope of the tangent line at $y = x^2$ at $x = 1$ to be 2.

13.5 Fermat's Method

Fermat also published a result of the tangent line problem in the 1630s, around the same time as Descartes, though his method was very different. Whereas Descartes' geometry-based method was exact, the method shown below by Fermat was not. For this reason Descartes' method was taken to be the mathematically sound method of computation of a tangent line. However, the method of Fermat would prove much more relevant to the ideas founding the basis of calculus.

Example 13.5.1. Compute the slope of the tangent line to $y = x^2$ at $x = 1$. Shown below is the graph of the function.
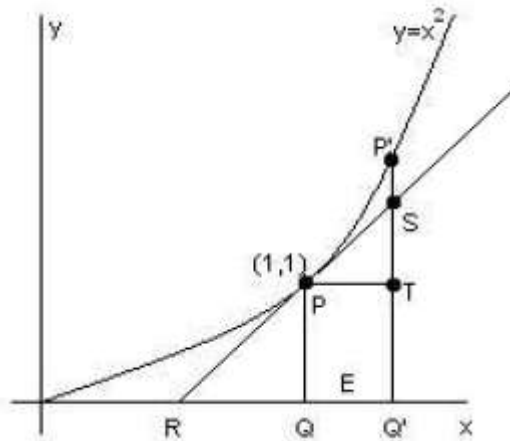
The tangent line at the point P is drawn. Fermat argued that if we choose a point P' a small distance along the curve from P then triangle PQR is similar to triangle PTS. Therefore we can relate the sides as follows,

$$\frac{RQ}{PQ}=\frac{PT}{ST}=\frac{E}{ST}.$$

And Fermat claimed that if E is small, then we have

$$\frac{RQ}{PQ}=\frac{E}{P'T} \quad \text{or} \quad RQ\approx\frac{E\cdot PQ}{P'T}.$$

Therefore, using the function $y=x^2$ we have
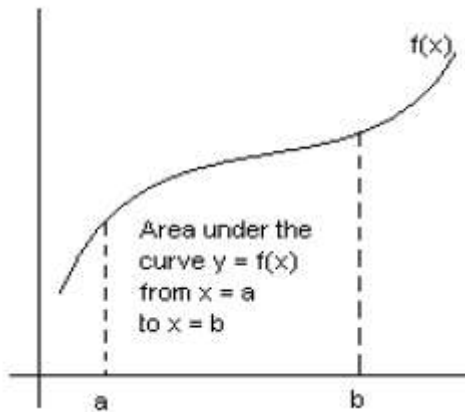
$$RQ\approx\frac{E\cdot 1}{(1+E)^2-1}=\frac{E}{1+2E+E^2-1}=\frac{E}{2E+E^2}=\frac{1}{2+E}.$$

Now, without justification, Fermat said that since E was very small already, go ahead and take E to be zero. The result is, of course

$$RQ=\frac{1}{2}$$

so that the slope of the tangent line is given by $\frac{1}{(1/2)}=2$, the same answer as that obtained by Descartes.


13.6 Areas

The problem of computing the area under a curve goes back to antiquity. Archimedes was the first do devise a method for computing such areas. His method was called the *method of exhaustion*. This method involved approximatingthe area using a series of rectangles like that shown below.
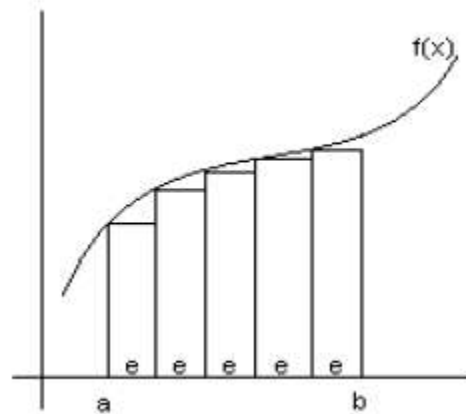
If we suppose that each of the *n* small rectangles has width *e*, given by $e = \dfrac{b-a}{n}$, then we may compute the area by summing up the areas of each of the small rectangles.

$$\text{area} = (f(a) \cdot e) + (f(a+e) \cdot e) + (f(a+2e) \cdot e) \cdots (f(b-e) \cdot e)$$

By computing the areas of the smaller rectangles we can get an approximation for the area lying underneath the graph of *y=f(x)*. Archimedes found that the more rectangles you use, the sum of the small rectangles gets closer and closer to some finite number.
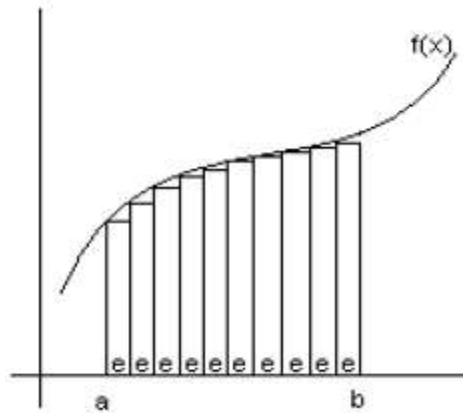
This number he claimed to be the *actual* area underneath the function *y=f(x)*. This would not be proven until the invention of calculus thousands of years later.

13.7 Arc Length
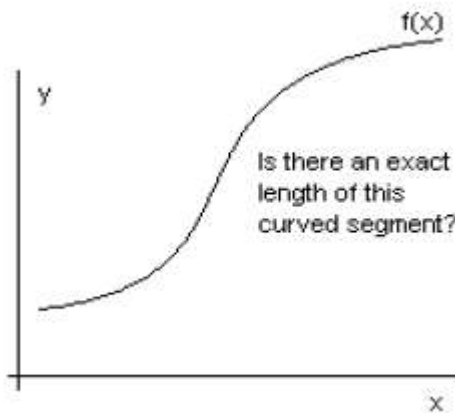
The problem of finding the length of a curve was considered impossible to solve for much of the history of thinking. The idea that a curved segment could have exactly the same length as a straight segment did not even seem possible to most mathematicians. That was until some people began to approximate the lengths of the curves by inscribing polygons about the curves (see below) finding that the lengths became increasingly more and more accurate the shorter the segment between polygon vertices became (between the big black dots in the figure).

In the process of his computation of the arc length of a segment of $y=x^{3/2}$, Fermat made a remarkable connection that linked three of the four major

problems facing the 17th century mathematicians. It was this discovery that would finally lead to the idea of the *derivative*. Unfortunately for Mr. Fermat, he was unable to realize the connection that he had made. And for this reason he is forever eclipsed by Newton and Leibnitz in the Calculus Hall of Fame. Better luck next time, Mr.
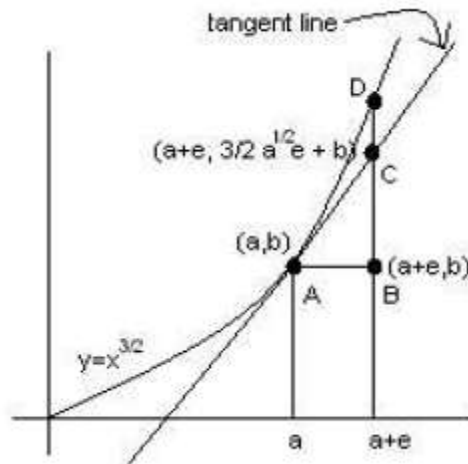
As shown previously, Fermat devised a technique for finding tangent lines. Using this method he was able to find that at the point $x=a$ the tangent line had the slope $\frac{3}{2}a^{1/2}$. The equation of the tangent line to the function $y=x^{3/2}$ then is given by

$$y=\frac{3}{2}a^{1/2}(x-a)+b.$$

Next, he made the argument that if we increase a by a small amount to $a+\epsilon$, then the arc length from A to D is approximately equal to the length of the segment AC. The length of the segment AC can be found using the Pythagorean Theorem,

$$AC^2=AB^2+BC^2=e^2+\left(\frac{3}{2}a^{1/2}e\right)^2=e^2\left(1+\frac{9}{4}a\right),$$

and upon taking the square root he obtained

$$AC = e\sqrt{1 + \frac{9}{4}a}.$$

Now if he wanted to compute a good approximation of the arc length from $x=0$ to $x=1$, he would sum up a sequence of short intervals, each of which described in the above manner. What Fermat realized is that by summing up the terms in this way he was actually approximating the area under the curve for the function

$$h(x) = \sqrt{1 + \frac{9}{4}x},$$

in the way of Archimedes.

Realizing that computing arc length of a function $y=f(x)$ is not only related to the computation of the tangent line to $f(x)$, Fermat noted a relation to the computation of the area under a function $h(x)$, which related $f(x)$ by

$$h(x) = \sqrt{1 + (\text{slope of tangent line})^2}$$

The discovery Newton and Leibnitz would make (independent and unknowingly of each other) was that the problem of computing the area under a curve is actually the inverse problem of finding the tangent line to a curve, and vice versa.