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Lecture notes of the undergraduate course

PHYS3050/7051

ELECTROMAGNETIC THEORY III

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Consultation Hours: Wednesday 2pm – 4pm

Preface

This lecture notes covers the principal elements of classical electromagnetic theory embodying Maxwell's equations with applications mainly to situations where electric charge can be treated as a continuous fluid. The intention is to introduce students to the background of classical field theory and the applications of the electromagnetic theory to solid state physics, classical optics, radiation theory and telecommunication.

The goal of this course is to provide a compact logical exposition of the fundamentals of the electromagnetic theory and the applications to various areas of physics and engineering. The treatment is quantitative throughout and an attempt has been made to imbue students with a sound understanding of the Maxwell's equations and with the ability to apply them to modern problems in physics.

The organization of the lectures is fairly standard and includes vector analysis, electrostatic, magnetostatic, mathematical techniques in the solution of the Maxwell's equations and the Laplace's equation, time varying fields and applications of the solution of the Maxwell's equations. The material on vector analysis gives greater emphasis to the relationship between fields and their sources.

Contents

Literature

1. J.D. Jackson, *Classical Electrodynamics*, 3rd ed. Wiley 1999.
2. R.K. Wangsness, *Electromagnetic Fields*, 2nd ed. Wiley 1986.
3. R. Plonsey and R.E. Collin, *Principles and Applications of Electromagnetic Fields*, McGraw Hill 1961.
4. B.I. Bleaney and B. Bleaney, *Electricity and Magnetism*, 3rd ed. Oxford U.P. 1983.
5. J.A. Stratton, *Electromagnetic Theory*, McGraw Hill, 1941.
6. W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. Addison-Wesley 1962.
7. R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynaman Lectures on Physics*, Vol. 2, Addison Wesley, 1964.

1 The Classical Theory of the Electromagnetic Field

Classical theory of the electromagnetic field or Classical Electrodynamics, formulated by Maxwell more than 100 years ago, is now a well established theory. In this context **'classical'** means **'non-quantum'**, but we would like to point out that the basic equations of electromagnetism, the Maxwell's equations, hold equally in quantum and classical field theory.

Electromagnetic interactions are ONE of FOUR fundamental types:

Type of interaction	Relative Strength
Strong interaction (nuclear)	1
Electromagnetic	10^{-2}
Weak interaction (e.g. β decay)	10^{-12}
Gravitation	10^{-40}

At our level of discussion there is no relation between these four types of interaction (i.e. they cannot be considered as different manifestations of a single FORCE).

Classical EM Theory is particularly interesting because of:

- Its historical role in the development of Physics in the late 19th and early 20th centuries.
- Numerous applications on 'our' scale of existence:
  Chemical bonding is due to EM forces.
  Communications via EM Waves (radio, TV, telephony, computers).
  Generation and transmission of electrical power.
  Knowledge of the rest of the universe from reception of EM waves.

1.1 Elementary Aspects of Electromagnetism

- EM forces are due to ELECTRIC CHARGE which is NOT in turn explicable in terms of anything else.

- Charges are of two kinds called positive and negative. In the static limit like charges repel and unlike attract.
- Charges are quantized in units of $e \simeq 1.6 \times 10^{-19}$ [Coulombs].
- In the static limit the inverse square (Coulomb) law of force holds:

$$\vec{F}_{q_2} = \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_1}{r^2}\,\hat{r}\ , \qquad q_1 \xrightarrow{\hat{r}} q_2\ ,$$

that the charge q_1 acts on the charge q_2 with the force $\vec{F}_{q_2}$. The parameter ε_0 determines the property of the medium and is called the electric permittivity.

The Coulomb law holds only for charges whose the spatial dimensions are small compared with the distance separating them.

The Coulomb law can be tested to great accuracy indirectly by showing that no charge rests on the inside of a statically charged hollow conductor[1].

If the exponent in the Coulomb's law

$$E \sim \frac{1}{r^n}$$

is not $n = 2$ but $n = 2(1 \pm \epsilon)$, the potential inside the hollow conductor would be large. Since the potential inside the hollow conductor found by the experiments was less in magnitude than a small detectable potential, then $\epsilon \simeq 10^{-9}$, the level of sensitivity of the detector.

- Electric charge is conserved (algebraically)

$$\sum_{\text{whole universe}} q = \text{constant}$$

In electromagnetic theory none of these things are explained in terms of anything else.

[1]S.J. Plimpton and W.E. Lawton, Phys. Rev. **50**, 1066 (1936).

- In the NON-STATIC case (moving charges) the force is no longer given by Coulomb's Law. In general is given by the Lorentz equation:

$$\vec{F}_{q_2} = q_2(\vec{E} + \vec{v} \times \vec{B}) \ ,$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields due to q_1.

The fields $\vec{E}$ and $\vec{B}$ will depend on the frame of reference of the observer ($\vec{F}$ must follow the required relativistic transformation law).
However, we do not think that q depends on the frame of reference.
i.e. q does not depend on its velocity with respect to an observer.

This is because in ordinary matter electrons move much faster than ions, their speeds depend on temperature, and electric fields are not observed to arise from changes in temperature.

Why there must be a $\vec{B}$ and how $\vec{E}$ and $\vec{B}$ are computed for arbitrary motion of charges is the substance of electromagnetic theory.

THE BASIC IDEA OF ELECTROMAGNETISM IS:

$$\text{CHARGE 1} \Longrightarrow \mathop{FIELD}\limits^{ELECTROMAGNETIC} \Longrightarrow \mathop{CHARGE}\limits^{FORCE\ ON}\ 2$$

Important conclusion

Fields are generated by *charges* - NOT by other fields.

1.2 Macroscopic Charges and Currents

We know that electric charge is quantized. The electron is a point charge on the smallest scale measurable. We may then speak, on a subatomic scale, of a microscopic theory of electromagnetism. On a subatomic scale there must be very strong and rapidly varying electric and magnetic fields on spatial scales $\sim 10^{-8}$ m and temporal scales $\sim 10^{-10}$ s.

When we measure the fields around a macroscopic circuit, clearly we are not looking at these fields. We are measuring fields on distance scales $\gg 10^{-8}$ m and time scales $\gg 10^{-10}$ s. The microscopic fields sum to (almost) zero. In the macroscopic context it is convenient and justifiable to regard the charge as a continuous fluid.

Charge density

When we encounter a large number of point charges in a finite volume, it is convenient to describe the source in terms of a charge density, defined as

$$\rho = \lim \frac{\Sigma q}{\Delta V} \ ,$$

where Σq is the algebraic sum of the charge in the volume ΔV.

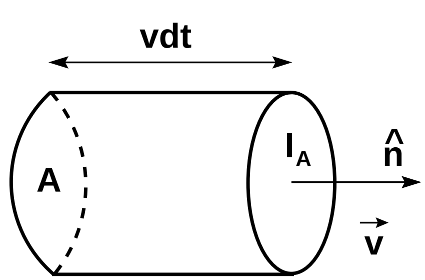
The limit is not to zero but to a $\Delta V \gg$ atomic scale size, which is still very small on the laboratory scale.

If the charge density is represented by a continuous function ρ, the total charge Q in a volume V is given by

$$Q = \int_V \rho dV \ .$$

Current density

For many purposes it is necessary to introduce the idea of current density.



Let $I_A = \partial q/\partial t$ is a current through the area A. Then the current density is defined by

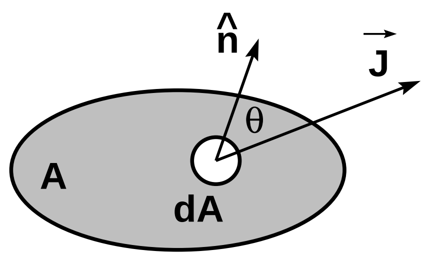
$$\vec{J} = \lim \frac{I}{A}\hat{n} = \lim \frac{\delta q}{\delta t\, A}\hat{n} = \lim \frac{\rho\, A\, v\, \delta t}{\delta t\, A}\hat{n}\vec{J} = \rho\vec{v}\ ,$$

where $\vec{v} = v\hat{n}$.

The limit is taken in the same sense as for ρ.

Total current through a surface area

If the current density $\vec{J}$ is known at every point of an arbitrary surface, we can obtain the total current through the area.



The current through δA is:

$$\delta I = J\cos(\theta)\delta A = \vec{J}\cdot\hat{n}\,\delta A\ .$$

Then the current through the total area A is the sum of the contributions from all elements of the area:

$$I_A = \int_A \vec{J}\cdot\hat{n}\ dA = \int_A \vec{J}\cdot d\vec{A}\ ,$$

where $d\vec{A}$ is a vector representing the element dA of the surface A. In vector analysis it is common to represent a surface by a vector whose length corresponds to the magnitude of the surface area and whose the direction is specified by the unit vector $\hat{n}$ normal to the surface.

In summary, when EM Theory is formulated in terms of ρ and $\vec{J}$ as the sources, we speak of a MACROSCOPIC THEORY.

2 Mathematical Description of Vector Fields

The study of electromagnetic theory requires considerable knowledge of vector analysis. In this lecture, we will introduce vector operations we will need for our study of electromagnetic theory. As we shall see, based on this lecture, it is possible to considerably simplify the formulation of electromagnetic theory.

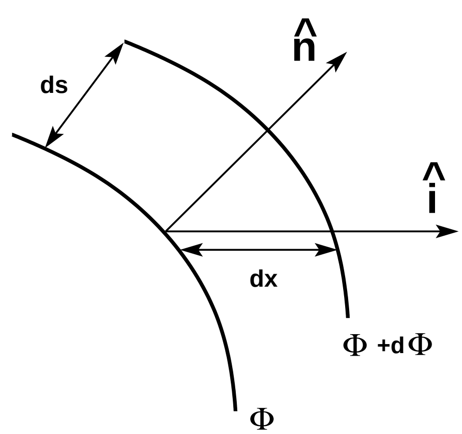
2.1 Gradient of a Scalar Function

Let us suppose that Φ represents a scalar field and that Φ is a single valued, continuous, and differentiable function of position.

The gradient of the scalar function Φ is defined as:

$$grad\ \Phi = \nabla\Phi = \frac{\partial\Phi}{\partial s}\ \hat{n}\ ,$$

where $\hat{n}$ is a unit vector in the direction the rate $\frac{\partial\Phi}{\partial s}$ has its maximum value. In other words, gradient tells us in which direction the change in Φ is maximal.



For some other direction $d\vec{X}$:

$$d\Phi = \nabla\Phi \cdot d\vec{X} = \frac{\partial \Phi}{\partial s}\ \hat{n} \cdot d\vec{X} = \frac{\partial \Phi}{\partial s} \cos\theta\ dX$$

In rectangular (cartesian) coordinates:

$$\begin{aligned}(\nabla\Phi)_x &= \nabla\Phi \cdot \hat{i} = \frac{\partial \Phi}{\partial s}\hat{n} \cdot \hat{i} = \frac{\partial \Phi}{\partial s}\cos\theta \\ &= \lim \frac{\delta\Phi}{\delta(s/\cos\theta)} = \lim \frac{\delta\Phi}{\delta x} = \frac{\partial \Phi}{\partial x}\end{aligned}$$

Hence

$$\nabla\Phi = \frac{\partial \Phi}{\partial x}\hat{i} + \frac{\partial \Phi}{\partial y}\hat{j} + \frac{\partial \Phi}{\partial z}\hat{k}$$

in cartesian coordinates.

The gradient is analogous to multiplication of a vector by a scalar. The result, of course, is a vector. We do not usually take a gradient of a vector, the result would be a tensor.

Example

Consider a scalar function $\Phi = xyz$. Gradient of $\Phi = xyz$ is

$$\begin{aligned}\nabla\Phi &= yz\frac{\partial x}{\partial x}\ \hat{i} + xz\frac{\partial y}{\partial y}\ \hat{j} + xy\frac{\partial z}{\partial z}\ \hat{k} \\ &= yz\hat{i} + xz\hat{j} + xy\hat{k}\ .\end{aligned}$$

2.2 Divergence Function

The divergence is the scalar which results from operation of ∇ upon a vector $\vec{F}$ in a fashion analogous to the dot product of two vectors. In Cartesian coordinates:

$$div\ \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Where there is a positive divergence, there is a source of a vector field. When $\nabla \cdot \vec{F} = 0$ everywhere, the field $\vec{F}$ is called **solenoidal**.

Example

Consider a vector $\vec{F} = x\hat{i}$.

$$div\ \vec{F} = \frac{\partial x}{\partial x}(\hat{i}\cdot\hat{i}) + \frac{\partial x}{\partial y}(\hat{j}\cdot\hat{i}) + \frac{\partial x}{\partial z}(\hat{k}\cdot\hat{i}) = 1\ .$$

Consider now a vector $\vec{F} = y\hat{i}$. In this case

$$div\ \vec{F} = \frac{\partial y}{\partial x}(\hat{i}\cdot\hat{i}) + \frac{\partial y}{\partial y}(\hat{j}\cdot\hat{i}) + \frac{\partial y}{\partial z}(\hat{k}\cdot\hat{i}) = 0\ .$$

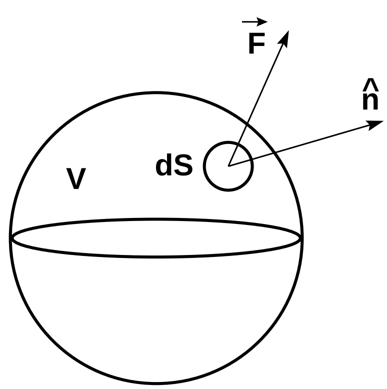
Thus, divergence of a given field is different from zero only if the field amplitude changes in the direction of the field. So the divergence is related to how the field changes as you move in the direction of the field.

2.3 Gauss' Divergence Theorem

Gauss' law, or sometimes called as the divergence theorem, is stated as

$$\int_V \nabla\cdot\vec{F}\,dV = \oint_S \vec{F}\cdot\hat{n}\,dS$$

volume integral   closed surface integral



Remember that $\hat{n}$ is the unit *outward* normal over S.

Definition of Flux

The FLUX of $\vec{F}$ through a surface S, not necessary a closed S, is defined as:

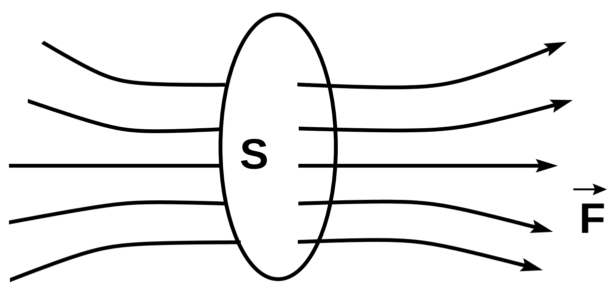
$$\int_S \vec{F} \cdot d\vec{S} = \Psi$$

Associated with this is the pictorial representation of fields by *lines of force*, the direction of $\vec{F}$ given by the tangent to a line and the strength of $\vec{F}$ given by the line density per unit area.

Note that from the divergence theorem:

$$\nabla \cdot \vec{F} = \lim_{dV \to 0} \left[\frac{1}{dV} \oint \vec{F} \cdot \hat{n} dS \right] = \lim_{dV \to 0} \left[\frac{\Psi}{dV} \right]$$

i.e. the divergence of a vector field is the emanating flux per unit volume.



A non-zero $\nabla \cdot \vec{F}$ is then implies a source (if positive) or a sink (if negative). And if $\nabla \cdot \vec{F} = 0$ there is no source or sink – the field lines have no beginnings or ends.

2.4 The Continuity Equation for Electric Current

Suppose we have some 'substance' of density ρ in a volume V enclosed by a surface S, as shown in Fig. 1.

Let $\vec{v}$ = macroscopic velocity of 'substance' .
Let q = rate of production of 'substance' per unit volume .
Let l = rate of annihilation of 'substance' per unit volume .

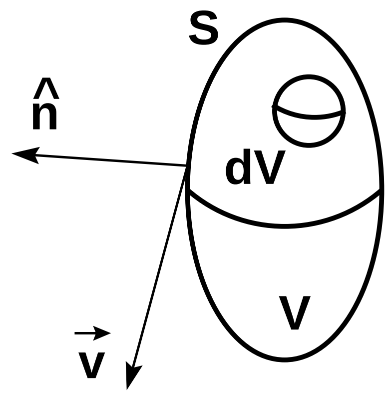


Figure 1:

Then, the rate of increase of total 'substance' in the volume V
=*Rate of production - rate of annihilation - rate of transport out through* S.

We can express this as

$$\frac{\partial}{\partial t}\int_V \rho\, dV = \int_V q\, dV - \int_V l\, dV - \oint_S \rho\vec{v}\cdot\hat{n}\, dS\ .$$

Using the Gauss' divergence theorem, we can write

$$\oint_S \rho\vec{v}\cdot\hat{n}\, dS = \int_V \nabla\cdot(\rho\vec{v})\, dV\ .$$

Thus

$$\int_V \frac{\partial\rho}{\partial t}\, dV = \int_V q\, dV - \int_V l\, dV - \int_V \nabla\cdot(\rho\vec{v})\, dV$$

Since this relation holds for arbitrary V:

$$\frac{\partial \rho}{\partial t} = q - l - \nabla \cdot (\rho \vec{v}) \ .$$

For a 'conserved' quantity such as electric charge: $q = l = 0$, and then

$$\rho \vec{v} = \vec{J} \ .$$

Hence

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \ ,$$

which is well known as the **continuity equation**.

When stationary currents are involved, then $\partial\rho/\partial t = 0$. In this case $\nabla \cdot \vec{J} = 0$, that is for stationary currents the current density is solenoidal.

2.5 Curl (Rotation) Function

Curl (or rotation) is a vector which results from the operation of ∇ operator upon a vector in a fashion analogous the cross product of two vectors

$$curl \ \vec{F} \equiv \nabla \times \vec{F} = \hat{i} \times \frac{\partial \vec{F}}{\partial x} + \hat{j} \times \frac{\partial \vec{F}}{\partial y} + \hat{k} \times \frac{\partial \vec{F}}{\partial z} \ ,$$

or

$$\begin{aligned} \nabla \times \vec{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \ . \end{aligned}$$

Curl is nonzero when the field increases (or decreases) in a different direction that the field pointed. If the field is pointed in the same direction as that in which is increased, the curl is zero. So the curl is related to how the field changes as you move across the field.

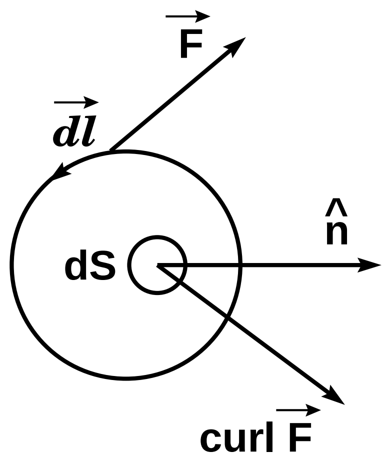
When $\nabla \times \vec{F} = 0$ everywhere, the field $\vec{F}$ is called **irrotational**.

2.6 Stokes' Theorem

Stokes' theorem may be stated in the form

$$\oint_l \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot \hat{n} dS$$

complete loop   surface not closed



However

$$\nabla \times \vec{F} \cdot \hat{n} = \lim_{dS \to 0} \left[\frac{1}{dS} \oint \vec{F} \cdot d\vec{l} \right] \ .$$

This gives an intuitive meaning to any component of $\nabla \times \vec{F}$ in terms of the line integral around a small element of surface. $\nabla \times \vec{F}$ is a measure of the vorticity of the field.

2.7 Successive Application of ∇

We can introduce scalar and vector products in which the operator ∇ appears more than once. For example, since the gradient of an arbitrary scalar function is a vector, we can take the divergence of the gradient

$$\nabla \cdot \nabla V = \nabla \cdot \left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \ .$$

The same result is obtained if we take $\nabla \cdot \nabla$ as a new operator ∇^2 with properties

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \; .$$

The operator ∇^2 is called the Laplacian and is a scalar.

The Laplacian may also be applied to a vector, with the result

$$\nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2} \; .$$

It is also possible to form the curl of the gradient, which is identically zero

$$\nabla \times \nabla V = 0 \; .$$

The divergence of the curl of a vector is also identically zero

$$\nabla \cdot \nabla \times \vec{F} = 0 \; .$$

The proof of the above properties is left to the students.

2.8 Electromagnetic Field Equations and Electric Potential

We will now illustrate some properties of the successive application of ∇, which will allow us to introduce the concept of vector and scalar potentials to the electromagnetic field theory.

2.8.1 Maxwell's Equations

Consider coupled differential equations for vector fields $\vec{E}$ and $\vec{B}$, the Maxwell's equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \; , \qquad (1)$$

$$\nabla \cdot \vec{B} = 0 \; , \qquad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \; , \qquad (3)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \; , \qquad (4)$$

By means of divergence theorem and Stokes's theorem, they can be easily transferred into an integral form

$$\begin{aligned}
\int_S \vec{E}\cdot\vec{n}dS &= \frac{q}{\varepsilon_0} \ , \\
\int_S \vec{B}\cdot\vec{n}dS &= 0 \ , \\
\oint_l \vec{E}\cdot\vec{dl} &= -\frac{\partial}{\partial t}\int_S \vec{B}\cdot\vec{n}dS \ , \\
\oint_l \vec{B}\cdot\vec{n}dS &= \mu_0\int_S \vec{J}\cdot\vec{n}dS + \frac{1}{c^2}\frac{\partial}{\partial t}\int_S \vec{E}\cdot\vec{n}dS \ .
\end{aligned}$$

According to the Helmholtz Theorem, an arbitrary vector $\vec{F}$ can always be written as

$$\begin{aligned}
\vec{F} &= -\frac{1}{4\pi}\nabla\int_V \frac{\nabla\cdot\vec{F}}{r}dV + \frac{1}{4\pi}\nabla\times\int_V \frac{\nabla\times\vec{F}}{r}dV \\
&= \vec{F_l} + \vec{F_t} \ .
\end{aligned}$$

Thus, specification of $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$ is necessary and sufficient to determine $\vec{F}$.

Hence, we need **four** equations of this type to determine $\vec{E}$ and $\vec{B}$.

In the following lectures, we will discuss how Maxwell arrived at these equations. We will also discuss the effect of Einstein's special theory of relativity on how we think about electromagnetic fields and Maxwell's equations. Much of our discussion will be about 'how to solve Maxwell's equations'.

2.8.2 Electric Potential

The solutions of the electromagnetic field equations are not in general simple and straightforward. Often their solution is aided by the use of potentials.

A potential is a quantity from which a vector field can be derived by some process of differentiation.

Examples

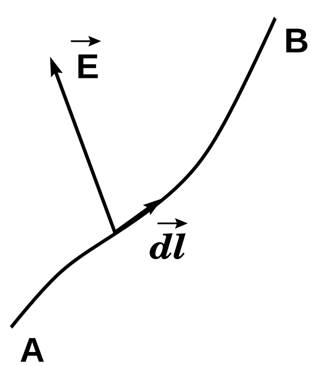
(1) Let $\nabla \times \vec{F} = 0$. Since $\nabla \times \nabla \equiv 0$ then the equation $\nabla \times \vec{F} = 0$ will have an integral of the form:

$$\vec{F} = \nabla\Phi \ ,$$

and Φ is called a **scalar potential**. Thus, the field may be derived from the gradient of the scalar potential function Φ.

In the electrostatic field $\nabla \times \vec{E} = 0$. Hence, $\vec{E} = -\nabla\Phi$. [2] Then the work done per unit charge q moving from point A to point B is:

$$\begin{aligned} \frac{W}{q} &= -\int_A^B \vec{E} \cdot d\vec{l} = \int_A^B \nabla\Phi \cdot d\vec{l} \\ &= \int_A^B \frac{\partial\Phi}{\partial r}\hat{r} \cdot d\vec{l} = \int_A^B d\Phi = \Phi_B - \Phi_A = \Delta\Phi_{AB} \ . \end{aligned}$$



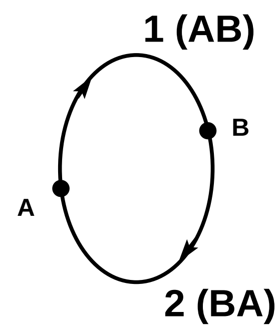
[2] The minus sign is inserted in the definition of Φ to agree with the definition of Φ as a potential ENERGY. The negative sign can also be understood physically from the fact that $\vec{E}$ is in the direction that a positive charge moves, hence in the direction of decreasing potential.

Note that if $\nabla \times \vec{E} = 0$ then:

$$\oint \vec{E} \cdot d\vec{l} = \int \nabla \times \vec{E} \cdot \hat{n} dS = 0 \ .$$

Hence

$$\Delta\Phi_{(AB)_1} = -\Delta\Phi_{(BA)_2} = \Delta\Phi_{(AB)_2} \ .$$



The work done is independent of the path chosen. Thus a field $\vec{F}$ with $\nabla \times \vec{F} = 0$ is a **conservative** field of force.

(2) Let $\nabla \cdot \vec{F} = 0$. Since $\nabla \cdot \nabla \times \vec{A} \equiv 0$ the equation $\nabla \cdot \vec{F} = 0$ will have an integral of the form:

$$\vec{F} = \nabla \times \vec{A}$$

and $\vec{A}$ is called a vector potential.

Note:

It turns out that $\nabla \cdot \vec{B} = 0$ always in electromagnetism so there will always be a vector field $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$ and such an $\vec{A}$ is referred to as THE vector potential in electromagnetism (though there may be other electromagnetic field functions with zero divergence).

Note also that just writing $\nabla \times \vec{A} = \vec{B}$ does not completely specify $\vec{A}$ even if $\vec{B}$ is known everywhere. One needs to specify $\nabla \cdot \vec{A}$ as well to completely determine $\vec{A}$. Equivalently we can say that defining $\nabla \times A = \vec{B}$ still leaves us free to define $\nabla \cdot \vec{A}$.

On the other hand $\nabla \times \vec{E} \neq 0$ in general in electromagnetism so it is not in general possible to write $\vec{E} = -\nabla\Phi$.

Exercise in class: *Vector analysis*

For a given scalar function Φ and vectors $\vec{A}, \vec{B}, \vec{C}$, indicate successive steps you would follow in the calculation of the following expressions:

(i) $\nabla \times \nabla \cdot \vec{A}$; $\vec{A} \cdot \nabla \vec{B}$; $\vec{A} \nabla \cdot \vec{B}$; $\vec{A} \times \nabla \cdot \vec{B}$; $\vec{A} \cdot \nabla \Phi$;

$\nabla \vec{A} \times \vec{B} \cdot \vec{C}$; $\nabla \Phi \cdot \vec{A} \times \vec{B}$; $\vec{A} \cdot \vec{B} \times \nabla \Phi$; $\nabla \cdot \vec{A} \times \nabla \vec{B} \cdot \vec{C}$;

$\nabla \times \vec{A} \cdot \vec{B} \times \vec{C}$.

(ii) Which of the expressions in (i) are vectors?

(iii) Which of the four statements is correct:

$$
\begin{aligned}
&1. \quad \nabla^2 \vec{A} = (\nabla \cdot \nabla)\vec{A} \,, \\
&2. \quad \nabla^2 \vec{A} = \nabla \cdot (\nabla \vec{A}) \,, \\
&3. \quad \nabla^2 \vec{A} = \nabla \cdot (\nabla \cdot \vec{A}) \,, \\
&4. \quad \nabla^2 \vec{A} = \nabla \times (\nabla \cdot \vec{A}) \,.
\end{aligned}
$$

Weekend exercises

(a) For fields of the form $r^n\hat{r}, (r \neq 0)$, find for which values of n the divergence is zero.

(b) For fields of the cylindrical form $\rho^n\hat{\phi}, (\rho \neq 0)$, find for which values of n the curl is zero.

(c) If the potential Φ satisfies the equation (Laplace equation) $\nabla^2\Phi = 0$, show that $\nabla\Phi$ is both solenoidal and irrotational.

3 The Experimental Basis of the Development of Electromagnetic Theory

In this lecture, we will present the basic properties of the electrostatic field in vacuum. The Coulomb law for the force between two point charges is the experimental basis for the development of electromagnetic theory.

3.1 Coulomb's Law – Force between Static Charges

In 1785, Coulomb investigated the nature of the force between charged bodies, and the results of his experiments can be formulated mathematically in what is known as Coulomb's law

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1\,q_2}{r^2}\ \hat{r}\ .$$

We can write Coulomb's law as

$$\vec{F}_2 = q_2\vec{E}_1\ .$$

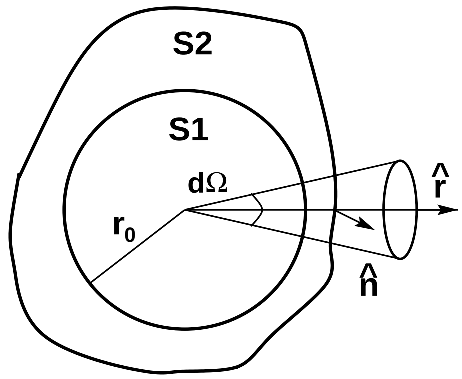
Hence

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\ \hat{r}\ .$$

The electric field is an example of a vector field. In principle, we can always calculate an electric field using Coulomb's law. However, there is an alternative way we can find the electric field. In particular, the field may be represented by means of the flux concept. The total flux of $\vec{E}$ from a point charge q_1 may be readily calculated by integrating $\vec{E}\cdot d\vec{S}$ over a surface enclosing q_1.

3.2 Derived Result – Gauss' Law

Consider a macroscopic charge q_1 closed by a surface S. We will show that the flux of the electric field produced by the charge q_1 is proportional to the charge q_1, and is independent of the shape of the surface closing the charge.



Consider first the flux through a spherical surface:

$$\begin{aligned}\Psi_{S1} &= \oint_{S1} \vec{E}\cdot\hat{n}dS = \oint_{S1} \frac{q_1}{4\pi\varepsilon_0 r_0^2}\ \hat{r}\cdot\hat{n}dS \\ &= \frac{q_1}{4\pi\varepsilon_0 r_0^2}\oint dS = \frac{q_1}{4\pi\varepsilon_0 r_0^2}\ 4\pi r_0^2 = \frac{q_1}{\varepsilon_0}\ .\end{aligned}$$

Next, consider the flux through an arbitrary surface

$$\Psi_{S2} = \oint_{S2} d\Psi_{S2} = \oint_{S2} \frac{q_1}{4\pi\varepsilon_0 r^2}\ \hat{r}\cdot\hat{n}dS\ ,$$

where $S2$ is an arbitrary surface enclosing q_1.

But from the inverse square law and some geometry:

$$\begin{aligned}d\Psi_{S2} &= \frac{q_1}{4\pi\varepsilon_0 r^2}\ \hat{r}\cdot\hat{n}dS \\ &= \frac{q_1}{4\pi\varepsilon_0}\frac{dS\ \cos\theta}{r^2} = \frac{q_1}{4\pi\varepsilon_0}\, d\Omega\ ,\end{aligned}$$

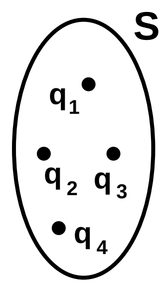
where $d\Omega$ is the solid angle subtended by dS at q_1.

The element of solid angle $d\Omega$ is independent of where we cut the bundle of electric lines of force.

Thus

$$d\Psi_{S2} = d\Psi_{S1} \ \text{ and } \ \oint d\Psi_{S2} = \oint d\Psi_{S1} = \frac{q_1}{\varepsilon_0}\ .$$

Furthermore for some arbitrary number of charges $q_1, q_2, q_3, \cdots, q_n$ within a surface S:



$$\begin{aligned}\oint d\Psi &= \oint(\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots) \cdot \hat{n}\ dS \\ &= \Psi_1 + \Psi_2 + \Psi_3 + \cdots \\ &= \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \frac{q_3}{\varepsilon_0} + \cdots = \frac{1}{\varepsilon_0}\sum q\ ,\end{aligned}$$

where $\sum q$ is the algebraic sum of all charges within the surface S.

If q is outside S, the surface integral vanishes since the total solid angle subtended at q by the surface is zero. Thus

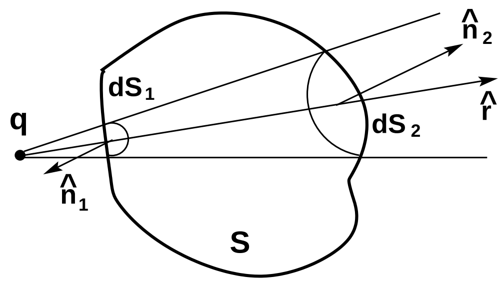
$$d\Psi_1 + d\Psi_2 = 0\ .$$

Proof:

$$\begin{aligned}d\Psi_1 &= \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\hat{n}_1\, dS_1}{r_1^2} \\ \frac{\hat{r}\cdot\hat{n}_1\, dS_1}{r_1^2} &= -\frac{dS_{1\perp}}{r_1^2} = -d\Omega\ .\end{aligned}$$

Similarly

$$d\Psi_2 = \frac{q}{4\pi\varepsilon_0}d\Omega = -d\Psi_1\ .$$



Since

$$\frac{\hat{r}\cdot\hat{n}_2 dS_2}{r_2^2} = +d\Omega \ ,$$

we find that $d\Psi_1 + d\Psi_2 = 0$ and integrating over all S:

$$\Psi = \oint_S d\Psi = 0 \ .$$

The physical interpretation of this result is that field lines originating from an external charge and entering the surface S must also leave this surface.

In summary, the Gauss' law says that the total electric flux through a closed surface S is:

$$\Psi = \frac{q}{\varepsilon_0} \ , \qquad \text{where} \ \ q = \sum \text{charges INSIDE} \ \ S \ .$$

Using the definition of the flux, we often write the Gauss' law as

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \ .$$

The power of the Gauss' law lies in the fact that we are free to apply it to any closed surface whose shape can be chosen arbitrary such that the evaluation of the surface integral becomes a simple straightforward task. The Gauss' law is particularly useful in simplifying the calculation of the electric field produced by certain symmetrical charge distributions. We illustrate this in the following examples.

Example of an application of the Gauss' Law

An infinitely long line is positively and uniformly charged with a constant linear charge density ρ_l. Use (a) Coulomb law, (b) Gauss' Law to find the electric field about the line.

(a) If we have to calculate the field due to a static macroscopic distribution of charge using the Coulomb law, we divide the macroscopic charge into infinitesimal (point) charges dq which produce an electric field $d\vec{E}$. The field $d\vec{E}$ is given by the Coulomb field

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} \ .$$

Then the total electric field is found by vector addition

$$\vec{E} = \int d\vec{E} \ .$$

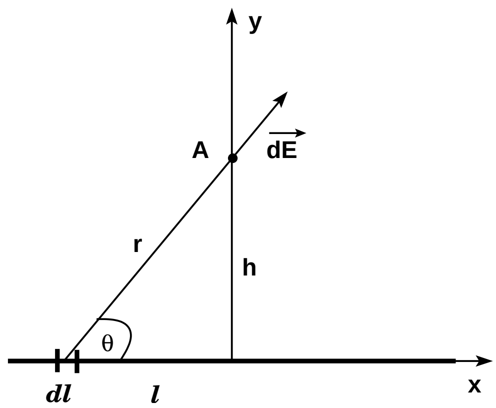
Since we are adding vectors, a caution must be employed. We use the following procedure, which is general and can be employed to any system:

1. Write the expression for the electric field $d\vec{E}$ produced by the infinitesimal (point) charge dq.
2. Resolve this vector into components dE_x,dE_y, and dE_z.
3. Calculate each component of E by integration, e.g. $E_x = \int dE_x$.
4. Find the resultant $\vec{E}$ from its components

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} \ .$$

Return now to our example of the charged infinitely long line.

Take a small element dl of the line containing a point charge dq. Electric



field produced by the point charge at A distance r from dl is given by the Coulomb field

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} \ .$$

We see from the figure that

$$r = \frac{h}{\sin\theta} \ , \qquad l = h\cot\theta \ .$$

Hence

$$dl = -\frac{h}{\sin^2\theta}d\theta \ , \qquad dq = \rho_l dl \ ,$$

and then

$$\begin{aligned} d\vec{E} &= \frac{\rho_l}{4\pi\varepsilon_0}\left(-\frac{h}{\sin^2\theta}\right)\frac{\sin^2\theta}{h^2}d\theta\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) \\ &= -\frac{\rho_l}{4\pi\varepsilon_0 h}\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)d\theta \ , \end{aligned}$$

where we have decomposed the unit vector $\hat{r}$ into two (x, y) components

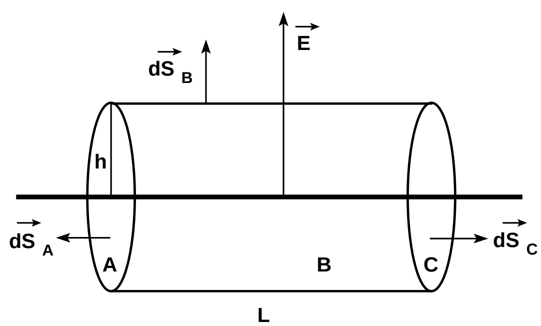
$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \ .$$

Integrating over θ from $\theta = 0$ to $\theta = \pi$, as the line is infinite, (at $x = -\infty, \theta = 0$ and at $x = \infty, \theta = \pi$), we obtain

$$\begin{aligned}\vec{E} &= \frac{\rho_l}{4\pi\varepsilon_0 h}\left[(-\sin 0 + \sin\pi)\,\hat{i} + (\cos 0 - \cos\pi)\,\hat{j}\right] \\ &= \frac{\rho_l}{2\pi\varepsilon_0 h}\;\hat{j}\ .\end{aligned}$$

Thus, the electric field produced by the charged line depends inversely on the distance from the line and points in the direction perpendicular to the line.

(b) Let us now calculate the field by the direct application of the Gauss's law.



The electric field near the uniformly charged line must be radially directed because of the symmetry of the problem. The field must have cylindrical symmetry because the problem is unchanged by rotating the line about its axis. The field must also be independent of position along the line because the distance to either end is infinite. This is the ideal situation for the application of Gauss's Law. We can apply a cylinder surface of radius h and length L centered about the line of charge, see the Figure above.

According to the Gauss's law

$$\int_S \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}\ ,$$

where $q = \rho_l L$ is the charge **closed** by the cylinder surface.

The flux through the cylinder surface splits into three fluxes

$$\int_S \vec{E} \cdot d\vec{S} = \int_A \vec{E} \cdot d\vec{S}_A + \int_B \vec{E} \cdot d\vec{S}_B + \int_C \vec{E} \cdot d\vec{S}_C \ .$$

Since $\vec{E} \perp d\vec{S}_A$, $\vec{E} \perp d\vec{S}_C$, $\vec{E} \parallel d\vec{S}_B$, and the magnitude of $\vec{E}$ is constant along the surface B, the flux through the cylinder reduces to

$$\int_S \vec{E} \cdot d\vec{S} = \int_B E dS_B = 2\pi h L E \ ,$$

i.e. a nonzero flux exists only through the surface B, and since the cylinder is symmetrically positioned about the line of charge, the magnitude of E is constant over the surface B.

Then, according to the Gauss's law

$$2\pi h L E = \frac{\rho_l L}{\varepsilon_0} \ ,$$

which gives

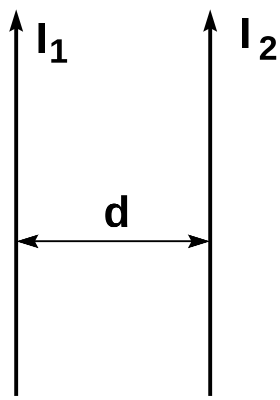
$$E = \frac{\rho_l}{2\pi\varepsilon_0 h} \ .$$

Note how simple are the calculations of the electric field using the Gauss's law. However, we were able to solve this problem because we knew the direction of the field at any point around the line.

3.3 Biot-Savart Law

In 1819, H.C. Oersted showed the magnetic effects of electric current.
In 1827, Ampere showed that quantitatively the magnetic forces in macroscopic circuits can be accounted for by what has come to be known as the Biot-Savart Law.

Experiment



Two long parallel wires. Force between the wires:

$$F = \frac{\mu_0}{4\pi}\frac{I_1\,I_2}{d}\ ,$$

where $\mu_0 = 4\pi \times 10^{-7}$ [H/m] in SI units, is the permeability of the vacuum. (This defines the SI unit of current, the Ampere.)
If $I_1 \| I_2$ then the force F is attractive.
If $I_1\, anti \|\, I_2$ then F is repulsive.
If one wire is rotated through $90°$ then $F = 0$.

All such observations are explained by the law between current elements being:

$$d\vec{F}_2 = \frac{\mu_0}{4\pi}\frac{I_2 d\vec{\ell}_2 \times (I_1 d\vec{\ell}_1 \times \hat{r})}{r^2}\ .$$

We can write the force as

$$d\vec{F}_2 = I_2 d\vec{\ell}_2 \times d\vec{B}\ ,$$

where

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I_1 d\vec{\ell}_1 \times \hat{r}}{r^2}\ ,$$

which is known as the Biot-Savart law for magnetic field produced by the current element $I_1 d\vec{\ell}_1$.

The Biot-Savart law allows to compute magnetic field produced by an arbitrary current distribution $Id\vec{\ell}$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{\ell} \times \hat{r}}{r^2} \ . \tag{5}$$

The method requires integration of small current elements.

We can simplify the calculations of $\vec{B}$ by using the following procedure.

If we replace $\hat{r}/r^2$ by $-\nabla(1/r)$, the integrand becomes

$$-d\vec{\ell} \times \nabla(1/r) \ .$$

However, using a vector identity that

$$\nabla \times \left(\frac{d\vec{\ell}}{r}\right) = \frac{1}{r}\nabla \times d\vec{\ell} - d\vec{\ell} \times \nabla\left(\frac{1}{r}\right) = -d\vec{\ell} \times \nabla\left(\frac{1}{r}\right) \ ,$$

since $\nabla \times d\vec{\ell} = 0$, we can write

$$\vec{B} = \nabla \times \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{\ell}}{r} \ . \tag{6}$$

Since we can write

$$\vec{B} = \nabla \times \vec{A} \ ,$$

as $\nabla \cdot \vec{B} = 0$ always, we can first calculate $\vec{A}$:

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_l \frac{d\vec{\ell}}{r} \tag{7}$$

and then using (6), we will find $\vec{B}$.

The integral for $\vec{A}$ is easier to calculate than the original expression (5) for $\vec{B}$. Since the curl operation is readily performed, we may use (7) as an intermediate step for finding $\vec{B}$ in a simpler way.

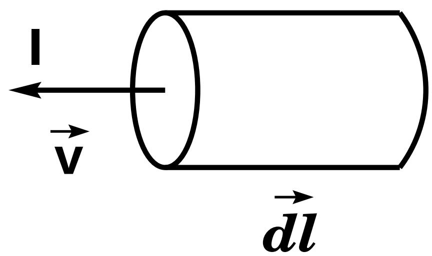
The vector $\vec{A}$ is called a vector potential, and will see later in the course many useful applications of $\vec{A}$ in electromagnetic theory.

3.4 Current Element and Charge Element

A charge dq moving with a velocity v is equivalent to an element of current $Id\ell$:

$$dq = Idt = I\frac{d\ell}{v} \ ,$$

where $dt =$ the time for all the charge in $d\ell$ to pass out of the volume.



Hence

$$dq\vec{v} = Id\vec{\ell} \ .$$

The Biot-Savart Law can then be written as

$$\begin{aligned} d\vec{F}_2 &= \frac{\mu_0}{4\pi} dq_2 \vec{v}_2 \times \frac{(dq_1 \vec{v}_1 \times \hat{r})}{r^2} \ , \\ d\vec{F}_2 &= dq_2 \, \vec{v}_2 \times d\vec{B}_1 \ , \end{aligned}$$

with

$$d\vec{B}_1 = \frac{\mu_0}{4\pi} dq_1 \frac{\vec{v}_1 \times \hat{r}}{r^2} \ ,$$

which shows that magnetic field is produced by a moving electric charge.

3.5 The Lorentz Force

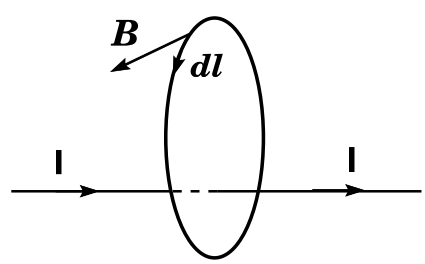
The Lorentz force is obtained putting Coulomb's Law and the Biot-Savart Law together: $\vec{F}_E = q\vec{E}$ and $\vec{F}_M = q\vec{v} \times \vec{B}$:

$$\vec{F}_{EM} = \vec{F}_E + \vec{F}_M = q(\vec{E} + \vec{v} \times \vec{B}) \ .$$

Thus, a motion of electric charges is modified by both the electric and magnetic forces. If the charge is stationary, the force depends only on $\vec{E}$, if it moves, there is an additional force proportional to $\vec{v}$.

3.6 Amperes Circuit Law

Amperes circuit law is a useful relation between currents and magnetic fields. This law allows us to calculate magnetic field produced by some currents in a very effective way.



The Amperes law says that for an arbitrary closed path around a current carrying conductor, the component of magnetic field tangent to the path is proportional to the net current passing through the surface bounded by the path

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

closed loop $\qquad I =$ total current through loop

Proof:

Consider a long wire of radius a carrying current I. Let P is a point on the integration path, see Figure 2.
The magnetic field at P is:

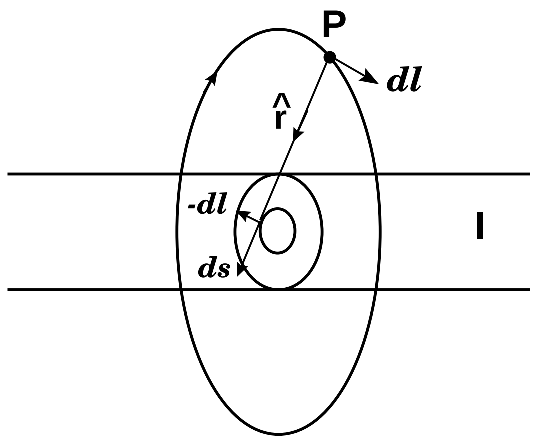


Figure 2: The source circuit and the integration path to prove the Ampere law.

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_s \frac{d\vec{s} \times (-\hat{r})}{r^2} \ .$$

Moving P by $d\vec{\ell}$ is equivalent to moving the current circuit by $-d\vec{\ell}$.
The solid angle subtended by $-d\vec{\ell}, d\vec{s}$ at P is:

$$\frac{(-d\vec{\ell} \times d\vec{s}) \cdot \hat{r}}{r^2} = \frac{-d\vec{\ell} \cdot d\vec{s} \times \hat{r}}{r^2} = d\vec{\ell} \cdot \frac{d\vec{s} \times -\hat{r}}{r^2} \ .$$

(The element of area normal to $\hat{r}$ is $-d\vec{\ell} \times \vec{s} \cdot \hat{r}$)

Thus due to the path element $d\vec{\ell}$, the change in solid angle subtended at P by the circuit is:

$$d\Omega = d\vec{\ell} \cdot \oint_s \frac{d\vec{s} \times (-\hat{r})}{r^2} \ .$$

Hence

$$d\Omega = d\vec{\ell} \cdot \frac{4\pi}{\mu_0 I} \oint_s d\vec{B} = \frac{4\pi}{\mu_0 I} \vec{B} \cdot d\vec{\ell} \ ,$$

where integration is around the circuit s giving the magnetic field $\vec{B}$ at some point P as shown in the diagram.

Now integrating round the closed path:

$$\oint_\ell \vec{B} \cdot d\vec{\ell} = \frac{\mu_0}{4\pi} I \oint_\ell d\Omega \ .$$

If P moves round a closed path (returning to its original position but not circulating *through* the current loop:

$$\oint_\ell d\Omega = 0 \ .$$

But if P circulates through the loop:

$$\oint_\ell d\Omega = 4\pi \ ,$$

and then

$$\oint_\ell \vec{B} \cdot \ d\vec{\ell} = \frac{\mu_0}{4\pi} \, I \, 4\pi = \mu_0 \, I \ .$$

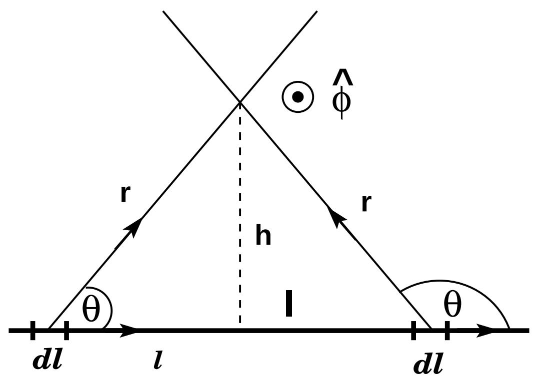
We conclude that the line integral of the magnetic field round a closed loop path is equal to $\mu_0 \, I$ where I is the current *passing through the loop.*

The Amperes law can be applied in highly symmetric situations to find the magnetic field more easily than by computing with the Biot-Savart law.

In either case, the result is the same. In case that lack the proper symmetry, Amperes law is not easily applied.

Example of an application of the Amperes Law

An infinitely long wire caries a constant current I. Use (a) Biot-Savart law, (b) Amperes law to find the magnetic field about the wire.



(a) The element of magnetic field $d\vec{B}$ due to an element $I\,d\vec{\ell}$ at distance $\vec{r} = r\hat{r}$ is found from the Biot-Savart formula:

$$d\vec{B} = \frac{\mu_0}{4\pi} I\,d\vec{\ell} \times \frac{\hat{r}}{r^2}$$

where we note that all $d\vec{B}$ are in the same $\hat{\phi}$ direction normal to the direction of the current. So we see from this symmetry that the field lines are circles concentric with the current. Furthermore, along any such circular path the field is constant in magnitude.

Let us calculate the magnitude of the magnetic field. Since

$$d\vec{\ell} \times \hat{r} = dl \sin\theta \hat{\phi} \ , \qquad r = \frac{h}{\sin\theta} \ , \qquad l = h\cot\theta \ ,$$

we have

$$dl = -\frac{h}{\sin^2\theta} d\theta$$

and then

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\sin^2\theta}{h^2}\left(-\frac{h}{\sin^2\theta}\right)\sin\theta d\theta\,\hat{\phi} = -\frac{\mu_0 I}{4\pi h}\sin\theta d\theta\hat{\phi}\ .$$

Integrating the above equation over the length of the wire, we obtain

$$\begin{aligned}\vec{B} &= -\frac{\mu_0 I}{4\pi h}\int_0^\pi \sin\theta d\theta\hat{\phi} = \frac{\mu_0 I}{4\pi h}\left(\cos 0 - \cos\pi\right)\hat{\phi} \\ &= \frac{\mu_0 I}{2\pi h}\hat{\phi}\ .\end{aligned}$$

(b) Let us now calculate the field using the Amperes law.

Since the field lines are circles concentric with the current, and along any such circular path the field is constant in magnitude, this is the ideal situation for the application of Amperes law:

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I \Rightarrow 2\pi h B = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi h}\ .$$

Note how simple are the calculations of the magnetic field using the Amperes law.

3.7 Faraday's Law of Electromagnetic Induction

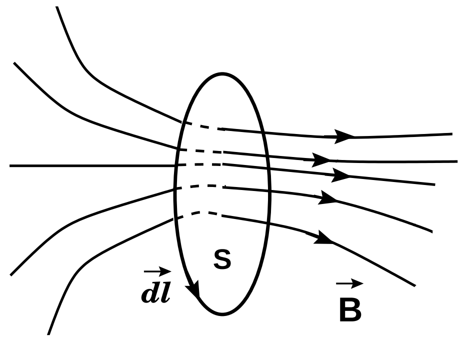
Faraday discovered electromagnetic induction by changing magnetic field. If we consider a closed stationary circuit located in a varying magnetic field, the induced electromotive force around this circuit is equal to the negative time rate of change of the magnetic flux through the circuit

$$\mathcal{E} = -\frac{d\Phi}{dt}\ ,$$

where Φ is the total magnetic flux through the circuit.

From the definition of the flux

$$\Phi = \int_S \vec{B}\cdot d\vec{S}\ ,$$



and from that emf $\mathcal{E}$ = work done per unit charge, we have

$$\frac{\mathcal{E}}{q} = \oint_\ell (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell} \ .$$

In a stationary circuit $\vec{v} = 0$ (and anyway $\vec{v} \times \vec{B} \perp d\vec{\ell}$ since $\vec{v} \parallel d\vec{\ell}$). Thus $\vec{v} \times \vec{B} \cdot d\vec{\ell} = 0$.

Hence

$$\mathcal{E} = \oint_\ell \vec{E} \cdot d\vec{\ell}$$

and finally

$$\oint_\ell \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} \ ,$$

or

$$\oint_\ell \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \ .$$

This is the Faraday's law written in the integral form.

Using the Stokes's theorem, we can rewrite the Faraday's law in the differential form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

The Faraday's law tells us that time-varying magnetic fields give rise to electric fields. This shows that the fields are related to each other, and we then must speak of **electromagnetic fields**, rather than separate electric and magnetic fields.

Questions:

1. Prove that the total electric flux through a closed surface S is proportional to the total charge inside the surface.

2. Prove the Amperes circuit law.

3. Derive the integral form of the Faraday's law and then transform it into the differential form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

Weekend exercises

(a) Find the pressure (force per unit area) between two infinite and opposite charged parallel planes of charge density $\pm\sigma$.

(b) Describe the magnetic field associated with $\vec{E} = x\hat{\jmath}$.

4 Differential Equations for the EM Field and Maxwell's Theory

We know now that electromagnetic forces are carried by electromagnetic fields that propagate at speed $c \simeq 3 \times 10^8$ ms^{-1}. Because of the finite propagation speed we are forced to assign energy & momentum to the fields i.e. we must think of them as real physical entities as against mere mathematical conveniences (as is the case for static fields). An electromagnetic system then qualifies as static only if all the charges have been at rest longer than the time taken to traverse the system at speed c.

In the 1830's Michael Faraday carried out experiments to measure a finite electromagnetic propagation speed. He was unsuccessful due to lack of time resolution in his apparatus. Faraday would have had no reason to think that the electromagnetic speed was the same as the speed of light (then known). In the 1860's, James Clerk Maxwell, seeking to advance Faraday's ideas about electromagnetic fields, by a brilliant process of intuition worked out how to generalize certain differential equations deduced from static experiments. He produced a set of field equations known by his name today. Maxwell also had a theory i.e. a set of qualitative ideas underpinning his equations. The theory (unlike the equations) has not stood the test of time.

4.1 Differential Equations for the EM Field

Let us take as the source of the electromagnetic field a continuous charge and current distribution (ρ and $\vec{J}$). Then we will produce macroscopic field equations

$$q = \int \rho \, dV \ , \qquad I = \int \vec{J} \cdot \vec{n} \, dS \ .$$

We find differential vector equations from the integral forms of observational results discussed above.

4.1.1 Divergence of $\vec{E}$

$$\int \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0} = \int \frac{\rho}{\varepsilon_0} dV \ .$$

Applying Gauss' Theorem to Coulomb's Law:

$$\int \nabla \cdot \vec{E} \, dV = \int \frac{\rho}{\varepsilon_0} dV \ .$$

Since this must hold for arbitrary V no matter how small:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \ .$$

4.1.2 Curl of $\vec{E}$

From Faraday's flux cutting rule we had:

$$\int \vec{E} \cdot d\vec{\ell} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, dS \ .$$

Applying Stokes's Theorem:

$$\int \nabla \times \vec{E} \cdot \hat{n} \, dS = -\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} \, dS \ .$$

For arbitrary S and $\hat{n}$ implies

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

4.1.3 Divergence of $\vec{B}$

From Biot-Savart law, we had:

$$\vec{B} = \frac{\mu_0}{4\pi} I \, d\vec{\ell} \times \frac{\hat{r}}{r^2} \ .$$

Applying the rule

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \ ,$$

we obtain

$$\nabla \cdot \vec{B} = \frac{\mu_0 I}{4\pi} \left(\frac{\hat{r}}{r^2} \cdot \nabla \times d\vec{\ell} - d\vec{\ell} \cdot \nabla \times \frac{\hat{r}}{r^2} \right) \ .$$

However, $\nabla \times d\vec{\ell} = 0$, since $d\vec{\ell}$ is a constant vector, and

$$\nabla \times \frac{\hat{r}}{r^n} \equiv 0 \quad \text{for any} \quad n \ ,$$

Hence

$$\nabla \cdot \vec{B} = 0 \ .$$

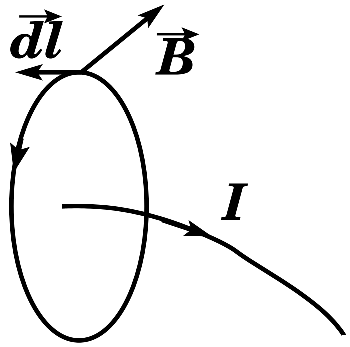
4.1.4 Curl of $\vec{B}$

From Ampere's Circuit Law:

$$\int \vec{B} \cdot d\vec{\ell} = \mu_0 I = \mu_0 \int \vec{J} \cdot \hat{n} \, dS \ .$$

Applying Stokes's Theorem, the relation

$$\int \nabla \times \vec{B} \cdot \hat{n} \, dS = \mu_0 \int \vec{J} \cdot \hat{n} \, dS$$



holds for arbitrary $\hat{n}$. Thus

$$\nabla \times \vec{B} = \mu_0 \vec{J} \ .$$

However, Maxwell realized that unlike the previous three differential equations, this one could not be generally true.

To see this, take its divergence and remember that $\nabla \cdot \nabla \times \vec{F} \equiv 0$ for any vector function $\vec{F}$:

$$\nabla \cdot \nabla \times \vec{B} = 0 = \mu_0 \nabla \cdot \vec{J} \ .$$

Thus

$$\nabla \cdot \vec{J} \equiv 0 \quad !!!$$

We have already seen that conservation of electric charge requires

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \ .$$

Thus $\nabla \cdot \vec{J} \equiv 0$ implies that

$$\frac{\partial \rho}{\partial t} \equiv 0 \ ,$$

i.e. we could never charge or discharge a capacitor.

Maxwell guessed at the right value of $\nabla \times \vec{B}$ as follows:

Since

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \rho = \varepsilon_0 \nabla \cdot \vec{E}$$

(from the first Maxwell equation for $\nabla \cdot \vec{E}$)
Hence

$$\nabla \cdot \vec{J} + \frac{\partial \varepsilon_0 \nabla \cdot \vec{E}}{\partial t} = 0 \quad \text{or} \quad \nabla \cdot (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = 0$$

If then we (after Maxwell) write:

$$\nabla \times \vec{B} = \mu_0 (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) \ ,$$

instead of

$$\nabla \times \vec{B} = \mu_0 \vec{J} \ ,$$

we obtain

$$\nabla \cdot \nabla \times \vec{B} \equiv 0 \quad \rightarrow \quad \nabla \cdot (\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = 0 \ ,$$

which is in accord with conservation and motion of charge.

Note from above that the term that Maxwell added

$$\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \ ,$$

has the dimensions of current density. Maxwell called it the **displacement current density**.
Maxwell had a theory underpinning his equations in which the displacement current was a real physical current - due to ‘polarization of the electromagnetic ether’. This theory has not survived. Nevertheless the above term is still referred to as the ‘displacement current’.

We write the fourth Maxwell equation then as:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ .$$

The differential form of Maxwell's equations is easier to interpret physically and is also useful in deriving the boundary conditions that the field vectors must satisfy.

The Maxwell's equations are self-consistent and no experimental evidence for requiring any further modifications has been found.

Exercise in class: *Fields within a capacitor*

A plane parallel capacitor is being charged with a current I. Show that the displacement current between the plates of the capacitor is equal to the conduction current I in the external charging circuit. Remember that the displacement current density is, by definition, $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ so the displacement current through a surface S is:

$$I_D = \int_S \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \hat{n}\, dS \ .$$

Assume that the external wires are perfect conductors so that $\vec{E}$ is zero in them. Assume the space between the plates is a perfect insulator so no conduction current flows within the capacitor.

Can you see any curious consequence in this case if the displacement current is assumed to be a real physical current (flow of charges)?

4.2 Maxwell's Equations and Prediction of Electromagnetic Waves

The basic equations to study electromagnetic theory are Maxwell's Equations, which govern the behavior of the time-varying electromagnetic field

$$\begin{aligned}
&\text{I.} && \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \ , \\
&\text{II.} && \nabla \cdot \vec{B} = 0 \ , \\
&\text{III.} && \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ , \\
&\text{IV.} && \nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ .
\end{aligned}$$

The first equation is just Gauss's law, the second tells us about the non-existence of magnetic charges, the third equation is Faraday's law, and the final equation is Ampere's law.

Maxwell's immediate triumph was to predict the existence of electromagnetic waves and their propagation speed. The calculated speed came (within experimental error) to be equal to the measured speed of light. This prediction obviously led to the conclusion that light was electromagnetic in nature. Thus arose a synthesis of electromagnetism and optics.

4.2.1 The Wave Equation for EM Waves in Vacuum

In a vacuum there are no sources i.e. $\rho = 0$ and $\vec{J} = 0$.
Hence, the Maxwell's equations III and IV reduce to the following:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ , \tag{8}$$

$$\nabla \times \vec{B} = = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ . \tag{9}$$

We eliminate $\vec{E}$ or $\vec{B}$ between equations (8) and (9) to obtain differential equations for $\vec{E}$ or $\vec{B}$ alone, using where required $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$.

Method: Think of $\nabla\times$ and $\frac{\partial}{\partial t}$ as linear (differential) operators. By analogy with methods of solving linear algebraic equations, applying $\nabla\times$ into (8) and $\partial/\partial t$ into (9), we obtain

$$\begin{aligned}\nabla\times\left(\nabla\times\vec{E}\right) &= -\frac{\partial\nabla\times\vec{B}}{\partial t}\ ,\\ \frac{\partial}{\partial t}\nabla\times\vec{B} &= \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}\ .\end{aligned}$$

Hence

$$\begin{aligned}\nabla\times\left(\nabla\times\vec{E}\right) &= -\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}\ ,\\ \nabla\nabla\cdot\vec{E}-\nabla^2\vec{E} &= -\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}\ .\end{aligned}$$

Because $\nabla\cdot\vec{E}=0$ in the vacuum, we finally obtain

$$\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}\ , \tag{10}$$

where

$$c^2=\frac{1}{\varepsilon_0\mu_0}\ .$$

The parameter c has the dimensions of velocity and is numerically equal to $3\times10^8\ \mathrm{ms}^{-1}$.

Equation (10) is the standard form of a three-dimensional vector wave equation. The field $\vec{B}$ satisfies the same equation.

4.2.2 Plane Wave Solution to the Wave Equation

The wave equation in a vacuum is

$$\nabla^2\vec{X}=\frac{1}{c^2}\frac{\partial^2\vec{X}}{\partial t^2}$$

for $\vec{X}\equiv\vec{E},\vec{B}$.

Look for plane wave solutions propagating in the z direction.
In this case, $\partial/\partial x = \partial/\partial y \equiv 0$, and then

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z^2} \ .$$

The differential equations for $\vec{E}$ and $\vec{B}$ both have the same form:

$$\frac{\partial^2 \vec{X}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} \ .$$

Such an equation has solutions of the form $X = f(z \pm ct)$ where f is an arbitrary function.

This solution represents a signal propagating with speed c as can be seen from the following discussion.

Let $X_0 = f(z_0 - ct_0)$ i.e. X at $t = t_0$ and $z = z_0$.
Now examine X at time Δt later and distance Δz further along in z. Since a harmonic wave does not change in vacuum, we have

$$\begin{aligned} X_1 &= f(z_0 + \Delta z - c(t_0 + \Delta t)) \\ &= f(z_0 - ct_0) = X_0 \qquad \text{when} \qquad \Delta z = c\Delta t \ , \end{aligned}$$

i.e. the signal propagates a distance $\Delta z = c\Delta t$ in time Δt i.e. it propagates with speed c.

Proof of solution:

Let f represent $f(z - ct)$.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial z} = \frac{\partial f}{\partial (z-ct)} = f' \ ,$$

where $'$ means differentiation wrt $z - ct$.
Similarly

$$\begin{aligned} \frac{\partial^2 f}{\partial z^2} &= \frac{\partial f'}{\partial z} = \frac{\partial f'}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial z} = f'' \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial t} = -cf' \end{aligned}$$

Similarly

$$\frac{\partial^2 f}{\partial t^2} = (-c)(-c)f'' \ .$$

Consequently

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \ .$$

Thus

$$c^2 = \frac{1}{\varepsilon_0 \mu_0} \rightarrow c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \ ,$$

which with the numerical values of the parameters

$$\varepsilon_0 \simeq 8.85 \times 10^{-12} \ \mathrm{Fm}^{-1} \qquad \mu_0 = 4\pi \times 10^{-7} \ \mathrm{Hm}^{-1}$$

gives

$$c \simeq 3 \times 10^8 \ \mathrm{ms}^{-1} \ .$$

What determines the form of the function f?

Boundary conditions do.
Source (charges & current), propagating EM in a vacuum, detectors (other charges)

4.2.3 Harmonic Waves

In vacuum, we chose a plane wave representation for the propagating EM wave. For a harmonic wave

$$c = f\lambda = \frac{\omega}{k}, \qquad \omega = 2\pi f, \qquad k = \frac{2\pi}{\lambda} \ ,$$

where f = frequency (Hz), λ = wavelength (m),
ω = radian frequency (radians s^{-1}), and k = propagation constant (m^{-1}).

$$f(z - ct) = f(z - \frac{\omega}{k}t) = f_1(\omega t - kz) \ .$$

A plane wave is represented by

$$\vec{E} = \vec{E}_0 \mathrm{e}^{-i(\omega t - kz)} \ .$$

Note that some textbooks on electromagnetism, for engineers in particular, use j instead of i for the imaginary number. We will use i throughout this lecture notes.

4.3 The Transverse Nature of Plane Waves in Vacuum

We now investigate the relations between the amplitudes and phases of the electric and magnetic fields in a plane harmonic wave. While it is true that the magnetic field satisfies the same wave equation as the electric field, it is not independent of the latter, since one must satisfy the Maxwell equations III and IV.

Since $\nabla \cdot \vec{B} = 0$ always in electromagnetism, and

$$\begin{aligned} \nabla \cdot \vec{B} &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \\ &= 0 + 0 + \frac{\partial B_z}{\partial z} \end{aligned}$$

for a plane wave propagating in the z direction, we have

$$\frac{\partial B_z}{\partial z} = 0 \ .$$

However, for a plane wave

$$\frac{\partial B_z}{\partial z} = -ikB_z \ .$$

Hence, the rhs must be zero, which means that either $k = 0$ (zero frequency) or $B_z = 0$ (transverse wave).
For a propagating wave $k \neq 0$, so the wave is transverse in $\vec{B}$.

In a vacuum $\nabla \cdot \vec{E} = 0$ and then by the same argument we conclude that the plane wave is also transverse in $\vec{E}$. In other cases in electromagnetism (e.g. for a plasma in a magnetic field) a plane wave may not be purely transverse in $\vec{E}$.

In addition: $\vec{E} \perp \vec{B}$ **for a plane EM wave in a vacuum.**

Proof:

Consider a harmonic wave propagating parallel to the z axis. In this case the field components are $\sim \mathrm{e}^{i(\omega t - kz)}$

We will expand Maxwell equation III

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

in Cartesian coordinates remembering that $\partial/\partial t \equiv i\omega$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -i\omega(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \tag{11}$$

For electromagnetic plane waves propagating along the z axis in a vacuum:

$$E_z = B_z = 0, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0, \quad \frac{\partial}{\partial z} = -ik \ .$$

Hence, Eq. (11) reduces to

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & -ik \\ E_x & E_y & 0 \end{vmatrix} = -i\omega(B_x\hat{i} + B_y\hat{j}) \ .$$

Comparing the left and right-hand sides:

$$x \text{ component} \quad \rightarrow \quad ikE_y = -i\omega B_x \rightarrow \ B_x = -\frac{k}{\omega}E_y$$

$$y \text{ component} \quad \rightarrow \quad -ikE_x = -i\omega B_y \rightarrow \ B_y = \frac{k}{\omega}E_x$$

Consider a scalar product

$$\begin{aligned} \vec{E} \cdot \vec{B} &= (\hat{i}E_x + \hat{j}E_y) \cdot (\hat{i}B_x + \hat{j}B_y) \\ &= (\hat{i}E_x + \hat{j}E_y) \cdot (\hat{i}\frac{-k}{\omega}E_y + \hat{j}\frac{k}{\omega}E_x) \\ &= -\frac{k}{\omega}E_xE_y + \frac{k}{\omega}E_xE_y = 0 \ , \end{aligned}$$

which means that $\vec{E} \perp \vec{B}$.

Note that

$$\frac{E}{B} = \frac{\omega}{k} = c \; .$$

In electromagnetic theory when E and B are related, their ratio is always a velocity characteristic of the problem in hand.

This is as far as Maxwell took the subject. It was for others like Heinrich Hertz 1884 to show how to solve Maxwell's equations with source terms ρ, $\vec{J}$ included (i.e. the *generation* of electromagnetic waves). We will consider this later.

You should be aware that we have not derived Maxwell's equations from the static limits like Coulomb's Law and the Biot-Savart Law. The solutions to Maxwell's equations include the static limits as special cases but many more. Maxwell's equations have the status of postulates suggested by experimental results.

In summary, we have the following important results for related electric and magnetic fields propagating in vacuum:

1. The electric and magnetic fields propagate in a form of plane waves, so-called electromagnetic (EM) waves.

2. The plane EM wave is transverse in $\vec{E}$ and $\vec{B}$.

3. The electric and magnetic fields are perpendicular to each other.

4. The ratio E/B is constant and equal to the velocity of the wave, that is equal to the speed of light in vacuum.

Questions:

(1) Starting from the Maxwell's equations derive the continuity equation, i.e. show that conservation of charge is built into the Maxwell's equations.

(2) Using the Maxwell's equations show that $\vec{B}$ satisfies the same wave equation as $\vec{E}$.

(3) Show, using the proof of solution of the wave equation, that $f(z+ct)$ represents a signal propagating in the negative z direction with speed c.

Weekend exercises

(a) Demonstrate that the Coulomb field $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ for a stationary point charge, follows from the Maxwell's equations.

5 EM Theory and Einstein's Special Theory of Relativity

Special relativity (1905) grew out of Einstein's meditation on electromagnetic theory and the properties of space and time. Historically, the insights of Einstein's theory follow after electromagnetism. Logically however, special relativity contains more general statements about nature than electromagnetism. Electromagnetic field theory is just one of a possible set of field theories that are compatible with the Einstein theory of space and time. It is evident that relativistic effects are important if we were to calculate the field of a charge moving with a speed comparable to that of light. What is not so obvious is that special relativity offers insights in to aspects of electromagnetic theory even in the case of the low speed charges we consider in this course.

Two such aspects are:

(1) The unity of the electromagnetic field i.e. the field is a single entity with 6 components (represented by two vectors $\vec{E}$ and $\vec{B}$, each with three components).

(2) Understanding the nature of causal relationships in electromagnetic theory, e.g.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

Does this mean that a time changing $\vec{B}$ *causes* a spatially changing $\vec{E}$?

5.1 Lorentz Transformation Equations for Space and Time

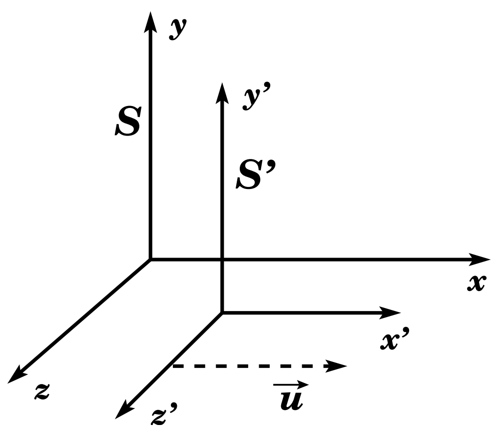
The principle of relativity

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light in vacuum is independent of the uniform motion of the observer or source.

The constancy of the velocity of light, independent of the motion of the source, gives rise to the relations between space and time coordinates in different inertial reference frames known as *Lorentz transformations.*

Consider a stationary reference frame S and a inertial frame S' moving with a velocity $\vec{u}$ parallel to the x axis.



Let x, y, z are coordinates in the S frame and x', y', z' are the coordinates in the S' frame which is moving with a constant velocity $\vec{u}$ parallel to the x axis with respect to S.

The time and space coordinates in S' are related to those in S by the **1D Lorentz transformations:**

$$\begin{aligned} x' &= \gamma(x - ut) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{ux}{c^2}\right) \ , \end{aligned}$$

where

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

is the Lorentz factor.

The above transformation corresponds to a situation of $\vec{u}$ parallel to the x axis. Later in the course, we will consider the general case of the velocity $\vec{u}$ of the frame S' in an arbitrary direction.

5.2 Force Transformation Equations

We will demonstrate how one might infer the law of Biot-Savart from application of special relativity to Coulomb law.

Consider a particle which is moving with velocity $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ in the S frame and is acted on by a force with components F_x, F_y and F_z. Then in the S' frame:

$$\begin{aligned} F'_x &= F_x - \frac{uv_y}{c^2(1 - \frac{uv_x}{c^2})}F_y - \frac{uv_z}{c^2(1 - \frac{uv_x}{c^2})}F_z \ , \\ F'_y &= \frac{(1 - \frac{v^2}{c^2})^{\frac{1}{2}}}{1 - \frac{uv_x}{c^2}}F_y = \frac{F_y}{\gamma(1 - \frac{uv_x}{c^2})} \ , \\ F'_z &= \frac{(1 - \frac{v^2}{c^2})^{\frac{1}{2}}}{1 - \frac{uv_x}{c^2}}F_z = \frac{F_z}{\gamma(1 - \frac{uv_x}{c^2})} \ . \end{aligned}$$

Suppose F_x, F_y, F_z represents the velocity independent Coulomb force. Then in the S' frame (source of the field now moving) the force is no longer velocity independent. In electromagnetic theory we say that there is now a magnetic force and we define a magnetic field $\vec{B}$ that determines the magnetic force.

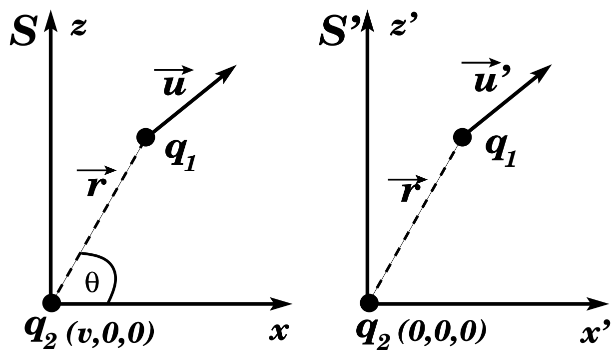
We now present a detailed calculation that illustrates how the form of the Maxwell's equations is determined by nature obeying Einstein's special theory of relativity.

5.2.1 The Force between Two Charges Moving with Constant Velocities

Invariance of electric charge

In ordinary matter, electrons move with much greater speeds than protons yet there is no associated electric field. This implies that electric charge is independent of velocity unless electromagnetic laws modified in some complicated way (see discussion by King in Physical Review Letters **5**, 562 (1960)).

Consider charges q_1 and q_2 moving with velocities $\vec{u}$ and $\vec{v}$ in an inertial frame S. No loss of generality occurs if $\vec{v}$ is taken parallel to the x axis.



Now consider a frame S' moving with velocity $\vec{v}$ along x axis, that q_2 is stationary in S'. Assume that at time $t = 0$, the frames S and S' overlap.

From the principle of relativity, in the S' frame Coulombs law holds. The force on q_1 seen in S' is therefore

$$\vec{F}\,' = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{r'\,^3}\,\vec{r}\,'\ .$$

We shall transform this expression to find the force observed in the S frame in which the source of the field (q_2) is moving.

We shall see that what we normally call the **MAGNETIC FIELD** arises as a natural consequence of relativistic invariance with no extra assumptions.

The x component of the force is

$$\vec{F}_x{}' = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r'^{\,3}}\, x'$$

and similarly for y and z.

The Lorentz transformations at $t = 0$ are

$$\begin{aligned} x' &= \gamma x \\ y' &= y \\ z' &= z \\ t' &= -\gamma \frac{vx}{c^2} \end{aligned}$$

To do the transformations, we also need transformation from $r\,'$ to r. Clearly, there is an "axial symmetry" here and we should involve the angle θ.

$$\begin{aligned} r'^2 &= x'^2 + y'^2 + z'^2 = \gamma^2 x^2 + y^2 + z^2 \\ &= \gamma^2 \left(x^2 + \frac{y^2 + z^2}{\gamma^2} \right) = \gamma^2 \left[x^2 + \left(1 - \frac{v^2}{c^2} \right) \left(y^2 + z^2 \right) \right] \\ &= \gamma^2 \left[x^2 + y^2 + z^2 - \frac{v^2}{c^2} \left(y^2 + z^2 \right) \right] \\ &= \gamma^2 \left(r^2 - \frac{v^2}{c^2} r^2 \sin^2\theta \right) = \gamma^2 r^2 \left(1 - \frac{v^2}{c^2} \sin^2\theta \right) \ , \end{aligned}$$

where $\sin\theta = \sqrt{(y^2 + z^2)}/r$.

Hence

$$r' = \gamma r \left(1 - \frac{v^2}{c^2} \sin^2\theta \right)^{\frac{1}{2}} \ .$$

Substituting into F_x', we obtain

$$\begin{aligned} F_x' &= \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2 \gamma x}{\gamma^3 r^3 \left(1 - \frac{v^2}{c^2} \sin^2\theta \right)^{\frac{3}{2}}} \\ &= q_1 g x \ , \end{aligned}$$

where

$$g = \frac{1}{4\pi\varepsilon_0} \frac{q_2}{\gamma^2 r^3 \left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{\frac{3}{2}}} \ .$$

Similarly

$$F'_y = \frac{q_1 g y}{\gamma} \qquad F'_z = \frac{q_1 g z}{\gamma}$$

The force transformations are

a) x component

$$F'_x = F_x - \frac{v u_y}{c^2\left(1 - \frac{v u_x}{c^2}\right)} F_y - \frac{v u_z}{c^2\left(1 - \frac{v u_x}{c^2}\right)} F_z$$

Thus

$$q_1 g x = F_x - \frac{v u_y}{c^2\left(1 - \frac{v u_x}{c^2}\right)} F_y - \frac{v u_z}{c^2\left(1 - \frac{v u_x}{c^2}\right)} F_z$$

b) y component

$$\begin{aligned} F'_y &= \frac{F_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)} \\ F_y &= \gamma\left(1 - \frac{v u_x}{c^2}\right) F'_y = \gamma\left(1 - \frac{v u_x}{c^2}\right) \frac{q_1 g y}{\gamma} \\ F_y &= q_1 g y \left(1 - \frac{v u_x}{c^2}\right) \end{aligned}$$

c) z component

$$F_z = q_1 g z \left(1 - \frac{v u_x}{c^2}\right) \ .$$

Substituting for $F_{y,z}$ in x equation

$$F_x = q_1 g x + q_1 g y \frac{v u_y}{c^2} + q_1 g z \frac{v u_z}{c^2} \ .$$

Note: Here is the germ of the magnetic field. The last two terms are typical second order relativistic terms $\approx v^2/c^2$. In a nonrelativistic calculation we would have $F_x = F'_x$.

We now combine results a), b) and c) into a single vector equation for the force $\vec{F}$. First, note that

$$vu_x = \vec{u} \cdot \vec{v} \ .$$

Next, we can write the x component as

$$\begin{aligned} F_x &= q_1 g x \left(1 - \frac{vu_x}{c^2}\right) + q_1 g x \frac{vu_x}{c^2} \\ &+ q_1 g y \frac{vu_y}{c^2} + q_1 g z \frac{vu_z}{c^2} \\ &= q_1 g \left(1 - \frac{vu_x}{c^2}\right) x + q_1 g \frac{v}{c^2} \left(\vec{u} \cdot \vec{r}\right) \end{aligned}$$

and with the y and z components

$$\begin{aligned} F_y &= q_1 g \left(1 - \frac{vu_x}{c^2}\right) y \\ F_z &= q_1 g \left(1 - \frac{vu_x}{c^2}\right) z \end{aligned}$$

these three components combine into

$$\begin{aligned} \vec{F} &= q_1 g \left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2}\right) \vec{r} + q_1 g \frac{\vec{v}}{c^2} \left(\vec{u} \cdot \vec{r}\right) \\ &= q_1 g \vec{r} + \frac{q_1 g}{c^2} \left[\vec{v} \left(\vec{u} \cdot \vec{r}\right) - \vec{r} \left(\vec{u} \cdot \vec{v}\right)\right] \\ \vec{F} &= q_1 g \vec{r} + \frac{q_1 g}{c^2} \vec{u} \times \left(\vec{v} \times \vec{r}\right) \end{aligned}$$

We can write this equation in the form of the Lorentz force

$$\vec{F} = q_1 \left(\vec{E} + \vec{u} \times \vec{B}\right)$$

where

$$\vec{E} = g\vec{r} = \frac{1}{4\pi\varepsilon_0} \, \frac{q_2}{\gamma^2 r^3 \left(1 - \frac{v^2}{c^2} \sin^2\theta\right)^{\frac{3}{2}}} \, \vec{r}$$

and

$$\vec{B} = \frac{\vec{v} \times g\vec{r}}{c^2} = \frac{\vec{v} \times \vec{E}}{c^2} \; .$$

Note that as $v \to 0, \gamma \to 1$, and then

$$\vec{E} \to \frac{1}{4\pi\varepsilon_0} \, \frac{q_2}{r^3} \, \vec{r} \; .$$

Moreover, the ration of magnitudes of magnetic to electric term in the force equation is uv/c^2, i.e. magnetic forces are second order relativistic effects.

5.3 Electric and Magnetic Field Lines of a Moving Charge

5.3.1 Electric Field Lines

Let $\beta = v/c$. Then, we can write the electric field as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \, \frac{q(1-\beta^2)}{r^3 \left(1 - \beta^2 \sin^2\theta\right)^{\frac{3}{2}}} \, \vec{r} \; .$$

For a given θ, the electric field E still varies as $1/r^2$, but the field lines are crowded in the direction perpendicular to $\vec{v}$.

In the forward direction $\theta = 0$ and then

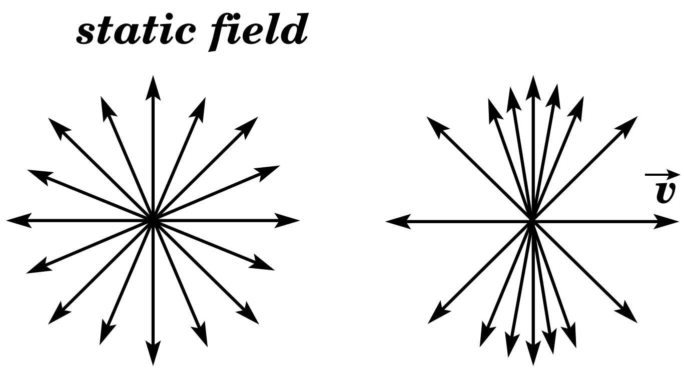
$$E = \frac{1}{4\pi\varepsilon_0} \, \frac{q(1-\beta^2)}{r^2} < E_s \; ,$$

where E_s is the static electric field (at $v = 0$).

In the perpendicular direction, $\theta = \pi/2$, and then

$$E = \frac{1}{4\pi\varepsilon_0} \, \frac{q}{r^2} \left(1 - \beta^2\right)^{-\frac{1}{2}} > E_s \; .$$

The electric field lines radiate from the present position of the charge.



5.3.2 Magnetic Field Lines

From the relation between electric and magnetic fields, we find

$$\begin{aligned}\vec{B} &= \frac{\vec{v}\times\vec{E}}{c^2}\\ &= \frac{1}{4\pi\varepsilon_0}\,\frac{q(1-\beta^2)\ \vec{v}\times\vec{r}}{c^2r^3\left(1-\beta^2\sin^2\theta\right)^{\frac{3}{2}}}\ .\end{aligned}$$

In spherical polar coordinates

$$\vec{B} = B_r\hat{r} + B_\theta\hat{\theta} + B_\phi\hat{\phi} = B_\phi\ \hat{\phi}$$

since $\vec{B}\sim\vec{v}\times\vec{r}$, and $\hat{r},\hat{\theta},\hat{\phi}$ are unit vectors.

The magnetic field lines form concentric rings about $\vec{v}$, and there is symmetry about the plane $\theta = \pi/2$.

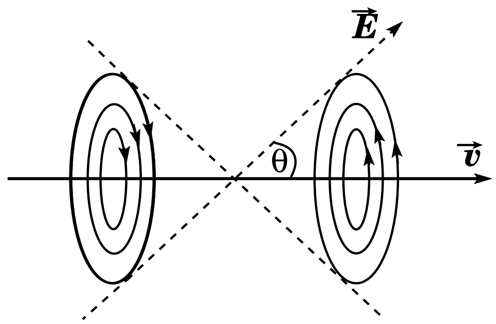
In the non-relativistic case of $v \ll 1,\ \beta\to 0$, and then

$$\vec{B} = \frac{1}{4\pi\varepsilon_0}\ \frac{q\ \vec{v}\times\vec{r}}{c^2r^3}\ ,$$

which is the Biot-Savart law.

Applied to a continuous line current I:

$$\vec{B} = \frac{1}{4\pi\varepsilon_0}\ \frac{I\ \vec{dl}\times\vec{r}}{c^2r^3}\ .$$



The constant $1/(\varepsilon_0 c^2)$ is normally written μ_0 - the magnetic permeability of free space.

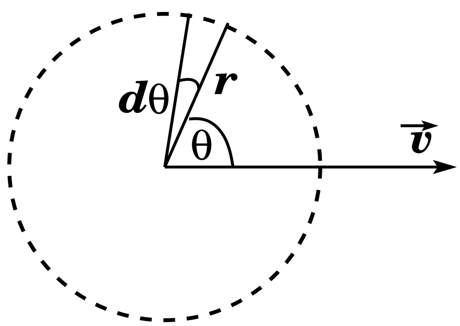
5.4 Field equations for an elementary point charge moving with uniform velocity

(1) Equation for the total electric flux

Consider the total electric flux

$$\Psi_E = \int \vec{E} \cdot \hat{n} \, dS \ .$$

We will use the axial symmetry and break sphere up into rings lying between θ and $\theta + d\theta$.



Since $\vec{E} \parallel \hat{n}$, the flux $d\Psi_E$ is

$$d\Psi_E = \vec{E} \cdot \hat{n}\ dS = EdS\ .$$

However

$$dS = 2\pi(r\sin\theta)\ rd\theta = 2\pi r^2 \sin\theta d\theta\ .$$

Hence

$$\begin{aligned} d\Psi_E &= \frac{1}{4\pi\varepsilon_0}\ \frac{q(1-\beta^2)}{r^2\left(1-\beta^2\sin^2\theta\right)^{\frac{3}{2}}}\ 2\pi r^2\sin\theta d\theta \\ &= \frac{q(1-\beta^2)}{2\varepsilon_0}\ \frac{\sin\theta d\theta}{\left(1-\beta^2\sin^2\theta\right)^{\frac{3}{2}}}\ . \end{aligned}$$

Thus

$$\Psi_E = \int d\Psi_E = \frac{q(1-\beta^2)}{2\varepsilon_0}\int_{\theta=0}^{\pi}\frac{\sin\theta d\theta}{\left(1-\beta^2\sin^2\theta\right)^{\frac{3}{2}}}\ .$$

To calculate the integral, put $\cos\theta = x$, so that $\sin\theta d\theta = -dx$, and then

$$\begin{aligned} I &= -\int\frac{dx}{(1-\beta^2+\beta^2x^2)^{3/2}} = -\frac{1}{\beta^3}\int\frac{dx}{\left(\frac{1-\beta^2}{\beta^2}+x^2\right)^{3/2}} \\ &= -\frac{1}{\beta^3}\int\frac{dx}{(a^2+x^2)^{3/2}}\ , \end{aligned}$$

where $a = \sqrt{1-\beta^2}/\beta$. Performing the integration, we obtain

$$I = -\frac{1}{a^2\beta^3}\ \frac{x}{(a^2+x^2)^{1/2}}\ .$$

Thus, including the limits of the integral

$$\begin{aligned} I &= -\int_{1}^{-1}\frac{dx}{(1-\beta^2+\beta^2x^2)^{3/2}} \\ &= -\frac{1}{a^2\beta^3}\frac{-1}{\sqrt{a^2+1}} + \frac{1}{a^2\beta^3}\frac{1}{\sqrt{a^2+1}} = \frac{2}{1-\beta^2} \end{aligned}$$

and then

$$\Psi_E = \frac{q(1-\beta^2)}{2\varepsilon_0} \frac{2}{1-\beta^2} = \frac{q}{\varepsilon_0} \ .$$

Hence, the electric field produced by a moving charge satisfies the Gauss's law.

(2) Magnetic flux through a closed surface

Consider the magnetic flux through a closed surface

$$\Psi_M = \int \vec{B} \cdot \hat{n} \ dS \ .$$

If we choose a sphere centered on q to calculate the integral, we find that $\vec{B}$ is perpendicular to $\hat{n}$ everywhere, and then $\Psi_M = 0$.

Similarly, $\nabla \cdot \vec{B} = 0$ always. It is easy to see that

$$\begin{aligned} \nabla \cdot \vec{B} &= \frac{1}{c^2} \nabla \cdot \left(\vec{v} \times \vec{E} \right) \\ &= \frac{1}{c^2} \left(\vec{E} \cdot \nabla \times \vec{v} - \vec{v} \cdot \nabla \times \vec{E} \right) = 0 \end{aligned}$$

since $\nabla \times \vec{v} = 0$ as $\vec{v}$ is constant, and $\vec{v}$ is perpendicular to $\nabla \times \vec{E}$.

(3) Spatial $\vec{E}$ derivative related to temporal $\vec{B}$ derivative.

We shall show that for a point charge under uniform velocity

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ .$$

In spherical polar coordinates

$$\begin{aligned} \nabla \times \vec{E} &= \frac{\hat{r}}{r \sin\theta} \left[\frac{\partial \left(\sin\theta E_\phi \right)}{\partial \theta} - \frac{\partial E_\theta}{\partial \phi} \right] \\ &+ \frac{\hat{\theta}}{r} \left[\frac{1}{\sin\theta} \frac{\partial E_r}{\partial \phi} - \frac{\partial \left(r E_\phi \right)}{\partial r} \right] \\ &+ \frac{\hat{\phi}}{r} \left[\frac{\partial \left(r E_\theta \right)}{\partial r} - \frac{\partial E_r}{\partial \theta} \right] \ . \end{aligned}$$

Since

$$E_\theta = E_\phi = 0 \ ,$$

then

$$\nabla \times \vec{E} = -\frac{1}{r}\ \frac{\partial E_r}{\partial \theta}\hat{\phi}$$

with

$$\begin{aligned} \vec{E}_r &= \frac{1}{4\pi\varepsilon_0}\ \frac{q(1-\beta^2)}{r^2\left(1-\beta^2\sin^2\theta\right)^{\frac{3}{2}}} \\ &= \frac{K}{\left(1-\beta^2\sin^2\theta\right)^{\frac{3}{2}}} \ . \end{aligned}$$

Next

$$\begin{aligned} \frac{\partial E_r}{\partial \theta} &= \left(-\frac{3}{2}\right) K \left(1-\beta^2\sin^2\theta\right)^{-\frac{5}{2}} \left(-2\beta^2\sin\theta\cos\theta\right) \\ &= \frac{3K\beta^2\sin 2\theta}{2\left(1-\beta^2\sin^2\theta\right)^{\frac{5}{2}}} \ . \end{aligned}$$

Hence

$$\nabla \times \vec{E} = -\frac{1}{4\pi\varepsilon_0}\ \frac{3q(1-\beta^2)}{2r^3}\frac{\beta^2\sin 2\theta}{\left(1-\beta^2\sin^2\theta\right)^{\frac{5}{2}}}\ \hat{\phi} \ .$$

To calculate $\partial\vec{B}/\partial t$, we use the theorem of partial derivatives. If

$$y = f(x,t)$$

then from the maximum change of y

$$dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial t}dt = 0 \ ,$$

we obtain

$$\frac{\partial y}{\partial t} = -\frac{\partial y}{\partial x}\frac{\partial x}{\partial t} \ .$$

Thus

$$\frac{\partial \vec{B}}{\partial t} = -v \frac{\partial \vec{B}}{\partial x} \ .$$

Alternatively, to see this "physically", remember that the field pattern moves with constant velocity $\vec{v}$. Let a stationary observer measure the change in the field $\vec{B}$ in a time interval dt. This change is the same as he would observed **at a fixed time** by moving a distance $dx = -vdt$, i.e.

$$d\vec{B} \quad \text{in} \quad dt \quad \equiv \quad d\vec{B} \quad \text{in} \quad dx = -vdt \ .$$

Hence

$$\frac{\partial \vec{B}}{\partial x} = -\frac{\partial \vec{B}}{v \partial t} \ ,$$

and then

$$\frac{\partial \vec{B}}{\partial t} = -v \frac{\partial \vec{B}}{\partial x} \ .$$

Now

$$\vec{B} = \frac{1}{4\pi\varepsilon_0} \frac{q(1-\beta^2) \ v \sin\theta}{c^2 r^2 \left(1 - \beta^2 \sin^2\theta\right)^{\frac{3}{2}}} \ \hat{\phi}$$

and

$$\frac{\partial \vec{B}}{\partial x} = \frac{\partial B_\phi}{\partial x} \ \hat{\phi} = \frac{\partial B_\phi}{\partial r} \frac{\partial r}{\partial x} \ \hat{\phi}$$

Since

$$\sin\theta = \frac{\sqrt{y^2 + z^2}}{r} = \frac{a}{r}$$

we can write B_ϕ as

$$B_\phi = \frac{Ka}{r^3 \left(1 - \frac{\beta^2 a^2}{r^2}\right)^{3/2}} = \frac{Ka}{\left(r^2 - \beta^2 a^2\right)^{3/2}}$$

where

$$K = \frac{1}{4\pi\varepsilon_0}\ \frac{q(1-\beta^2)v}{c^2} \quad \text{and} \quad a = \sqrt{y^2+z^2}$$

Next

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{\partial\left(x^2+y^2+z^2\right)^{1/2}}{\partial x} \\
&= \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}\ 2x = \frac{x}{r} = \cos\theta \\
\frac{\partial B_\phi}{\partial r} &= Ka\left(-\frac{3}{2}\right)\left(r^2-\beta^2 a^2\right)^{-5/2}\ 2r \\
&= -\frac{3Ka\ r}{\left(r^2-\beta^2 a^2\right)^{5/2}}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial B_\phi}{\partial x} &= -\frac{1}{4\pi\varepsilon_0}\ \frac{3q(1-\beta^2)\ vr^2\sin\theta\cos\theta}{c^2 r^5\left(1-\beta^2\sin^2\theta\right)^{\frac{5}{2}}} \\
&= -\frac{1}{4\pi\varepsilon_0}\ \frac{3q(1-\beta^2)\ v\sin 2\theta}{2c^2 r^3\left(1-\beta^2\sin^2\theta\right)^{\frac{5}{2}}}
\end{aligned}$$

and then

$$\begin{aligned}
\frac{\partial \vec{B}}{\partial t} &= -v\frac{\partial B_\phi}{\partial x}\ \hat{\phi} = \frac{1}{4\pi\varepsilon_0}\ \frac{3q(1-\beta^2)\ v^2\sin 2\theta}{2c^2 r^3\left(1-\beta^2\sin^2\theta\right)^{\frac{5}{2}}}\ \hat{\phi} \\
&= \frac{1}{4\pi\varepsilon_0}\ \frac{3q(1-\beta^2)\beta^2\ \sin 2\theta}{2r^3\left(1-\beta^2\sin^2\theta\right)^{\frac{5}{2}}}\ \hat{\phi}
\end{aligned}$$

Comparing with $\nabla\times\vec{E}$, we see that

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\ .$$

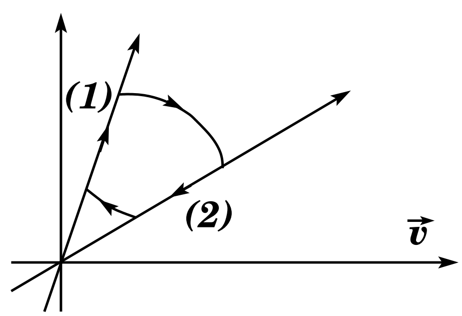
5.4.1 Electromagnetic Induction

It has been known since about 1831 when Faraday first waved a magnet near an electric circuit and played with transformers that when the magnetic flux through a circuit changes, an electromotive force (emf) $\mathcal{E}$ appears in it. Faraday gave the rule

$$\mathcal{E} = -\frac{\partial \Phi_M}{\partial t} \ .$$

The nature of this "phenomenon" is, however often **misinterpreted**.

Consider a point charge moving near a closed circuit as shown in the figure.



Because of the θ dependence of E, the electric field on side (1) is larger than that on side (2). Thus, there is a net driving force round the circuit.

Calculate the resulting electromotive force in the circuit, which is equal to the work done on a charge in the circuit

$$\mathcal{E} = W_q = \int \frac{\vec{F} \cdot \vec{dl}}{q} = \int (\vec{E} + \vec{u} \times \vec{B}) \cdot \vec{dl} \ ,$$

where u is the velocity of the charge q.

Now since $\vec{u} \times \vec{B}$ is perpendicular to both $\vec{u}$ and $\vec{B}$, it is also perpendicular to $\vec{dl}$, and then

$$\int \left(\vec{u} \times \vec{B} \right) \cdot \vec{dl} = 0 \ .$$

Hence

$$\begin{aligned}
\mathcal{E} &= \int \vec{E}\cdot d\vec{l} = \int \nabla\times\vec{E}\cdot\hat{n}\; dS = \int \nabla\times\vec{E}\cdot d\vec{S} \\
&= -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S} = -\frac{\partial\Psi_M}{\partial t}\ .
\end{aligned}$$

However, it is obvious in this example that the changing magnetic flux is not the **CAUSE** of the emf. The changing magnetic field and the electric field have a common **CAUSE** through the charge q.

We can conclude that: **Electric and magnetic fields do not produce each other** - they are both due to electric charges.

It is often thought in the textbooks, however that e.g. in a transformer the changing Ψ_M produces $\mathcal{E}$. It happens because the flux cutting rule is an extremely powerful one for calculating the integrated electric field of electric currents.

The Faraday's rule $\mathcal{E} = -\partial\Phi_M/\partial t$, which arises from $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$ should not be thought of as a casual relationship. What it means is that if a charge moving with a constant velocity produces a time varying magnetic field then that charge also produces a spatially varying electric field.

(4) Relation between spatial variation of $\vec{B}$ and temporal variation of $\vec{E}$.

Since

$$\begin{aligned}
\vec{B} &= \frac{\vec{v}\times\vec{E}}{c^2}\ , \\
\nabla\times\left(\vec{v}\times\vec{E}\right) &= \left(\vec{E}\cdot\nabla\right)\vec{v} - \left(\vec{v}\cdot\nabla\right)\vec{E} + \vec{v}\left(\nabla\cdot\vec{E}\right) - \vec{E}\left(\nabla\cdot\vec{v}\right)
\end{aligned}$$

and $\vec{v}$ is constant, we obtain

$$\nabla\cdot\vec{v} = 0\ , \qquad \left(\vec{E}\cdot\nabla\right)\vec{v} = 0\ ,$$

and then

$$\begin{aligned}
\nabla\times\vec{B} &= \frac{1}{c^2}\nabla\times\left(\vec{v}\times\vec{E}\right) \\
&= \frac{1}{c^2}\left\{-\left(\vec{v}\cdot\nabla\right)\vec{E} + \vec{v}\left(\nabla\cdot\vec{E}\right)\right\}
\end{aligned}$$

Now

$$\begin{aligned} (\vec{v}\cdot\nabla)\,\vec{E} &= \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\vec{E} \\ &= v\frac{\partial\vec{E}}{\partial x} = -\frac{\partial\vec{E}}{\partial t} \end{aligned}$$

as

$$v_x = v \qquad v_y = v_z = 0 \quad \text{and} \quad \frac{\partial}{\partial t} = -v\frac{\partial}{\partial x}$$

and then

$$\nabla\times\vec{B} = \frac{1}{c^2}\left(\vec{v}\;\nabla\cdot\vec{E} + \frac{\partial\vec{E}}{\partial t}\right) \; .$$

For points in free space

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \; .$$

In summary

Maxwell's equations for a point charge moving with uniform velocity are

$$\begin{aligned} \nabla\cdot\vec{E} &= 0 \\ \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{B} &= \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \end{aligned}$$

These equations arise from the necessity for the correct relativistic transformations between frames in uniform relative motion. If the postulates of relativity are correct and Coulomb's law gives the field of a stationary charge, these equations follow, and the force on a charge is

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \; .$$

Questions

(1) Show that magnetic and electric fields of a charge moving with a constant velocity $\vec{v}$ are related by

$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}$$

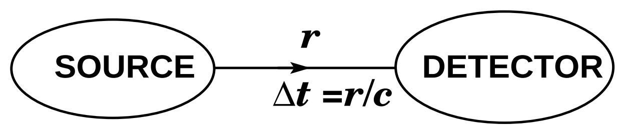
(2) Show that the magnetic field lines produced by a charge moving with a constant velocity $\vec{v}$ form concentric rings about $\vec{v}$.

(3) Show that the electric field produced by a moving charge satisfies the Gauss's law.

(4) Explain the statement: Electric and magnetic fields do not produce each other - they are both due to electric charges.

6 Energy and Momentum in the Electromagnetic Field

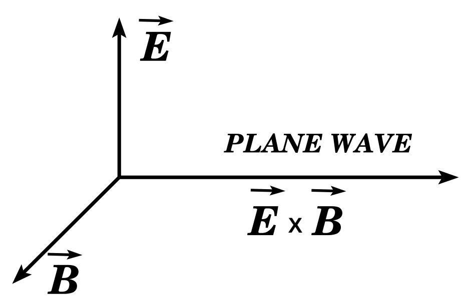
Since the electromagnetic field propagates at finite speed it must be assigned energy and momentum or else we must say that energy and momentum are not conserved in the finite time of propagation.



The field takes a time $\Delta t = r/c$ to travel from the source charge to the detector (another charge).

6.1 The Energy Conservation Theorem – Poyntings' Theorem

Energy may be transported through space by means of electromagnetic waves. We expect energy flow in the direction of propagation of the wave: $\vec{E} \times \vec{B}$.



We will show that the power flow across an element of area $d\vec{S}$ is given by $c^2\varepsilon_0\vec{E} \times \vec{B} \cdot d\vec{S}$. To derive this relation consider a volume V bounded by a closed surface S, and containing a charged material.

Consider the expression

$$\nabla \cdot (\vec{E} \times \vec{B}) \ . \tag{12}$$

If we employ the vector identity

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{B}$$

and use the Maxwell equations III and IV:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{and} \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \ ,$$

we can write Eq. (12) as

$$\nabla \cdot (\vec{E} \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \mu_0 \vec{E} \cdot \vec{J} - \varepsilon_0 \mu_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \ ,$$

or

$$\frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B}) = -\vec{E} \cdot \vec{J} - \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \frac{B^2}{\mu_0} \right) \ .$$

On lhs we put $1/\mu_0 = \varepsilon_0 c^2$ and integrate the equation over some closed surface S enclosing a volume V. Then, we obtain

$$\oint_V \varepsilon_0 c^2 \nabla \cdot (\vec{E} \times \vec{B}) \, dV = -\oint_V \vec{E} \cdot \vec{J} \, dV - \frac{\partial}{\partial t} \oint_V \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \frac{B^2}{\mu_0} \right) dV \ . \tag{13}$$

Now we apply Gauss' theorem to the lhs of the above equation, and find

$$\oint_S \varepsilon_0 c^2 \, (\vec{E} \times \vec{B}) \cdot \hat{n} \, dS \qquad \text{-- Energy flux}$$

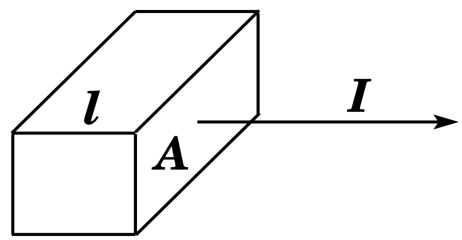
$$= -\oint_V \vec{E} \cdot \vec{J} \, dV \qquad \text{-- Rate of doing work by field on the current}$$

$$-\frac{\partial}{\partial t} \oint_V \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \frac{B^2}{\mu_0} \right) dV \qquad \text{-- Field energy.}$$

This interpretation of the terms in this equation is suggested by some special cases:

6.1.1 Ohmic Heating

From the circuit theory, we know that in a resistive medium Ohm's Law is: $V = IR$ and $R = \mathcal{R}\ell/A$, where $\mathcal{R}$ is the resistivity of the medium.



Thus, $I = V/R = (VA)/(\mathcal{R}\ell)$. Since $E = V/\ell$ ($\vec{E} = -\nabla V$), we get

$$I/A = \frac{1}{\mathcal{R}}E \ .$$

Defining the conductivity $\sigma = 1/\mathcal{R}$, Ohm's Law becomes:

$$I/A = J = \sigma E$$

The rate of conversion of electromagnetic field energy in to heat is then:

$$\mathcal{H} = IV = \frac{V^2}{R} = \frac{E^2\,\ell^2}{\mathcal{R}\frac{\ell}{A}} = \sigma E^2 \ell A = \sigma E^2 \mathcal{V} \ ,$$

where $\mathcal{V} = \ell A$ is the volume of the resistive medium.

Hence

$$\frac{\mathcal{H}}{\mathcal{V}} = \sigma E^2 = E\,J = \vec{E}\cdot\vec{J} \ .$$

Thus, $\vec{E}\cdot\vec{J}$ is the rate of heating per unit volume in this case.

6.1.2 Electrostatic Field Energy Density

It is well known that the work required to charge a capacitor of a capacitance C to a voltage V is $W = \frac{1}{2}CV^2$. According to the field theory of electromagnetism this work done corresponds to conversion from some other form of energy into electrostatic field energy

$$\mathcal{E} = W = \frac{1}{2}CV^2 \text{ and } C = \frac{\varepsilon_0 A}{d} \ ,$$

and the electric field in the capacitor is given by $V = Ed$. Thus:

$$\mathcal{E} = \frac{1}{2}\frac{\varepsilon_0 A}{d}E^2 d^2 = \frac{1}{2}\varepsilon_0 E^2 \mathcal{V} \ ,$$

where $\mathcal{V} = Ad$ is the volume of the capacitor.

Hence

$$\frac{\mathcal{E}}{\mathcal{V}} = \frac{1}{2}\varepsilon_0 E^2 \ .$$

6.1.3 Magnetostatic Field Energy Density

It is well known that the work required to energize an inductor of L [henrys] to a current I is $\mathcal{E} = \frac{1}{2}LI^2$ and that the magnetic field within a long solenoid of self-inductance $L = \mu_0 n^2 A\ell$ is $B = \mu_0 nI$. According to the field theory of electromagnetism this work done corresponds to transformation of some other form of energy to magnetic field energy.

Thus, for a solenoid of length ℓ and area of cross section A we have energy:

$$\begin{aligned} \mathcal{E} &= W = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 A\ell I^2 = \frac{1}{2}(\mu_0 nI)^2 A\ell \frac{1}{\mu_0} \\ &= \frac{1}{2}\frac{B^2}{\mu_0}\mathcal{V} \end{aligned}$$

or

$$\frac{\mathcal{E}}{\mathcal{V}} = \frac{1}{2}\frac{B^2}{\mu_0} \ .$$

All our observations are in accord with supposing that the expressions for energy density found in electrostatic and magnetostatic situations are applicable in general.

The expression

$$\vec{N} = \varepsilon_0 c^2 (\vec{E} \times \vec{B})$$

is referred to as the **Poynting vector**.

It represents the energy flux in the electromagnetic field, i.e. the energy flow per unit area (measured normal to the flow) per unit time.

The units of N are thus watts/square meter.

The energy flow equation can be converted into the form of a differential continuity equation or **energy conservation law**. From Eq. (13), we have

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{N} = -\vec{J} \cdot \vec{E} \ ,$$

where

$$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$$

is the energy density of the EM field.

The physical meaning of the differential continuity equation is that the time rate of change of electromagnetic energy within a certain volume, plus the energy flowing out through the boundary surface of the volume is equal to the negative of the total work done by the fields on the source inside the volume. Thus. $\vec{J} \cdot \vec{E}$ is a conversion of electromagnetic energy into heat energy.

Example

An Earth satellite transmits a power $P = 5$ W. What are the typical $\vec{E}$ and $\vec{B}$ in the radiation field at a distance of 1000 km from the transmitter?

The mean flux (Poynting vector magnitude) at a distance r meters is:

$$N = \frac{P}{4\pi r^2} = \frac{5}{4\pi 10^{12}} = 4 \times 10^{-13} \ [\mathrm{Wm}^{-2}] \ .$$

Since

$$\vec{E} \perp \vec{B} \quad \text{and} \quad B = \frac{E}{c} \ , \qquad N = \varepsilon_0 c E^2$$

$$4 \times 10^{-13} = 8.85 \times 10^{-12} \times (3 \times 10^8)\ E^2$$

Hence

$$E \simeq 1.2 \times 10^{-4}\ \text{Vm}^{-1}\ .$$

Thus, an antenna consisting of a straight piece of wire 10 cm long would have a voltage $V \simeq E\ell \simeq 1.23 \times 10^{-4} \times 0.1$ or $\simeq 12\ \mu$V induced in it if it were placed parallel to $\vec{E}$.

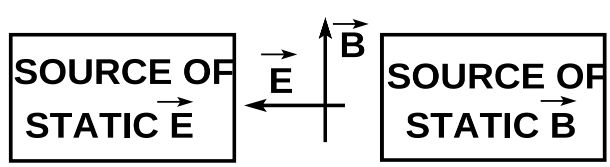
Then

$$B = E/c = 1.2 \times 10^{-4}/(3 \times 10^8) \simeq 4 \times 10^{-13}\ [\text{T}]\ .$$

It is a very weak field. For, example magnetic field produced by a hair dryer is of order 10^{-3} T. Magnetic field produced by a large laboratory electromagnet is about 5 T.

6.1.4 No Fluxes from Static Fields

Consider a source of electrostatic field $\vec{E}$ and magnetostatic field $\vec{B}$. If it is arranged so that $\vec{E} \perp \vec{B}$ it is sometimes asked whether we should expect to see an energy flux of $\varepsilon_0 c^2 \vec{E} \times \vec{B}$.



The question is easily answered in the negative by noting that the electrostatic field is $\vec{E}, \vec{0}$ and the magnetostatic field is $\vec{0}, \vec{B}$. Each of these has

$\vec{E} \times \vec{B} = 0$. The $\vec{E}, \vec{B}$ obtained by mentally combining the electrostatic $\vec{E}$ from one source and the magnetostatic $\vec{B}$ *does not constitute an electromagnetic field.*

Electromagnetic fields do not interact with each other

The fields interfere (as the term is used in physical optics). Beyond the interference region the two plane waves are the same as they would have been had there been no interference. If there are *charges* in the region of interference then this is a different matter. The *charges* may absorb and re-emit radiation and so permanently modify the radiation fields.

6.2 Phase Relationships in Harmonic Waves

Only the in-phase components of $\vec{E}$ and $\vec{B}$ contribute to net energy flow averaged over a whole cycle of the radiation.
If $\vec{E} = E_0 \cos(\omega t)\hat{i}$ and $\vec{B} = B_0 \cos(\omega t)\hat{j}$ then:

$$\vec{N} = \varepsilon_0 c^2 \vec{E} \times \vec{B} = \varepsilon_0 c^2 E_0 B_0 \cos^2(\omega t)\ \hat{k}\ .$$

Since, $\overline{\cos^2(\omega t)} = \frac{1}{2}$, we obtain

$$\overline{N} = \frac{1}{2}\varepsilon_0 c^2 E_0 B_0 = \varepsilon_0 c^2 E_{rms} B_{rms}\ ,$$

where $E_{rms} = E_0/\sqrt{2}$ and $B_{rms} = B_0/\sqrt{2}$.

If however $\vec{E} = E_0\ \cos(\omega t)\hat{i}$ and $\vec{B} = B_0\ \sin(\omega t)\hat{j}$ then:

$$\vec{N} = \varepsilon_0 c^2 \vec{E} \times \vec{B} = \varepsilon_0 c^2 E_0 B_0 \cos(\omega t)\ \sin(\omega t)\ \hat{k}\ .$$

Since, $\overline{\cos(\omega t)\ \sin(\omega t)} = 0$, we have $\overline{N} = 0$.

6.3 Momentum Flux

To obtain an expression for the momentum carried by the electromagnetic field, we may employ the relativistic energy-momentum relationship

$$\mathcal{E}^2 = p^2c^2 + m_0^2c^4\ .$$

Since for electromagnetic radiation $m_0 = 0$, we obtain

$$p = \frac{\mathcal{E}}{c} \ .$$

Thus the momentum flux of the electromagnetic field is:

$$\mathcal{M} = \frac{\varepsilon_0 c^2 \vec{E} \times \vec{B}}{c} = \varepsilon_0 c \vec{E} \times \vec{B} \ .$$

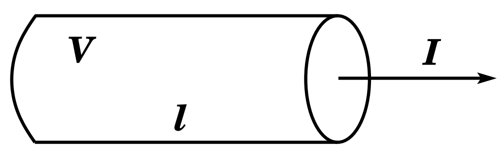
6.4 Electromagnetic Energy Flow

We now consider two simple examples illustrating applications of the Poynting vector to circuit theory to show how the field theory provides an alternative way of viewing some circuit problems.

6.4.1 Energy Flow into a Resistive Wire

Consider a wire (resistor) of length ℓ, carrying a current I. Let V is a potential difference applied along a wire.

Circuit theory calculation

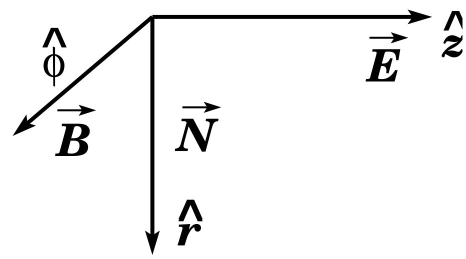


According to circuit theory the power dissipated in the wire is $P = VI$, where I is the current flowing through the wire. Thus, according to the circuit theory, energy flows along the wire.

Field theory calculation

Lets look at the same problem from the point of view of the field theory.

According to electromagnetic field theory the flow of energy should be described by the Poynting vector $\vec{N} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$.



Calculate the electric and magnetic field produced by the current. The electric field propagates along the wire, and is given by

$$\vec{E} = -\nabla V = \frac{V}{\ell}\ \hat{z}\ .$$

From the Amperes line integral theorem, we find the magnetic field around the wire

$$\vec{B} = \frac{\mu_0 I}{2\pi a}\ \hat{\phi}\ .$$

Hence

$$\vec{N} = \varepsilon_0 c^2 \vec{E} \times \vec{B} = \varepsilon_0 c^2 \frac{V}{\ell} \frac{\mu_0 I}{2\pi a}\ \hat{r}$$

since $\hat{z} \times \hat{\phi} = \hat{r}$.

Since $\varepsilon_0 \mu_0 c^2 = 1$, we finally obtain

$$\vec{N} = \frac{VI}{2\pi \ell a}\ \hat{r}\ .$$

Thus, the field theory predicts that energy flows into the wire from the air *not* along the wire. The energy is in the fields, the wire provides boundary conditions and guides the fields.

If the energy is in the fields, it means that the electromagnetic energy goes out of a battery into the air, and then goes into the wire from the air. This is exactly the case we will show in the following example.

The total rate at which field energy flows in to the surface is given by

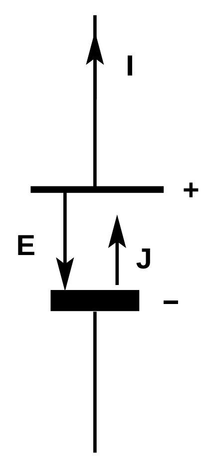
$$\oint_S \vec{N} \cdot d\vec{S} = \oint \vec{N} \cdot \hat{r}\, dS = \oint N\, dS$$
$$= N \oint dS_{side} = \frac{VI}{2\pi \ell a} 2\pi \ell a = VI \ ,$$

($\vec{N} \perp d\vec{S}$ on the ends of the cylinder), which is in agreement with the result of the circuit theory. This result demonstrates quantitatively that the power which heats the resistor enters through the sides not through the wires.

6.4.2 Energy Flow out of Battery

In the above example, we have shown that the energy enters the resistor from the air. Then, a question arises: If the energy enters the resistor from the air, how does the energy get out to the air from a source of energy (battery)?

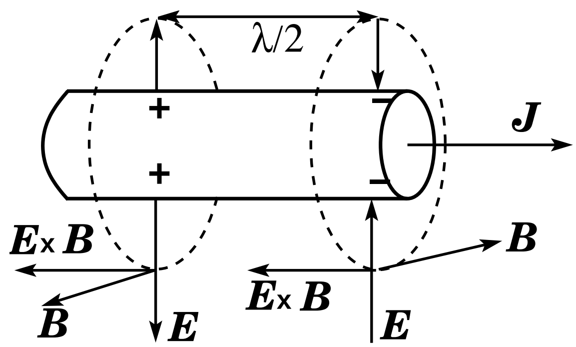
Consider a battery, which provides energy to the resistor.



Inside the battery $\vec{J}$ and $\vec{E}$ are in opposite directions. The magnetic field circulates around the battery, so we see that the Poynting vector $\vec{N}$ points out into the air, not along the wire.

6.4.3 Propagation of Energy along a Wire

Above we showed that field energy flows in to a wire so that it can be dissipated as heat. It is well known that electromagnetic energy can be transmitted along a wire with very little loss. This is a different situation. If we



consider the case of a perfect conductor we find that there will be a current wave along the wire with surface charges induced producing a *radial* electric field. Then $\vec{E} \times \vec{B}$ is parallel to the wire and the field description of energy transmission is that it is transmitted in the space around the wire. In the space around the wire $\vec{E}$ and $\vec{B}$ are in phase and $\vec{N}$ is always in the same direction. Within a perfect conductor we will show that $\vec{E}$ and $\vec{B}$ are $\pi/2$ out of phase so the mean N averaged over a cycle is 0. So there is no net energy transmission within the perfectly conducting wire.

Exercise in class: *Energizing of a capacitor*

Consider the energizing of a plane parallel capacitor with circular plates. Show that circuit and field calculations agree as to the rate of energizing of the capacitor, i.e. $P_c = P_f$ where:

$P_c = VI$ = rate of doing work (by current I and voltage V between the plates) in charging the capacitor according to circuit theory.

$P_f = \int\limits_S \varepsilon_0 c^2 \vec{E} \times \vec{B} \cdot d\vec{S}$ = rate of energy flow into the surface of the capacitor according to field theory.

From which direction does the energy flow in to the capacitor according to field theory?

We will keep the calculation simple assuming the plates of the capacitor to be uniformly charged. Under what conditions is this assumption likely to be true?

Questions:

(1) Using the Maxwell's equations derive the continuity equation for the Poynting vector.

(2) Show, using the field theory calculation, that the power dissipated along a resistive wire is $P = VI$, the same predicted by the circuit theory.

7 General Solution of the Maxwell's Equations

The electromagnetic fields of charges in uniform motion are effectively bound to the charges. The fields of accelerated charges on the other hand can propagate as electromagnetic (EM) waves at the speed c and can have a life of their own (until absorbed by some other charges).

Consider a simple model showing why only accelerating charges can emit electromagnetic waves.

Charges in uniform motion produce a constant magnetic field, i.e. when $v = \text{const}$ then $B = \text{const}$. EM wave is produced when $B \neq \text{const}$. From the Faraday induction law

$$\mathcal{E}(t) = -\frac{d\Psi}{dt} \neq \text{const} \ ,$$

where $\Psi = \vec{B}(t) \cdot \vec{S}$ is the magnetic flux.

$\vec{B} \neq \text{const}$ when $\vec{v} \neq \text{const}$ (acceleration).

7.1 Difficulty of the Direct Solution of Maxwell's Equations with Time Varying Sources

The existence of an electromagnetic field implies the presence of currents and charges. If the currents and charges are known, we can find the fields solving the Maxwell's equations.

Consider the Maxwell's equations

$$\begin{aligned}
\text{I.} \quad & \nabla \cdot \vec{E} = \rho/\varepsilon_0 \\
\text{II.} \quad & \nabla \cdot \vec{B} = 0 \\
\text{III.} \quad & \nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \\
\text{IV.} \quad & \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} \ .
\end{aligned}$$

In general case, $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ depend on $(\vec{r},t)$, the charge and current densities also depend on $(\vec{r},t)$.

First, we will try to separate the Maxwell's equations into an equation for $\vec{E}$ alone or $\vec{B}$ alone.

Assuming in the usual way that space and time operators commute, we act with $\frac{1}{c^2}\frac{\partial}{\partial t}$ on III and $\nabla\times$ on IV, and obtain

$$\begin{aligned} \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\vec{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} &= 0 \\ \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\vec{E} - \nabla\times(\nabla\times\vec{B}) &= -\mu_0\nabla\times\vec{J} \,. \end{aligned}$$

Eliminating $\vec{E}$ by subtraction

$$\nabla\times(\nabla\times\vec{B}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = \mu_0\nabla\times\vec{J}$$

and using the vector identity for double $\times$ product, and II

$$\nabla\times(\nabla\times\vec{B}) = -\nabla^2\vec{B} + \nabla(\nabla\cdot\vec{B}) = -\nabla^2\vec{B} \ ,$$

we obtain

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = -\mu_0\nabla\times\vec{J} \,. \tag{14}$$

Similarly, elimination of $\vec{B}$ gives

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{1}{\varepsilon_0}\nabla\rho + \mu_0\frac{\partial}{\partial t}\vec{J} \,. \tag{15}$$

Equations (14) and (15) are in the form of coupled wave equations known as inhomogeneous Helmholtz equations. We see that the current density $\vec{J}$ enters into these equations in a relatively complicated way, and for this reason these equations and are not readily soluble in general.

In the absence of currents and charges, $\vec{J}=0$, $\rho=0$, and then the above equations describe a free EM field, and can be solved separately.

Solution of the wave equations in the absence of currents and charges is given in a form of plane transverse waves

$$\vec{U} = \sum_k \vec{U}_k \ \mathrm{e}^{-i\left(\omega_k t - \vec{k}\cdot\vec{r}\right)} \ , \tag{16}$$

where $\vec{U} \equiv (\vec{E}, \vec{B})$, $\left|\vec{k}\right| = \omega_k/c$, and $\vec{U}_k$ is the amplitude of the $\vec{E}$ or $\vec{B}$ wave propagating in the $\vec{k}$ direction.

The general solution of the wave equations, in the presence of space and time varying currents and charges, is more readily attained via the electromagnetic potentials.

7.2 Scalar and Vector Potentials

Generally, we do not find fields $\vec{E}$ and $\vec{B}$ directly by integration of Eqs. (14) and (15). We rather first compute scalar and vector potentials from which the fields may be found.

We will illustrate here the advantage of working with the scalar and vector potentials.

Scalar potential is a quantity from which a field can be derived by a process of differentiation, e.g. in electrostatics

$$\vec{E} = -\nabla\Phi \ ,$$

where Φ is the electrostatic potential.

Introduce the vector potential $\vec{A}$ defined such that the Maxwell's equation II remains unchanged. Since $\nabla \cdot \vec{B} = 0$, we can always write

$$\vec{B} = \nabla \times \vec{A} \ .$$

Substitute this relation to the Maxwell's equation III, and since $\nabla \times \nabla\Phi = 0$, where Φ is an arbitrary scalar function, we find

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\Phi \ . \tag{17}$$

In the static limit of $\partial\vec{A}/\partial t = 0$, the scalar function Φ reduces to the familiar electrostatic potential.

The above equation shows that the electric field depends on the specific choice of the potentials.

We can define new potentials without changing $\vec{E}$ and $\vec{B}$

$$\begin{aligned} \vec{A}' &= \vec{A} + \nabla\Lambda \ , \\ \Phi' &= \Phi - \frac{\partial}{\partial t}\Lambda \ . \end{aligned} \tag{18}$$

Proof:

$$\begin{aligned} \vec{E}' &= -\frac{\partial}{\partial t}\vec{A} - \nabla\Phi' \\ &= -\frac{\partial}{\partial t}\left(\vec{A}' + \nabla\Lambda\right) - \nabla\left(\Phi - \frac{\partial}{\partial t}\Lambda\right) \\ &= -\frac{\partial}{\partial t}\vec{A} - \nabla\Phi = \vec{E} \ . \end{aligned}$$

$$\begin{aligned} \vec{B}' &= \nabla \times \vec{A}' = \nabla \times \vec{A} + \nabla \times (\nabla\Lambda) \\ &= \nabla \times \vec{A} = \vec{B} \ . \end{aligned}$$

as required.

The transformation (18) is called a *gauge transformation*, and the invariance of the fields under such transformations is called *gauge invariance*.

The definition $\vec{B} = \nabla \times \vec{A}$ does not completely define $\vec{A}$ despite the fact that $\vec{B}$ is completely defined. The vector potential $\vec{A}$ is arbitrary to the extent that the gradient of some scalar function can be added. Thus, infinite set of possible potentials Φ corresponds to an infinite set of possible vector potentials. However, the Maxwell's equations should be independent of the specific choice of potentials.

7.2.1 Lorentz Gauge

Recall the **Helmholtz Theorem** which says that any vector field can be written as a sum two terms

$$\begin{aligned}\vec{F} &= -\frac{1}{4\pi}\nabla \int_V \frac{\nabla \cdot \vec{F}}{r} dV + \frac{1}{4\pi}\nabla \times \int_V \frac{\nabla \times \vec{F}}{r} dV \\ &= \vec{F}_l + \vec{F}_t \ ,\end{aligned}$$

where $\vec{F}_l$ is called the *longitudinal* part of the field and has $\nabla \times \vec{F}_l = 0$, while $\vec{F}_t$ is called the *transverse* part and has $\nabla \cdot \vec{F}_t = 0$.

We see that $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ together determine $\vec{F}$ but neither do alone. Thus, if we define $\nabla \cdot \vec{A}$, we complete the definition of $\vec{A}$. This is called "choosing the gauge of the potential". The above is an excellent illustration of the power of the Helmholtz theorem. This theorem enables us to recognize basic common properties of vector fields independent of their individual physical properties.

Derive differential equations for $\vec{A}$ and Φ.

From the Maxwell's equation I and (17), we find

$$\nabla \cdot \vec{E} = -\frac{\partial}{\partial t}\nabla \cdot \vec{A} - \nabla^2 \Phi \ .$$

The electric field $\vec{E}$ will satisfy the Maxwell's equation I when

$$-\frac{\partial}{\partial t}\nabla \cdot \vec{A} - \nabla^2 \Phi = \rho/\varepsilon_0 \ .$$

From the Maxwell's equation IV and $\vec{B} = \nabla \times \vec{A}$, we have

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\vec{A} - \nabla \Phi\right) \ ,$$

which can be written as

$$-\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A}) = \mu_0 \vec{J} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \frac{1}{c^2}\nabla\frac{\partial}{\partial t}\Phi \ .$$

Hence

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0 \vec{J} + \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi\right) \ .$$

We have already shown that

$$-\frac{\partial}{\partial t}(\nabla\cdot\vec{A}) - \nabla^2\Phi = \rho/\varepsilon_0 \ . \tag{19}$$

The freedom of choosing $\vec{A}$ and Φ means that we can choose a set of potentials to satisfy the condition

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0 \ .$$

This is called the *Lorentz gauge* and defines $\nabla\cdot\vec{A}$. This equation is sometimes called the Lorentz equation.

Under the Lorentz gauge, the Maxwell equations reduce to two uncoupled wave equations

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{J} \ ,$$

and

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\varepsilon_0 \ .$$

7.2.2 Coulomb Gauge

Another useful gauge of the potentials is the *Coulomb gauge* or *transverse gauge*

$$\nabla\cdot\vec{A} = 0 \ .$$

The origin of the name "Coulomb gauge" is in equation (19) that under the condition $\nabla\cdot\vec{A} = 0$ reduces to the Poisson equation

$$\nabla^2\Phi = -\rho/\varepsilon_0 \ ,$$

that determines the Coulomb potential due to the charge density ρ.

Before we proceed further, we will show that the solution of the Poisson equation is of the form

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r} dV \ .$$

It can be proved in the following way.

From the Coulomb's law

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho \hat{r}}{r^2} dV \ .$$

and using the relation

$$-\nabla \frac{1}{r} = \frac{\hat{r}}{r^2} \ ,$$

we can write the electric field as

$$\vec{E} = -\frac{1}{4\pi\varepsilon_0} \nabla \int \frac{\rho}{r} dV = -\nabla \Phi \ ,$$

where

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r} dV \ . \tag{20}$$

Since the electric field satisfies the Maxwell's equation I, we find

$$\nabla \cdot \vec{E} = -\nabla^2 \Phi = \frac{\rho}{\varepsilon_0} \ ,$$

as required.

Now, we can find the wave equation for $\vec{A}$ under the Coulomb gauge,

Under the Coulomb gauge, the vector potential satisfies the inhomogeneous wave equation

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} = -\mu_0 \vec{J} + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} \ .$$

The term involving the scalar potential is called "longitudinal" as it has vanishing $\nabla\times$. This suggests that it may cancel the longitudinal part of the current density $\vec{J}$. According to the Helmholtz Theorem, the current density can be written as

$$\begin{aligned}\vec{J} &= -\frac{1}{4\pi}\nabla\int_V \frac{\nabla\cdot\vec{J}}{r}dV + \frac{1}{4\pi}\nabla\times\int_V \frac{\nabla\times\vec{J}}{r}dV \\ &= \vec{J}_l + \vec{J}_t \ .\end{aligned}$$

Using the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

and the solution of the Poisson equation, we find the longitudinal part of the current density

$$\begin{aligned}\vec{J}_l &= -\frac{1}{4\pi}\nabla\int_V \frac{\nabla\cdot\vec{J}}{r}dV = -\frac{1}{4\pi}\nabla\int_V \frac{-\frac{\partial\rho}{\partial t}}{r}dV \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\nabla\int_V \frac{\rho}{r}dV = \varepsilon_0\frac{\partial}{\partial t}\nabla\Phi \ .\end{aligned}$$

Then

$$\mu_0\vec{J}_l = \mu_0\varepsilon_0\frac{\partial}{\partial t}\nabla\Phi = \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} \ .$$

Hence, the inhomogeneous term in the wave equation for $\vec{A}$ can be expressed entirely in terms of the transverse current.

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\mu_0\vec{J}_t \ .$$

This is the origin of the name "**transverse gauge**".

7.3 Solution of the Inhomogeneous Wave Equations

We have shown that the Maxwell equations can be reduced to two independent wave equations for the potentials $\vec{A}$ and Φ. In fact, we have four scalar equations for (A_x, A_y, A_z, Φ). Each of these equations has the same form. We will illustrate the solution on one of the equations, Φ:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\varepsilon_0 \ .$$

A general solution may be found by considering two limiting cases:

(a) Electrostatic limit $\partial/\partial t \equiv 0$

In this limit the wave equation for Φ reduces to the Poisson equation whose the solution is

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{r'}\, dV' \ .$$

(b) Source free limit $\rho = 0$

In this case, the wave equation reduces to the homogeneous equation

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \ .$$

This equation has a spherically symmetric solution of the form

$$\Phi(r,t) = \frac{f(t - r/c)}{r} \ ,$$

where $f(t - r/c)$ is an arbitrary function of the retarded time $t - r/c$. The retardation r/c is equal to the time needed for the electromagnetic wave to pass the distance from the source to a given point in space.

Proof:

If there are no boundary surfaces, the potential can depend only on r, and must in fact be spherically symmetric. Thus, in spherical coordinates only the radial part of the Laplacian will contribute to the wave equation

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\Phi}{\partial r}\right] \ .$$

Hence, we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial t_r}\frac{\partial t_r}{\partial r} = -\frac{1}{c}f' \ ,$$

where $t_r = t - r/c$ and $f' = \partial f/\partial t_r$.

Thus

$$\begin{aligned}
\nabla^2\Phi &= -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\frac{f'}{cr} + \frac{f}{r^2}\right)\right] \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{rf'}{c} + f\right) \\
&= -\frac{1}{r^2}\left[\frac{f'}{c} + \frac{r}{c}\left(-\frac{1}{c}\right)f'' + \left(-\frac{1}{c}\right)f'\right] \\
&= \frac{1}{rc^2}f'' \ ,
\end{aligned}$$

where $f'' = \partial^2 f/\partial t_r^2$.

Since $\partial/\partial t = \partial/\partial t_r$, we have

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{1}{c^2 r}f'' \ ,$$

as required.

We can construct a general solution of the wave equation by noting that it must represent a spherical wave outside the source and reduce to the appropriate static limit. This solution is

$$\Phi(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(t - r/c)}{r}\, dV \ ,$$

where r is the distance coordinate from the source (from the charge ρdV) at the time when the potential wave left it. This exhibits the causal behavior associated with the wave disturbance. The argument of ρ shows that an effect observed at the point r at time t is caused by the action of the source a distant r away at an earlier or retarded time $t' = t - r/c$. The time r/c is the time of propagation of the disturbance from the source to the point r. Thus, the Maxwell's equations satisfy the causality principle.

7.4 Alternative Solution: Green Functions Method

The wave equations all have the basic structure

$$\nabla^2\Phi(r,t) - \frac{1}{c^2}\frac{\partial^2\Phi(r,t)}{\partial t^2} = -4\pi f(r,t) \ ,$$

where $f(r,t)$ is a known (source distribution) function.

To solve this equation, we will introduce the Green function of the equation and solve it as an inhomogeneous Helmholtz equation.

Suppose that $\Phi(r,t)$ and $f(r,t)$ have the Fourier integrals

$$\begin{aligned}\Phi(r,\omega) &= \int_{-\infty}^{\infty}\Phi(r,t)\mathrm{e}^{i\omega t}dt \ ,\\ f(r,\omega) &= \int_{-\infty}^{\infty} f(r,t)\mathrm{e}^{i\omega t}dt \ .\end{aligned}$$

When we insert it into the wave equation, we find that the Fourier transform $\Phi(r,\omega)$ satisfies the inhomogeneous Helmholtz wave equation

$$\left(\nabla^2 + k^2\right)\Phi(r,\omega) = -4\pi f(r,\omega) \ ,$$

where $k=\omega/c$ is the wave number.

The advantage of working in Fourier components is to remove the derivative over time.

Green function

For a unit point source the potential satisfies the Poisson equation

$$\nabla^2\frac{1}{r} = -4\pi\delta(r) \ .$$

The function $1/r = G(r)$ is called a Green function of the above differential equation.

In analogy, we can define the Green function of the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(r,t-t_0) = -4\pi\delta(r)\delta(t-t_0) \ .$$

The Fourier transform gives

$$\left(\nabla^2 + k^2\right) G_k = -4\pi\delta(r)\mathrm{e}^{i\omega t_0} \ .$$

where G_k is the Fourier transform of the Green function $G(r, t - t_0)$, which we are trying to find.

If there are no boundary surfaces, the Green function depends only on r, and then the Laplacian operator in spherical coordinates depends only on r giving

$$\frac{1}{r}\frac{d^2}{dr^2}\left(rG_k\mathrm{e}^{-i\omega t_0}\right) + k^2 G_k \mathrm{e}^{-i\omega t_0} = -4\pi\delta(r) \ .$$

Everywhere except $r = 0$, the function $rG_k\mathrm{e}^{-i\omega t_0}$ satisfies the homogeneous equation

$$\frac{d^2}{dr^2}\left(rG_k\mathrm{e}^{-i\omega t_0}\right) + k^2(rG_k\mathrm{e}^{-i\omega t_0}) = 0 \ ,$$

whose the solution is

$$rG_k\mathrm{e}^{-i\omega t_0} = A\mathrm{e}^{ikr} + B\mathrm{e}^{-ikr} \ .$$

Thus, the general solution for the Green function is

$$G_k = \frac{\mathrm{e}^{\pm ikr}\mathrm{e}^{i\omega t_0}}{r} \ .$$

Using the inverse Fourier transform, we find

$$G(r,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\pm ikr}}{r}\mathrm{e}^{-i\omega\tau}d\omega \ ,$$

where $\tau = t - t_0$.

The integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-i\omega(\tau\mp r/c)}d\omega$$

is the delta function $\delta(\tau \mp r/c)$. Thus

$$G(r,\tau) = \frac{1}{r}\delta(\tau \mp r/c) \ .$$

The Green function is a casual response function, and has the same property as the scalar potential of a point source.

In summary

The general (retarded) solutions of the Maxwell's equations are

$$\vec{B} = \nabla \times \vec{A} \ , \qquad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \ , \tag{21}$$

with

$$\Phi(r,t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(t - r/c)}{r} \, dV \ , \tag{22}$$

$$\vec{A}(r,t) = \frac{1}{4\pi\varepsilon_0 c^2} \int \frac{\vec{J}(t - r/c)}{r} \, dV \ . \tag{23}$$

In practice, we calculate the scalar and vector potentials from Eqs. (22) and (23), and then find the electric and magnetic fields from Eqs. (21).

Questions

(1) Find a differential equation for the vector potential $\vec{A}$ in the Lorentz gauge.

(2) Explain, why the Coulomb gauge is often called "Transverse gauge".

(3) Prove that the homogeneous wave equation

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

has the solution of the form

$$\Phi(r,t) = \frac{f(t - r/c)}{r}$$

where $f(t - r/c)$ is an arbitrary function of the retarded time $t - r/c$.

8 Solution of Laplace's Equation and Boundary Value Problem

In the previous lecture, we have shown that electric and magnetic fields are readily calculated with a help of the scalar and vector potentials. In this lecture, we will illustrate applications of the scalar potential to physical problems involving bounded fields.

There is a class of problems in electromagnetism in which a field can be derived without involvement of the complete set of the Maxwell's equations as the gradient of a scalar potential which satisfies Laplace's equation

$$\nabla^2\Phi = 0 \ .$$

The condition for this to happen is that

$$\nabla \cdot \vec{F} = 0 \qquad \text{and} \qquad \nabla \times \vec{F} = 0 \ . \tag{24}$$

Since, $\nabla \times \vec{F} = 0$ is the condition for $\vec{F} = \nabla\Phi$, where Φ is an arbitrary scalar function, then $\nabla \cdot \vec{F} = 0$ means that

$$\nabla \cdot (\nabla\Phi) = \nabla^2\Phi = 0 \ .$$

Thus, the scalar potential Φ contains all the necessary information to completely specify the field of the properties (24).

Examples

1. Electrostatic problems involving Laplace's equation

Since in general $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ and $\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$ we see that the requirement for Laplace's equation to be relevant is that $\rho = 0$ and $\partial/\partial t = 0$, i.e. a source-free region and static conditions. Of course there must be a source of charges somewhere or there would be no field anywhere. The typical situation where solution of Laplace's equation is relevant in electrostatic is where we have source-free non-conducting regions between statically charged conductors.

2. Magnetostatic problems involving Laplace's equation

Since in general $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}$, we see that the requirement for Laplace's equation to be relevant is that $\vec{J} = 0$ and $\partial/\partial t = 0$, i.e. a source-free region and static conditions. Again, there must be currents somewhere or there would be no fields anywhere. The typical situation is to be calculating the magnetic field in the non-conducting region between constant currents.

8.1 Uniqueness of the Solution of Laplace's Equation

As we will see below, general solutions of Laplace's equation are in terms of some constants which are usually found from boundary conditions for a given problem. A question arises: **What boundary conditions are appropriate for the Laplace equation to ensure that a unique and well-behaved (physically reasonable) solution will exist inside the bounded region?** Our experience leads to believe that specification of the potential on a closed surface defines a unique potential problem. This is called Dirichlet theorem or Dirichlet boundary conditions.

8.1.1 Dirichlet theorem

Consider a volume V completely bounded by a closed surface S. Within S there is a potential Φ satisfying $\nabla^2\Phi = 0$. The Dirichlet theorem says that the value of Φ is uniquely determined by the variation of Φ over S.

Proof:

Suppose, to the contrary, that there exist two solutions Φ_1 and Φ_2 satisfying the same boundary condition, i.e. $\nabla^2\Phi_1 = 0$ and $\nabla^2\Phi_2 = 0$ within S, but $\Phi_1 = \Phi_2$ on S.

Let $U = \Phi_1 - \Phi_2$ is the difference between the solutions. Since Φ_1 and Φ_2 are known to be solutions of the Laplace equation, then from the linearity of the ∇^2 operator $\nabla^2 U = 0$, i.e. U is also a solution of the Laplace equation.

We will prove that $U = 0$ inside the volume. To show this, we introduce a vector $\vec{F} = U\nabla U$. Then using the property

$$\nabla \cdot \vec{F} = \nabla \cdot U\nabla U = U\nabla \cdot (\nabla U) + \nabla U \cdot \nabla U$$

and the Gauss' Divergence Theorem, we get

$$\int_V \nabla \cdot \vec{F} dV = \int_V U\nabla \cdot \nabla U dV + \int_V \nabla U \cdot \nabla U dV$$

$$= \int_V U\nabla^2 U dV + \int_V (\nabla U)^2 \, dV$$

$$= \int_S U\nabla U \cdot d\vec{S} \ .$$

Now RHS$= 0$ because $U = 0$ over S.
Also, the integral

$$\int_V U\nabla^2 U dV = 0 \ ,$$

because Φ_1 and Φ_2 both satisfy the Laplace equation throughout V.

Hence

$$\int_V (\nabla U)^2 \, dV = 0 \ .$$

Since the integral from a positive function is always positive, ∇U must be zero for the integral to be zero. Thus $\nabla U = 0$ and consequently, inside V, U is constant. Since $U = 0$ on S, so that inside V, $\Phi_1 = \Phi_2$.

8.2 Solutions of Laplace's Equation

There are different methods of solving the Laplace equation

- Method of Images
- Green functions method
- Variational method
- Method of lattices
- Numerical Monte-Carlo simulations method
- Method of separation of variables
- Solution in spherical coordinates

We will illustrate last two methods which can be applied to a large class of problems in electromagnetism. The other methods can be applied to specific problems. For these methods it is necessary that the boundaries over which the potential is specified coincide with the constant bounding surfaces.

8.2.1 Method of Separation of Variables

In cartesian coordinates the Laplace equation for the scalar potential can be written as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \ . \tag{25}$$

Since x, y, z are independent variables, the solution of the Laplace equation is of the form

$$\Phi(x, y, z) = X(x)Y(y)Z(z) \ .$$

Substituting this into the Laplace equation and dividing both sides by XYZ, we obtain

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0 \ .$$

This equation can be separated into three independent equations. To show this, we write this equation as

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} \ .$$

Both sides of the above equation depend on different (independent) variables, thus are equal to a constant, say $-\alpha^2$:

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= -\alpha^2 \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} &= -\alpha^2 \ . \end{aligned}$$

The second equation can be written as

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha^2 - \frac{1}{Z}\frac{d^2Z}{dz^2} \ .$$

Again, both sides depend on different variables, thus are equal to a constant, say $-\beta^2$:

$$\begin{aligned} \frac{1}{Y}\frac{d^2Y}{dy^2} &= -\beta^2 \\ \alpha^2 - \frac{1}{Z}\frac{d^2Z}{dz^2} &= -\beta^2 \ . \end{aligned}$$

Hence, after the separation of the variables, we get three independent ordinary differential equations

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} + \alpha^2 &= 0 \ , \\ \frac{1}{Y}\frac{d^2Y}{dy^2} + \beta^2 &= 0 \ , \\ \frac{1}{Z}\frac{d^2Z}{dz^2} - (\alpha^2 + \beta^2) &= 0 \ . \end{aligned}$$

The solutions of these equations depend on whether α^2 and β^2 are positive or negative. If we choose α^2 and β^2 to be positive, the solutions of the differential equations are

$$
\begin{aligned}
X(x) &= \sum_k \left(A_k e^{i\alpha x} + B_k e^{-i\alpha x}\right) \\
Y(y) &= \sum_l \left(C_l e^{i\beta y} + D_l e^{-i\beta y}\right) \\
Z(z) &= \sum_p \left(E_p e^{\sqrt{\alpha^2+\beta^2} z} + F_p e^{-\sqrt{\alpha^2+\beta^2} z}\right)
\end{aligned}
$$

The solutions can also be written in the form

$$
\begin{aligned}
X(x) &= \sum_k \left[A_k \sin(\alpha x) + B_k \cos(\alpha x)\right] \\
Y(y) &= \sum_l \left[C_l \sin(\beta y) + D_l \cos(\beta y)\right] \\
Z(z) &= \sum_p \left[E_p \sinh\left(\sqrt{\alpha^2+\beta^2} z\right) + F_p \cosh\left(\sqrt{\alpha^2+\beta^2} z\right)\right] .
\end{aligned}
$$

The above solutions are in a general form, where the constants $\alpha, \beta, A_k, B_k, C_l, D_l, E_p$ and F_p can be found from specific boundary conditions.

Consider two examples:
1. We have a solution with known boundary conditions, find the problem.
2. We have a problem with specific boundary conditions, find the solution.

Example 1.

Consider the following two-dimensional solution of the Laplace equation

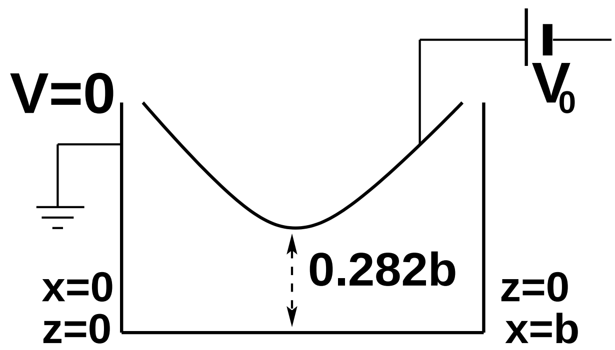
$$\Phi(x, z) = X(x)Z(z) = V_0 \sin(\alpha x) \sinh(\alpha z)$$

with the lower boundary $\Phi_{min} = 0$ and the upper boundary $\Phi_{max} = V_0$.

In what circumstance is the above the solution?

Consider $\Phi(x, z)$ in some limits. $X = 0$ for $x = 0$ or $\alpha x = \pi$, i.e. $x = \pi/\alpha$.

The solution thus satisfies the boundary conditions along the "vertical" lines for $\alpha = \pi/b$.



Since $\sinh(\alpha z) = 0$ for $z = 0$, the boundary condition along the lower boundary is satisfied.

For the solution to satisfy the upper boundary condition, the shape of the upper boundary must be such that

$$V_0 \sin(\alpha x) \sinh(\alpha z) = V_0$$

for all points x, z on the line, i.e.

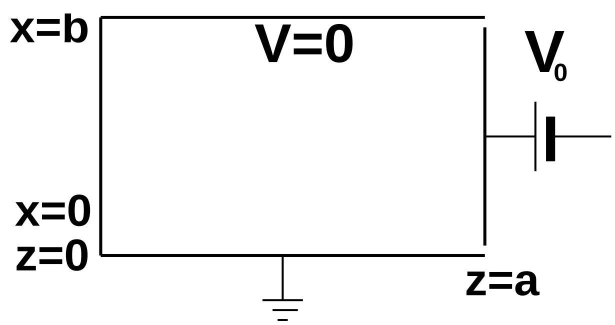
$$\sin\left(\frac{\pi x}{b}\right) \sinh\left(\frac{\pi z}{b}\right) = 1 \ .$$

Since $\sin\left(\frac{\pi x}{b}\right) \to 0$ at the edges, it is greatest $(= 1)$ at the center $x = b/2$. Hence, $\sinh\left(\frac{\pi z}{b}\right)$ must be equal to one at $x = b/2$. This happens when

$$\begin{aligned} \frac{\pi z}{b} &= \operatorname{arc}\sinh(1) \approx 0.885 \\ z &= \frac{0.885b}{\pi} = 0.282b \end{aligned}$$

We usually have reverse problems to the above that we have a set of electrodes which constitute equipotential lines or surfaces, and need to find the appropriate solution of the Laplace equation. This is illustrated in the next example, where we will try to find potential inside a rectangular box whose three sides have potential equal to zero, and the remaining side has a potential V_0.

Example 2.



This two-dimensional problem has a general solution

$$\Phi(x,z) = \sum_n [A_n \sin(\alpha x) + B_n \cos(\alpha x)] \times [E_n \sinh(\alpha z) + F_n \cosh(\alpha z)] \;.$$

The boundary condition $\Phi = 0$ at $x = 0$ can be satisfied by $B_n = 0$.
The boundary condition $\Phi = 0$ at $z = 0$ can be satisfied by $F_n = 0$.
To have $\Phi = 0$ at $x = b$ we must have $\alpha b = n\pi$.
Hence the solution reduces to

$$\Phi(x,z) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi z}{b}\right) \;,$$

where $K_n = A_n E_n$.

To find K_n we apply the remaining boundary condition $\Phi = V_0$ at $z = a$

$$\Phi(x,a) = V_0 = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi a}{b}\right)$$

This is a Fourier series in x and in the usual way we use the orthogonality properties of sine functions to calculate K_n.

$$\int_0^{2\pi} \sin(m\phi)\sin(n\phi)\; d\phi = \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$

$$\int_0^{2\pi} \cos(m\phi)\cos(n\phi)\ d\phi \quad = \quad \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases}$$

$$\int_0^{2\pi} \sin(m\phi)\cos(n\phi)\ d\phi \quad = \quad 0 \quad \text{for all} \quad m \text{ and } n$$

Multiplying both sides by $\sin(m\pi x/b)$ and integrating over $x = 0 \to b$, we get

$$\begin{aligned} \int_0^b V_0 \sin\left(\frac{m\pi x}{b}\right) dx \quad &= \quad \sum_{n=1}^{\infty} K_n \sinh\left(\frac{n\pi a}{b}\right) \\ &\quad \times \int_0^b \sin\left(\frac{m\pi x}{b}\right) \sin\left(\frac{n\pi x}{b}\right) dx \ . \end{aligned}$$

All integrals on the RHS are equal to zero except for $m = n$

$$V_0 \left[-\cos\left(\frac{n\pi x}{b}\right)\right]_0^b \frac{b}{n\pi} \quad = \quad K_n \sinh\left(\frac{n\pi a}{b}\right) \int_0^b \sin^2\left(\frac{n\pi x}{b}\right) dx \ ,$$

which we can write as

$$\begin{aligned} \frac{V_0 b}{n\pi}\left[1 - \cos(n\pi)\right] \quad &= \quad K_n \sinh\left(\frac{n\pi a}{b}\right) \\ &\quad \times \int_0^b \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi x}{b}\right)\right] dx \ . \end{aligned}$$

The $\cos\left(\frac{2n\pi x}{b}\right)$ integrates to zero over the range $0 \to b$, giving

$$\frac{V_0 b}{n\pi}\left[1 - \cos(n\pi)\right] \quad = \quad K_n \sinh\left(\frac{n\pi a}{b}\right) \frac{b}{2}$$

and finally

$$K_n = \frac{2V_0}{n\pi} \ \frac{1 - \cos(n\pi)}{\sinh\left(\frac{n\pi a}{b}\right)} \ .$$

If n is an even number then $\cos(n\pi) = 1$ and $K_n = 0$.
If n is an odd number then $\cos(n\pi) = -1$ and $1 - \cos(n\pi) = 2$.
Hence

$$K_n = \frac{4V_0}{n\pi} \ \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \qquad \text{for odd } n$$

and then

$$\Phi(x,z) = \sum_{\text{odd } n} \frac{4V_0}{n\pi} \frac{\sinh\left(\frac{n\pi z}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left(\frac{n\pi x}{b}\right)$$

8.2.2 Solution of the Laplace Equation in Spherical Coordinates

In this lecture, we continue the discussion of boundary-value problems and will illustrate solution of the Laplace equation for general problems of spherical symmetry.

In spherical coordinates (r, θ, ϕ), the Laplace equation can be written as

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) \\ &\quad + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \ .\end{aligned}$$

Multiplying by r^2, the Laplace equation can be written as a sum of two separate parts

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \ .$$

The first part depends only on r, whereas the second part depends only on θ, ϕ. Thus, the solution is of the separable form

$$\Phi = R(r)Y(\theta,\phi) \ .$$

Hence, substituting $\Phi = R(r)Y(\theta,\phi)$ and dividing by $R(r)Y(\theta,\phi)$, we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] \ .$$

Both sides of the above equation depend on different variables, thus must be equal to the same constant, say $-\alpha$:

$$\begin{aligned} \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \alpha R &= 0 \\ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} - \alpha Y &= 0 \ . \end{aligned}$$

Thus, the Laplace equation splits into two independent differential equations. We will call them **(A)** and **(B)**.

(A) Consider the equation for Y.

Multiplying both sides by $\sin^2\theta$, we get

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) - \alpha\sin^2\theta Y + \frac{\partial^2 Y}{\partial\phi^2} = 0 \ .$$

This equation contains two separate parts, one dependent only on θ and the other dependent only on ϕ. Therefore, the solution will be of the form

$$Y(\theta,\phi) = X(\theta)\Psi(\phi) \ .$$

Hence, we get

$$\frac{1}{X}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dX}{d\theta}\right) - \alpha\sin^2\theta = -\frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} \ .$$

As before, both sides must be equal to a constant, say m^2:

$$\begin{aligned} &\frac{1}{X}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dX}{d\theta}\right) - \alpha\sin^2\theta = m^2 \\ &\frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = -m^2 \end{aligned}$$

(A1) First, we will solve the equation for Ψ, which we can write as

$$\frac{d^2\Psi}{d\phi^2} = -m^2\Psi \ .$$

It is the familiar differential equation for a harmonic motion. The solution of this equation is of simple exponent form:

$$\Psi(\phi) = A \exp(im\phi) \ ,$$

where A is a constant.

To determine the constant m note that in rotation, ϕ and $\phi + 2\pi$ correspond to the same position in space: $\Psi(\phi) = \Psi(\phi + 2\pi)$, which is satisfied when

$$\exp(im\phi) = \exp[im(\phi + 2\pi)] \ .$$

This leads to

$$\exp(i2\pi m) = 1 \ ,$$

that is satisfied when $m = 0, \pm 1, \pm 2, \ldots$.
Hence, the constant m^2 is not an arbitrary number, is an integer.

(A2) Now, we will find $X(\theta)$.

Using the solution for Ψ, the differential equation for $X(\theta)$ can be written as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dX}{d\theta}\right) - \left(\alpha + \frac{m^2}{\sin^2\theta}\right) X = 0 \ .$$

Introducing a new variable $z = \cos\theta$, we can rewrite this equation as

$$\left(1 - z^2\right)\frac{d^2X}{dz^2} - 2z\frac{dX}{dz} - \left(\alpha + \frac{m^2}{1 - z^2}\right) X = 0$$

or

$$\frac{d}{dz}\left[\left(1 - z^2\right)\frac{dX}{dz}\right] - \left(\alpha + \frac{m^2}{1 - z^2}\right) X = 0 \ .$$

The above equation is known as the *generalized Legendry* differential equation, and its solutions are the *associated Legendry polynomials*. For $m = 0$, the equation is called the *ordinary Legendry* differential equation whose solution is given by the *Legendry polynomials*.

Lets look into the solution procedure of the above equation. This will allow us to find α and $X(\theta)$.
We assume that the whole range of z ($\cos\theta$), including the north and south poles ($z = \pm 1$), is in the region of interest. The desired solution should be single valued, finite, and continuous on the interval $-1 \leq z \leq 1$ in order to represent a physical potential.

The differential equation for X has poles at $z = \pm 1$. In order to find the solution of this equation, we first check what solution could be continuous near the poles.

Lets check a possible solution near $z = 1$. Substituting $x = 1 - z$, then $dx = -dz$ and in terms of x the equation takes a form

$$\frac{d}{dx}\left[x\,(2-x)\,\frac{dX}{dx}\right] - \left(\alpha + \frac{m^2}{x(2-x)}\right) X = 0 \ .$$

We look for a solution in the trial form of power series in x

$$X(x) = x^s \sum_{n=0}^{\infty} a_n x^n \ .$$

Substituting this into the differential equation for X, we get

$$\begin{aligned} &2s^2 a_0 x^{s-1} + (s+1)(2sa_1 - sa_0 + 2a_1)x^s + \ldots \\ &- \left(\alpha + \frac{m^2}{x(2-x)}\right)(a_0 + a_1 x + \ldots)x^s = 0 \ . \end{aligned}$$

Near $x \approx 0$, we can replace $x(2-x)$ by $2x$, and obtain

$$\left(2s^2 a_0 - \frac{m^2}{2} a_0\right) x^{s-1} + (\ldots)x^s \ldots = 0 \ .$$

This equation is satisfied for all x only if the coefficients at $x^s,\ x^{s\pm 1}, \ldots$ are zero. From this, we find that

$$s = \pm \frac{1}{2}|m| \ .$$

We take only $s = +\frac{1}{2}|m|$ as for $s = -\frac{1}{2}|m|$ the solution for $X(x)$ at $x = 0$ would go to infinity. We require the solution to be finite at any point x.

Thus, the solution that is continuous near $x = 0$ is of the form

$$X(x) = x^{\frac{1}{2}|m|} \sum_{n=0}^{\infty} a_n x^n$$

or in terms of z

$$X(x) = (1-z)^{\frac{1}{2}|m|} \sum_{n=0}^{\infty} a'_n z^n \ .$$

Using the same procedure, we can show that near the pole $z = -1$, the continuous solution is

$$X(x) = (1+z)^{\frac{1}{2}|m|} \sum_{n=0}^{\infty} a''_n z^n \ .$$

Hence, we will try to find the solution in the form

$$X(x) = \left(1 - z^2\right)^{\frac{1}{2}|m|} \sum_{n=0}^{\infty} b_n z^n \ .$$

Substituting this equation into the differential equation for $X(z)$ and collecting all terms at the same powers of z^n, we obtain

$$\begin{aligned} &\sum_n \{(n+1)(n+2)b_{n+2} - n(n-1)b_n \\ &- 2(|m|+1)nb_n - (\alpha + |m| + m^2)b_n\Big\} z^n = 0 \ . \end{aligned}$$

Hence, we get a recurrence relation for the coefficients b_n

$$b_{n+2} = \frac{(n+|m|)(n+|m|+1)+\alpha}{(n+1)(n+2)} \ b_n \ .$$

We have two separate solutions for even and odd n. For $b_0 \neq 0$, we put $b_1 = 0$, and the solution is given in terms of even n. For $b_0 = 0$, we put $b_1 \neq 0$, and the solution is given in terms of odd n.
We cannot accept both the even and odd solutions at the same time, because in this case the solution $X(z)$ would not be a single valued function.
For example, for $b_0 \neq 0$, we have $\alpha = -|m| - m^2$, but for $b_1 \neq 0$, we have $\alpha = -2 - 3|m| - m^2$. If we would accept both of the solutions at the same time, the potential would have two different values.

We check now whether the series is converting when $n \to \infty$ which would ensure that the potential is finite.

Since $b_{n+2} > b_n$, the series diverges for $z = \pm 1$. Therefore, in order to get the potential finite everywhere in the space, we have to terminate the series at some $n = n_0$. In other words, we assume that $b_{n_0+1} = b_{n_0+2} = \ldots = 0$.

The series terminating at $n = n_0$ indicates that

$$(n_0 + |m|)(n_0 + |m| + 1) + \alpha = 0 \ .$$

Introducing

$$l = n_0 + |m| \ ,$$

we see that $l \geq |m|$, and

$$\alpha = -l(l+1) \ , \qquad l = 0, 1, 2, \ldots$$

Thus, the solution for $X(z)$ is

$$X_{lm}(z) = \left(1 - z^2\right)^{\frac{1}{2}|m|} \sum_{n}^{l-|m|} b_n z^n \ ,$$

where the sum is over even n when $l - |m|$ is an even number, and over odd n when $l - |m|$ is an odd number.

First few solutions

$$\begin{aligned} X_{00}(z) &= b_0 = b_0 P_0^0(z) \\ X_{10}(z) &= b_1 z = b_1 P_1^0(z) \\ X_{11}(z) &= b_0 \sqrt{1 - z^2} = b_0 P_1^1(z) \ , \end{aligned}$$

where $P_0^0(z) = 1$, $P_1^0(z) = z$, $P_1^1(z) = \sqrt{1 - z^2}, \ldots$ are the associate Legendry polynomials of the order l.

Useful examples [in terms of θ ($z = \cos\theta$)]:

$$\begin{aligned} P_0^0(\cos\theta) &= 1 \\ P_1^0(\cos\theta) &= \cos\theta \\ P_1^1(\cos\theta) &= \sin\theta \end{aligned}$$

$$\begin{aligned}
P_2^0(\cos\theta) &= \frac{1}{4}\left[3\cos(2\theta)+1\right] \\
P_2^1(\cos\theta) &= \frac{3}{2}\sin(2\theta) \\
P_2^2(\cos\theta) &= \frac{3}{2}\left[1-\cos(2\theta)\right]
\end{aligned}$$

Orthogonality of the Legendry polynomials is an important property:

$$\int_{-1}^{1} P_l^m(\cos\theta)P_k^n(\cos\theta)\ d(\cos\theta) = 0$$

unless $m = n$ and $l = k$, for which

$$\int_{-1}^{1} \left[P_l^m(\cos\theta)\right]^2\ d(\cos\theta) = \frac{2}{2l+1}\ \frac{(l+m)!}{(l-m)!}$$

With the above notation, the solution for $Y(\theta,\phi)$ is of the form

$$Y(\theta,\phi) = \sum_l \sum_m A_{lm} P_l^m(\cos\theta)\mathrm{e}^{im\phi}$$

(B) What left is to find the radial part $R(r)$.

With $\alpha = -l(l+1)$, the differential equation for R is of the form

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)R = 0\ .$$

Dividing by r and introducing a new function $U(r) = rR(r)$, we obtain

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0\ .$$

Lets first check the asymptotic solution for $r \gg 1$. In this case we can ignore the second term in the differential equation, and find that the asymptotic equation has a solution $U(r \gg 1) = Cr$, where C is a constant.

Following this asymptotic behavior, we will try the general solution of a form

$$U(r) = r^s \ .$$

Substituting this into the differential equation, we obtain

$$\left[s(s-1) - l(l+1)\right] r^{s-2} = 0 \ .$$

This equation is satisfied for any r when

$$s = (l+1) \qquad \text{or} \qquad s = -l \ .$$

Thus, the general solution is of the form

$$U(r) = C_1 r^{l+1} + C_2 r^{-l} \ ,$$

or

$$R(r) = C_1 r^l + C_2 r^{-(l+1)} \ .$$

Thus, solution of the Laplace equation in spherical polar coordinates is of the form:

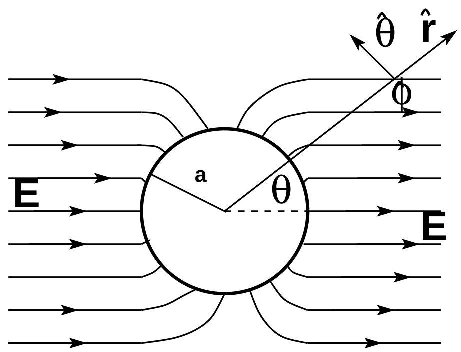
$$\begin{aligned} \Phi(r,\theta,\phi) &= \sum_l \sum_m \left(C_{1l} r^l + C_{2l} r^{-(l+1)} \right) \\ &\quad \times A_{lm} P_l^m(\cos\theta) \mathrm{e}^{im\phi} \ . \end{aligned}$$

The solution can be written as

$$\begin{aligned} \Phi(r,\theta,\phi) &= \sum_l \sum_m \left\{ \left(C_{1l} r^l + C_{2l} r^{-(l+1)} \right) \right. \\ &\quad \left. \times \left[a_{lm} \cos(m\phi) + b_{lm} \sin(m\phi) \right] P_l^m(\cos\theta) \right\} \ . \end{aligned}$$

Example 1: *Potential inside an uniformly charged sphere*

Consider an example of boundary-value problem with azimuthal symmetry: A conducting sphere of a radius a in an uniform electric field.



The conducting sphere is an equipotential volume (else there would be electric fields driving currents till it became so). Take the zero potential on the sphere.

The boundary conditions to be satisfied are:

1. The potential on the surface of the sphere $V(a, \theta, \phi) = 0$.
2. The potential at infinity is the uniform field potential (no effect of the sphere at infinity), so $\Phi = -Er\cos\theta$ at infinity.

Since the applied potential is independent of the angle *phi*, the induced potential will also be independent of ϕ. Thus, we can set $m = 0$ in the general solution and get

$$\Phi(r,\theta) = \sum_l \left[C_{1l} r^l + C_{2l} r^{-(l+1)}\right] P_l^0(\cos\theta) \ .$$

The boundary condition at infinity is satisfied for all constants $C_{1l} = 0$ except for $l = 1$ (remember $P_1 = \cos\theta$).

$$\Phi(r,\theta) = C_{11} r P_1(\cos\theta) + \sum_l \frac{C_{2l} P_l(\cos\theta)}{r^{l+1}} \ .$$

As $r \to \infty$, the potential $\Phi(r,\theta) \to C_{11} r\cos\theta = -Er\cos\theta$.
Therefore $C_{11} = -E$.

The other boundary condition to be satisfied is $\Phi(a, \theta) = 0$ on the surface of the sphere.

$$\Phi(a, \theta) = 0 = -EaP_1(\cos\theta) + \sum_l \frac{C_{2l} P_l(\cos\theta)}{a^{l+1}} \ .$$

We can determine the coefficients C_{2l} using the orthogonality properties of the Legendry polynomials. Multiplying the above equation by $P_k(\cos\theta)$ and integrating over $\cos\theta$, we obtain

$$\begin{aligned} 0 &= -aE \int_{-1}^{1} P_1(\cos\theta) P_k(\cos\theta)\ d(\cos\theta) \\ &\quad + \sum_l \frac{C_{2l}}{a^{l+1}} \int_{-1}^{1} P_l(\cos\theta) P_k(\cos\theta)\ d(\cos\theta) \end{aligned}$$

If $k \neq 1$:

The first term vanishes by orthogonality of the Legendry polynomials. All terms in the summation vanish except that for $l = k$. Thus

$$0 = \frac{C_{2k}}{a^{k+1}} \int_{-1}^{1} P_k(\cos\theta) P_k(\cos\theta)\ d(\cos\theta)$$

Since the integral is nonzero, then $C_{2k} = 0$ for $k \neq 1$.

For $k = 1$:

$$\begin{aligned} &\int_{-1}^{1} P_l^m(\cos\theta) P_l^m(\cos\theta)\ d(\cos\theta) \\ &= \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \end{aligned}$$

which for $l = 1$ and $m = 0$ is equal to $2/3$. Thus

$$0 = -aE\frac{2}{3} + \frac{C_{21}}{a^2}\frac{2}{3}$$

from which, we find $C_{21} = Ea^3$. Hence

$$\Phi(r, \theta) = -Er\cos\theta + Ea^3\ \frac{\cos\theta}{r^2} \ .$$

The first term is just the potential of a uniform field E. The second term is the potential due to the induced surface charges or, equivalently, is the potential of the induced dipole moment $p = 4\pi\varepsilon_0 E a^3$

$$\Phi_{dip} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \ .$$

More useful exercises: Before you approach the tutorial problems on this subject (tutorial set number 6), try to solve the following problems

1. Consider a two-dimensional region with boundaries at $x = 0, b$ and $z = 0, a$, as shown in the figure 3. The boundary conditions are

$$\begin{aligned} \frac{\partial\Phi}{\partial z} &= 0 \qquad \text{at} \qquad z = 0 \ , \\ \Phi &= 0 \qquad \text{at} \qquad x = 0, b \ , \\ \Phi &= V_0 \qquad \text{at} \qquad z = a \ . \end{aligned}$$

   Find the potential at any point inside the two-dimensional region.

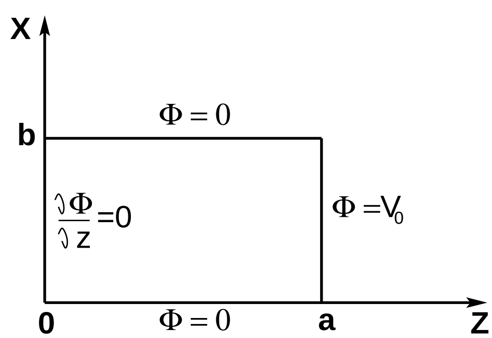


Figure 3:

2. Consider a conducting sphere of radius R. The surface of the sphere is kept at a potential

$$\Phi(R, \theta, \phi) = V_0 \sin\theta \sin\phi \ .$$

Using the above as a boundary condition, find the potential at any point inside the sphere.

Questions

(1) Explain, why in the two-dimensional case, we take from the general three-dimensional solution of the Laplace equations

$$\Phi = X(x)Z(z) \ ,$$

or

$$\Phi = Y(y)Z(z) \ ,$$

but not

$$\Phi = X(x)Y(y) \ .$$

9 Electromagnetic Antennas: Hertzian Dipole

In this lecture we will illustrate an application of the vector potential $\vec{A}$ in calculations of the electric and magnetic fields produced by a source system containing time varying charges and currents.

Consider the retarded solutions of the Maxwell's equations

$$\vec{B} = \nabla \times \vec{A} \ , \qquad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \ ,$$

with

$$\begin{aligned} \Phi(r,t) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(t-r/c)}{r}\, dV \ , \\ \vec{A}(r,t) &= \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{\vec{J}(t-r/c)}{r}\, dV \ . \end{aligned}$$

The above solution holds for the Lorentz gauge in which

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t} \ .$$

Assume that the charges and currents vary sinusoidally in time

$$\begin{aligned} \rho(r,t) &= \rho(r)\mathrm{e}^{i\omega t} \\ \vec{J}(r,t) &= \vec{J}(r)\mathrm{e}^{i\omega t} \ . \end{aligned}$$

In this case

$$\nabla \cdot \vec{A} = -\frac{i\omega}{c^2}\Phi \ ,$$

which gives

$$\Phi = -\frac{c^2}{i\omega}\nabla \cdot \vec{A} \ .$$

Thus, the scalar potential can be eliminated from the field equations leaving only the dependence on $\vec{A}$. Hence, we can express both $\vec{E}$ and $\vec{B}$ in terms of the vector potential $\vec{A}$ alone. We have

$$\vec{E} = \frac{c^2}{i\omega}\nabla(\nabla \cdot \vec{A}) - \frac{\partial \vec{A}}{\partial t} \ , \qquad \vec{B} = \nabla \times \vec{A} \ .$$

This result may seem rather strange at first, since normally we should expect to need both the scalar and vector potentials in order to completely determine the field. The explanation and in fact an another way of saying the same thing is that time varying charge must satisfy the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -i\omega\rho \ ,$$

so that

$$\rho = -\frac{\nabla \cdot \vec{J}}{i\omega} \ .$$

Then

$$\begin{aligned}
\Phi(r,t) &= \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(t - r/c)}{r} \, dV \\
&= -\frac{1}{4\pi\varepsilon_0 i\omega} \int \frac{\nabla \cdot \vec{J}}{r} \, dV \\
&= -\frac{1}{4\pi\varepsilon_0 i\omega} \nabla \cdot \int \frac{\vec{J}}{r} \, dV = -\frac{c^2}{i\omega} \nabla \cdot \vec{A} \ .
\end{aligned}$$

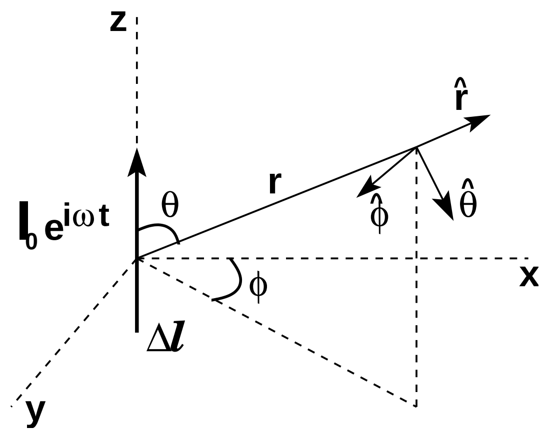
Thus, specification of $\vec{J}$ alone is sufficient to completely determine all sources, and hence a solution for $\vec{A}$ in terms of $\vec{J}$ contains all the necessary information to completely specify the time-varying field.

9.1 Field of an Element of Alternating Current

Consider a linear element Δl carrying an alternating current $I = I_0 \exp(i\omega t)$. The current element may be viewed as two charges Q and $-Q$ oscillating back and forth.

Assume that Δl is much smaller than the wavelength $\lambda = 2\pi c/\omega$. In this case we can ignore the phase variation along Δl.

An understanding of the properties of such an antenna is of great interest since, in principle, all radiating structures can be considered as a sum of small radiating elements. Moreover, many practical antennas working at low



frequencies are very short compared with the wavelength.

If $\Delta l \ll \lambda$, then there are three spatial regions of interest:

- The near field (static) zone $\Delta l \ll r \ll \lambda$
- The intermediate field (induction) zone $\Delta l \ll r \sim \lambda$
- The far field (radiation) zone $\Delta l \ll \lambda \ll r$

We will see that the fields have different properties in the different zones. In the near zone the fields have the character of static fields, with a strong dependence on the properties of the source. In the far zone, the fields are transverse to the radius vector and fall of as r^{-1}, typical of radiation fields.

The retarded current element

$$\vec{J}(t - r/c)dV = \vec{I}(t - r/c)dl = \vec{dl} I_0 \mathrm{e}^{i(\omega t - kr)} \ ,$$

where $k = \omega/c$.

Thus, the vector potential is

$$\vec{A} = \vec{\Delta l} \frac{I_0}{4\pi\varepsilon_0 c^2} \frac{\mathrm{e}^{i(\omega t - kr)}}{r} \ ,$$

where $\vec{\Delta l} = \int \vec{dl}$.

In the near zone, where $r \ll \lambda$ (or $kr \ll 1$), the exponent $\exp(-ikr)$ can be replaced by unity. In the far field zone $kr \gg 1$, the exponential oscillates rapidly, and in this region it is sufficient to approximate $\exp(-ikr) \approx 1-ikr$. In the intermediate zone, all powers of kr must be retained.

In cartesian coordinates the vector potential has only one component, say A_z along the current element.

Referred to spherical coordinate system, $\vec{A}$ has components

$$\begin{aligned} A_r &= A_z \cos\theta = \frac{I_0 \Delta l \cos\theta}{4\pi\varepsilon_0 c^2} \frac{\mathrm{e}^{i(\omega t - kr)}}{r} \ , \\ A_\theta &= -A_z \sin\theta = -\frac{I_0 \Delta l \sin\theta}{4\pi\varepsilon_0 c^2} \frac{\mathrm{e}^{i(\omega t - kr)}}{r} \ , \\ A_\phi &= 0 \ . \end{aligned} \tag{26}$$

In order to find the fields $\vec{E}$ and $\vec{B}$, we have to calculate $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$, that in spherical polar coordinates are given by

$$\begin{aligned} \nabla \cdot \vec{A} &= \frac{1}{r^2} \frac{\partial \left(r^2 A_r\right)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \left(A_\theta \sin\theta\right)}{\partial \theta} \\ &+ \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \end{aligned}$$

$$\begin{aligned} \nabla \times \vec{A} &= \frac{\hat{r}}{r \sin\theta} \left[\frac{\partial \left(A_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] \\ &+ \frac{\hat{\theta}}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial \left(r A_\phi\right)}{\partial r} \right] \\ &+ \frac{\hat{\phi}}{r} \left[\frac{\partial \left(r A_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \end{aligned}$$

Since $A_\phi = 0$ and there is no ϕ dependence of A_r and A_θ, i.e. $\partial A_{r,\theta}/\partial\phi = 0$, the above equations reduce to

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial \left(r^2 A_r\right)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \left(A_\theta \sin\theta\right)}{\partial \theta} \ ,$$

$$\nabla \times \vec{A} = \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \frac{\hat{\phi}}{r} \; .$$

Hence, the magnetic field of the current element is

$$\vec{B} = \nabla \times \vec{A} = B_\phi \hat{\phi} \; ,$$

where

$$B_\phi = \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \; , \tag{27}$$

and $B_r = B_\theta = 0$.

Thus, the magnetic field is perpendicular to the radius vector at all distances.

Calculate the magnitude B_ϕ. Substituting Eq. (26) into Eq. (27), we obtain

$$\begin{aligned} B_\phi &= \frac{-I_0 \Delta l}{4\pi \varepsilon_0 c^2 r} \left[\frac{\partial \left(\sin\theta e^{i(\omega t - kr)} \right)}{\partial r} + \frac{e^{i(\omega t - kr)}}{r} \frac{\partial \cos\theta}{\partial \theta} \right] \\ &= \frac{I_0 \Delta l}{4\pi \varepsilon_0 c^2 r} \left[ik \sin\theta e^{i(\omega t - kr)} + \sin\theta \frac{e^{i(\omega t - kr)}}{r} \right] \end{aligned}$$

Hence

$$B_\phi = \frac{I_0 \Delta l}{4\pi \varepsilon_0 c^2} \left[\frac{ik}{r} + \frac{1}{r^2} \right] \sin\theta e^{i(\omega t - kr)} \; .$$

The magnetic field is composed of two terms: the near zone term $\sim 1/r^2$ and the far zone term $\sim 1/r$. In the limit of $\omega \to 0$, the near zone term reduces to the Biot-Savart formula. The far zone term is only present for an oscillating field ($\omega \neq 0$) and therefore it is radiation field arising from accelerated (oscillating) charge.

Calculate electric field of the current element:

$$\vec{E} = \frac{c^2}{i\omega} \nabla (\nabla \cdot \vec{A}) - \frac{\partial \vec{A}}{\partial t} \; ,$$

where

$$\frac{\partial \vec{A}}{\partial t} = i\omega \vec{A} = i\omega \left(A_r \hat{r} + A_\theta \hat{\theta} \right) ,$$

and

$$\begin{aligned}
\nabla \cdot \vec{A} &= \frac{1}{r^2}\frac{\partial \left(r^2 A_r\right)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial \left(\sin\theta A_\theta\right)}{\partial \theta} \\
&= \frac{I_0 \Delta l}{4\pi\varepsilon_0 c^2} \left\{ \frac{1}{r^2} \frac{\partial \left(r \cos\theta \mathrm{e}^{i(\omega t - kr)} \right)}{\partial r} \right. \\
&- \left. \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin^2\theta \frac{\mathrm{e}^{i(\omega t - kr)}}{r} \right) \right\} \\
&= \frac{I_0 \Delta l}{4\pi\varepsilon_0 c^2} \left\{ \frac{\cos\theta}{r^2} \left[\mathrm{e}^{i(\omega t - kr)} - ikr\mathrm{e}^{i(\omega t - kr)} \right] \right. \\
&- \left. \frac{2 \sin\theta \cos\theta \mathrm{e}^{i(\omega t - kr)}}{r^2 \sin\theta} \right\} \\
&= -\frac{I_0 \Delta l}{4\pi\varepsilon_0 c^2} \left[\frac{1}{r^2} + \frac{ik}{r} \right] \cos\theta \mathrm{e}^{i(\omega t - kr)} .
\end{aligned}$$

Since in spherical coordinates

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin\theta}\frac{\partial}{\partial \phi} ,$$

we obtain

$$\left[\nabla \left(\nabla \cdot \vec{A} \right) \right]_r = -\frac{I_0 \Delta l \cos\theta}{4\pi\varepsilon_0 c^2} \left[-\frac{ik}{r^2} - \frac{2}{r^3} + \frac{k^2}{r} - \frac{ik}{r^2} \right] \mathrm{e}^{i(\omega t - kr)}$$
$$= \frac{I_0 \Delta l \cos\theta}{4\pi\varepsilon_0 c^2} \left[\frac{2}{r^3} + \frac{2ik}{r^2} - \frac{k^2}{r} \right] \mathrm{e}^{i(\omega t - kr)} ,$$

$$\left[\nabla \left(\nabla \cdot \vec{A} \right) \right]_\theta = \frac{I_0 \Delta l}{4\pi\varepsilon_0 c^2} \left[\frac{1}{r^3} + \frac{ik}{r^2} \right] \sin\theta \ \mathrm{e}^{i(\omega t - kr)} ,$$

$$\left[\nabla \left(\nabla \cdot \vec{A} \right) \right]_\phi = 0 .$$

Hence the radial part of the electric field, E_r, is

$$\begin{aligned}
E_r &= \frac{c^2}{i\omega}\left[\nabla(\nabla\cdot\vec{A})\right]_r - \left[\frac{\partial\vec{A}}{\partial t}\right]_r \\
&= \frac{I_0\Delta l\cos\theta}{4\pi\varepsilon_0 i\omega}\left[\frac{2}{r^3}+\frac{2ik}{r^2}-\frac{k^2}{r}\right]\mathrm{e}^{i(\omega t-kr)} \\
&- i\omega\frac{I_0\Delta l\cos\theta}{4\pi\varepsilon_0 c^2}\frac{\mathrm{e}^{i(\omega t-kr)}}{r} \\
&= \frac{I_0\Delta l\cos\theta}{4\pi\varepsilon_0 c}\left[\frac{2}{ikr^3}+\frac{2}{r^2}+\frac{ik}{r}-\frac{i\omega}{cr}\right]\mathrm{e}^{i(\omega t-kr)}
\end{aligned}$$

Since $k=\omega/c$, the $1/r$ terms cancel and then E_r simplifies to

$$E_r = \frac{I_0\Delta l\cos\theta}{4\pi\varepsilon_0 c}\left[\frac{2}{ikr^3}+\frac{2}{r^2}\right]\mathrm{e}^{i(\omega t-kr)} \ .$$

Similarly, we find the θ component of E as

$$\begin{aligned}
E_\theta &= \frac{c^2}{i\omega}\left[\nabla(\nabla\cdot\vec{A})\right]_\theta - \left[\frac{\partial\vec{A}}{\partial t}\right]_\theta \\
&= \frac{I_0\Delta l\sin\theta}{4\pi\varepsilon_0 i\omega}\left[\frac{1}{r^3}+\frac{ik}{r^2}\right]\mathrm{e}^{i(\omega t-kr)} + i\omega\frac{I_0\Delta l\sin\theta}{4\pi\varepsilon_0 c^2}\ \frac{\mathrm{e}^{i(\omega t-kr)}}{r} \\
&= \frac{I_0\Delta l\sin\theta}{4\pi\varepsilon_0 c}\left[\frac{1}{ikr^3}+\frac{1}{r^2}+\frac{ik}{r}\right]\mathrm{e}^{i(\omega t-kr)}
\end{aligned}$$

The radial part of the electric field contributes only to the near and intermediate zones, whereas the angular part contributes to all of the zones.

The $1/r^3$ part is the Coulomb type contribution. It is similar in nature to a static field surrounding a small linear-current element and an electric dipole.

Proof:

The Coulomb or static field is for $\omega\to 0$. In this limit the $1/r^3$ contribution is

$$E_\theta = \frac{I_0\Delta l\sin\theta}{4\pi\varepsilon_0 c}\frac{1}{ikr^3}\mathrm{e}^{i(\omega t-kr)}$$

$$
\begin{aligned}
&= \frac{I_0 \Delta l \sin\theta}{4\pi\varepsilon_0 c} \frac{1}{ikr^3} \left(1 - ikr + \frac{1}{2}(-ikr)^2 + \cdots\right) \\
&= -\frac{I_0 \Delta l \sin\theta}{4\pi\varepsilon_0 c} \frac{1}{r^2}
\end{aligned}
$$

where we have taken only the real (physical) part of the field.

Since $I_0 = \Delta q/\Delta t$ and $\Delta l/\Delta t = c$, we get

$$
E_\theta = -\frac{\Delta q \sin\theta}{4\pi\varepsilon_0} \frac{1}{r^2} \ ,
$$

as required.

Electric and magnetic fields in near and far field zones

Consider first the near field zone ($r \ll \lambda$). In this limit the magnetic and electric fields are

$$
\begin{aligned}
\vec{B}_{near} &= \frac{I_0 \Delta l}{4\pi\varepsilon_0 c^2} \frac{1}{r^2} \sin\theta \hat{\phi} \mathrm{e}^{i(\omega t - kr)} \\
\vec{E}_{near} &= -i \frac{I_0 \Delta l}{4\pi\varepsilon_0 c} \frac{2}{kr^3} \left(\cos\theta \hat{r} + \sin\theta \hat{\theta}\right) \mathrm{e}^{i(\omega t - kr)}
\end{aligned}
$$

Since the magnetic field is real and the electric field is imaginary, the Pointing vector involving the near-zone field components is a pure imaginary quantity. It does not represent any flow of energy. This imaginary quantity represents energy that oscillates back and forth between the source and the region of space surrounding the source.

Consider now the far zone or radiation components of the magnetic and electric fields.

$$
\begin{aligned}
\vec{E}_{rad} &= E_\theta^R \hat{\theta} \ , \qquad E_\theta^R = \frac{I_0 \Delta l \sin\theta}{4\pi\varepsilon_0 c} \frac{ik}{r} \ \mathrm{e}^{i(\omega t - kr)} \\
\vec{B}_{rad} &= B_\phi^R \hat{\phi} \ , \qquad B_\phi^R = \frac{I_0 \Delta l \sin\theta}{4\pi\varepsilon_0 c^2} \frac{ik}{r} \ \mathrm{e}^{i(\omega t - kr)} \ .
\end{aligned}
$$

Note that:

1. The electric and magnetic fields oscillate in phase.
2. The electric and magnetic fields are orthogonal to each other.
3. The ratio $\frac{E_\theta^R}{B_\phi^R} = c$, the value for plane waves in free space.
4. The electric and magnetic fields are transverse to the radius vector at all distances.
5. The Poynting vector $\vec{N} = c^2\varepsilon_0 \vec{E}_{rad} \times \vec{B}_{rad}$ is a real quantity and is in the direction of the radius vector, indicating that the energy of the field propagates away from the current element.

These properties show that in the far zone the field is in a form of plane waves.

9.2 Power Radiated from the Current Element

The power flux at any point is given by the Poynting vector

$$\vec{N} = c^2\varepsilon_0 \vec{E} \times \vec{B} \ .$$

Then, the total power radiated across a sphere of radius r is

$$W = \int_S \vec{N} \cdot d\vec{S} \ ,$$

where

$$dS = r^2 \sin\theta \ d\theta d\phi \ .$$

Only those partial products in $\vec{E} \times \vec{B}$ which vary as $1/r^2$ will have net radiated power. The other partial products are small as they fall off more rapidly than $1/r^2$. Thus, the only part of the fields entering into the expression for the radiated power is the far field zone part (radiation component) consisting of the terms varying as $1/r$.

The volume $Id\vec{l}$ is energized and the energy flows but there is no net energy loss over a cycle in the equilibrium situation.

The radiation components of $\vec{E}$ and $\vec{B}$ are in phase and average over time is

$$E^{R}\bar{B}^{R} = \frac{1}{2}\left(E_{\theta}^{R}\right)_{0}\left(B_{\phi}^{R}\right)_{0} \ ,$$

where $\left(E_{\theta}^{R}\right)_{0}$ and $\left(B_{\phi}^{R}\right)_{0}$ are amplitudes of E_{θ}^{R} and B_{ϕ}^{R}.

Hence, the time averaged Poynting vector is

$$\begin{aligned}\bar{N} &= \frac{1}{2}c^{2}\varepsilon_{0}\frac{I_{0}^{2}\Delta l^{2}}{16\pi^{2}\varepsilon_{0}^{2}c^{3}}\frac{k^{2}}{r^{2}}\sin^{2}\theta \\ &= \frac{I_{0}^{2}\Delta l^{2}}{32\pi^{2}\varepsilon_{0}c}\frac{4\pi^{2}}{\lambda^{2}r^{2}}\sin^{2}\theta = \frac{I_{0}^{2}}{8\varepsilon_{0}c}\left[\frac{\Delta l}{\lambda}\right]^{2}\frac{\sin^{2}\theta}{r^{2}} \ .\end{aligned}$$

Thus, the total power is

$$\begin{aligned}W &= \int_{0}^{\pi}\int_{0}^{2\pi} r^{2}\bar{N}\sin\theta \ d\theta d\phi \\ &= \int_{0}^{2\pi}\frac{I_{0}^{2}}{8\varepsilon_{0}c}\left[\frac{\Delta l}{\lambda}\right]^{2} d\phi \int_{0}^{\pi}\sin^{3}\theta d\theta \ .\end{aligned}$$

Integrating, we get

$$W = 2\pi\frac{I_{0}^{2}}{8\varepsilon_{0}c}\left[\frac{\Delta l}{\lambda}\right]^{2}\frac{4}{3} \ = \frac{\pi I_{0}^{2}}{3\varepsilon_{0}c}\left[\frac{\Delta l}{\lambda}\right]^{2} \ .$$

We can write the total power radiated in terms of the power absorbed in an equivalent resistance, called the radiation resistance:

$$W = \frac{1}{2}\frac{2\pi}{3\varepsilon_{0}c}\left[\frac{\Delta l}{\lambda}\right]^{2} I_{0}^{2} \ = \frac{1}{2}RI_{0}^{2} \ .$$

where

$$R = \frac{2\pi}{3\varepsilon_{0}c}\left[\frac{\Delta l}{\lambda}\right]^{2}$$

is the radiation resistance.

Since $1/(\varepsilon_0 c) = \sqrt{\mu_0/\varepsilon_0} = 377$ or 120π, we obtain

$$R = 80\pi^2 \left[\frac{\Delta l}{\lambda}\right]^2 .$$

For example, if $\Delta l/\lambda \approx 0.1$, then $R = 0.8\pi^2 \approx 8$ ohms.
This example shows that for a current element which is 10% of the wavelength long, the resistance is very small.

Thus, if $\Delta l/\lambda \ll 1$, the radiation losses are negligible, that the radiated power is very small.

This also explains why ordinary circuit theory often works so well although it ignores the loss of energy by radiation from AC currents.

Appreciable power would be radiated only if the current amplitude I_0 were very large. A large current, on the other hand, would lead to large amounts of power dissipation in the conductor, and hence a very low efficiency.

We can conclude that current carrying systems that have linear dimensions small compared with the wavelength radiate negligible power. An efficient antenna should have dimensions comparable to or greater than the wavelength.

9.3 Gain of the Dipole Antenna

A further property of the dipole antenna that is worthy of consideration is the directional property of power radiated in different directions.

The gain or directivity function of a transmitting antenna is the ratio of the Poynting flux to the flux due to an isotropic radiator emitting the same total power W:

$$g_T(\theta, \phi) = \frac{N(\theta, \phi)}{N_{iso}} ,$$

where

$$N_{iso} = \frac{W}{4\pi r^2}$$

is the energy flux uniform in all directions.

For the infinitesimal dipole:

$$\begin{aligned} N(\theta,\phi) &= \frac{I_0^2}{8\varepsilon_0 c}\left[\frac{\Delta l}{\lambda}\right]^2 \frac{\sin^2\theta}{r^2} \ , \\ W &= \frac{\pi I_0^2}{3\varepsilon_0 c}\left[\frac{\Delta l}{\lambda}\right]^2 \ . \end{aligned}$$

Hence

$$g_T(\theta,\phi) = g_T(\theta) = \frac{3}{2}\sin^2\theta \ .$$

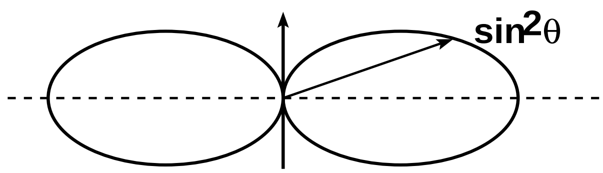


Figure 4: The polar radiation pattern of the dipole antenna.

The directivity function $g_T(\theta,\phi)$ defines a three-dimensional surface called the polar radiation pattern of the antenna. The function varies as $\sin^2\theta$, and hence the radiation is most intense in the $\theta = \pi/2$ direction and zero in the directions $\theta = 0, \pi$. The maximum gain then is 1.5 for directions defined by $\theta = \pi/2$, in the equatorial plane of the dipole. The gain function is independent of ϕ.

We can conclude, that the directivity function $g_T(\theta,\phi)$ is a measure of how effective the antenna is in concentrating the radiated power in a given direction.

Questions:

(1) Show that in spherical polar coordinates, the magnetic field of a short current element $I\vec{\Delta l} = \vec{\Delta l} I_0 \exp(i\omega t)$ has only an azimuthal component of the form

$$\vec{B} \;=\; \frac{I_0 \Delta l}{4\pi\varepsilon_0 c^2} \left[\frac{ik}{r} + \frac{1}{r^2} \right] \sin\theta \; \mathrm{e}^{i(\omega t - kr)} \; \hat{\phi} \; .$$

(2) Show that in the far field zone of a radiating short current element, the electric and magnetic fields oscillate in phase and are orthogonal to each other.

(3) Given the expressions for the EM field of a Hertzian dipole, show that the total radiated power from the dipole is

$$W \;=\; \frac{\pi I_0^2}{3\varepsilon_0 c} \left[\frac{\Delta l}{\lambda} \right]^2 \; .$$

(4) Show that the time averaged Poynting vector of the EM field emitted by a short current element is maximal in the equatorial plane of the element.

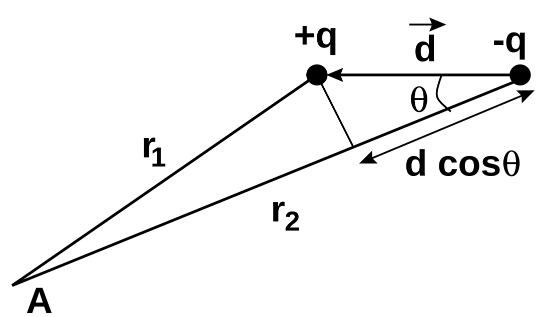
10 Electromagnetic Theory of Polarizable Materials

If an uncharged dielectric (insulator) is placed in an electric field, the field within the dielectric will be found to be modified by charges induced by the external field. The charge is induced by rearrangement of bound charges within the molecules of the dielectric. In dielectrics these charges are a set of molecular dipoles. The dipole may exist permanently or may be induced by the external field. Dielectrics with permanent dipole moments are usually electrically neutral due to random orientation, in the absence of electric fields, of the dipole moments. An example is H_2O. We will study electromagnetic theory of polarizable materials in terms of dipoles, which we will treat as building blocks of dielectric materials.

10.1 Potential and Electric Field of a Single Dipole

Mathematically, it is convenient to deal with the dipole not as just a pair of individual plus and minus charges, but as a separate object on its own rights.

Suppose that opposite charges $\pm q$ are separated by a distance d. We will find the potential Φ at a distance r and angle θ under the assumption that $r \gg d$, called the dipole potential.



We define dipole moment as the product of the charge times the separation

$$\vec{p} = q\vec{d}\,.$$

Since $r_2 - r_1 \approx d\cos\theta$ at $r \gg d$, we get for the potential at point A

$$\begin{aligned}\Phi &= \frac{q}{4\pi\varepsilon_0 r_1} - \frac{q}{4\pi\varepsilon_0 r_2} \\ &= \frac{q}{4\pi\varepsilon_0}\,\frac{r_2 - r_1}{r_1 r_2} = \frac{q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{r_1 r_2} \approx \frac{1}{4\pi\varepsilon_0}\,\frac{\vec{p}\cdot\hat{r}}{r^2}\ .\end{aligned} \tag{28}$$

We can find electric field of a dipole using $\vec{E} = -\nabla\Phi$. Since the potential of the dipole depends on r and θ, it is convenient to work in the spherical coordinates in which the electric field is given by

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta} + E_\phi\hat{\phi}\ ,$$

where

$$\begin{aligned}E_r &= -\frac{\partial\Phi}{\partial r} = \frac{1}{4\pi\varepsilon_0}\,\frac{2p\cos\theta}{r^3}\ , \\ E_\theta &= -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{1}{4\pi\varepsilon_0}\,\frac{p\sin\theta}{r^3}\ , \\ E_\phi &= -\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} = 0\ .\end{aligned}$$

Hence

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta}\right)\ .$$

Figure 5 shows a sketch of the electric field lines of an electric dipole moment.

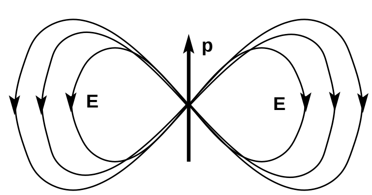


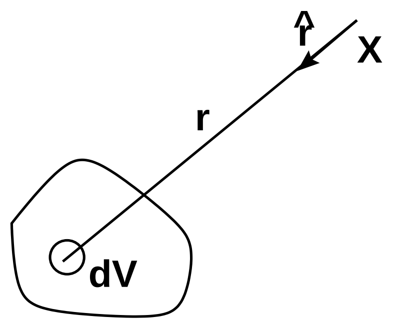
Figure 5: The electric field lines of a dipole moment.

10.2 Polarization Vector

If there are N dipoles per unit volume, the dipole moment per unit volume is:

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i \ ,$$

and is called the **polarization**.



The electric potential set up by an arbitrary volume distribution of electric dipoles can be calculated by using the potential produced by a single dipole, Eq. (28), and the above definition of $\vec{P}$. The electric potential at an arbitrary distance r from a volume element dV containing such dipoles is

$$d\Phi = \frac{1}{4\pi\varepsilon_0} \frac{\vec{P} \cdot \hat{r}}{r^2} dV \ ,$$

where $\hat{r}$ is the unit vector from dV toward X, and we have assumed that r is much larger than the extent of the volume element dV.

Let $\hat{r}$ be the unit vector from X toward dV. (We want to integrate over V with the position of X fixed). In this case, we change $\hat{r} \to -\hat{r}$, and obtain

$$d\Phi = \frac{1}{4\pi\varepsilon_0} \vec{P} \cdot \left(-\frac{\hat{r}}{r^2} \right) dV$$

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int_V \vec{P} \cdot \left(-\frac{\hat{r}}{r^2} \right) dV \ .$$

The result can be transformed into one that has an interesting physical interpretation.

Noting that

$$-\frac{\hat{r}}{r^2} = \nabla\frac{1}{r} \ ,$$

we have

$$\vec{P}\cdot\left(-\frac{\hat{r}}{r^2}\right) = \vec{P}\cdot\nabla\frac{1}{r} \ .$$

Applying a vector identity $\nabla\cdot(\Phi\vec{A}) = \Phi\nabla\cdot\vec{A} + \vec{A}\cdot\nabla\Phi$, we thus obtain

$$\vec{P}\cdot\nabla\left(\frac{1}{r}\right) = \nabla\cdot\left(\frac{\vec{P}}{r}\right) - \frac{\nabla\cdot\vec{P}}{r} \ .$$

Hence

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int_V \nabla\cdot\left(\frac{\vec{P}}{r}\right)\, dV + \frac{1}{4\pi\varepsilon_0}\int_V \left(-\frac{\nabla\cdot\vec{P}}{r}\right)\, dV \ .$$

Transforming the first term using Gauss' divergence theorem, the potential becomes

$$\Phi = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\vec{P}\cdot\hat{n}}{r}\, dS + \frac{1}{4\pi\varepsilon_0}\int_V \left(-\frac{\nabla\cdot\vec{P}}{r}\right)\, dV \ . \tag{29}$$

On comparing of Eq. (29) with the general form of the potential, Eq. (20), we deduce that the first term on the rhs of the above equation is the potential of a surface charge density $\vec{P}\cdot\hat{n}$. The second term is the potential of a volume charge density $-\nabla\cdot\vec{P}$.

10.3 Maxwell's Equation for $\nabla\cdot\vec{E}$ in a Dielectric

In general, the electric field in a dielectric can be found from the Maxwell's equation I:

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \ .$$

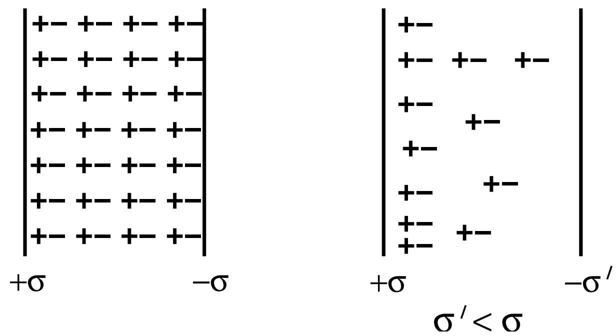


Figure 6: Surface (left picture) and volume charges (right picture). Surface charges exist because there are no neighboring charges at the end surfaces of the material to cancel them out. Volume charges exist because the number of dipoles per unit volume changes, that there is an incomplete cancellation of charge density from adjacent dipoles.

In the dielectric it is convenient to (mentally) separate the polarization charges from whatever other charges might be there also. The other charges are usually referred to as the *free charges* or conducting charges to distinguish them from the *bound charges* in the dielectric.

We can write

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho_f + \rho_p \ ,$$

where ρ_f is the "free" charge density and ρ_p is the polarization charge density throughout the volume. If we express ρ_p in terms of $\vec{P}$ ($\rho_p = -\nabla \cdot \vec{P}$), we have

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho_f - \nabla \cdot \vec{P} \ .$$

Hence

$$\nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f \ . \tag{30}$$

Now it is common practice to drop the subscript f, but one must remember that ρ now stands for *the charge density not counting the polarization charges.*

It is common practice to define

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \ , \tag{31}$$

where $\vec{D}$ is called the *dielectric displacement*.

The reason for the name "dielectric displacement" can be easily understood if we refer to the Maxwell's theory.

Take time derivative of both sides of Eq. (31):

$$\frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} \ .$$

We know from the Maxwell's theory that the first term on the rhs of the above equation represents displacement current density, and the second term is the polarization current density. Therefore, $\partial \vec{D}/\partial t$ can be called a generalization of the displacement current density, and then $\vec{D}$ can be regarded as the dielectric displacement.

Introducing the dielectric displacement, Eq. (30) becomes

$$\nabla \cdot \vec{D} = \rho \ .$$

We can read this equation that the source for the field $\vec{D}$ is the free charge density ρ.

We turn now to a consideration of the macroscopic effects of the polarizability of dielectric materials. We will consider only ideal dielectrics.

Ideal dielectrics can be divided into following categories:

1. Homogeneous – properties independent of the position.
2. Isotropic – properties independent of direction.
3. Linear – polarization proportional to $\vec{E}$.
4. Stationary – properties independent of time.

Case of simple isotropic and linear dielectrics

Ordinary dielectrics (glass, teflon, plastics etc.) are linear in polarization for fields not strong enough to cause dielectric breakdown i.e. $\vec{P} \propto \vec{E}$.
For these materials, we can write

$$\vec{P} = \chi\varepsilon_0\vec{E}$$

where χ is called the dielectric susceptibility. Then, we can write

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0(1+\chi)\vec{E} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon\vec{E} \ ,$$

where ε_r is the *relative permittivity* or *dielectric constant*, and ε is the *permittivity*. Hence

$$\nabla \cdot \vec{D} = \nabla \cdot (\varepsilon\vec{E}) = \rho$$

or

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \ ,$$

if ε is independent of position i.e. if ε is a permittivity of a homogeneous dielectric.

The same result would be obtained by replacing ε_0 in Coulomb's law by ε.

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}\hat{r} \ .$$

The ratio $\varepsilon/\varepsilon_0$ then represents the relative shielding of q_1 from q_2 by the polarization charges induced in the medium.

The theory of the molecular structure of a material will yield an estimate of χ and ε. The dipole moment $\vec{p}$ of a molecule will be proportional to the local electric field $\vec{E}$ so we can define a *molecular susceptibility* χ_m such that $p = \chi_m\varepsilon_0 E$.
Then

$$\vec{P} = \sum_i \vec{p}_i = N\chi_m\varepsilon_0\vec{E} \ ,$$

where N is the number of molecules per unit volume.

Thus

$$\varepsilon = \varepsilon_0 \varepsilon_r = \varepsilon_0 (1 + N\chi_m)$$

and ε is a quantity directly measurable from the measurement of capacitance

$$C = \frac{\varepsilon A}{d} = \varepsilon_r \frac{\varepsilon_0 A}{d} = \varepsilon_r C_0 \ ,$$

where C_0 is the capacitance without the dielectric.

We can summarize, that filling a capacitor with dielectric multiplies its capacitance by ε_r.

Exercise in class: *Capacitor filled with a homogeneous dielectric*

A plane parallel capacitor has charges $+\sigma$ and $-\sigma$ per unit area on its plates. The capacitor is filled with a homogeneous and linear dielectric of dielectric constant $\varepsilon_r = 1 + \chi$.

Show that:

1. The electric field within the dielectric is:

$$E = \frac{\sigma}{\varepsilon_r \varepsilon_0} \ .$$

2. The polarization charge per unit area on the surface of the dielectric adjacent to the surface of the negatively charged plate is:

$$\sigma_s = \frac{\chi \sigma}{1 + \chi} \ .$$

3. The capacitance of the capacitor is $C = \varepsilon_r C_0$ where C_0 is the capacitance of the same capacitor without the dielectric (i.e. a vacuum or air between the plates).

10.4 Dense Dielectrics: The Clausius-Mossotti Relation

The Lorentz theory of polarizability of dense dielectric materials distinguishes between the mean electric field $\vec{E}$ and the local electric field $\vec{E}_{loc}$ as seen by a typical dipole. The typical dipole is considered to be at the centre of a small sphere that has been excavated from the dielectric. $\vec{E}_{loc}$ is thus the mean field $\vec{E}$ minus $\vec{E}_{plug}$ where $\vec{E}_{plug}$ is the field of the spherical volume excavated. We will show that:

$$\vec{E}_{loc} = \vec{E} + \frac{\vec{P}}{3\varepsilon_0} \ .$$

If there are N molecules per unit volume each of polarizability α the poarization vector:

$$\vec{P} = N\alpha\vec{E}_{loc} = N\alpha\left(\vec{E} + \frac{\vec{P}}{3\varepsilon_0}\right) \ ,$$

and by definition:

$$\vec{P} = (\varepsilon_r - 1)\varepsilon_0\vec{E} \ .$$

Let $\vec{E}$ be the mean field throughout the dielectric.
Let $\vec{E}_{plug}$ be the field due to the spherical plug alone.
Let $\vec{E}_{loc}$ be the field in the spherical hole.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{plug} \quad \text{and from above} \quad \vec{E}_{plug} = -\frac{\vec{P}}{3\varepsilon_0} \ .$$

Thus

$$\vec{E}_{loc} = \vec{E} + \frac{\vec{P}}{3\varepsilon_0} \ .$$

The argument now is that each molecule is at the centre of a small hole and the field acting on the molecule is thus $\vec{E}_{loc}$. If α is the molecular polarizability, its induced dipole is thus:

$$\vec{p} = \alpha\vec{E}_{loc}$$

If there are N molecules per unit volume then:

$$\vec{P} = N\vec{p} = N\alpha\vec{E}_{loc} = N\alpha\left(\vec{E} + \frac{\vec{P}}{3\varepsilon_0}\right) \ .$$

By definition $\vec{P} = \chi\varepsilon_0\vec{E} = (\varepsilon_r - 1)\varepsilon_0\vec{E}$. Substituting for $\vec{P}$:

$$\begin{aligned}
(\varepsilon_r - 1)\varepsilon_0\vec{E} &= N\alpha\left[\vec{E} + \frac{(\varepsilon_r - 1)\varepsilon_0}{3\varepsilon_0}\vec{E}\right] \\
(\varepsilon_r - 1)\varepsilon_0 &= N\alpha\left(1 + \frac{\varepsilon_r - 1}{3}\right) = \frac{N\alpha}{3}(\varepsilon_r + 2) \\
\frac{\varepsilon_r - 1}{\varepsilon_r + 2} &= \frac{N\alpha}{3\varepsilon_0} \ ,
\end{aligned}$$

which is known as the Clausius-Mossotti relation for a dense dielectric.

We can solve the Clausius-Mossotti equation for ε_r, and obtain

$$\begin{aligned}
\varepsilon_r - 1 &= (\varepsilon_r + 2)\frac{N\alpha}{3\varepsilon_0} \\
\varepsilon_r\left[1 - \frac{N\alpha}{3\varepsilon_0}\right] &= 1 + \frac{2N\alpha}{3\varepsilon_0} \ .
\end{aligned}$$

Note that

$$\varepsilon_r = \frac{1 + \frac{2N\alpha}{3\varepsilon_0}}{1 - \frac{N\alpha}{3\varepsilon_0}} \to \infty \ \text{ as } \ \frac{N\alpha}{3\varepsilon_0} \to 1 \ ,$$

which is called the Clausius-Mossotti catastrope.

Removal of the plug leaves polarization charges, whose field tends to line up the dipole parallel to the external field. The system is self-polarizing (Clausius-Mossotti catastrope) if $\frac{N\alpha}{3\varepsilon_0} \to 1$.

10.5 Time Dependent Fields and the Complex Dielectric Susceptibility

The induced polarization charges do not produce any currents inside the dielectric. There is no DC current in response to a DC electric field, but if $\vec{P}$

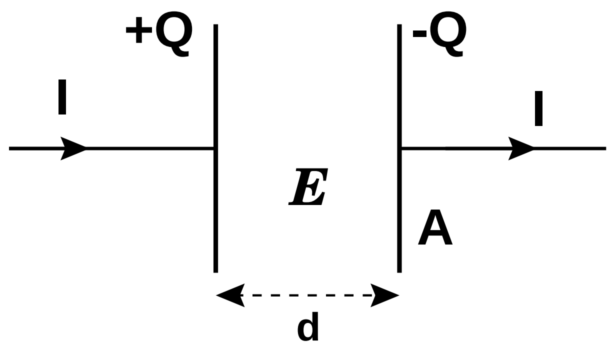
is changing with time (because $\vec{E}$ is changing with time) there will be an AC current density:

$$\vec{J} = \frac{\partial \sigma_s}{\partial t}\hat{n} = \frac{\partial \vec{P}}{\partial t} \ .$$

Thus, $\partial\vec{P}/\partial t$ plays the role of polarization current density.

Let us consider what will happen if a dielectric is introduced into a time-varying electric field.

With an alternating electric field (due say to an AC voltage across a capacitor filled with a dielectric) the polarization $\vec{P}$ may lag in phase behind the driving field $\vec{E}$. This means there is internal friction and heat dissipation as is discussed below. The "capacitor" will exhibit resistive as well as capacitive properties.



Consider the work done in charging a capacitor. We start from the circuit theory quantities and will express them in terms of the field quantities.

The work done in charging the capacitor is

$$\frac{dW}{dt} = VI = Ed\frac{dQ}{dt} \ ,$$

where V is the voltage.

Since

$$Q = CV = CEd = \frac{\varepsilon A}{d} Ed \ , = A\varepsilon E = AD$$

we obtain

$$Q = A(\varepsilon_0 E + P) \ ,$$

and then

$$\frac{dW}{dt} = Ed\frac{d}{dt}A(\varepsilon_0 E + P) = E\,dA\frac{d}{dt}(\varepsilon_0 E + P) \ .$$

Since $dA = \mathcal{V}$ is the volume of the capacitor, we can write

$$\frac{dW}{dt} = \int\limits_{\mathcal{V}} \left[E\frac{d}{dt}(\varepsilon_0 E) + E\frac{dP}{dt}\right] d\mathcal{V} \ ,$$

or

$$\frac{dW}{dt} = \int\limits_{\mathcal{V}} \frac{d}{dt}\left(\frac{1}{2}\varepsilon_0 E^2\right) d\mathcal{V} + \int\limits_{V} E\frac{dP}{dt}\, d\mathcal{V} \ ,$$

where we took into account a possibility that the electric field and polarization can vary across the capacitor's plates.

The first term in the above equation is the rate of doing work building up $\vec{E}$ field. The second term is the rate of doing work on the dipoles by $\vec{E}$.

Thus, the supplied energy to the capacitor is used to build up the electric field inside the capacitor and to polarize the dielectric. Consider separately both terms.

First term:

If $E = E_0\ \cos(\omega t)$, the first term takes the form

$$\int\limits_{\mathcal{V}} -E_0 \cos(\omega t)\, \varepsilon_0 \omega E_0 \sin(\omega t)\, d\mathcal{V} \ .$$

The element of work done per unit volume unit time is

$$\frac{dW}{d\mathcal{V}} = -\varepsilon_0 \omega E_0^2\ \cos(\omega t)\ \sin(\omega t)$$

Averaging over a cycle, we get

$$\frac{dW}{d\mathcal{V}} = -\varepsilon_0 \omega E_0^2 \int\limits_{t=0}^{2\pi/\omega} \cos(\omega t)\, \sin(\omega t)\, dt = 0\ .$$

Work is done building up the field in one part of the cycle but the stored energy is given back in another part.

Second term:

If $P = \chi\varepsilon_0 E = \chi\varepsilon_0\, E_0 \cos(\omega t)$ the same zero net energy conversion averaged over a cycle will happen with this term. If there is internal friction there will be a phase difference between P and E.
Write

$$P = \chi\varepsilon_0 E_0 \cos(\omega t + \phi)$$

Then

$$P = \chi\varepsilon_0 E_0\, \cos\phi\, \cos(\omega t) - \chi\varepsilon_0 E_0\, \sin\phi\, \sin(\omega t)$$

and

$$\frac{dP}{dt} = -\omega\chi\varepsilon_0 E_0\, \cos\phi\, \sin(\omega t) - \omega\chi\varepsilon_0 E_0\, \sin\phi\, \cos(\omega t)$$

Hence, the work done per unit volume per cycle will be

$$\begin{aligned}\frac{dW}{d\mathcal{V}} &= -\int\limits_0^{2\pi/\omega} \omega\chi\varepsilon_0 E_0^2 \cos\phi\, \cos(\omega t)\, \sin(\omega t)\, dt \\ &\quad -\int\limits_0^{2\pi/\omega} \omega\chi\varepsilon_0 E_0^2\, \sin\phi\, \cos^2(\omega t)\, dt\ .\end{aligned}$$

Since the first integral is zero, we get

$$\frac{dW}{d\mathcal{V}} = -\omega\chi\varepsilon_0 E_0^2\, \sin\phi \int\limits_0^{2\pi/\omega} \cos^2(\omega t)\, dt\ .$$

The integral on the rhs of the above equation is positive and $dW/d\mathcal{V}$ must be positive corresponding to energy dissipation (or the dielectric would keep getting energy from its interior and building up the field with it).

Thus, $\sin\phi$ must be negative, so $-\pi < \phi < 0$.

10.6 The Complex Susceptibility and Permittivity

In the usual way use complex exponentials to represent amplitude and phase. We can write

$$E = E_0 \mathrm{e}^{i\omega t} \quad \text{then} \quad P = P_0 \mathrm{e}^{i(\omega t - \phi)} \qquad (\phi \text{ is positive}) \ .$$

We can write the complex polarization in different forms

$$\begin{aligned} P &= P_0 \mathrm{e}^{-i\phi}\,\mathrm{e}^{i\omega t} = (P_0\cos\phi - iP_0\sin\phi)\mathrm{e}^{i\omega t} \ , \\ P &= \left(\frac{P_0}{E_0}\cos\phi - i\frac{P_0}{E_0}\sin\phi\right)E_0\mathrm{e}^{i\omega t} \ , \\ P &= \varepsilon_0(\chi' - i\chi'')E_0\mathrm{e}^{i\omega t} = \varepsilon_0\chi_c\,E_0\,\mathrm{e}^{i\omega t} = \varepsilon_0\chi_c\,E \ , \end{aligned}$$

where $\chi_c = \chi' - i\chi''$ is a complex susceptibility.

With the complex polarization, the dielectric displacement takes the form

$$D = \varepsilon_0 E + P = \varepsilon_0 E + \varepsilon_0\chi_c E = \varepsilon_0(1 + \chi_c)E \ ,$$

or

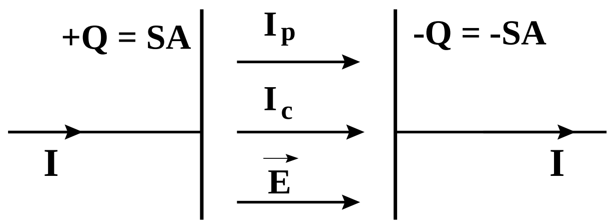
$$D = \varepsilon_0(1 + \chi' - i\chi'')E = \varepsilon_0\varepsilon_r E = \varepsilon_c E \ ,$$

where ε_c is a complex permittivity, and ε_r is a complex relative permittivity or dielectric constant

$$\varepsilon_c = \varepsilon_0(1 + \chi') - i\varepsilon_0\chi'' \ .$$

Example - equivalent circuit of the lossy capacitor

The imaginary part of the dielectric susceptibility corresponds to net energy dissipation i.e. in the circuit theory language the dielectric susceptibility adds a resistive component to the capacitor. The material filling the capacitor could also have some ordinary ohmic conductivity (due to the presence of 'free' charges as well as 'bound' charges in the material). Let us calculate the magnitude of the resistance.



Let I be the charging current in the external circuit, and

$$I_p = \int \vec{J}_p \cdot d\vec{A} = A\frac{dP}{dt}$$

be the polarization current in the dielectric. Let

$$I_c = \int \vec{J}_c \cdot d\vec{A} = A\sigma E = A\sigma\frac{V}{d}$$

be the conduction current in the dielectric due to its finite conductivity σ.

If S is the charge per unit area on the plates then:

$$\begin{aligned} S &= (\text{charge supplied by } I) \\ &- (\text{charge removed by } I_p) \\ &- (\text{charge removed by } I_c)\ . \end{aligned}$$

Then

$$Q = SA = \int I\,dt - \int I_p\,dt - \int I_c\,dt$$

and

$$E = \frac{S}{\varepsilon_0} = \frac{V}{d}$$

Hence

$$S = \varepsilon_0 \frac{V}{d}$$

and also remember that $P = \chi \varepsilon_0 E = \chi \varepsilon_0 \frac{V}{d}$, we obtain

$$Q = SA = \varepsilon_0 \frac{V}{d} A = \int I dt - \int I_p \, dt - \int I_c \, dt$$

or taking a derivative in time, we get

$$\begin{aligned} \varepsilon_0 \frac{A}{d} \frac{dV}{dt} &= I - I_p - I_c \\ &= I - A \frac{\chi \varepsilon_0}{d} \frac{dV}{dt} - A \frac{\sigma V}{d} \end{aligned}$$

Putting

$$\frac{dV}{dt} = i\omega V \quad \text{and} \quad \chi = \chi' - i \, \chi''$$

and solving for I, we get

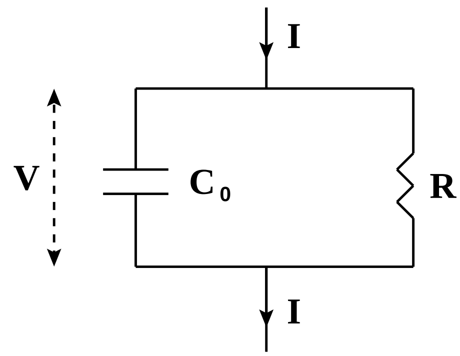
$$I = \frac{\varepsilon_0 A}{d} \left[(1 + \chi' - i\chi'') i\omega + \frac{\sigma}{\varepsilon_0} \right] V \ .$$

Separating real and imaginary parts and putting $C_0 = \varepsilon_0 A/d$ (the capacitance there would be if the dielectric were lossless):

$$I = \left[C_0 \left(\frac{\sigma}{\varepsilon_0} + \omega \chi'' \right) + i\omega C_0 \left(1 + \chi' \right) \right] V \ .$$

Since the capacitor transmits some charges through the internal dielectric, in the circuit theory this system is equivalent to a parallel circuit

$$I = \left(\frac{1}{R} + i\omega C \right) V \ .$$



Comparing with the above result for current flow in the lossy capacitor we see that the effective capacitance is $C_0(1+\chi')$ and the effective resistance is

$$R = \frac{1}{C_0\left(\frac{\sigma}{\varepsilon_0} + \omega\chi''\right)} \ ,$$

where we remember that C_0 is the capacitance in the absence of losses.

10.7 Added Note - The Loss Tangent

The properties of a dielectric material are usually specified by giving its dielectric constant K, and its loss tangent $\tan\delta$.

If we write:

$$I = i\omega C_0\left[(1+\chi') - i\left(\chi'' + \frac{\sigma}{\varepsilon_0\omega}\right)\right] V \ .$$

The quantity in [ ] brackets can be defined as a complex relative permittivity ε_r:

$$\varepsilon_r = 1 + \chi' - i(\chi'' + \frac{\sigma}{\varepsilon_0\omega}) \ .$$

This is a generalization on the previous definition of complex relative permittivity to include the effects of ohmic conductivity. Then we can define

a generalized permittivity and dielectric constant $\varepsilon = \varepsilon_0 \varepsilon_r$.
Now if we write:

$$\begin{aligned} \varepsilon_r &= K' e^{-i\delta} = K'(\cos\delta - i\,\sin\delta) \\ &= K'\cos\delta(1 - i\,\tan\delta) = K(1 - i\,\tan\delta)\ . \end{aligned}$$

Then the standard form is

$$\varepsilon = \varepsilon_0 K(1 - i\,\tan\delta)\ ,$$

where $\tan\delta$ is the 'loss tangent' and is read out on some AC bridges as an alternative to reading out the resistive property of a lossy capacitor.
Since $K'\,\cos\delta = (1+\chi')$ and $K'\,\sin\delta = (\chi'' + \frac{\sigma}{\varepsilon_0\omega})$ it follows that:

$$\tan\delta = \frac{\left[\chi'' + \frac{\sigma}{\varepsilon_0\omega}\right]}{[1+\chi']}\ .$$

In this equation, $\tan\delta$ includes the effects of finite conductivity and the effects of polarization damping force.

Questions:

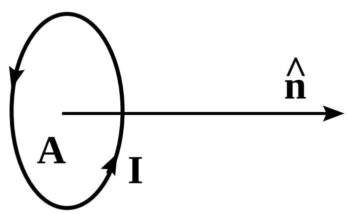
(1) Show that the electrostatic potential due to a distribution of electric dipoles of moment per unit volume $\vec{P}$ throughout a volume V enclosed by a surface S is that of a volume charge density $-\nabla\cdot\vec{P}$ together with a surface charge density $\vec{P}\cdot\hat{n}$.

(2) Show that the polarization of a dielectric driven by a time varying electric field lags in phase the driving field.

(3) Show that the phase difference between the polarization and a time-varying electric field results in a complex permittivity of the dielectric.

11 Magnetic Fields in Magnetizable Materials

We have previously discussed how the polarization of dielectrics by an externally applied electric field is equivalent to volume and surface distribution of charge. Analogously, a magnetic field can act on molecular scale current loops to produce macroscopic effects. It was Ampère who first suggested that the magnetism of matter was due to the cooperative effects of currents circulating in atoms (and not, as previously thought, due to a separate magnetic charge called *poles*).



11.1 Magnetic Polarization Currents

We define the magnetic moment of a current loop as

$$\vec{\mu} = IA\hat{n} \ ,$$

which is equal to the product of the area of the plane loop and the magnitude of the circulating current. The vector direction $\hat{n}$ of the moment is perpendicular to the plane of the loop and along the direction set by the right-hand rule.

In a material body we define a macroscopic dipole moment per unit volume

$$\vec{M} = \sum_i \vec{\mu}_i \ ,$$

which is called **magnetization**.

Theorem

If a volume V enclosed by surface S has a magnetic dipole moment per unit volume $\vec{M}$ (which may be a function of position), the macroscopic magnetic fields so produced are equivalent to those of:

- A volume current density $\vec{J}_V = \nabla \times \vec{M}$.
- A surface current density $\vec{J}_S = \vec{M} \times \hat{n}$.

Proof:

The theorem is proved by showing that the vector potential due to the dipole distribution in a volume V closed by a surface S can be written in the form

$$\vec{A} = \frac{1}{4\pi\varepsilon_0 c^2} \int\limits_V \frac{\nabla \times \vec{M}}{r}\, dV + \frac{1}{4\pi\varepsilon_0 c^2} \oint\limits_S \frac{\vec{M} \times \hat{n}}{r}\, dS \ .$$

From the solution of the Maxwell's equations **(static) vector potential of a current loop** is given by

$$\vec{A} = \frac{1}{4\pi\varepsilon_0 c^2} \int \frac{\vec{J}}{r}\, dV = \frac{I}{4\pi\varepsilon_0 c^2} \int \frac{d\vec{l}}{r} \ .$$

Consider a current loop of radius a, shown in Fig. 7.

In polar spherical coordinates

$$d\vec{l} \ = \ a d\phi \ \hat{\phi} = -a \sin\phi \ d\phi \ \hat{i} + a\cos\phi \ d\phi \ \hat{j} \ .$$

However

$$r = \left[(x - a\cos\phi)^2 + (y - a\sin\phi)^2 + z^2\right]^{1/2} \ ,$$

which for $a \ll R$ can be written as

$$\begin{aligned} r &= \left(x^2 + y^2 + z^2 - 2ax\cos\phi + a^2 - 2ay\sin\phi\right)^{1/2} \\ &\approx \left(R^2 - 2ax\cos\phi - 2ay\sin\phi\right)^{1/2} \\ &\approx R\left(1 - \frac{ax\cos\phi}{R^2} - \frac{ay\sin\phi}{R^2}\right) \ . \end{aligned}$$

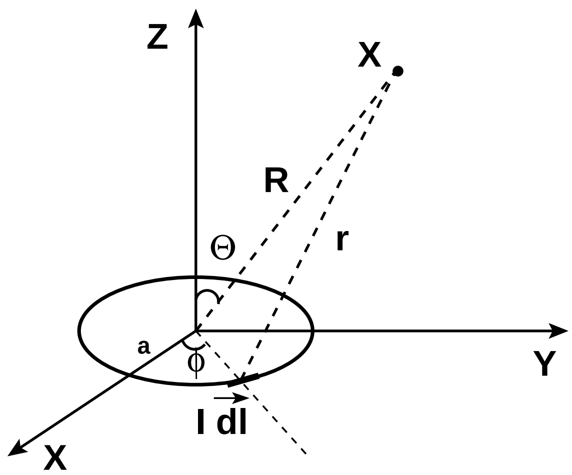


Figure 7: Current loop of radius a.

Hence

$$r^{-1} = R^{-1}\left(1 + \frac{ax\cos\phi}{R^2} + \frac{ay\sin\phi}{R^2}\right) \ ,$$

where we have used the Taylor expansion of $1/(1-x) = 1 + x + \ldots$.

Thus

$$\begin{aligned}\vec{A} &= \frac{I}{4\pi\varepsilon_0 c^2 R}\int_0^{2\pi}\left(1 + \frac{ax\cos\phi}{R^2} + \frac{ay\sin\phi}{R^2}\right) \\ &\qquad \times\left(-a\sin\phi\ d\phi\ \hat{i} + a\cos\phi\ d\phi\ \hat{j}\right) \ .\end{aligned}$$

Since

$$\int_0^{2\pi}\sin\phi\ d\phi = \int_0^{2\pi}\cos\phi\ d\phi = \int_0^{2\pi}\sin\phi\cos\phi\ d\phi = 0 \ ,$$

the formula for $\vec{A}$ simplifies to

$$\vec{A} = \frac{I}{4\pi\varepsilon_0 c^2 R^3}\int_0^{2\pi}\left(-a^2 y\sin^2\phi\ \hat{i} + a^2 x\cos^2\phi\ \hat{j}\right)d\phi \ ,$$

which can be written as

$$\vec{A} \;=\; \frac{Ia^2}{4\pi\varepsilon_0 c^2 R^3}\left[-y\hat{\imath}\int_0^{2\pi}\sin^2\phi\;d\phi + x\hat{\jmath}\int_0^{2\pi}\cos^2\phi\;d\phi\right]\;.$$

Next, since

$$\int_0^{2\pi}\sin^2\phi\;d\phi = \int_0^{2\pi}\cos^2\phi\;d\phi = \pi\;,$$

we obtain

$$\vec{A} \;=\; \frac{Ia^2\pi}{4\pi\varepsilon_0 c^2 R^3}\left[-y\hat{\imath} + x\hat{\jmath}\right]\;.$$

Using the relation

$$\hat{k}\times\hat{R} = \hat{k}\times\left[\frac{x}{R}\hat{\imath} + \frac{y}{R}\hat{\jmath} + \frac{z}{R}\hat{k}\right] = \frac{x\hat{\jmath}}{R} - \frac{y\hat{\imath}}{R}$$

we get

$$\begin{aligned}\vec{A} \;&=\; \frac{Ia^2\pi}{4\pi\varepsilon_0 c^2 R^2}\;\hat{k}\times\hat{R} = \frac{\mu}{4\pi\varepsilon_0 c^2}\;\hat{k}\times\frac{\hat{R}}{R^2}\\ &=\; -\frac{1}{4\pi\varepsilon_0 c^2}\;\vec{\mu}\times\nabla\frac{1}{R}\end{aligned}$$

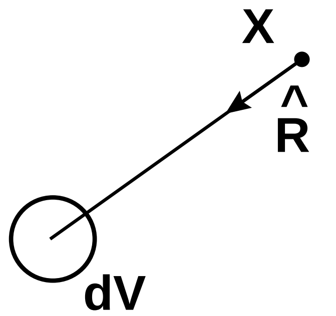
where we have used the result

$$\nabla\frac{1}{R} = -\frac{\hat{R}}{R^2}$$

If we consider a set of dipole moments and change, for a convenience, the direction of $\hat{R}$ into $-\hat{R}$, we can write that the vector potential $d\vec{A}$ produced by a set of dipole moments contained in a volume dV is

$$d\vec{A} \;=\; \frac{1}{4\pi\varepsilon_0 c^2}\;\vec{M}\times\nabla\frac{1}{R}\;dV$$

where $\vec{M}$ is the magnetic dipole moment per unit volume.



Then, the vector potential produced by the whole dipole moments contained in the volume V is

$$\vec{A} \;=\; \frac{1}{4\pi\varepsilon_0 c^2} \int \vec{M} \times \nabla \frac{1}{R}\, dV$$

Using a vector identity

$$\nabla \times (\phi \vec{A}) = \nabla \phi \times \vec{A} + \phi \nabla \times \vec{A}$$

we have

$$\vec{M} \times \nabla \frac{1}{R} = -\nabla \frac{1}{R} \times \vec{M} = \frac{\nabla \times \vec{M}}{R} - \nabla \times \frac{\vec{M}}{R}$$

and then the vector potential is

$$\vec{A} \;=\; \frac{1}{4\pi\varepsilon_0 c^2} \int \frac{\nabla \times \vec{M}}{R}\, dV - \frac{1}{4\pi\varepsilon_0 c^2} \int \nabla \times \frac{\vec{M}}{R}\, dV$$

In order to proceed further, we introduce a **Theorem**

$$-\int_V \nabla \times \frac{\vec{M}}{R}\, dV = \int_S \frac{\vec{M} \times \hat{n}}{R}\, dS$$

This is an application of the more general theorem

$$-\int_V \nabla \times \vec{F}\, dV = \int_S \vec{F} \times \hat{n}\, dS$$

Proof:

Let $\vec{C}$ be a constant vector. Then

$$\begin{aligned}\nabla \cdot (\vec{F} \times \vec{C}) &= (\nabla \times \vec{F}) \cdot \vec{C} - (\nabla \times \vec{C}) \cdot \vec{F} \\ &= \vec{C} \cdot (\nabla \times \vec{F}) \qquad (32)\end{aligned}$$

We will prove the general theorem by using the divergence theorem

$$\int_V \nabla \cdot (\vec{F} \times \vec{C})\ dV = \int_S (\vec{F} \times \vec{C}) \cdot \hat{n}\ dS$$

For an arbitrary constant vector $\vec{C}$, and using (32), we get

$$\begin{aligned}\vec{C} \cdot \int_V \nabla \times \vec{F}\ dV &= \int_V \vec{C} \cdot \nabla \times \vec{F}\ dV \\ &= \int_V \nabla \cdot (\vec{F} \times \vec{C})\ dV = \int_S \vec{F} \times \vec{C} \cdot \hat{n}\ dS\ .\end{aligned}$$

Hence

$$\vec{C} \cdot \int_V \nabla \times \vec{F}\ dV = -\int_S \vec{C} \times \vec{F} \cdot \hat{n}\ dS\ .$$

However

$$\vec{C} \times \vec{F} \cdot \hat{n} = \vec{C} \cdot \vec{F} \times \hat{n}$$

and then we obtain

$$\vec{C} \cdot \int_V \nabla \times \vec{F}\ dV = -\vec{C} \cdot \int_S \vec{F} \times \hat{n}\ dS$$

Since this is true for arbitrary $\vec{C}$, we finally have

$$\int_V \nabla \times \vec{F}\ dV = -\int_S \vec{F} \times \hat{n}\ dS$$

as required.

Thus

$$\vec{A} = \frac{1}{4\pi\varepsilon_0 c^2} \int_V \frac{\nabla \times \vec{M}}{R}\ dV + \frac{1}{4\pi\varepsilon_0 c^2} \int_S \frac{\vec{M} \times \hat{n}}{R}\ dS\ ,$$

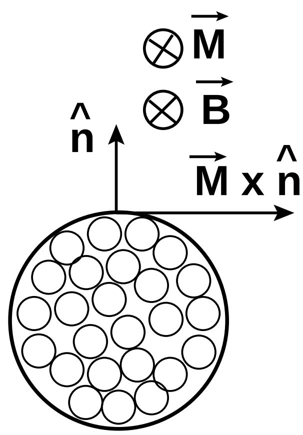
or

$$\vec{A} \;=\; \frac{1}{4\pi\varepsilon_0 c^2}\int_V \frac{\vec{J}_V}{R}\,dV + \frac{1}{4\pi\varepsilon_0 c^2}\int_S \frac{\vec{J}_S}{R}\,dS\;.$$

The effective (Ampere) currents associated with a dipole moment per unit volume $\vec{M}$ are

(i) A current density $\vec{J}_V = \nabla \times \vec{M}$ throughout the volume.

(ii) A surface currents $\vec{J}_S = \vec{M} \times \hat{n}$.



11.2 The Magnetic Intensity Vector $\vec{H}$

When we were dealing with dielectric materials in the presence of electric fields, it was convenient to introduce the displacement vector $\vec{D}$ in order to eliminate the necessity of taking the electric dipole polarization $\vec{P}$ of the material into account explicitly.

A similar procedure is used for the magnetic materials, and we will illustrate it here for both static and time-varying fields.

For a static current distribution, the Maxwell's equation IV reduces to

$$\nabla \times \vec{B} = \mu_0 \vec{J}\;.$$

In a medium where there are magnetic polarization currents as well as conduction currents we can write

$$\begin{aligned}\vec{J} &= \vec{J}_c + \vec{J}_m \ , \\ \vec{J} &= \vec{J}_c + \nabla \times \vec{M} \ .\end{aligned}$$

Hence

$$\nabla \times \vec{B} = \mu_0(\vec{J}_c + \nabla \times \vec{M}) \ ,$$

which can be written as

$$\nabla \times (\vec{B} - \mu_0\vec{M}) = \mu_0\vec{J}_c \ ,$$

or

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_c \ .$$

This shows that the vector $\vec{B}/\mu_0 - \vec{M}$ has as its source only the conduction current $\vec{J}_c$. Therefore, to eliminate the necessity of dealing directly with the magnetization $\vec{M}$, we can define a new vector

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \ ,$$

which is called the magnetic (field) intensity vector.

Then the Maxwell's equation IV for static fields in magnetic materials takes the form

$$\nabla \times \vec{H} = \vec{J}_c \ .$$

In dealing with magnetic materials we often know $\vec{J}_c$ but not $\vec{M}$ (well not directly anyway.) Think e.g. of an inductor filled with some magnetizable material like iron. Then $\vec{H}$ becomes useful. It is a way of avoiding a detailed calculation of the polarization currents. The magnetic intensity $\vec{H}$ is the magnetic analogue of the dielectric displacement $\vec{D}$ in the electric case. We may drop the subscript c and write

$$\nabla \times \vec{H} = \vec{J} \ ,$$

but we should remember that $\vec{J}$ is now not the total electric current density everywhere.

11.3 Linear Isotropic Magnetic Materials

For most materials (excluding ferromagnetics) the magnetization $\vec{M}$ is proportional to the applied external field. Hence, at any point the vectors $\vec{B}$, $\vec{M}$, and $\vec{H}$ will be in the same direction and we may write

$$\vec{B} = \mu_0\mu_r\vec{H} = \mu_0(1+\chi_m)\vec{H} = \mu\vec{H} \ ,$$

where μ_r is the *relative permeability*, χ_m is the *magnetic susceptibility*, and $\mu = \mu_0(1+\chi_m)$ is called the magnetic permeability.

Since

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \ ,$$

we have defined χ_m such that

$$\mu_0\vec{M} = \mu_0\chi_m\vec{H} = \frac{\mu_0\chi_m\vec{B}}{\mu} = \frac{\chi_m\vec{B}}{1+\chi_m} \ .$$

Hence, we get

$$\vec{H} = \frac{\vec{B}}{\mu_0(1+\chi_m)} \ .$$

Thus, if we know the material we use, we can find $\vec{H}$.

Example *A solenoid filled with magnetizable material*

In the expression $B = \mu_0 NI$ we should include the Ampere surface currents as well as the conduction currents in the wire. Since, $B = \mu_0 I^{'}$, where $I^{'}$ is the total current per unit length

$$B = \mu_0(NI + M) = \mu_0(NI + \chi_m H) = \mu_0(NI + \frac{\chi_m B}{\mu}) \ ,$$

which can be written as

$$B(1 - \frac{\mu_0\chi_m}{\mu}) = \mu_0 NI \ ,$$

or

$$B = \frac{\mu_0 NI}{1 - \frac{\mu_0 \chi_m}{\mu}} \ .$$

Note that if χ_m is positive then B is greater than it would have been in the absence of the magnetizable material. Evidently in this case the macroscopic Ampere current is in the same sense as the conduction current in the solenoid.

From the definitions

$$1 - \frac{\mu_0 \chi_m}{\mu} = 1 - \frac{\mu_0 \chi_m}{\mu_0(1+\chi_m)}$$
$$1 - \frac{\chi_m}{1+\chi_m} = \frac{1}{1+\chi_m} = \frac{1}{\mu_r}$$

$$B = \mu_r \mu_0 NI = \mu NI = \mu H \ .$$

The effect of filling the solenoid with a material of relative permeability μ_r is to multiply B by a factor μ_r (assuming the current in the wire remains the same).

Note: H is *independent of the presence or absence of the magnetic material.*

Consider a solenoid filled with a magnetic material. By the definition

$$H = \frac{B}{\mu_0} - M \ .$$

When we remove the magnetic material, $M = 0$, and then

$$H = \frac{B}{\mu_0} = \frac{\mu_0 NI}{\mu_0} = NI \ .$$

With the material present

$$H = \frac{B}{\mu_0} - M = \frac{\mu NI}{\mu_0} - \chi_m H = \mu_r NI - \chi_m H \ .$$

Hence

$$H = (1+\chi_m)NI - \chi_m H$$

from which, we find

$$H(1+\chi_m) \;=\; (1+\chi_m)NI$$

and then

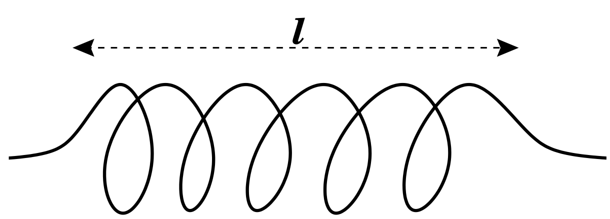
$$H \;=\; NI$$

as before.

This gives rise to the notion of H as an inducing field and B as a resultant field. This concept is much used in the study of magnetic properties of materials.

The effect of magnetic materials on inductors

Because the relative permeability μ_r multiplies the magnetic field by μ_r for the same current, the self inductance of a solenoid is multiplied by μ_r.



Let N is a number of turns/meter. Hence, Nl is the number of turns, and then the flux is

$$\Psi_o = L_o I = nB_o A = (N\ell)(\mu_0 NI)A \ .$$

Thus

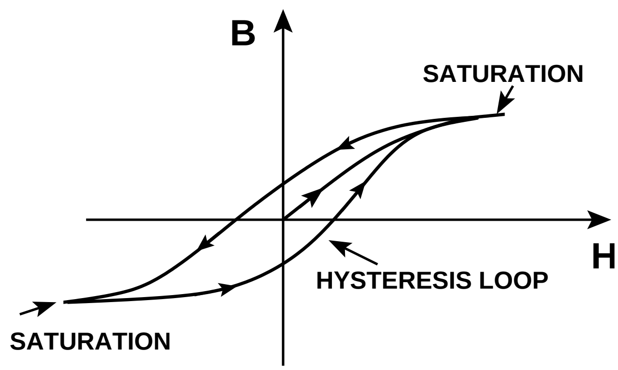
$$L_o = \mu_0 N^2 A\ell$$

with relative permeability μ_r, so that $B \Rightarrow \mu_r B_o$.

$$L = \mu_r\mu_0 N^2 A\ell = \mu_r L_o \ .$$

11.4 The Magnetization of Iron

Iron is not a linear isotropic material. It is subject to saturation of the internally produced magnetic fields because all the internal current loops are lined up. Iron does not have a unique value of magnetic susceptibility because of strong nonlinearities. In ferromagnetic materials the relation between B and H is usually presented graphically in terms of hysteresis.



Work done in magnetization

Think of the case of the solenoid of length ℓ and cross-section area A filled with magnetic material. The applied voltage V is:

$$V = -\mathcal{E} = \frac{d\Psi}{dt} = \ell N \frac{d}{dt} BA \ ,$$

where Nl is the number of turns.

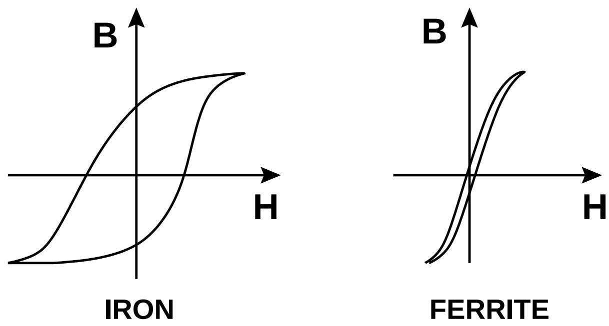
The rate of doing work is:

$$P = VI = A\ell NI \frac{dB}{dt} = \mathcal{V}\ H \frac{dB}{dt}$$

where $\mathcal{V} = Al$ is the volume of the material.

In the time dt that it takes to change B by dB, the work done is $dW = Pdt = \mathcal{V}\ H\, dB$. Thus, the work done per unit volume is $dW = H\, dB$.

Hence, the work done per unit volume in one cycle of the hysteresis loop of a non-linear material such as iron is given by the area of the loop.



It is sometimes useful to write

$$\begin{aligned}\int H\,dB &= \mu_0 \int H\,d(H+M)\\ &= \mu_0 \int H\,dH + \mu_0 \int H\,dM\\ &= \frac{1}{2}\mu_0 H^2 + \mu_0 \int H\,dM\ ,\end{aligned}$$

where the first term on the rhs is the work to establish magnetic field, and the second terms is the work by the field H to establish magnetization dM.

11.5 Time Dependent Magnetic Fields and Energy Loss

In a time-dependent magnetic field the magnetization $\vec{M}$ may not stay in phase with the driving field $\vec{H}$. This corresponds to internal friction and heat dissipation.

Let

$$H = H_0 \cos(\omega t)\ .$$

Then

$$M = M_0 \cos(\omega t + \phi)\ ,$$

where M_0 and ϕ represent the amplitude and phase of the magnetization response to the magnetizing field.
Thus

$$M = M_0 \cos\phi \cos(\omega t) - M_0 \sin\phi \sin(\omega t) \ .$$

Now the work done in magnetization per cycle of the AC current producing the magnetizing field is

$$W = \mu_0 \int H \, dM = \mu_0 \int\limits_{t=0}^{2\pi/\omega} H \frac{dM}{dt} \, dt \ .$$

Since

$$\frac{dM}{dt} = -\omega M_0 \ \cos\phi \ \sin(\omega t) - \omega M_0 \ \sin\phi \ \cos(\omega t) \ ,$$

we have $W = W_1 + W_2$. Consider the term W_1:

$$W_1 = -\mu_0 H_0 M_0 \omega \cos\phi \int\limits_{t=0}^{2\pi/\omega} \cos(\omega t) \ \sin(\omega t) \, dt \ .$$

This term is zero - it represents reversible energy conversion to and from H.

Consider now the term W_2:

$$W_2 = -\mu_0 H_0 M_0 \omega \sin\phi \int\limits_{0}^{2\pi/\omega} \cos^2(\omega t) \, dt \ .$$

This term represents work done against internal friction during magnetizing and demagnetizing the material.

Since $\int \cos^2(\omega t) \, dt$ is positive, $\sin\phi$ must be negative so that work is done on the material i.e. ϕ is negative.

The complex magnetic susceptibility etc

Let

$$H = H_0 e^{i\omega t} \ ,$$

which for a physical field can be written as

$$H = H_0 \cos(\omega t) = \mathrm{Re}\, H_0 \mathrm{e}^{i\omega t} \; .$$

Then

$$M = M_0\, \mathrm{e}^{i(\omega t - \phi)} = M_0\, \mathrm{e}^{-i\phi}\, \mathrm{e}^{i\omega t}$$

or

$$\begin{aligned} M &= (M_0 \cos\phi - i M_0 \sin\phi)\mathrm{e}^{i\omega t} \\ &= \left(\frac{M_0}{H_0}\cos\phi - i\frac{M_0}{H_0}\sin\phi\right) H_0 \mathrm{e}^{i\omega t} \end{aligned}$$

This result can be written in terms of real and imaginary susceptibility

$$M = (\chi' - i\chi'')H_0 \mathrm{e}^{i\omega t} = (\chi' - i\chi'')H \; .$$

Using this complex number notation, we find

$$\begin{aligned} B &= \mu_0(H + M) = \mu_0 H + \mu_0(\chi' - i\chi'')H \\ &= [\mu_0(1 + \chi') - i\mu_0\chi'']H \end{aligned}$$

and then

$$B = (\mu^{'} - i\mu^{''})H = \mu H \; ,$$

where $\mu^{'} = \mu_0(1 + \chi')$, $\mu^{''} = mo\chi''$, and μ is the complex permeability.

11.6 The Ferromagnet

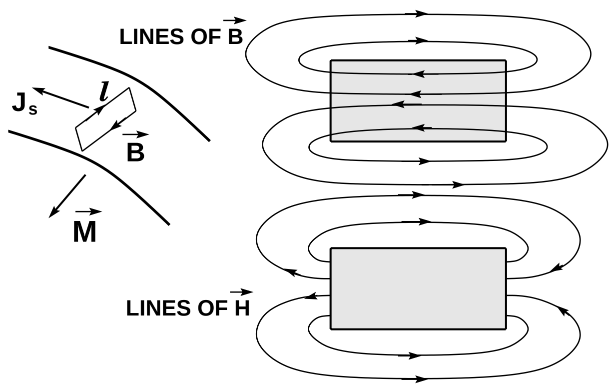
We have so far considered magnetic materials, diamagnetics and paramagnetics in which the magnetization is a function of the external field, i.e. $\vec{M} \sim \vec{B}$. There is a class of materials, called ferromagnetics in which macroscopic magnetization exists even in the absence of the external field.

Consider a homogeneous ferromagnetic material. In this case $\vec{B}$ inside the material is due solely to the $\vec{M}$

$$\vec{J}_s = \vec{M} \times \hat{n}$$

or in terms of the magnitudes

$$\vec{J}_s = \vec{M} \times \hat{n} \ .$$



Using Ampère's law $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$, we obtain

$$B\ell = \mu_0 M \ell \qquad \text{i.e.} \qquad B = \mu_0 M$$

Thus

$$H = \frac{B}{\mu_0} - M = \frac{\mu_0 M}{\mu_0} - M = 0$$

in the region where $M \neq 0$, i.e. inside the magnet.

Outside the magnet

$$H = \frac{B}{\mu_0} - 0 = \frac{B}{\mu_0} \ ,$$

i.e. $\vec{H}$ is just a scaled replica of $\vec{B}$.

Note:

$\nabla \cdot \vec{B} = 0$ always and the lines of $\vec{B}$ form closed loops. Then

$$\nabla \cdot \vec{H} = \nabla \cdot \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = -\nabla \cdot \vec{M} \ .$$

Thus $\vec{H}$ has a source (field lines start and stop) where $\vec{M}$ varies i.e. at the ends of the magnet.

For a ferromagnet

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} = \rho_m \ ,$$

so we can think of ρ_m as a volume density of 'magnetic charge' giving rise to the $\vec{H}$ field. It must be stressed that this equivalence is purely mathematical, and does not prove a physical existence of magnetic charges.

Moreover, for the ferromagnet

$$\nabla \times \vec{H} = \vec{J}_c = 0 \ .$$

Note the similar mathematical properties of $\vec{H}$ here to those of the $\vec{E}$ field in electrostatics

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla \times \vec{E} = 0 \ .$$

Historically, magnetostatics of permanently magnetized materials (ferromagnets) developed via use of the $\vec{H}$ field and 'magnetic charges' or 'poles'. One obtains a law analogous to Coulomb's Law for the force between magnetic poles.

Exercise in class: *Plane magnetized material*

An infinite plane surface divides the universe into a vacuum on one side and a magnetic material on the other. Within the magnetic material there exists a uniform magnetic moment per unit volume $\vec{M}$ which is parallel to the surface.

(a) Show that while the direction of the magnetic induction vector $\vec{B}$ is different on the two sides of the surface, its magnitude is given everywhere by:

$$B = \frac{M}{2\varepsilon_0 c^2} \ .$$

(b) Find the magnitude and direction of the magnetic field $\vec{B}$ everywhere due to an infinite plane parallel slab of material of thickness d which is permanently uniformly magnetized with dipole moment per unit volume $\vec{M}$ lying parallel to the bounding surfaces.

(c) Find the magnetic intensity $\vec{H}$ everywhere.

11.7 Maxwell's Equations in Dielectric and Magnetic Materials

The Maxwell equation containing a current density term is

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \ .$$

We think this is always true provided $\vec{J}$ is the *total electric current density*. Applying this in a region where there may be electric and magnetic polarization effects we can write

$$\vec{J} = \vec{J}_c + \vec{J}_E + \vec{J}_M = \vec{J} + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M} \ .$$

Thus

$$\nabla \times \vec{B} = \mu_0 \vec{J}_c + \mu_0 \frac{\partial \vec{P}}{\partial t} + \mu_0 \nabla \times \vec{M} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \ ,$$

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_c + \frac{\partial}{\partial t} (\varepsilon_0 \vec{E} + \vec{P}) \ ,$$

and finally

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} ,$$

where we must remember that $\vec{J}$ represents the conduction current only.

In summary: The Maxwell's equations in materials are of the following form

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho , \\ \nabla \cdot \vec{B} &= \nabla \cdot \vec{H} = 0 , \\ \nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} , \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} , \end{aligned}$$

but in general

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M} .$$

These equations are supplemented by appropriate constitutive relations which connect the electric field $\vec{E}$ and the magnetic induction $\vec{B}$ with the displacement field $\vec{D}$ and the magnetic field $\vec{H}$

$$\vec{D} = \varepsilon \vec{E} \qquad \text{and} \qquad \vec{B} = \mu \vec{H} .$$

These relations carry information about the material.

Questions:

(1) Show that inside a ferromagnet $H = 0$.

(2) Derive the Maxwell's equations for the EM fields in electric and magnetic materials.

12 Poynting's Theorem Revisited

We have seen in Section 6.1 how energy of the electromagnetic field may be transported through vacuum (empty space) by means of electromagnetic waves. We have shown that the direction of propagation of energy is determined by the Poynting vector. In this lecture, we will reconsider the Poynting theorem taking into account propagation of the electromagnetic field in magnetizable materials.

12.1 Poynting Vector in Terms of $\vec{E}$ and $\vec{H}$

In a vacuum

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{\vec{B}}{\mu_0} = \varepsilon_0 c^2 \vec{B} \ .$$

Thus the Poynting vector is

$$\vec{N} = \varepsilon_0 c^2 \vec{E} \times \vec{B} = \vec{E} \times \vec{H} \ .$$

The cross product $\vec{E} \times \vec{H}$ also turns out to be the correct expression for the Poynting vector when magnetizable materials are involved and is the expression most commonly quoted for it.

Consider, as before in Section 6.1, a flow of the energy through a surface S. Using the Gauss's theorem and a vector identity

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \ ,$$

we obtain

$$\begin{aligned} &\int_s (\vec{E} \times \vec{H}) \cdot d\vec{S} = \int_V \nabla \cdot (\vec{E} \times \vec{H}) \ dV \\ &= \int \vec{H} \cdot (\nabla \times \vec{E}) \ dV - \int \vec{E} \cdot (\nabla \times \vec{H}) \ dV \end{aligned}$$

Now substitute from the Maxwell equations

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B} \ , \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial}{\partial t}\vec{D} \ , \end{aligned}$$

and obtain

$$\int (\vec{E} \times \vec{H}) \cdot d\vec{S} = -\int \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\, dV - \int \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\, dV - \int \vec{E} \cdot \vec{J}\, dV \ .$$

The lhs is the rate of flow of field energy out of volume V.

1. First term on the rhs is the rate of work in establishing the magnetic field in V.

2. Second term is the rate of doing work in establishing the electric field in V.

3. Third term is the rate of doing work on the currents in V.

Note that the rate of doing work in establishing the magnetic field is

$$\int \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\, dV \ .$$

Thus, to change the field by an amount $d\vec{B}$ we have to do work per unit volume

$$dW_M = \vec{H} \cdot d\vec{B}$$

and then the total work is

$$\begin{aligned} W_M &= \int \vec{H} \cdot d\vec{B} = \int \vec{H} \cdot d(\mu_0 \vec{H} + \mu_0 \vec{M}) \\ &= \int \vec{H} \cdot \frac{d(\mu_0 \vec{H})}{dt} dt + \int \mu_0 \vec{H} \cdot \frac{d\vec{M}}{dt} dt \ . \end{aligned}$$

The first term on the rhs is work in energizing field, and the second term is work in aligning magnetic dipoles.

In a previous discussion we saw that $\int \vec{H} \cdot d\vec{M}$ can involve loss of energy from the field to the material (c.f. magnetization of iron). A similar result holds for the electric field.

We go further with Poynting's Theorem

Put $\vec{B} = \mu\vec{H}$, then we get

$$\int \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\, dV = \int \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)\, dV = \frac{\partial}{\partial t}\int \frac{1}{2}\frac{B^2}{\mu}\, dV$$

$$\int \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\, dV = \int \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\, dV = \frac{\partial}{\partial t}\int \frac{1}{2}\varepsilon E^2\, dV\ .$$

Hence

$$\begin{aligned}\int_s (\vec{E} \times \vec{H}) \cdot d\vec{S} &= -\frac{\partial}{\partial t}\int \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\frac{B^2}{\mu}\right)\, dV \\ &\quad -\int \vec{E} \cdot \vec{J}\, dV\ .\end{aligned}$$

Thus, the rate of flow of field energy out of volume V is equal to the rate of changing energy of the EM field plus the arte of doing work on the currents in V.

12.2 Poynting Vector for Complex Sinusoidal Fields

It is well known that the electromagnetic field (e.g. light) is a real physical quantity (observable). However, in the electromagnetic theory it is advantageous to represent the real electromagnetic field by complex sinusoidal quantities because of its mathematical simplicity. In addition, what we usually measure is the average intensity of the field, $\langle \vec{E}^* \cdot \vec{E} \rangle$, which is a real quantity. We usually write

$$\vec{E} = \vec{E}_0 \mathrm{e}^{i\omega t} \qquad \text{and} \qquad \vec{H} = \vec{H}_0 \mathrm{e}^{i\omega t}\ ,$$

where $\vec{E}_0$ and $\vec{H}_0$ are complex quantities including both amplitude and phase information. We understand that the electric and magnetic fields are given by the **REAL PARTS** of $\vec{E}$ and $\vec{H}$.

The power of the complex exponential scheme lies in the fact that for operations such as summation, subtraction, integration etc., we take real parts **AFTER** the operation. For example:

$$\begin{aligned}
\mathrm{Re}\vec{E}_1 + \mathrm{Re}\vec{E}_2 &= \mathrm{Re}\left(\vec{E}_1 + \vec{E}_2\right) \\
\mathrm{Re}\frac{d\vec{E}_1}{dt} + \mathrm{Re}\frac{d\vec{E}_2}{dt} &= \mathrm{Re}\frac{d}{dt}\left(\vec{E}_1 + \vec{E}_2\right)
\end{aligned}$$

Some care has to be taken in evaluating the Poynting vector.

The Poynting vector $\vec{N} = \vec{E} \times \vec{H}$, but if we write complex exponential expressions for $\vec{E} \times \vec{H}$, we must remember that

$$\vec{N} = \mathrm{Re}\vec{E}_c \times \mathrm{Re}\vec{H}_c \neq \mathrm{Re}\left(\vec{E}_c \times \vec{H}_c\right) \ ,$$

where we use a subscript c to indicate that we are writing a complex exponential.

Proof:

We can write

$$\begin{aligned}
\vec{E}_c &= \vec{E}_0 \mathrm{e}^{i\omega t} = \left(\vec{E}_r + i\vec{E}_i\right)(\cos\omega t + i\sin\omega t) \\
&= \left(\vec{E}_r \cos\omega t - \vec{E}_i \sin\omega t\right) + i\left(\vec{E}_r \sin\omega t + \vec{E}_i \cos\omega t\right) \ .
\end{aligned}$$

Similarly

$$\begin{aligned}
\vec{H}_c &= \vec{H}_0 \mathrm{e}^{i\omega t} = \left(\vec{H}_r + i\vec{H}_i\right)(\cos\omega t + i\sin\omega t) \\
&= \left(\vec{H}_r \cos\omega t - \vec{H}_i \sin\omega t\right) + i\left(\vec{H}_r \sin\omega t + \vec{H}_i \cos\omega t\right) \ ,
\end{aligned}$$

where $\vec{E}_r,\ \vec{E}_i,\ \vec{H}_r$ and $\vec{H}_i$ are real vectors.
However

$$\begin{aligned}
\mathrm{Re}\vec{E}_c &= \vec{E}_r \cos\omega t - \vec{E}_i \sin\omega t \\
\mathrm{Re}\vec{H}_c &= \vec{H}_r \cos\omega t - \vec{H}_i \sin\omega t
\end{aligned}$$

Clearly, if we calculate $\mathrm{Re}(\vec{E}_c \times \vec{H}_c)$ we get extra terms in addition to those in the expression for $\mathrm{Re}\vec{E}_c \times \mathrm{Re}\vec{H}_c$, and then

$$\mathrm{Re}\vec{E}_c \times \mathrm{Re}\vec{H}_c \neq \mathrm{Re}(\vec{E}_c \times \vec{H}_c) \ ,$$

as required.

There is however a useful expression for the **MEAN POYNTING VECTOR** in terms of the complex exponential $\vec{E}_c$ and $\vec{H}_c$

$$\bar{\vec{N}} = \frac{1}{2}\mathrm{Re}\left(\vec{E}_c \times \vec{H}_c^*\right) = \frac{1}{2}\mathrm{Re}\left(\vec{E}_c^* \times \vec{H}_c\right) \ ,$$

where "bar" over $\vec{N}$ means average over whole cycles of the sinusoidal field.

Proof:

Since

$$\begin{aligned} & \vec{N} = \mathrm{Re}\vec{E}_c \times \mathrm{Re}\vec{H}_c \\ = & \left(\vec{E}_r \cos\omega t - \vec{E}_i \sin\omega t\right) \times \left(\vec{H}_r \cos\omega t - \vec{H}_i \sin\omega t\right) \\ = & \vec{E}_r \times \vec{H}_r \cos^2\omega t + \vec{E}_i \times \vec{H}_i \sin^2\omega t \\ & -(\vec{E}_r \times \vec{H}_i + \vec{E}_i \times \vec{H}_r)\cos\omega t \sin\omega t \end{aligned}$$

and

$$\begin{aligned} \frac{1}{T}\int_0^T \cos^2\omega t \ dt &= \frac{1}{T}\int_0^T \sin^2\omega t \ dt = \frac{1}{2} \\ \int_0^T \cos\omega t \sin\omega t \ dt &= 0 \end{aligned}$$

we obtain

$$\bar{\vec{N}} = \frac{1}{2}(\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i) \ .$$

On the other hand, take

$$\begin{aligned} \vec{H}_c^* &= \left(\vec{H}_r - i\vec{H}_i\right)\mathrm{e}^{-i\omega t} \ , \\ \vec{E}_c &= \left(\vec{E}_r + i\vec{E}_i\right)\mathrm{e}^{i\omega t} \ , \end{aligned}$$

and then

$$(\vec{E}_c \times \vec{H}_c^*) = (\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i) + i(\vec{E}_i \times \vec{H}_r - \vec{E}_r \times \vec{H}_i)\ .$$

Hence

$$\frac{1}{2}\mathrm{Re}\left(\vec{E}_c \times \vec{H}_c^*\right) = \frac{1}{2}(\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i)\ , = \bar{\vec{N}}$$

as required.

In summary: The average Poynting vector $\vec{N}$ of complex exponential fields satisfies the relation

$$\bar{\vec{N}} = \mathrm{Re}\vec{E}_c \mathbin{\bar{\times}} \mathrm{Re}\vec{H}_c = \frac{1}{2}\mathrm{Re}\left(\vec{E}_c \times \vec{H}_c^*\right)\ .$$

13 Plane Wave Propagation in Dielectric and Magnetic Media

Now, we shall examine in some detail how existing radiation is modified by the material it passes through. We will find that the conductivity is the most significant parameter.

We have seen that in a lossy dielectric the properties of the dielectric can be described using a complex permittivity and similarly in a lossy magnetic material use a complex permeability.

Thus, for a lossy material the Maxwell equations can be written as

$$\begin{aligned}
\nabla \cdot \vec{E} &= \rho/\varepsilon \ , \\
\nabla \cdot \vec{B} &= 0 \ , \qquad (\nabla \cdot \vec{H} = 0) \ , \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \ , \\
\nabla \times \vec{H} &= \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \ ,
\end{aligned}$$

where ε, μ are complex quantities that characterize the material, and $\vec{J}$ is conduction current only, i.e. $\vec{J} = \sigma \vec{E}$.

Consider a plane wave propagating in the z direction

$$\vec{E} = \vec{E}_0 \mathrm{e}^{i(\omega t - kz)} \ .$$

Then

$$\nabla \times \vec{H} = \sigma \vec{E} + i\omega\varepsilon \vec{E} = (\sigma + i\omega\varepsilon)\vec{E}$$

Since for a plane wave propagating in the z direction the derivatives $\partial/\partial x$ and $\partial/\partial y$ of $\vec{E}$ and $\vec{H}$ are zero, we have

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = (\sigma + i\omega\varepsilon)\vec{E}$$

From the LHS, we see that

$$(\nabla \times \vec{H})_z = 0 \;\rightarrow\; (\sigma + i\omega\varepsilon)E_z = 0 \ .$$

Hence $E_z = 0$ unless $(\sigma + i\omega\varepsilon) = 0$. Thus, $\vec{E} \perp \hat{k}$.

Since $E_z = 0$, we have that $\nabla \cdot \vec{E} = 0$.

Also

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

can be used to show that $\vec{E} \perp \vec{H}$.

In addition, we can find from the above equation that $H_z = 0$.

Since $\nabla \cdot \vec{E} = 0$, the quantity ε occurs only in the equation for $\nabla \times \vec{H}$, and it is common to proceed as follows:

$$\begin{aligned} \nabla \times \vec{H} &= \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + i\omega\varepsilon\vec{E} \\ \nabla \times \vec{H} &= i\omega \left(\frac{\sigma}{i\omega} + \varepsilon \right) \vec{E} = i\omega \left(\varepsilon - i\frac{\sigma}{\omega} \right) \vec{E} \end{aligned}$$

Now

$$\varepsilon - i\frac{\sigma}{\omega} = \varepsilon' - i\varepsilon'' - i\frac{\sigma}{\omega} = \varepsilon' - i\left(\varepsilon'' + \frac{\sigma}{\omega}\right) = \bar{\varepsilon}$$

which gives

$$\nabla \times \vec{H} = i\omega\bar{\varepsilon}\vec{E} \ .$$

Physically what has been done is to lump together the conduction current and the lossy dielectric current term. To an external observer they are inseparable. Only using some theory of the internal structure of the dielectric, they can be separated.

We can summarize our findings of the propagation of the EM wave in a conducting material that the electric and magnetic fields of the propagating wave satisfy the following equations

$$\nabla \cdot \vec{E} = 0 \tag{33}$$

$$\nabla \cdot \vec{B} = 0 \tag{34}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -i\omega\mu\vec{H} \tag{35}$$

$$\nabla \times \vec{H} = i\omega\varepsilon\vec{E} \tag{36}$$

where we have left the bar off ε.

We now proceed to solve Eqs. (33)–(36). We will see that the solution leads to a dispersive equation. We look for plane wave solutions, and will try to find how k behaves.

The procedure is as follows: Taking $\nabla\times$ of (35) and using (36), we obtain

$$\begin{aligned} \nabla \times (\nabla \times \vec{E}) &= \omega^2\mu\varepsilon\vec{E} \\ -\nabla^2\vec{E} &= \omega^2\mu\varepsilon\vec{E} \end{aligned}$$

Hence

$$-\frac{\partial^2\vec{E}}{\partial z^2} = \omega^2\mu\varepsilon\vec{E} \ .$$

Since

$$\frac{\partial^2\vec{E}}{\partial z^2} = -\ k^2\vec{E}$$

we finally obtain a dispersion equation

$$k^2 = \omega^2\mu\varepsilon$$

This dispersion equation is not as simple as it looks. We cannot just say that phase velocity is

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

as ε and μ are complex quantities and then k is complex.

What does an imaginary value of k mean?

We can write the complex number k as

$$k = \alpha - i\beta \ ,$$

where α and β are real.
Then

$$E = E_0 \mathrm{e}^{i(\omega t - kz)} = E_0 \mathrm{e}^{i(\omega t - \alpha z + i\beta z)} = E_0 \mathrm{e}^{-\beta z} \ \mathrm{e}^{i(\omega t - \alpha z)} \ .$$

Clearly from this, the phase velocity is

$$v_p = \frac{\omega}{\alpha}$$

and $\beta = -\ \mathrm{Im}(k)$ is the attenuation coefficient.

Physically the losses come from (1) Conduction currents, (2) Lossy dielectric, and (3) Lossy magnetic material, which are lumped together in ε.

We now find α and β.

We can write

$$\begin{aligned} k &= \omega \left(\mu\varepsilon\right)^{\frac{1}{2}} = \omega \left\{ \left[\varepsilon' - i \left(\varepsilon'' + \frac{\sigma}{\omega} \right) \right] \left(\mu' - i\mu'' \right) \right\}^{\frac{1}{2}} \\ k &= \omega \left\{ \left[\varepsilon'\mu' - \mu'' \left(\varepsilon'' + \frac{\sigma}{\omega} \right) \right] \right. \\ & \quad \left. - \ i \left[\mu' \left(\varepsilon'' + \frac{\sigma}{\omega} \right) + \varepsilon'\mu'' \right] \right\}^{\frac{1}{2}} \end{aligned}$$

Let

$$\begin{aligned} p &= \varepsilon'\mu' - \mu'' \left(\varepsilon'' + \frac{\sigma}{\omega} \right) \\ q &= \mu' \left(\varepsilon'' + \frac{\sigma}{\omega} \right) + \varepsilon'\mu'' \end{aligned}$$

Then

$$
\begin{aligned}
k &= \omega\,(p - iq)^{\frac{1}{2}} \\
k &= \omega\left[\left(p^2 + q^2\right)^{\frac{1}{2}} \mathrm{e}^{-i\arctan(q/p)}\right]^{\frac{1}{2}} \\
k &= \omega\left(p^2 + q^2\right)^{\frac{1}{4}} \mathrm{e}^{-i\theta}
\end{aligned}
$$

where $\theta = \frac{1}{2}\arctan\, q/p$.
Hence

$$
\begin{aligned}
\alpha &= \mathrm{Re}(k) = \omega\left(p^2 + q^2\right)^{\frac{1}{4}} \cos\theta \\
\beta &= -\mathrm{Im}(k) = \omega\left(p^2 + q^2\right)^{\frac{1}{4}} \sin\theta
\end{aligned}
$$

which in general is quite complicated.

13.1 Wave Refraction and Skin Effect

We will illustrate propagation of an EM wave in dielectrics and conducting materials on two examples:

Example 1: *The low loss dielectric and no magnetic effects*

For dielectrics, σ is negligible and then we disregard magnetic properties.

$$
\varepsilon'' + \frac{\sigma}{\omega} \ll \varepsilon'
$$

$$
\mu = \mu_0 = \frac{1}{\varepsilon_0 c^2}
$$

Hence

$$
\begin{aligned}
k &= \frac{\omega}{\sqrt{\varepsilon_0} c}\left[\varepsilon' - i\left(\varepsilon'' + \frac{\sigma}{\omega}\right)\right]^{\frac{1}{2}} \\
&= \frac{\omega}{\sqrt{\varepsilon_0} c}\left[\left((\varepsilon')^2 + \left(\varepsilon'' + \frac{\sigma}{\omega}\right)^2\right)^{\frac{1}{2}} \mathrm{e}^{-i\Theta}\right]^{\frac{1}{2}} \quad ,
\end{aligned}
$$

where

$$\Theta = \arctan\left(\frac{\varepsilon'' + \frac{\sigma}{\omega}}{\varepsilon'}\right) \approx \frac{\varepsilon'' + \frac{\sigma}{\omega}}{\varepsilon'}$$

Then

$$\begin{aligned} k &= \frac{\omega}{c}\frac{\sqrt{\varepsilon'}}{\sqrt{\varepsilon_0}}\mathrm{e}^{-i\Theta/2} \\ k &= \frac{\omega}{c}\sqrt{\frac{\varepsilon'}{\varepsilon_0}}\ \mathrm{e}^{-i\left(\frac{\varepsilon''+\sigma/\omega}{2\varepsilon'}\right)} \end{aligned}$$

Hence

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\varepsilon'}{\varepsilon_0}}\ \cos\left(\frac{\varepsilon'' + \sigma/\omega}{2\varepsilon'}\right) \approx \frac{\omega}{c}\ \frac{\sqrt{\varepsilon'}}{\sqrt{\varepsilon_0}}$$

We can find phase velocity and refractive index

$$\begin{aligned} v_p &= \frac{\omega}{\alpha} = c\sqrt{\frac{\varepsilon_0}{\varepsilon'}} < c \qquad \text{for} \ \ \varepsilon' > \varepsilon_0\ , \\ n &= \frac{c}{v_p} = \sqrt{\frac{\varepsilon'}{\varepsilon_0}} > 1\ , \end{aligned}$$

which are not affected by the losses.

Thus, inside the dielectric, the EM wave will propagate with a phase velocity $v_p < c$ and will be refracted.

Similarly, we can find β

$$\beta = -\mathrm{Im}(k) = \frac{\omega}{c}\sqrt{\frac{\varepsilon'}{\varepsilon_0}}\ \sin\left(\frac{\varepsilon'' + \sigma/\omega}{2\varepsilon'}\right)$$

Since for small θ, $\sin\theta \approx \theta$, we get

$$\begin{aligned} \beta &\approx \frac{\omega}{c}\sqrt{\frac{\varepsilon'}{\varepsilon_0}}\ \left(\frac{\varepsilon'' + \sigma/\omega}{2\varepsilon'}\right) \\ \beta &\approx \frac{\omega}{c}\ \frac{\varepsilon'' + \sigma/\omega}{2\sqrt{\varepsilon'\varepsilon_0}} \end{aligned}$$

Thus, losses (absorption of the wave) are small.

We can summarize, that the theory predicts that the refractive index for a lossless dielectric is given by

$$n = \sqrt{n_r} \ .$$

Table below compares theoretical values of n with that obtained experimentally. An excellent agreement is observed, except for polar molecules (e.g. water).

	$\sqrt{n_r}$	n (experimental)
Air	1.00029	1.00029
Argon	1.00028	1.00028
CO_2 gas	1.00047	1.00045
Benzene	1.49	1.48
Ethanol	5.3	1.36
NaCl	2.47	1.54
Water	9.0	1.33

Example 2: *Good conductor*

A good conductor, e.g. Cu, Ag.

(1) There are no dielectric and magnetic losses $\varepsilon'' = \mu'' = 0$.

Consider the Maxwell equation IV:

$$\nabla \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + i\omega\varepsilon \vec{E} \ . \tag{37}$$

Definition of a Good Conductor

Good conductor is when the conduction term on the rhs of the Maxwell equation (37) dominates over the displacement current, i.e. when $\sigma \gg \omega\varepsilon$.

For example, for copper at $\omega = 1$ MHz $= 2\pi \times 10^6$ rad

$$\frac{\sigma}{\omega\varepsilon_0} = \frac{5.8 \times 10^7}{2\pi \times 10^6 \times 8.85 \times 10^{-12}} = 1.0 \times 10^{12}\ .$$

Consider general expression for k:

$$\begin{aligned} k &= \omega\left\{\left[\varepsilon'\mu' - \mu''\left(\varepsilon'' + \frac{\sigma}{\omega}\right)\right]\right. \\ &\qquad \left. - i\left[\mu'\left(\varepsilon'' + \frac{\sigma}{\omega}\right) + \varepsilon'\mu''\right]\right\}^{\frac{1}{2}} \\ k &= \omega\left\{[\varepsilon'\mu' - 0] - i\left[\mu'\frac{\sigma}{\omega} + 0\right]\right\}^{\frac{1}{2}} \\ k &= \omega\left\{\varepsilon'\mu' - i\mu'\frac{\sigma}{\omega}\right\}^{\frac{1}{2}} \end{aligned}$$

Now we might as well drop the dashes on ε, μ understanding that they are real quantities, and obtain

$$\begin{aligned} k &= \omega\left\{\varepsilon\mu - i\mu\frac{\sigma}{\omega}\right\}^{\frac{1}{2}} \\ k &= \omega\sqrt{\varepsilon\mu}\left\{1 - i\frac{\sigma}{\varepsilon\omega}\right\}^{\frac{1}{2}} \\ k &= \omega\sqrt{\varepsilon\mu}\left\{\left[1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{2}} \mathrm{e}^{-i\arctan\frac{\sigma}{\varepsilon\omega}}\right\}^{\frac{1}{2}} \end{aligned}$$

Remembering that for a good conductor $\frac{\sigma}{\varepsilon\omega} \gg 1$, we get

$$\begin{aligned} k &= \omega\sqrt{\varepsilon\mu}\left[\frac{\sigma}{\varepsilon\omega}\mathrm{e}^{-i\frac{\pi}{2}}\right]^{\frac{1}{2}} \\ &= \omega\sqrt{\varepsilon\mu}\sqrt{\frac{\sigma}{\varepsilon\omega}}\mathrm{e}^{-i\frac{\pi}{4}} \end{aligned}$$

Since $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, we obtain

$$k = \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\begin{aligned} k &= \sqrt{\frac{\omega\ \mu\ \sigma}{2}}\,(1-i) \\ \alpha &= \mathrm{Re}(k) = \sqrt{\frac{\omega\mu\sigma}{2}} = \beta = -\mathrm{Im}(k)\ . \end{aligned}$$

Note a very heavy attenuation of this wave

$$E = E_0 \mathrm{e}^{-\beta z}\ \mathrm{e}^{i(\omega t - \alpha z)}$$

Since

$$\alpha = \beta = \frac{2\pi}{\lambda}$$

the distance for attenuation "e" fold (i.e. amplitude falls to a factor $\frac{1}{\mathrm{e}}$ of its original value in a distance $\frac{1}{\beta}$) is

$$z = \frac{1}{\beta} = \frac{\lambda}{2\pi}$$

i.e.

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\mu\sigma}} \approx \frac{\lambda}{6}\ \ !!$$

where δ is called the skin depth in the conductor. It is the distance the wave must propagate in order to decay by an amount e^{-1}. This effect is sometimes called "skin effect" as with an increasing σ the current flows in a narrower and narrower layer, until in the limit of $\sigma \to \infty$ a true current exists only on the surface of the conductor.

Questions:

(1) Derive the special form of Poynting's Theorem applicable in certain material media

$$\int (\vec{E} \times \vec{H}) \cdot d\vec{S} = -\int \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\, dV - \int \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\, dV - \int \vec{E} \cdot \vec{J}\, dV$$

Interpret the above equation in terms of energy storage and energy flow etc. State qualitative meaning of each term in the equation.

(2) Prove that the useful formula for the mean Poynting vector for sinusoidal fields is

$$\bar{\vec{N}} = \frac{1}{2}\mathrm{Re}\left(\vec{E}_c \times \vec{H}_c^*\right)$$

where $\vec{E}_c = \vec{E}_0 \exp(i\omega t)$ and $\vec{H}_c^* = \vec{H}_0 \exp(-i\omega t)$.

(3) Show that the amplitude of a plane wave propagating in a non-conducting material is damped with the rate β which arises from the imaginary parts of the complex permittivity and permeability.

(4) Show that in a good conductor an EM wave propagates on the surface of the conductor.

Weekend exercises

Find the skin depth for cooper at 60 MHz.

14 Transitions Across Boundaries for Electromagnetic Fields

14.1 Applications in dielectrics

Across boundaries between different material media there are sharp changes in electrical properties ε, μ, σ. On a macroscopic scale the fields may have to be regarded as varying discontinuously across such boundaries. The source of such discontinuities will be the surface polarization charges and currents $\vec{P} \cdot \hat{n}$ and $\vec{M} \times \hat{n}$ discussed previously.

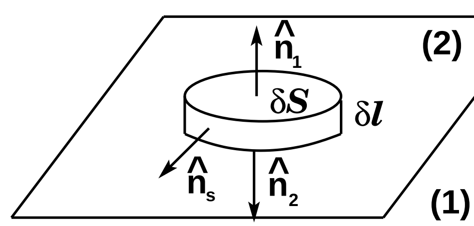
We will use the Maxwell's divergence equations I and II to investigate the transition of normal field components, and the curl equations III and IV to investigate the transition of tangential field components.

14.1.1 Normal Component of $\vec{B}$

Consider the Maxwell equation II: $\nabla \cdot \vec{B} = 0$.
Using the Gauss' divergence theorem, the Maxwell's equation can be written as

$$\oint_S \vec{B} \cdot \hat{n} \, dS = 0 \ ,$$

where S is an arbitrary surface closing some area on the boundary plane.



In order to evaluate the integral, we consider a thin cylinder of area δS and thickness $\delta\ell$ including the boundary

$$(\vec{B}_2 \cdot \hat{n}_1)\,\delta S + (\vec{B}_1 \cdot \hat{n}_2)\,\delta S + \int_{sides} \vec{B} \cdot \hat{n}_s\, dS = 0 \ .$$

Since $\vec{B}$ is finite everywhere and we are interested in the transformation of the field at the boundary, $\delta\ell \to 0$, the integral

$$\int_{sides} \vec{B} \cdot \hat{n}\, dS \to 0 \quad \text{as} \quad \delta\ell \to 0 \ .$$

Hence

$$\vec{B}_2 \cdot \hat{n}_1 + \vec{B}_1 \cdot \hat{n}_2 = 0 \ .$$

However

$$\hat{n}_1 = -\hat{n}_2 = \hat{n} \ ,$$

and then

$$\hat{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0 \qquad \text{or} \qquad B_{2\perp} = B_{1\perp} \ .$$

Thus, **the normal component of $\vec{B}$ is continuous across a boundary between two different dielectric materials.**

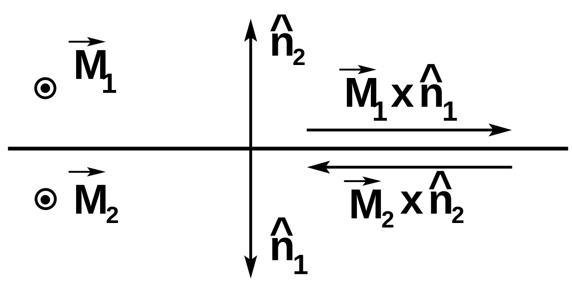


Figure 8: Polarization currents at the boundary between two different materials.

We can understand this result intuitively by noting that the $\vec{B}$ fields of polarization currents $\vec{M} \times \hat{n}$ are parallel to the boundary and so do not affect the normal component of $\vec{B}$, see Figure 8.

14.1.2 Normal Component of $\vec{H}$

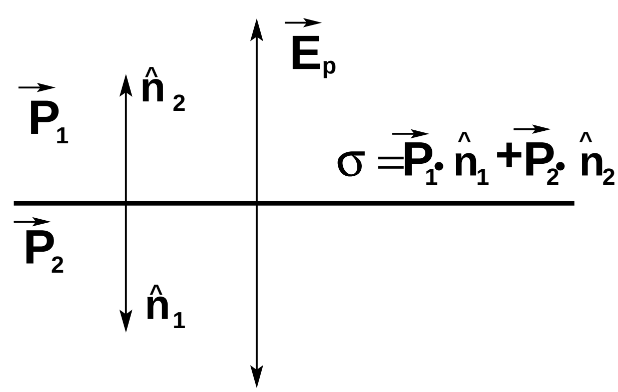
Since $\vec{B} = \mu\vec{H}$, we have

$$B_{1\perp} = \mu_1 H_{1\perp} = \mu_2 H_{2\perp} = B_{2\perp} \ .$$

Thus, when $\mu_1 \neq \mu_2$, the normal component of $\vec{H}$ is not continuous across a boundary. This result for the normal component of $\vec{H}$ is the consequence of different magnetizations of the materials.

14.1.3 Normal Component of $\vec{E}$

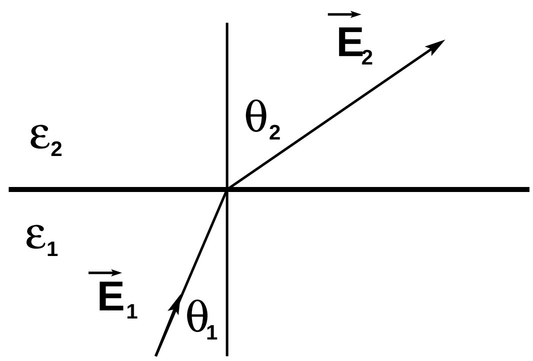
Since in nonconducting dielectrics $\nabla \cdot \vec{D} = 0$, an identical argument to the above will show that the **normal component of $\vec{D}$ is continuous across a boundary.**
In the case of dielectrics we write $\vec{D} = \varepsilon\vec{E}$.
Then $\varepsilon_1 E_{1\perp} = \varepsilon_2 E_{2\perp}$ can be a useful boundary condition.



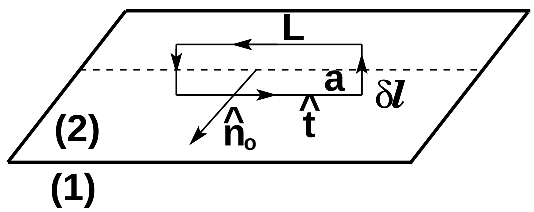
The electric field of dielectric surface charge $\vec{P} \cdot \hat{n}$ is normal to the boundary. This causes the discontinuity in the $\vec{E}$ field.

14.1.4 Tangential Component of $\vec{E}$

We will use the Maxwell's curl equations III and IV to investigate the transition of tangential field components.



For the tangential component of $\vec{E}$, we will apply the Faraday induction law (Maxwell equation) to a closed path such as in the Figure, in which the sides perpendicular to the boundary are made infinitely short compared to the parallel sides L.



Consider the Maxwell equation III:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Integrating both sides of this equation over the surface a and applying the Stokes's theorem, we obtain

$$\int_l \vec{E} \cdot d\vec{l} = -\int_a \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}_0 \; da \; .$$

Hence

$$\left(\vec{E}_2 \cdot \hat{t}_2 + \vec{E}_1 \cdot \hat{t}_1\right) \; L + \int_{ends} \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}_0 \; da \; , \tag{38}$$

where $\hat{n}_o$ is the unit vector normal to the surface a, and $\hat{t}_1$ and $\hat{t}_2$ are unit vectors along the paths L on the side (1) and (2), respectively.

As $dl \to 0$ the rhs and the second term on lhs of Eq. (38) go to zero, since $\vec{E}$ and $\vec{B}$ are finite everywhere. In this limit, the area enclosed by the path approaches zero.

Since, $\hat{t}_2 = -\hat{t}_1 = \hat{t}$, we then have

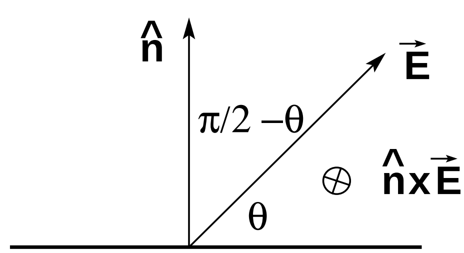
$$\left(\vec{E}_2 - \vec{E}_1\right) \cdot \hat{t} = 0 \ .$$

But $\vec{E} \cdot \vec{t}$ is the component of $\vec{E}$ tangential to the surface. Since this is true for any $\hat{t}$, we obtain $E_{1\|} = E_{2\|}$, and we conclude that:

The tangential component of $\vec{E}$ is continuous across a boundary between two different dielectric materials.

The continuity of a tangential component can be written in an equivalent form as

$$\hat{n} \times \left(\vec{E}_2 - \vec{E}_1\right) = 0 \ .$$



Explanation

From the Figure above, we see that the cross product

$$\left|\hat{n} \times \vec{E}\right| = E \sin(\pi/2 - \theta) = E \cos\theta \ ,$$

i.e. it is the tangential component of $\vec{E}$.

14.1.5 Tangential Component of $\vec{H}$

Consider the Maxwell's equation IV:

$$\nabla \times \vec{H} = \vec{J}_c + \varepsilon \frac{\partial \vec{E}}{\partial t} \ ,$$

where $\vec{J}_c$ is the conduction current, i.e. not counting polarization currents.

Stokes's theorem then gives

$$\int \vec{H} \cdot d\vec{l} = \int_a \vec{J}_c \cdot \hat{n} da + \varepsilon \int_a \frac{\partial \vec{E}}{\partial t} \cdot \hat{n} da \ .$$

Both terms on the rhs go to zero as $\delta l \to 0$ because $\vec{J}_c$ and $\partial \vec{E}/\partial t$ are finite.

Hence

$$\left(\vec{H}_2 - \vec{H}_1\right) \cdot \hat{t} = 0 \ ,$$

or

$$\hat{n} \times \left(\vec{H}_2 - \vec{H}_1\right) = 0 \ .$$

Thus, **the tangential component of $\vec{H}$ is continuous across a boundary.**

14.1.6 Tangential component of $\vec{B}$

The tangential component of $\vec{B}$ is not in general continuous across a boundary because of the presence of the magnetic polarization surface currents $\vec{M} \times \hat{n}$, which do not have a finite current density as they flow in an infinitely thin surface layer. Thus, if we examine the corresponding Maxwell equation for $\nabla \times \vec{B}$, the term in the integral involving $\vec{J}$ may stay finite as $\delta j \to 0$.

Since

$$\vec{H} = \varepsilon_0 c^2 \vec{B} - \vec{M} \ ,$$

we have

$$\vec{H}_2 - \vec{H}_1 = \varepsilon_0 c^2 \left(\vec{B}_2 - \vec{B}_1\right) - \left(\vec{M}_2 - \vec{M}_1\right) \ .$$

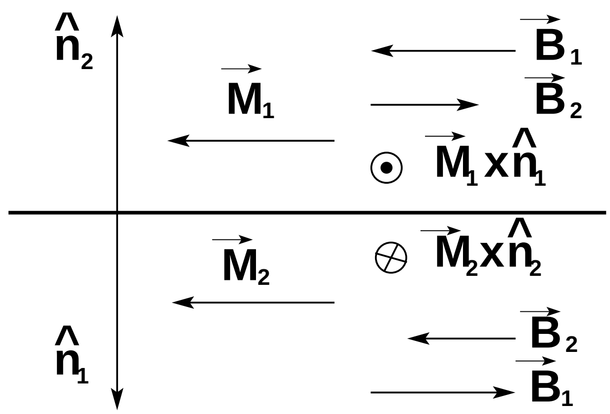
Take scalar product of both sides with $\hat{t}$, and using the fact that

$$\left(\vec{H}_2 - \vec{H}_1\right) \cdot \hat{t} = 0 \ ,$$

we obtain

$$\left(\vec{B}_2 - \vec{B}_1\right) \cdot \hat{t} = \frac{1}{\varepsilon_0 c^2} \left(\vec{M}_2 - \vec{M}_1\right) \cdot \hat{t}$$

If $\vec{M}_1 \neq \vec{M}_2$ the difference between the two surface currents generates a discontinuity in the $\vec{B}$ field (see Figure below).



Summary: Field components that are continuous across a boundary

1. The normal component of $\vec{D}$.

2. The tangential component of $\vec{E}$.

3. The normal component of $\vec{B}$.

4. The tangential component of $\vec{H}$.

Questions:

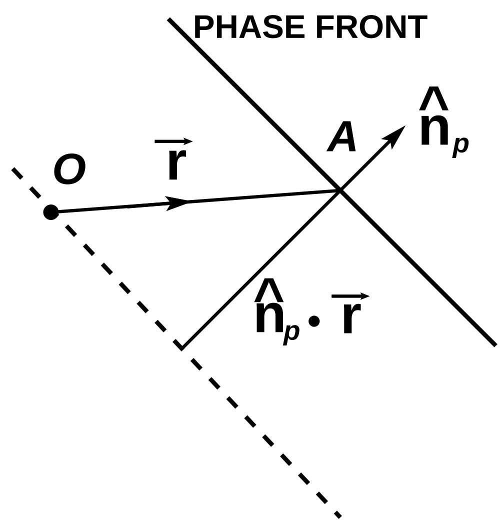
(1) Prove the following boundary conditions at a bounding surface between two dielectrics:

(a) The normal component of $\vec{B}$ is continuous across the boundary.
(b) The tangential component of $\vec{E}$ is continuous across the boundary.
(c) The tangential component of $\vec{H}$ is continuous across the boundary.

15 Reflection and Transmission of Waves Across a Boundary

At a boundary between two different media, the general boundary conditions cannot be satisfied by a transmitted wave alone. There has to be a reflected wave also. What happens is that the incident radiation is absorbed by charges in the boundary and reradiated in all directions. The waves interfere destructively except in two directions along those of the reflected and transmitted waves. The directions, amplitudes and phases of the reflected and transmitted waves can be derived from the general boundary conditions already obtained. We consider linear isotropic materials characterized by electric and magnetic constants ε, μ, and σ.

15.1 Representation of Plane Waves in Different Directions



Let $\hat{n}_p$ is the unit normal to the phasefront (wavefront) of a propagating wave, and $\vec{r}$ is a position vector that is independent of $\hat{n}_p$. The vector $\vec{r}$ determines position of a wave-front as seen by an observer O.

Suppose that the wave propagates in the z direction. Then

$$\vec{E} \;=\; \vec{E}_0 \exp[i\,(\omega t - kz)]\ .$$

Since

$$z \;=\; \hat{n}\cdot\vec{r}\ ,$$

we obtain

$$\vec{E} \;=\; \vec{E}_0 \exp[i\,(\omega t - \hat{n}\cdot\vec{r}\ k)]\ .$$

Thus, the observer can distinguish different waves at the point A by different $\hat{n}$'s.

15.1.1 Representation of $\vec{B}$ in terms of $\vec{E}$

For the field of a plane wave, we have

$$\vec{B} = \frac{k}{\omega}\hat{n}_p \times \vec{E}\ , \tag{39}$$

where $\hat{n}_p$ is the unit vector in the direction of propagation of the field.

Recall that the relation (39) arises from the Maxwell equation III:

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\ .$$

To show this, we expand both sides of this equation, and obtain

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix} = -\frac{\partial\vec{B}}{\partial t} = -\hat{i}\frac{\partial B_x}{\partial t} - \hat{j}\frac{\partial B_y}{\partial t}\ ,$$

where we have used the fact that the wave propagates in the z direction.

Comparing the coefficients standing at the same unit vectors, we find that the x component gives

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

from which, we obtain

$$ikE_y = -i\omega B_x \qquad \text{or} \qquad \frac{E_y}{B_x} = \frac{-\omega}{k} \ .$$

The y component gives:

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

from which, we obtain

$$-ikE_x = -i\omega B_y \qquad \text{or} \qquad \frac{E_x}{B_y} = \frac{\omega}{k} \ .$$

Thus

$$E = \left(E_x^2 + E_y^2\right)^{\frac{1}{2}} = \frac{\omega}{k}\left(B_x^2 + B_y^2\right)^{\frac{1}{2}} = \frac{\omega}{k} B \ .$$

Since $\vec{E}$, $\vec{B}$, $\hat{n}_p$, are mutually orthogonal, $\vec{E} \times \vec{B}$ gives the direction of $\hat{n}_p$ (Poynting result) of the propagation direction. Then $\hat{n}_p \times \vec{E}$ gives the direction of $\vec{B}$. Hence

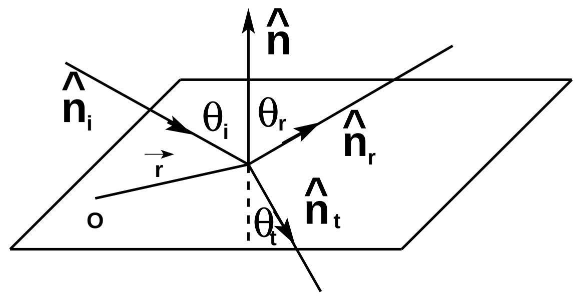
$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{k}{\omega\mu}\hat{n}_p \times \vec{E} \ . \tag{40}$$

In the next few sections, we will use the continuity conditions for $\vec{E}$ and $\vec{H}$. With the relation (40), we will be able to limit the analysis to the electric field alone, as knowing properties of $\vec{E}$, we can find from Eq. (40) properties of $\vec{H}$.

15.2 Directions of Reflected and Transmitted Waves

Here, we will show that some familiar elementary results on reflection and transmission can be derived from the Maxwell equations.

The most useful results in this connection concern the continuity of tangential components of the $\vec{E}$ and $\vec{H}$ fields. However, most of the results can be derived from the fact that the tangential component of $\vec{E}$ is continuous across a boundary.



Consider properties of an EM wave at a boundary between two materials. Let the origin of position coordinates $\vec{r}$ be located on the boundary S.

The electric field for incident, reflected and transmitted waves are

$$\begin{aligned}\vec{E}_i &= \vec{E}_0 \exp\left[i\left(\omega t - \hat{n}_i \cdot \vec{r}\ k_1\right)\right]\ ,\\ \vec{E}_r &= \vec{E}_1 \exp\left[i\left(\omega t - \hat{n}_r \cdot \vec{r}\ k_1\right)\right]\ ,\\ \vec{E}_t &= \vec{E}_2 \exp\left[i\left(\omega t - \hat{n}_t \cdot \vec{r}\ k_2\right)\right]\ .\end{aligned}$$

These equations determine the three electric fields relative to the direction of observation $\vec{r}$.

Why do we need the presence of the reflected wave in the propagation of an incident wave through the boundary?

The answer to this question is provided by the requirement that the tangential components of $\vec{E}$ and $\vec{H}$ must be continuous through the boundary.

Suppose that $\vec{E}$ is polarized in the plane of incidence. Then

$$E_i \cos\theta_i - E_r \cos\theta_r = E_t \cos\theta_t\ , \tag{41}$$

and

$$H_i + H_r = H_t\ . \tag{42}$$

Since

$$H = \frac{k}{\omega\mu} E\ ,$$

the relation (42) takes the form

$$E_i + E_r = \frac{k_2}{k_1} E_t \ , \tag{43}$$

as for a dielectric $\mu_1 = \mu_2 = \mu_0$.

Now we see that without E_r, Eq. (41) gives

$$E_i = \frac{\cos\theta_t}{\cos\theta_i} E_t \ ,$$

while Eq. (43) gives

$$E_i = \frac{k_2}{k_1} E_t \ .$$

Thus, without E_r we would get two different values for E_i or E_t, which we cannot accept as both continuity conditions Eqs. (41) and (42) must be satisfied at the same moment. Hence, we conclude that the continuity conditions for $\vec{E}$ and $\vec{H}$ will be satisfied only if $E_r \neq 0$.

What are the relative directions of $\hat{n}_i, \hat{n}_r$ and $\hat{n}_t$?

From the boundary condition for the tangential component of $\vec{E}$:

$$\hat{n} \times \left(\vec{E}_i + \vec{E}_r \right) = \hat{n} \times \vec{E}_t$$

i.e.

$$\begin{aligned} &\hat{n} \times \left\{ \vec{E}_0 \exp\left[i \left(\omega t - \hat{n}_i \cdot \vec{r}\, k_1 \right) \right] + \vec{E}_1 \exp\left[i \left(\omega t - \hat{n}_r \cdot \vec{r}\, k_1 \right) \right] \right\} \\ &= \hat{n} \times \left\{ \vec{E}_2 \exp\left[i \left(\omega t - \hat{n}_t \cdot \vec{r}\, k_2 \right) \right] \right\} \end{aligned}$$

This relation must hold over the whole surface S for all $\vec{r}$ (subject to $\hat{n}\cdot\vec{r} = 0$). Thus, the exponential phase factors must all be the same. Otherwise if it was true for one $\vec{r}$ it would not be true for other $\vec{r}$'s, but we have a freedom of choosing $\vec{r}$. Hence

$$k_1 \hat{n}_i \cdot \vec{r} \;\; = \;\; k_1 \hat{n}_r \cdot \vec{r} = k_2 \hat{n}_t \cdot \vec{r} \ ,$$

from which we have

$$\hat{n}_i \cdot \vec{r} \ = \ \hat{n}_r \cdot \vec{r} \ ,$$

and

$$\hat{n}_i \cdot \vec{r} = \frac{k_2}{k_1} \hat{n}_t \cdot \vec{r} \ .$$

These relations will help us to prove that
Incident, reflected and transmitted rays are coplanar
i.e. $\hat{n}_i, \ \hat{n}_r, \ \hat{n}_t$ are coplanar.

To show this, we use the relation

$$\vec{r} = -\hat{n} \times (\hat{n} \times \vec{r})$$

Proof:

Since

$$\hat{n} \times (\hat{n} \times \vec{r}) = (\hat{n} \cdot \vec{r})\hat{n} - (\hat{n} \cdot \hat{n})\vec{r}$$

and $\hat{n} \cdot \vec{r} = 0$ as the vector $\vec{r}$ lies on the plane S, we obtain

$$\hat{n} \times (\hat{n} \times \vec{r}) = -\vec{r}$$

as required.

Hence

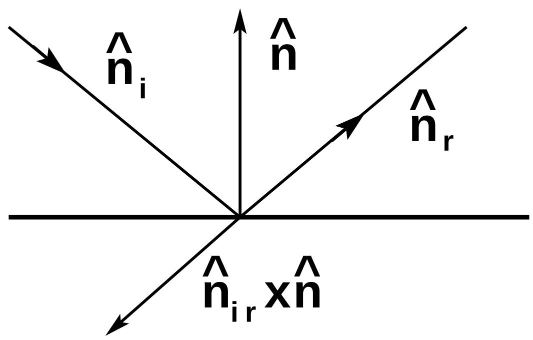
$$\hat{n}_i \cdot [\hat{n} \times (\hat{n} \times \vec{r})] = \hat{n}_r \cdot [\hat{n} \times (\hat{n} \times \vec{r})] \ .$$

Interchange $(\cdot)$ and $(\times)$ products

$$(\hat{n}_i \times \hat{n}) \cdot (\hat{n} \times \vec{r}) = (\hat{n}_r \times \hat{n}) \cdot (\hat{n} \times \vec{r}) \ .$$

This must be true for all $\vec{r}$ in plane S. Thus:

$$\hat{n}_i \times \hat{n} = \hat{n}_r \times \hat{n}$$



This implies that $\hat{n}_r$ is in the "plane of incidence", i.e. the plane containing $\hat{n}$ and $\hat{n}_i$.

Similarly

$$k_1\hat{n}_i \cdot \vec{r} = k_2\hat{n}_t \cdot \vec{r}$$

implies that

$$k_1 \ \hat{n}_i \cdot \hat{n} \times (\hat{n} \times \vec{r}) = k_2 \ \hat{n}_t \cdot \hat{n} \times (\hat{n} \times \vec{r}) \ .$$

Interchange $(\cdot)$ and $(\times)$ products, we find that

$$k_1 \ (\hat{n}_i \times \hat{n}) \cdot (\hat{n} \times \vec{r}) = k_2 \ (\hat{n}_t \times \hat{n}) \cdot (\hat{n} \times \vec{r})$$

for all $\vec{r}$ in S. Thus

$$k_1\hat{n}_i \times \hat{n} = k_2\hat{n}_t \times \hat{n} \ ,$$

which implies that $\hat{n}_i$, $\hat{n}$, and $\hat{n}_t$ are coplanar.

The coplanar property of the waves is observed in any experiment. This is an another example of a remarkable triumph of the Maxwell's electromagnetic theory.

15.3 Snell's Law of Refraction and Angle of Reflection

Since

$$\hat{n}_i \times \hat{n} = \hat{n}_r \times \hat{n}$$

we have that

$$\sin\theta_i = \sin\theta_r$$

Thus

$$\theta_i = \theta_r \ .$$

The angle of incidence equals the angle of reflection, another familiar law of elementary optics.

Moreover

$$k_1 \ \hat{n}_i \times \hat{n} = k_2 \ \hat{n}_t \times \hat{n}$$

from which we have

$$k_1 \sin\theta_i = k_2 \sin\theta_t$$

and then

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{k_2}{k_1} = n_{12} \ .$$

This is the well-known law of refraction in optics, called the **Snell's law**.

In the case where k_1 and k_2 are purely real (e.g. in dielectrics), the refractive index has a simple physical interpretation

$$n_{12} = \frac{k_2}{k_1} = \frac{\omega/k_1}{\omega/k_2} \ , = \frac{v_1}{v_2}$$

i.e. the refractive index in equal to the ratio of phase velocities.

Questions:

(1) Prove that in the reflection and refraction at a bounding surface, the direction of incident, reflected and refracted waves are coplanar.

(2) Using the results of Question 1, derive the familiar laws of elementary optics:
(a) Angle of reflection equals to the angle of incidence.

(b) Snell's law of refraction.

(3) Show, using the continuity conditions for $\vec{E}$ and $\vec{H}$ that both reflection and refraction takes place in the incidence of light on a boundary between two dielectrics.

16 Fresnel's Equations

The boundary condition on tangential $\vec{E}$ does not give sufficient information to calculate $\vec{E}_r$ and $\vec{E}_t$ in terms of $\vec{E}_i$. For a given $\vec{E}_i$ there are still two unknowns in the equation for continuity of tangential $\vec{E}$ viz $\vec{E}_r$ and $\vec{E}_t$. We need a second relation between $\vec{E}_i$, $\vec{E}_r$ and $\vec{E}_t$. This can be obtained from the continuity of the magnetic field.

We know that $\vec{B}$ tangential is not continuous across a boundary because of the presence of $\vec{M} \times \hat{n}$ surface currents in magnetized materials. To allow for such possibilities we can use the more general condition that tangential component of $\vec{H}$ is continuous, i.e.

$$\hat{n} \times \left(\vec{H}_i + \vec{H}_r\right) = \hat{n} \times \vec{H}_t \ .$$

Since

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{k}{\omega\mu}\hat{n}_p \times \vec{E} \ ,$$

where $\hat{n}_p$ is a unit ray vector in the direction of propagation, we have an equation

$$\begin{aligned} &\hat{n} \times \left(\frac{k_1}{\omega\mu_1}\hat{n}_i \times \vec{E}_i + \frac{k_1}{\omega\mu_1}\hat{n}_r \times \vec{E}_r\right) \\ &= \hat{n} \times \left(\frac{k_2}{\omega\mu_2}\hat{n}_t \times \vec{E}_t\right) \ , \end{aligned} \tag{44}$$

which together with

$$\hat{n} \times \left(\vec{E}_i + \vec{E}_r\right) = \hat{n} \times \vec{E}_t \ , \tag{45}$$

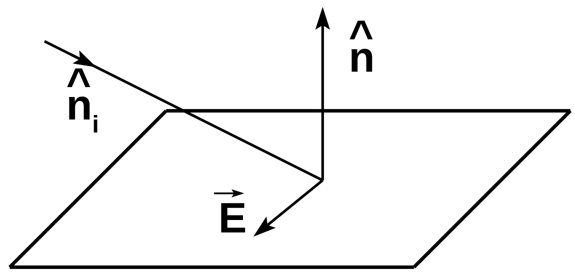
contain sufficient information to determine $\vec{E}_r$ and $\vec{E}_t$ in terms of $\vec{E}_i$.

The solution of these two simultaneous vector equations is greatly facilitated by considering separately, electric field components parallel and normal to the plane of incidence.

Equations (44) and (45) also provide a simple explanation of why we need reflected field at the boundary to obtain the correct results for the field amplitudes.

16.1 $\vec{E}_i$ normal to plane of incidence

In this case, the incident electric field $\vec{E}_i$ is purely tangential to the boundary. Since the materials are isotropic, the induced fields $\vec{E}_r$ and $\vec{E}_t$ will also be tangential to the the boundary.



Thus the condition

$$\hat{n} \times \left(\vec{E}_i + \vec{E}_r\right) = \hat{n} \times \vec{E}_t$$

gives

$$\vec{E}_0 + \vec{E}_1 = \vec{E}_2 \ .$$

Note then that

$$\hat{n} \cdot \vec{E}_0 = \hat{n} \cdot \vec{E}_1 = \hat{n} \cdot \vec{E}_2 = 0 \ .$$

The other equation for $\vec{E}_0$, $\vec{E}_1$ and $\vec{E}_2$ comes from the $\vec{H}$ condition through the relation

$$\hat{n} \times \left(\frac{k_1}{\omega\mu_1}\hat{n}_i \times \vec{E}_i + \frac{k_1}{\omega\mu_1}\hat{n}_r \times \vec{E}_r\right)$$
$$= \hat{n} \times \left(\frac{k_2}{\omega\mu_2}\hat{n}_t \times \vec{E}_t\right)$$

and since the phase factors in $\vec{E}_\alpha \exp i(\omega t - \hat{n}_\alpha \cdot \vec{r}\, k)$ are the same, we obtain

$$\frac{k_1}{\omega\mu_1}\left[\hat{n} \times \left(\hat{n}_i \times \vec{E}_0\right) + \hat{n} \times \left(\hat{n}_r \times \vec{E}_1\right)\right]$$
$$= \frac{k_2}{\omega\mu_2}\hat{n} \times \left(\hat{n}_t \times \vec{E}_2\right) \ .$$

Using the relation $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$ and the fact that $\hat{n} \cdot \vec{E}_0 = \hat{n} \cdot \vec{E}_1 = \hat{n} \cdot \vec{E}_2 = 0$, we obtain

$$\frac{k_1}{\omega\mu_1}\left(\hat{n} \cdot \hat{n}_i \vec{E}_0 + \hat{n} \cdot \hat{n}_r \vec{E}_1\right) = \frac{k_2}{\omega\mu_2}\hat{n} \cdot \hat{n}_t \vec{E}_2$$

However

$$\begin{aligned}
\hat{n} \cdot \hat{n}_i &= \cos(\pi - \theta_i) = -\cos\theta_i \\
\hat{n} \cdot \hat{n}_r &= \cos\theta_r = \cos\theta_i \\
\hat{n} \cdot \hat{n}_t &= \cos(\pi - \theta_t) = -\cos\theta_t
\end{aligned}$$

and then, we obtain

$$\frac{k_1}{\omega\mu_1}\left(\vec{E}_0 \cos\theta_i - \vec{E}_1 \cos\theta_i\right) = \frac{k_2}{\omega\mu_2}\vec{E}_2 \cos\theta_t \ .$$

Since $\vec{E}$'s are all in the same direction, we might as well drop the vector signs. Thus

$$E_0 + E_1 = E_2 \tag{46}$$

$$E_0 \cos\theta_i - E_1 \cos\theta_i = \frac{k_2\mu_1}{k_1\mu_2} E_2 \cos\theta_t \ . \tag{47}$$

Eliminating E_1 using (46), $E_1 = E_2 - E_0$, we get

$$E_0 \cos\theta_i - (E_2 - E_0)\cos\theta_i = \frac{k_2\mu_1}{k_1\mu_2} E_2 \cos\theta_t$$

$$2E_0 \cos\theta_i = \left(\cos\theta_i + \frac{k_2\mu_1}{k_1\mu_2}\cos\theta_t\right) E_2$$

$$2k_1\mu_2 E_0 \cos\theta_i = (k_1\mu_2 \cos\theta_i + k_2\mu_1 \cos\theta_t)\, E_2$$

and then

$$E_2 = \frac{2k_1\mu_2 \cos\theta_i}{k_1\mu_2\cos\theta_i + k_2\mu_1\cos\theta_t}\ E_0$$

Since

$$k_1 \sin\theta_i = k_2 \sin\theta_t$$

we can eliminate θ_t

$$k_2 \cos\theta_t = \sqrt{k_2^2 - k_1^2 \sin^2\theta_i}$$

and obtain

$$E_2 = \frac{2k_1\mu_2\cos\theta_i}{k_1\mu_2\cos\theta_i + \mu_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}\ E_0\ . \tag{48}$$

Similarly, eliminating $E_2 = E_0 + E_1$ from (47) above, we obtain

$$E_1 = \frac{k_1\mu_2\cos\theta_i - \mu_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}{k_1\mu_2\cos\theta_i + \mu_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}\ E_0\ . \tag{49}$$

Equations (48) and (49) are called **Fresnel equations** for the electric field amplitudes.

The corresponding $\vec{H}$ fields are not parallel to each other, but their relative magnitudes can be deducted from equations of the form

$$\vec{H} = \frac{k}{\mu\omega}\hat{n} \times \vec{E} \qquad \text{i.e.} \qquad H = \frac{kE}{\mu\omega}\ .$$

16.2 $\vec{E}_i$ in the plane of incidence

In this case $\vec{H}_i$ is tangential to the boundary plane and then $\vec{H}_r$ and $\vec{H}_t$ are tangential too. Thus

$$\hat{n} \times \left(\vec{H}_i + \vec{H}_r\right) = \hat{n} \times \vec{H}_t$$

becomes

$$\vec{H}_0 + \vec{H}_1 = \vec{H}_2\ .$$

The continuity of tangential $\vec{E}$ is given by

$$\hat{n} \times \left(\vec{E}_0 + \vec{E}_1\right) = \hat{n} \times \vec{E}_2\ .$$

It is simpler to work in terms of $\vec{H}$ in this case.
We express $\vec{E}$ in terms of $\vec{H}$

$$\vec{E} = -\frac{\omega}{k}\ \hat{n} \times \vec{B} = -\frac{\mu\omega}{k}\ \hat{n} \times \vec{H}\ .$$

Thus

$$\begin{aligned} &\hat{n} \times \left[\frac{\mu_1}{k_1}\hat{n}_i \times \vec{H}_0 + \frac{\mu_1}{k_1}\hat{n}_r \times \vec{H}_1\right] \\ &= \hat{n} \times \left(\frac{\mu_2}{k_2}\hat{n}_t \times \vec{H}_2\right)\ . \end{aligned}$$

By a procedure similar to the case of $\vec{E}_i$ normal to plane of incidence, we obtain

$$H_1 = \frac{k_2^2\mu_2\cos\theta_i - \mu_1 k_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}{k_2^2\mu_1\cos\theta_i + \mu_2 k_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}\ H_0\ , \qquad (50)$$

$$H_2 = \frac{2k_2^2\mu_2\cos\theta_i}{k_2^2\mu_1\cos\theta_i + \mu_2 k_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}\ H_0\ , \qquad (51)$$

Equations (50) and (51) are called **Fresnel equations** for the magnetic field amplitudes.

This time the $\vec{E}$'s are not parallel, but their relative amplitudes can be deducted from $E = (\mu\omega/k)H$.

16.3 Fresnel Equations for dielectric media

In a dielectric: conductivities $\sigma_1 = \sigma_2 = 0$, $\mu_1 = \mu_2 = \mu_0$, $k = 2\pi/\lambda$ =real, and

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu_0}}\ .$$

Since $k \sim 1/\lambda \sim 1/v_p \sim \sqrt{\varepsilon}$, the Snell's law ($k_1\sin\theta_i = k_2\sin\theta_t$) becomes

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t$$

and then

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{v_1}{v_2} = n_{12}$$

Consider two examples:

(1) E normal to the plane of incidence. In general

$$\frac{E_2}{E_0} = \frac{2k_1\mu_2\cos\theta_i}{k_1\mu_2\cos\theta_i + k_2\mu_1\cos\theta_t}$$

Since

$$\mu_1 = \mu_2 = \mu_0 \qquad \text{and} \qquad \frac{k_2}{k_1} = \frac{\sin\theta_i}{\sin\theta_t} \ ,$$

we obtain

$$\begin{aligned}\frac{E_2}{E_0} &= \frac{2\cos\theta_i}{\cos\theta_i + \frac{k_2}{k_1}\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} \\ &= \frac{2\cos\theta_i\sin\theta_t}{\cos\theta_i\sin\theta_t + \sin\theta_i\cos\theta_t} \ .\end{aligned}$$

Hence

$$\frac{E_2}{E_0} = \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_t+\theta_i)} \ .$$

Similarly

$$\frac{E_1}{E_0} = \frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)} \ .$$

(2) For E in the plane of incidence

$$\frac{E_2}{E_0} = \frac{2\cos\theta_i\cos\theta_t}{\sin(\theta_t+\theta_i)\cos(\theta_i-\theta_t)} \ ,$$

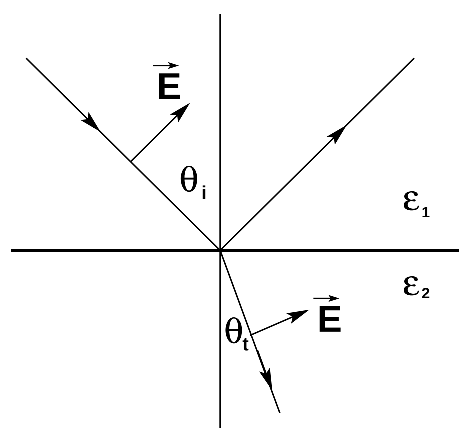
$$\frac{E_1}{E_0} = \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_t+\theta_i)} \ .$$

17 Applications of the Boundary Conditions and the Fresnel Equations

Let us now examine some of the consequences of the Snell's law. There are two cases possible: $n_2 > n_1$ and $n_1 > n_2$. In the first case, an optical wave travels from an optically "rarer" to optically "denser" medium. In the second case, we have the inverse situation. We will consider these two cases separately for dielectrics and conductors.

17.1 Applications in dielectrics

17.1.1 Polarization by reflection



For the case of $\vec{E}$ in the plane of incidence the ratio of reflected to incident amplitude is

$$\frac{E_1}{E_0} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \ .$$

If $\theta_i + \theta_t = \frac{\pi}{2}$ then $\tan(\theta_i + \theta_t) = \infty$ and consequently $E_1 = 0$.

At the same time E_1 will not be zero for the electric field component normal to the plane of incidence.

Thus if $\vec{E}_i$ has arbitrary polarization then $\vec{E}_r$ will be plane polarized with $\vec{E}_r$ normal to the plane of incidence.

If $\theta_i + \theta_t = \frac{\pi}{2}$ then $\theta_t = \frac{\pi}{2} - \theta_i$. Thus

$$\frac{\sin\theta_i}{\sin\theta_t} = n_{21} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{\sin\theta_i}{\sin(\frac{\pi}{2} - \theta_i)} = \tan\theta_i \ .$$

Hence, the angle of incidence for total linear polarization of the reflected wave is

$$\theta_i = \arctan\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \ .$$

It is known in the literature as the Brewster's angle.

An alternative proof that if $\vec{E}$ is in the plane of incidence, then $E_1 = 0$ (no reflected field polarized in the plane of incidence).

We can use the continuity conditions for the tangential components of $\vec{E}$ and $\vec{H}$, from which we have

$$E_0\cos\theta - E_1\cos\theta = E_2\cos\theta_t \ , \tag{52}$$

and

$$H_0 + H_1 = H_2 \ . \tag{53}$$

Since

$$H = \frac{k}{\omega\mu}E \ ,$$

and $\theta_t = \frac{\pi}{2} - \theta$, we get

$$\begin{aligned} E_0 - E_1 &= E_2\tan\theta \ , \\ E_0 + E_1 &= n_{12}E_2 \ , \end{aligned} \tag{54}$$

where $n_{12} = k_2/k_1$.

However, from the Snell's law we have that

$$\frac{\sin\theta}{\sin\theta_t} = \frac{\sin\theta}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{\sin\theta}{\cos\theta} = \tan\theta = n_{12} \ .$$

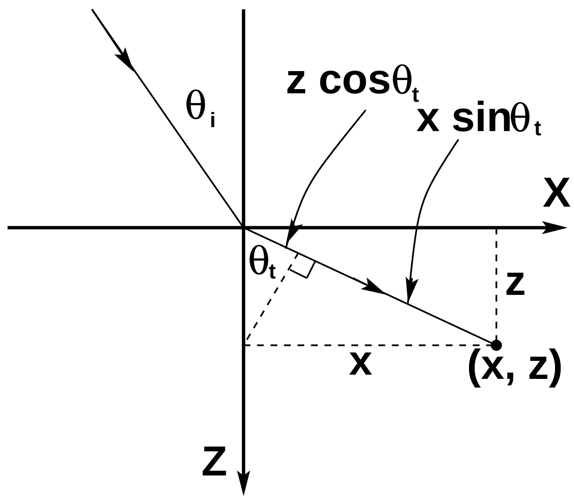
Thus, Eqs. (54) will be satisfied simultaneously only if $E_1 = 0$, i.e. there is no reflected field in the plane of incidence.

17.1.2 Total internal reflection

Consider the Snell's law

$$\sin\theta_t = \frac{\sin\theta_i}{n_{21}} , \qquad \text{where} \;\; n_{21} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \; .$$

In the case $n_{21} < 1$ ($\varepsilon_2 < \varepsilon_1$), going from an optically more dense to optically less dense medium e.g. from water in to air, real angles θ_t are obtained only for $\sin\theta_i \leq n_{21}$.



For greater θ_i, $\sin\theta_t > 1$, and then the angle of refraction θ_t becomes imaginary. In this case, there is no real refracted wave, only a reflected wave. The Fresnel equations are still capable of giving a formal solution in this case

$$\begin{aligned} \cos\theta_t &= \sqrt{1-\sin^2\theta_t} = \sqrt{1-\frac{\sin^2\theta_i}{n_{21}^2}} \\ &= i\sqrt{\frac{\sin^2\theta_i}{n_{21}^2}-1} = -i\beta \; , \end{aligned}$$

where β is real positive. Put, e.g.

$$\sin\theta_t = \frac{\sin\theta_i}{n_{21}} = \alpha \; .$$

Now consider the propagation of the transmitted wave in the less optically dense medium, for which

$$E_t = E_2\,\mathrm{e}^{i(\omega t - \hat{n}_t\cdot\vec{r}k)} \ ,$$

with

$$\hat{n}_t\cdot\vec{r} = z\,\cos\theta_t + x\,\sin\theta_t = -i\beta z + \alpha x \ ,$$

and then

$$E_t = E_2\mathrm{e}^{i(\omega t + i\beta' z - \alpha' x)} = E_2\,\mathrm{e}^{-\beta' z}\,\mathrm{e}^{i(\omega t - \alpha' x)} \ .$$

Here $\beta' = k\beta$ and $\alpha' = k\alpha$, i.e. there is attenuation in the z direction but no phase propagation. Phase propagation occurs in the x direction along the boundary.

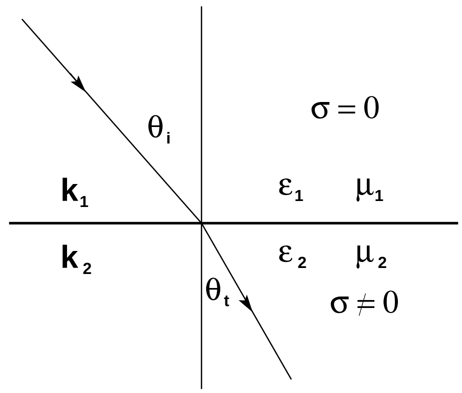
This illustrates a general method of applying the Fresnel equations. For only a limited range of circumstances will all the angles θ_i, θ_r, θ_t be real. We can however always apply a *generalized Snell's Law* $k_2\,\sin\theta_t = k_1\,\sin\theta_i$ to find $\sin\theta_t$ and $\cos\theta_t$ and proceed as above.
Note that the planes of constant phase are normal to the boundary (i.e. they have their normals tangential to the boundary). The phase of the transmitted wave below the boundary must match the incident wave above. The wavelength in the second medium is

$$\lambda' = \frac{2\pi}{\alpha'} = \frac{2\pi}{k\alpha} = \frac{2\pi}{\frac{2\pi}{\lambda_0}\frac{\sin\theta_i}{n_{21}}} = \frac{\lambda_0 n_{21}}{\sin\theta_i} \ ,$$

where λ_0 is the wavelength of freely propagating waves in this medium.
The planes of constant amplitude in the transmitted medium are parallel to the boundary.

17.2 Transmission and Reflection at a Conducting Surface

Propagation of EM fields in conductors (metals) is more complicated phenomenon than in dielectrics. Consider a propagation of an EM wave in a



medium composed of a dielectric and a conductor, and assume that the incident wave originates in the dielectric.

From previous work, we know that in the dielectric

$$k_1^2 = \varepsilon_1 \mu_1 \omega^2$$

and in the conductor

$$k_2^2 = \varepsilon_2 \mu_2 \omega^2 - i\omega\sigma\mu_2 = \varepsilon_2 \mu_2 \omega^2 \left(1 - \frac{i\sigma}{\varepsilon_2 \omega}\right) \ .$$

Moreover, from the Snell's law

$$\sin\theta_t = \frac{k_1}{k_2} \sin\theta_i \ .$$

Assume a good conductor. Since $k_2 \to \infty$ as $\frac{\sigma}{\varepsilon_2 \omega} \gg 1$, we see that $\sin\theta_t \to 0$ independent of θ_i. Thus, the direction of the transmitted wave is normal to the surface **independent** of the angle of incidence θ_i.
It follows that in the conductor the field vectors $\vec{E}$ and $\vec{H}$ lie tangential to the boundary and so the normal components of these vectors on the conductor side of the boundary are zero.

It follows that:

- Since the normal component of $\vec{B}$ (or $\vec{H}$) is continuous across the boundary, the normal component of $\vec{H}$ or $\vec{B}$ is zero on the dielectric

side also.
Thus the normal component of $\vec{B}$ of the reflected wave must be equal and opposite to that of the incident wave.

- In the conductor

$$\left(\frac{E}{H}\right)_2 = Z_2 = \frac{\mu_0\omega}{k_2} \ll \left(\frac{E}{H}\right)_1 = Z_1 \ .$$

  This happens because

$$\left|\frac{k_2^2}{k_1^2}\right| = K\left|1 - \frac{i\sigma}{\varepsilon\omega}\right| \to \infty \qquad \text{as } \frac{\sigma}{\varepsilon\omega} \gg 1 \ .$$

  This means that the electric field in the conductor (which is tangential to the surface) $\to 0$.
  Since $E_\parallel$ is continuous across the boundary $E_\parallel$ is zero also in the dielectric at the boundary.

In summary, we have two useful special boundary conditions at the surface between a dielectric and a perfect conductor:

1. The tangential component of $\vec{E} = 0$.
2. The normal component of $\vec{B}$ or $\vec{H} = 0$.

17.2.1 Field vectors at normal incidence

We now consider the special case of normal incidence at a boundary, i.e. when the wave propagation vector coincides with the normal to the boundary. Figure 9 shows how the field vectors must look in the plane of the surface between dielectric and conductor.

Since for a good conductor

$$\frac{\sigma}{\varepsilon\omega} \to \infty \ ,$$

the field components

$$E_t = E_i - E_r \quad \to \ 0 \ ,$$

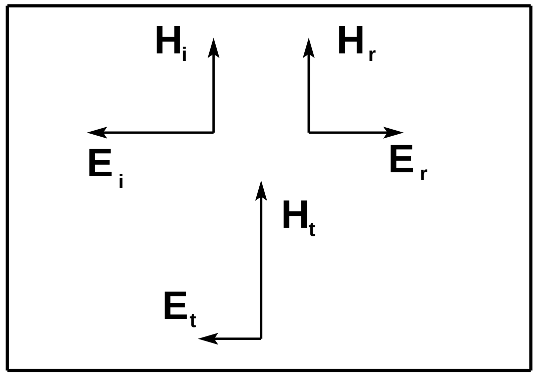


Figure 9: The field vectors in the plane of the surface between dielectric and conductor.

$$H_t = H_i + H_r \quad \rightarrow \; 2H_i \; ,$$

In this case, the power reflection coefficient

$$\alpha_p = \frac{E_r\, H_r}{E_i\, H_i} = \frac{E_r^2}{E_i^2} \quad \rightarrow \quad 1 \;\; \text{as} \;\; \frac{\sigma}{\varepsilon\omega} \; \rightarrow \infty \; .$$

One could think that under the normal incidence, there is only the incident and transmitted wave with no reflected wave. Here, we prove the necessity of assuming the existence of the reflected wave.

From the continuity of the tangential components, we have

$$E_i - E_r \;\; = \;\; E_t \; , \tag{55}$$

$$H_i + H_r \;\; = \;\; H_t \; . \tag{56}$$

However

$$H = \sqrt{\frac{\varepsilon}{\mu_0}} E = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \sqrt{\frac{\varepsilon_0}{\mu_0}} E = n \sqrt{\frac{\varepsilon_0}{\mu_0}} E \; .$$

Thus, Eq. (56) can be written as

$$n_1 E_i + n_1 E_r = n_2 E_t \; ,$$

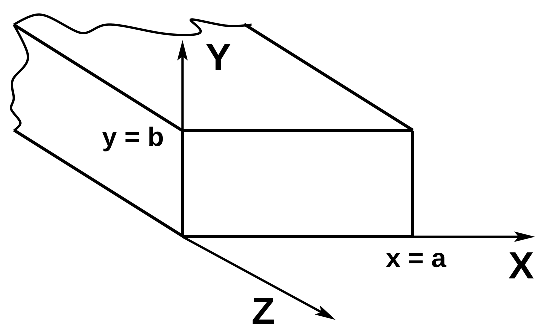
and then we get two equations for the amplitudes of the electric field

$$\begin{aligned} E_i - E_r &= E_t \ , \\ E_i + E_r &= \frac{n_2}{n_1} E_t \ . \end{aligned}$$

If E_r is missing, we could not simultaneously satisfy both equations. Thus, there always is a reflected wave in the normal incidence.

17.3 Wave Propagation in a Conducting Rectangular Pipe (The Rectangular Waveguide)

In this lecture, we discuss the propagation of bounded EM waves by considering the propagation of radiation through a waveguide where the radiation is fully confined in the transverse plane. We will consider the case where the bounding walls are planar and cross section is rectangular.



From the nature of the problem we see that we are likely to get standing waves in the x direction due to reflections between the walls $x = 0$ and $x = a$, standing waves in the y direction due to reflections between the walls $y = 0$ and $y = b$ and waves propagating in the z direction.

We solve Maxwell's equations subject to the good conductor boundary conditions being satisfied at $x = 0$, $x = a$, $y = 0$ and $y = b$. We write the z dependence of any field component in the form

$$e^{-\gamma z} \qquad \text{Thus} \ \frac{\partial}{\partial z} \equiv -\gamma \ ,$$

where γ describes the propagation conditions, e.g. γ purely imaginary describes a wave propagating without loss.
We describe the electromagnetic field by the vector pair $\vec{E}, \vec{H}$, and we use Maxwell's equations in the form

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon} = 0 \\
\nabla \cdot \vec{B} &= 0 \;\text{ or }\; \nabla \cdot \vec{H} = 0 \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \\
\nabla \times \vec{H} &= \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}
\end{aligned}$$

Certain characteristic modes of propagation are found. In general a *characteristic mode* is one which propagates with constant polarization.

In a rectangular waveguide, one may have TE (transverse electric) waves, or TM (transverse magnetic) wave, but not TEM wave. If both E and B fields are transverse, the wave would be going straight down the guide. However, such a wave would not satisfy various boundary conditions.

17.3.1 Transverse Electric (TE) Modes

It is possible to propagate a wave with the electric field polarized in the xy plane by lifting the restriction that both the electric and magnetic fields are transverse.

Look for solutions to Maxwell's equations with $E_z = 0$ that satisfy the good conductor boundary conditions.

For the TE modes we use Maxwell's equations to derive a wave equation for H_z, the longitudinal component of $\vec{H}$. This equation is used to satisfy the boundary conditions.

Write out Maxwell's equations in Cartesian coordinates.

- From $\nabla \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ E_x & E_y & 0 \end{vmatrix} + i\omega\mu\vec{H} = 0$$

$$\text{x component} \qquad \gamma E_y + i\omega\mu H_x = 0 \tag{57}$$

$$\text{y component} \qquad -\gamma E_x + i\omega\mu H_y = 0 \tag{58}$$

$$\text{z component} \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + i\omega\mu H_z = 0 \tag{59}$$

- From $\nabla \cdot \vec{H} = 0$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Hence:

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} - \gamma H_z = 0 \tag{60}$$

- From $\nabla \times \vec{H} - (\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}) = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\gamma \\ H_x & H_y & H_z \end{vmatrix} - (\sigma + j\omega\varepsilon)\vec{E} = 0$$

$$\frac{\partial H_z}{\partial y} + \gamma H_y - (\sigma + i\omega\varepsilon)E_x = 0 \tag{61}$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} - (\sigma + i\omega\varepsilon)E_y = 0 \tag{62}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \qquad (E_z = 0) \tag{63}$$

- From $\nabla \cdot \vec{E} = \rho/\varepsilon = 0$

  And using the TE condition $E_z = 0$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad (E_z = 0) \tag{64}$$

From Eqs. (57) and (58)

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{i\omega\mu}{\gamma} = Z \tag{65}$$

where Z is called the wave impedance.
In Eq. (59), we can then express E_x and E_y in terms of H_y and H_x and show that it becomes identical to Eq. (60).

Use Eq. (65) in Eqs. (61) and (62)
In Eq. (61) substitute for E_x:

$$\frac{\partial H_z}{\partial y} + \gamma H_y - (\sigma + i\omega\varepsilon)\frac{i\omega\mu}{\gamma} H_y = 0$$

Hence

$$\begin{aligned} H_y &= \frac{-1}{\gamma - (\sigma + i\omega\varepsilon)\frac{i\omega\mu}{\gamma}} \frac{\partial H_z}{\partial y} \\ &= \frac{-\gamma}{\gamma^2 - i\omega\mu(\sigma + i\omega\varepsilon)} \frac{\partial H_z}{\partial y} \end{aligned} \tag{66}$$

In Eq. (62) substitute for E_y and similarly get

$$H_x = \frac{-\gamma}{\gamma^2 - i\omega\mu(\sigma + i\omega\varepsilon)} \frac{\partial H_z}{\partial x} \tag{67}$$

Using Eqs. (65), (66), and (67), we find that Eqs. (63) and (64) are automatically satisfied.

Putting Eqs. (66) and (67) into Eq. (60), we obtain

$$\frac{-\gamma}{\gamma^2 - i\omega\mu(\sigma + i\omega\varepsilon)} \frac{\partial^2 H_z}{\partial x^2}$$

$$-\frac{\gamma}{\gamma^2 - i\omega\mu(\sigma + i\omega\varepsilon)} \frac{\partial^2 H_z}{\partial y^2} - \gamma H_z = 0$$

Dividing by $-\gamma$ and multiplying by $k^2 = \gamma^2 - i\omega\mu(\sigma + i\omega\varepsilon)$, we obtain

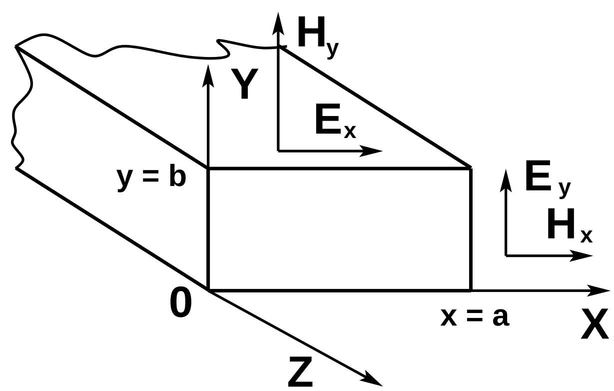
$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k^2 H_z = 0 \tag{68}$$

If we can solve Eq. (68) for H_z then we can solve Eqs. (66) and (67) for H_x and H_y. Then we can solve Eq. (65) for E_x and E_y, and then we will know all the field components.

17.3.2 Boundary Conditions

The boundary conditions at the surface of a good conductor are

1. $\vec{H}$ normal to boundary (in xy plane)$= 0$.
2. $\vec{E}$ tangential to boundary (in xy plane)$= 0$.



Thus, we must have at the boundaries $x = 0$ and $x = a$, $H_x = 0$ and $E_y = 0$.
Looking at Eq. (62), we see that this means that $\partial H_z/\partial x$ must equal zero at $x = 0$ and $x = a$.

Similarly, at the boundaries $y = 0$ and $y = b$ we must have $H_y = 0$ and $E_x = 0$.
Looking at Eq. (61), we see that this means that $\partial H_z/\partial y$ must equal zero at $y = 0$ and $y = b$.

The solution of Eq. (68), which satisfies these boundary conditions is of the form

$$H_z = H_0 \cos(k_x x) \cos(k_y y)\, \mathrm{e}^{i\omega t - \gamma z}$$

with

$$k_x a = m\pi \qquad \text{and} \qquad k_y b = n\pi$$

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i\omega t - \gamma z} \ .$$

Here $m, n = 0, 1, 2 \ldots$. An m, n combination represents a possible TE mode of propagation. The modes are designated in the form TE_{mn}.

Then Eq. (68) gives

$$\left[-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + k^2\right] H_z = 0$$

For a non-trivial solution, $H_z \neq 0$, and then

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 - i\mu\omega(\sigma + i\varepsilon\omega)$$

$$\gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + i\mu\omega(\sigma + i\varepsilon\omega)$$

In general γ is complex. We can write $\gamma = \beta + i\alpha$.
Then we can write

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{-\beta z}\, \mathrm{e}^{i(\omega t - \alpha z)}$$

Then $\alpha = 2\pi/\lambda_g$, where λ_g is the wavelength *in the waveguide* at frequency ω. The phase velocity in the waveguide is $v_p = \omega/\alpha$.

The parameter β is the attenuation coefficient describing losses in the waveguide. Energy loss may be due to:

- Ohmic resistivity (σ of the medium finite).
- Dielectric losses (described by ε having an imaginary component)
- Magnetic losses (described by μ having an imaginary component)

17.3.3 TE Modes in a Lossless Waveguide

Assume a lossless propagation for which we have $\sigma = 0$ and ε and μ both purely real. In this case

$$\gamma^2 = -\varepsilon\mu\omega^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \ ,$$

or introducing the phase velocity, we obtain

$$\gamma^2 = -\frac{\omega^2}{v_0^2} + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

where v_0 is **the phase velocity of propagation of waves in an infinite (unbounded) medium of the type filling the waveguide.**

$$\frac{\omega}{v_m} = \frac{2\pi f}{f\lambda_0} = \frac{2\pi}{\lambda_0}$$

where λ_0 is the infinite medium wavelength at that frequency.

$$\gamma^2 = -\left(\frac{2\pi}{\lambda_0}\right)^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

The parameter γ^2 can be negative or positive, and the nature of the propagation changes according as:

1. Consider γ^2 negative. Then $\gamma = ik_g$ is purely imaginary and we have a propagating wave

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i(\omega t - k_g z)} \ ,$$

where k_g is the guide propagation constant.

The wave then propagates with a guide wavelength λ_g given by

$$\frac{2\pi}{\lambda_g} = k_g = \sqrt{\left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

There is a maximum λ_0 (=λ_c say) such that k_g is real

$$\frac{1}{\lambda_c^2} = \left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2$$

Thus, there is a minimum frequency f_{mn} such that the TE_{mn} mode will propagate down the waveguide

$$f_{mn} = v_m\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2} \ .$$

This can be derived from the cut-off condition $f_{mn}\,\lambda_c = v_m$.

Thus, the waveguide acts as a high-pass filter for any m, n mode.

2. Consider γ^2 positive. Then $\gamma = \beta$ say is purely real and

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{-\beta z}\, \mathrm{e}^{i\omega t} \ .$$

There is no phase propagation but amplitude attenuation. Note there is no energy loss mechanism available. This is an evanescent mode at frequencies less that f_{mn} analogous to the case of total internal reflection.

Field components in the TE modes.

Since

$$H_z \sim \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i\omega t - \gamma z} \ .$$

From Eq. (66)

$$H_y \sim \left(\frac{\partial H_z}{\partial y}\right) \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i\omega t - \gamma z} \ .$$

From Eq. (67)

$$H_x \sim \left(\frac{\partial H_z}{\partial x}\right) \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i\omega t - \gamma z} \ .$$

And from Eq. (65)

$$E_x \sim H_y \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i\omega t - \gamma z}$$

$$E_y \sim H_x \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \mathrm{e}^{i\omega t - \gamma z}$$

Allowed values of m and n for TE modes

$m = n = 0$ is not allowed since E_x, E_y, H_x, H_y all contain *sine* terms so all fields vanish for $m = n = 0$. All other TE_{mn} modes are allowed.

$m = n = 0$ mode is never possible in a transmission line consisting of a single closed conductor like a rectangular waveguide. It is possible in 2-conductor lines e.g. the coaxial line or the twin wire transmission line.

TM (transverse magnetic) modes

Put $H_z = 0$ and go through the whole procedure again. $E_z \neq 0$ now. Equations analogous to Eqs. (66), (67), and (68) appear for components of the $\vec{E}$ vector this time. Consequently the previous discussion about modes and their cut-off frequencies for TE modes is also true for TM modes. The only difference is that more modes are not allowed.

TM modes with either $m = 0$ OR $n = 0$ are NOT ALLOWED.

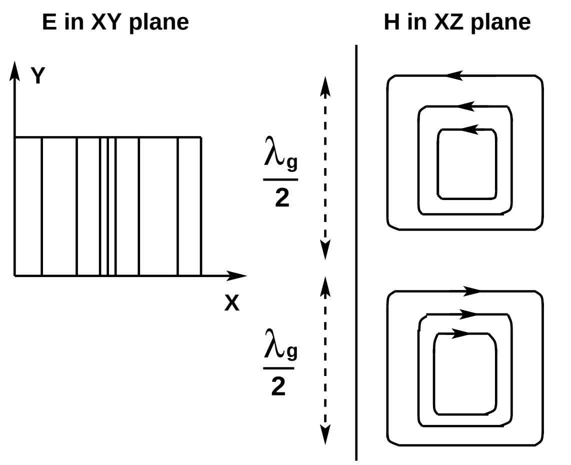
Special properties of the TE_{10} mode

If the transverse dimensions of the rectangular waveguide are different ($a \neq b$) there is a finite range of frequencies over which the TE_{10} mode is the only allowed mode. This means that a waveguide can be designed to allow propagation in one mode only. We have

$$f_{mn} = v_m \sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}$$

$$f_{10} = \frac{v_m}{2a} < f_{01} = \frac{v_m}{2b}$$

where we adopt the convention that $a > b$. In the frequency range $f_{10} \to f_{01}$ the TE_{10} mode is the only mode allowed.



17.3.4 Phase and Group Velocities for TE and TM Modes

The phase velocity

$$v_p = f\lambda_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{\left(\frac{2\pi}{\lambda_0}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

Thus

$$v_p = \frac{v_0}{\sqrt{1 - \left(\frac{m\lambda_0}{2a}\right)^2 - \left(\frac{n\lambda_0}{2b}\right)^2}}$$

where $v_0 = \omega/k_0$ is the phase velocity in the unbounded medium and $k_0 = 2\pi/\lambda_0$.

We see that the phase velocity of the wave inside the waveguide is greater than in an unbounded medium, and so may be greater than the speed of light in vacuum.

Alternately we could write

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{2\pi}{\lambda_c}\right)^2 = k_c^2$$

where λ_c is the infinite medium wavelength at the cut-off frequency for the m, n mode.

$$v_p = \frac{\omega}{\sqrt{k_0^2 - k_c^2}} = \frac{\omega}{k_0\sqrt{1 - \left(\frac{k_c}{k_0}\right)^2}} = \frac{v_0}{\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}} \ ,$$

$$v_p = \frac{v_0}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \ ,$$

provided $f > f_c$ i.e. in the pass-band for that mode.
As $f \to f_c$, the phase velocity $v_p \to \infty$.
If a vacuum or air fills the waveguide then $v_0 = c$ and $v_p > c$.

The group velocity

Since $v_p > c$ typically we see that v_p is not the velocity of propagation of energy or information down the waveguide. The waveguide is a dispersive medium. The group velocity $v_g = \frac{d\omega}{dk_g}$ is then different from the phase velocity $v_p = \omega/k_g$.

The group velocity is the velocity of propagation of some modulation of the wave that carries information. A single frequency harmonic wave carries no information. It is just there. A finite bandwidth is required to carry in formation. Consider the sum of two frequencies ω and $\omega + d\omega$

$$\begin{aligned}
&\cos(\omega t - kz) + \cos[(\omega + d\omega)t - (k + dk)z] \\
&= 2\,\cos\left(\frac{d\omega}{2}t - \frac{dk}{2}z\right)\,\cos\left[\frac{2\omega + d\omega}{2}t - \frac{2k + dk}{2}z\right] \\
&= 2\cos\left(\frac{d\omega}{2}t - \frac{dk}{2}z\right)\,\cos(\omega t - kz) \ ,
\end{aligned}$$

where we have assumed that $d\omega \ll 2\omega$ and $dk \ll 2k$.

The velocity of the amplitude modulation is

$$v_g = \frac{\frac{d\omega}{2}}{\frac{dk}{2}} = \frac{d\omega}{dk}$$

In the rectangular waveguide

$$k_g = \frac{2\pi}{\lambda_g} = \sqrt{k_0^2 - k_c^2} = \frac{\sqrt{\omega^2 - \omega_c^2}}{v_m}$$

where $k_0 = \omega/v_m$ and $k_c = \omega_c/v_m$.

Remember v_m is the infinite medium phase velocity and ω_c is the cut-off (angular) frequency of the m, n mode. Then differentiating

$$\frac{dk_g}{d\omega} = \frac{1}{v_m}\frac{1}{2}(\omega^2 - \omega_c^2)^{-\frac{1}{2}}\, 2\omega = \frac{\omega}{v_m\sqrt{\omega^2 - \omega_c^2}}$$

Finally

$$\begin{aligned} v_g &= \frac{d\omega}{dk_g} = \frac{v_m\sqrt{\omega^2 - \omega_c^2}}{\omega} \\ &= v_m\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = v_m\sqrt{1 - \left(\frac{f_c}{f}\right)^2} < v_m \ . \end{aligned}$$

Thus

$$v_g\, v_p = v_m\sqrt{1 - \left(\frac{f_c}{f}\right)^2}\frac{v_m}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = v_m^2$$

In a vacuum-filled waveguide $v_g\, v_p = c^2$. Thus, relativity is still all right.

Questions:

(1) Show that under the Brewster's angle of incidence there is no reflected electric field in the plane of the incidence.

(2) Show that in a vacuum-filled rectangular waveguide $v_p > c$ and $v_g\, v_p = c^2$.

18 Relativistic Transformation of the Electromagnetic Field

18.1 The Principle of Relativity

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light in vacuum is independent of the uniform motion of the observer or source.

The constancy of the velocity of light, independent of the motion of the source, gives rise to the relations between space and time coordinates in different inertial reference frames known as *Lorentz transformations.*

Consider a stationary reference frame S and a inertial frame S' moving with a velocity $\vec{u}$ parallel to the x axis.

The time and space coordinates in S' are related to those in S by the Lorentz transformations

$$\begin{aligned} x' &= \gamma(x - \beta ct) \ , \\ y' &= y \ , \\ z' &= z \ , \\ ct' &= \gamma(ct - \beta x) \ , \end{aligned}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ is the Lorentz factor, and $\beta = u/c$.

The above transformation corresponds to a situation of $\vec{u}$ parallel to the x axis. If the axis in S and S' remain parallel, but the velocity $\vec{u}$ of the frame S' is in an arbitrary direction, the generalization of the above transformations is

$$\begin{aligned} \vec{r}' &= \vec{r} + (\gamma - 1)\,\frac{(\vec{r} \cdot \vec{\beta})\vec{\beta}}{\beta^2} - \gamma\vec{\beta}ct \\ ct' &= \gamma\left(ct - \vec{\beta} \cdot \vec{r}\right) \ , \end{aligned}$$

where $\vec{\beta} = \vec{u}/c$

Proof:

Decompose the vector $\vec{r}$ into two components: parallel and normal to $\vec{\beta}$

$$\vec{r} = \vec{r}_{\|} + \vec{r}_{\perp}$$

Then, using the one dimensional Lorentz transformations, we have

$$\begin{aligned} \vec{r}'_{\|} &= \gamma \left(\vec{r}_{\|} - \vec{\beta} ct \right) \\ \vec{r}'_{\perp} &= \vec{r}_{\perp} \end{aligned}$$

We can write the parallel and normal components as

$$\begin{aligned} \vec{r}_{\|} &= \frac{(\vec{r} \cdot \vec{\beta})\vec{\beta}}{\beta^2} \\ \vec{r}_{\perp} &= \vec{r} - \vec{r}_{\|} \end{aligned}$$

Hence

$$\begin{aligned} \vec{r}' &= \vec{r}'_{\|} + \vec{r}'_{\perp} \\ &= \gamma \left(\vec{r}_{\|} - \vec{\beta} ct \right) + \vec{r} - \vec{r}_{\|} \\ &= \vec{r} + (\gamma - 1) \frac{(\vec{r} \cdot \vec{\beta})\vec{\beta}}{\beta^2} - \gamma \vec{\beta} ct \end{aligned}$$

Similarly

$$ct' = \gamma (ct - \beta x) = \gamma \left(ct - \beta r_{\|} \right)$$

However

$$\beta r_{\|} = \vec{\beta} \cdot \vec{r}_{\|} = \frac{(\vec{r} \cdot \vec{\beta})}{\beta^2} \vec{\beta} \cdot \vec{\beta} = \vec{r} \cdot \vec{\beta}$$

which gives

$$ct' = \gamma \left(ct - \vec{\beta} \cdot \vec{r} \right)$$

as required.

The inverse Lorentz transformation is

$$\begin{aligned}\vec{r} &= \vec{r}' + (\gamma - 1)\frac{(\vec{r}' \cdot \vec{\beta})\vec{\beta}}{\beta^2} + \gamma\vec{\beta}ct' \\ ct &= \gamma\left(ct' + \vec{\beta} \cdot \vec{r}'\right)\end{aligned}$$

The principle of relativity indicates that the Maxwell equations and the continuity equation should be invariant under the Lorentz transformation.

In the frame S:

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{J} &= -\frac{\partial \rho}{\partial t}\end{aligned}$$

In the frame S':

$$\begin{aligned}\nabla' \cdot \vec{D}' &= \rho' \\ \nabla' \cdot \vec{B}' &= 0 \\ \nabla' \times \vec{E}' &= -\frac{\partial \vec{B}'}{\partial t'} \\ \nabla' \times \vec{H}' &= \vec{J}' + \frac{\partial \vec{D}'}{\partial t'} \\ \nabla' \cdot \vec{J}' &= -\frac{\partial \rho'}{\partial t'}\end{aligned}$$

where the prime variables are functions of the transformed variables (t' and $\vec{r}'$).

First, we will illustrate the transformation of an arbitrary scalar function Ψ and the divergence $\nabla \cdot \vec{F}$.

Consider the transformation of a time derivative $\partial\Psi/\partial(ct)$

$$\begin{aligned}\frac{\partial\Psi}{\partial(ct)} &= \frac{\partial\Psi}{\partial(ct')}\frac{\partial ct'}{\partial(ct)} + \frac{\partial\Psi}{\partial x'}\frac{\partial x'}{\partial(ct)} + \frac{\partial\Psi}{\partial y'}\frac{\partial y'}{\partial(ct)} \\ &+ \frac{\partial\Psi}{\partial z'}\frac{\partial z'}{\partial(ct)} = \frac{\partial\Psi}{\partial(ct')}\frac{\partial ct'}{\partial(ct)} + \nabla'\Psi\cdot\frac{\partial\vec{r}'}{\partial(ct)}\end{aligned}$$

However

$$\frac{\partial ct'}{\partial(ct)} = \gamma\ , \qquad \frac{\partial\vec{r}'}{\partial(ct)} = -\gamma\vec{\beta}$$

which gives

$$\frac{\partial\Psi}{\partial(ct)} = \gamma\left(\frac{\partial}{\partial(ct')} - \vec{\beta}\cdot\nabla'\right)\Psi$$

Consider now the divergence

$$\nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Since

$$\begin{aligned}\frac{\partial F_x}{\partial x} &= \frac{\partial F_x}{\partial(ct')}\frac{\partial ct'}{\partial x} + \frac{\partial F_x}{\partial x'}\frac{\partial x'}{\partial x} \\ &+ \frac{\partial F_x}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial F_x}{\partial z'}\frac{\partial z'}{\partial x}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial ct'}{\partial x} &= -\gamma\beta_x \\ \frac{\partial x'}{\partial x} &= 1 + (\gamma-1)\frac{\beta_x^2}{\beta^2} \\ \frac{\partial y'}{\partial x} &= \frac{\partial z'}{\partial x} = 0\end{aligned}$$

we obtain

$$\frac{\partial F_x}{\partial x} = -\gamma\beta_x\frac{\partial F_x}{\partial(ct')} + \alpha_x\frac{\partial F_x}{\partial x'}$$

where

$$\alpha_x = 1 + (\gamma - 1)\beta_x^2/\beta^2$$

Similarly, for F_y and F_z, and finally we get

$$\nabla \cdot \vec{F} = \bar{\alpha} * (\nabla' \cdot \vec{F}) - \gamma\vec{\beta} \cdot \frac{\partial \vec{F}}{\partial(ct')}$$

where $\bar{\alpha}$ is a 3×3 diagonal matrix

$$\bar{\alpha} = \begin{vmatrix} 1 + (\gamma - 1)\frac{\beta_x^2}{\beta^2} & 0 & 0 \\ 0 & 1 + (\gamma - 1)\frac{\beta_y^2}{\beta^2} & 0 \\ 0 & 0 & 1 + (\gamma - 1)\frac{\beta_z^2}{\beta^2} \end{vmatrix}$$

Using the above transformations, we can derive transformations for the current density $\vec{J}$ and the charge density ρ.
In order to do it, we consider the continuity equation, that can be written as

$$\nabla \cdot \vec{J} = -\frac{\partial c\rho}{\partial(ct)}$$

Hence

$$\begin{aligned} \bar{\alpha} * (\nabla' \cdot \vec{J}) - \gamma\vec{\beta} \cdot \frac{\partial \vec{J}}{\partial(ct')} &= -\gamma \left(\frac{\partial}{\partial(ct')} - \vec{\beta} \cdot \nabla' \right) c\rho \\ \nabla' \cdot (\bar{\alpha} * \vec{J}) - \gamma\vec{\beta} \cdot \nabla'(c\rho) &= -\gamma \frac{\partial}{\partial(ct')} \left(c\rho - \vec{\beta} \cdot \vec{J} \right) \end{aligned}$$

Since

$$\vec{\beta} \cdot \nabla'(c\rho) = \nabla' \cdot (c\rho\vec{\beta})$$

we obtain

$$\nabla' \cdot \left(\bar{\alpha} * \vec{J} - \gamma c\rho\vec{\beta} \right) = -\frac{\partial}{\partial(ct')} \left[\gamma \left(c\rho - \vec{\beta} \cdot \vec{J} \right) \right]$$

Thus, the continuity equation will be invariant under the Lorentz transformation if

$$\begin{aligned} c\rho' &= \gamma\left(c\rho - \vec{\beta}\cdot\vec{J}\right) \ , \\ \vec{J}' &= \bar{\alpha} * \vec{J} - \gamma c\rho\vec{\beta} \ . \end{aligned} \tag{69}$$

In order to understand the physical meaning of these equations, consider the following example.

Example

Assume that in the S frame there is a stationary volume charge of density $\rho \neq 0$. Since ρ is stationary, there are no currents in the S frame ($\vec{J} = 0$). What are the charge and current densities as seen in the S' frame?

In the S frame

$$\vec{J} = 0 \ , \qquad \rho \neq 0$$

According to Eq. (69), in the S' frame

$$\vec{J}' = -\gamma c\rho\vec{\beta} \ , \qquad \rho' = \gamma\rho$$

Thus, there is a current in the S' frame. As seen from S' a given part of the charge is length contracted in the direction of motion so the charge density is correspondingly increased by the factor $\gamma > 1$. The length contracted charge density appears from S' to move in the opposite direction. We can understand this result: The stationary charge in the S frame moves with velocity $-\vec{u}$ in the S' frame.

Less obvious and more interesting is the following situation.
In the S frame

$$\vec{J} \neq 0 \ , \qquad \rho = 0 \ .$$

Then, someone will see charge density $\rho' \neq 0$ in the S' frame.
This is a pure relativistic effect.

18.2 Transformation of Electric and Magnetic Field Components

To find the transformation rules for electric and magnetic field components we will use the transformations of the time and space derivatives derived above.

Consider two of the Maxwell equations that in the S frame are

$$\begin{aligned}\nabla \cdot \vec{D} &= \rho \\ \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t}\end{aligned}$$

These equations should be equivalent of two equations

$$\begin{aligned}\nabla' \cdot \vec{D}\,' &= \rho' \\ \nabla' \times \vec{H}\,' &= \vec{J}' + \frac{\partial \vec{D}\,'}{\partial t'}\end{aligned}$$

in the S' frame.

Using the transformations of the time and space derivatives, we have

$$\begin{aligned}&\left[\bar{\alpha} * \nabla' - \gamma\vec{\beta}\frac{\partial}{\partial(ct')}\right] \times \vec{H} \\ &\qquad -\gamma\left(\frac{\partial}{\partial(ct')} - \vec{\beta}\cdot\nabla'\right) c\vec{D} = \vec{J} \\ &\left[\bar{\alpha} * \nabla' - \gamma\vec{\beta}\frac{\partial}{\partial(ct')}\right] \cdot c\vec{D} = c\rho\end{aligned}$$

Substituting the transformations of $\vec{J}$ and ρ, we find that the $\vec{D}$ and $\vec{H}$ vectors transform as

$$\begin{aligned}c\vec{D}\,' &= \gamma\left(\bar{\alpha}^{-1} * c\vec{D} + \vec{\beta} \times \vec{H}\right) \\ \vec{H}\,' &= \gamma\left(-\vec{\beta} \times c\vec{D} + \bar{\alpha}^{-1} * \vec{H}\right)\end{aligned}$$

where $\bar{\alpha}^{-1}$ is the inverse of the matrix $\bar{\alpha}$

$$\bar{\alpha}^{-1} = \begin{vmatrix} 1+(\frac{1}{\gamma}-1)\frac{\beta_x^2}{\beta^2} & 0 & 0 \\ 0 & 1+(\frac{1}{\gamma}-1)\frac{\beta_y^2}{\beta^2} & 0 \\ 0 & 0 & 1+(\frac{1}{\gamma}-1)\frac{\beta_z^2}{\beta^2} \end{vmatrix}$$

From the Maxwell equations

$$\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$

we find that the $\vec{E}$ and $\vec{B}$ vectors transform as

$$\begin{aligned} \vec{E}' &= \gamma \left(\bar{\alpha}^{-1} * \vec{E} + \vec{\beta} \times c\vec{B} \right) \\ c\vec{B}' &= \gamma \left(-\vec{\beta} \times \vec{E} + \bar{\alpha}^{-1} * c\vec{B} \right) \end{aligned}$$

18.3 Transformation Rules in Terms of Parallel and Normal Components

Suppose that the frame S' is moving with speed u in the direction parallel to the z axis. In this case, $\beta_x = \beta_y = 0$, $\beta_z = \beta \neq 0$, and then the transformations take the form

$$\begin{aligned} c\vec{D}' &= \gamma c D_x \hat{\imath} + \gamma c D_y \hat{\jmath} + c D_z \hat{k} + \gamma\beta \hat{k} \times \vec{H} \\ \vec{H}' &= -\gamma\beta c \hat{k} \times \vec{D} + \gamma H_x \hat{\imath} + \gamma H_y \hat{\jmath} + H_z \hat{k} \\ \vec{E}' &= \gamma E_x \hat{\imath} + \gamma E_y \hat{\jmath} + E_z \hat{k} + \gamma c\beta \hat{k} \times \vec{B} \\ c\vec{B}' &= -\gamma\beta \hat{k} \times \vec{E} + \gamma c B_x \hat{\imath} + \gamma c B_y \hat{\jmath} + c B_z \hat{k} \end{aligned}$$

It is useful to rephrase the transformation rules in terms of components parallel and normal to $\vec{u}$. The parallel components are the z components and the normal components lie in the xy plane. For example

$$\vec{E} = \vec{E}_\perp + \vec{E}_\parallel = \left(E_x \hat{\imath} + E_y \hat{\jmath} \right) + E_z \hat{k}$$

and the same for $\vec{D}$, $\vec{H}$ and $\vec{B}$.

18.3.1 Rules for Parallel Components

$$\vec{E}'_{\parallel} = \vec{E}'_z\hat{k} = \vec{E}_z\hat{k} = \vec{E}_{\parallel}$$

and similarly

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \ , \qquad \vec{H}'_{\parallel} = \vec{H}_{\parallel} \ , \qquad \vec{D}'_{\parallel} = \vec{D}_{\parallel}$$

18.3.2 Rules for Normal Components

$$\begin{aligned} c\vec{D}'_{\perp} &= cD'_x\hat{\imath} + cD'_y\hat{\jmath} \\ &= \gamma cD_x\hat{\imath} + \gamma cD_y\hat{\jmath} + \gamma\beta H_x\hat{\jmath} - \gamma\beta H_y\hat{\imath} \\ &= \gamma\left(cD_x - \beta H_y\right)\hat{\imath} + \gamma\left(cD_y + \beta H_x\right)\hat{\jmath} \end{aligned}$$

$$\begin{aligned} \vec{H}'_{\perp} &= H'_x\hat{\imath} + H'_y\hat{\jmath} \\ &= \gamma\left(H_x + \beta cD_y\right)\hat{\imath} + \gamma\left(H_y - \beta cD_x\right)\hat{\jmath} \end{aligned}$$

$$\begin{aligned} \vec{E}'_{\perp} &= E'_x\hat{\imath} + E'_y\hat{\jmath} \\ &= \gamma\left(E_x - \beta cB_y\right)\hat{\imath} + \gamma\left(E_y + \beta cB_x\right)\hat{\jmath} \end{aligned}$$

$$\begin{aligned} c\vec{B}'_{\perp} &= cB'_x\hat{\imath} + cB'_y\hat{\jmath} \\ &= \gamma\left(cB_x + \beta E_y\right)\hat{\imath} + \gamma\left(cB_y - \beta E_x\right)\hat{\jmath} \end{aligned}$$

In general

$$\begin{aligned} c\vec{D}'_{\perp} &= \gamma\left(c\vec{D}_{\perp} + \vec{\beta}\times\vec{H}_{\perp}\right) \\ \vec{H}'_{\perp} &= \gamma\left(\vec{H}_{\perp} - \vec{\beta}\times c\vec{D}_{\perp}\right) \\ \vec{E}'_{\perp} &= \gamma\left(\vec{E}_{\perp} + \vec{\beta}\times c\vec{B}_{\perp}\right) \\ c\vec{B}'_{\perp} &= \gamma\left(c\vec{B}_{\perp} - \vec{\beta}\times\vec{E}_{\perp}\right) \end{aligned}$$

Example 1 - purely electric field in S

Suppose that in S, $\vec{E} \neq 0$ but $\vec{B} = 0$.
Then from the transformation rules, in S':

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel} \qquad \vec{E}_{\perp}' = \gamma \vec{E}_{\perp}$$

$$\vec{B}_{\parallel}' = 0 \qquad \vec{B}_{\perp}' = -\frac{\gamma}{c^2}\,\vec{u} \times \vec{E}_{\perp}$$

Thus

$$\vec{B}\,' = \vec{B}_{\perp}' = -\frac{\vec{u} \times \vec{E}_{\perp}'}{c^2} = -\frac{\vec{u} \times \vec{E}\,'}{c^2}$$

since $\vec{u} \times \vec{E}_{\parallel}' = 0$.

Thus what appears to be purely an electric field to one observer is seen as both an electric and a magnetic field to a second observer moving with respect to the first.

Example 2 - purely magnetic field in S

Now suppose that in S, $\vec{E} = 0$ while $\vec{B} \neq 0$.

Then using the transformation rules, in $S^{'}$:

$$\vec{B}_{\parallel}' = \vec{B}_{\parallel} \qquad \vec{B}_{\perp}' = \gamma \vec{B}_{\perp}$$

$$\vec{E}_{\parallel}' = 0 \qquad \vec{E}_{\perp}' = \gamma \vec{u} \times \vec{B}_{\perp}$$

Thus

$$\vec{E}\,' = \vec{E}_{\perp}' = \vec{u} \times \vec{B}_{\perp}' = \vec{u} \times \vec{B}\,'$$

We see that what appears to be a purely magnetic field for one observer will appear to be both an electric and a magnetic field to a relatively moving

observer.
This result could be used to calculate the emf in an electric dynamo from the point of view of an observer watching the conductor move in a magnetic field or from the point of view of an observer moving with the conductor.

18.4 Transformation of the Components of a Plane EM Wave

Example 1

Suppose that a plane wave propagates in vacuum along the z axis. Then

$$\begin{aligned} \vec{E} &= \hat{\imath} E \mathrm{e}^{i(\omega t - kz)} = \hat{\imath} E_0 \\ \vec{B} &= \hat{\jmath} B \mathrm{e}^{i(\omega t - kz)} = \hat{\jmath} B_0 \end{aligned}$$

Hence from the transformation rules, in S' moving in the same direction:

$$\begin{aligned} \vec{E}' &= \gamma \left(E_0 \hat{\imath} - \beta c B_0 \hat{\imath} \right) = \gamma \left(E_0 - u B_0 \right) \hat{\imath} \\ c\vec{B}' &= \gamma \left(c B_0 \hat{\jmath} - \beta E_0 \hat{\jmath} \right) = \gamma \left(c B_0 - \beta E_0 \right) \hat{\jmath} \end{aligned}$$

Since in vacuum

$$cB_0 = E_0$$

we obtain

$$\begin{aligned} \vec{E}' &= \gamma \left(1 - \frac{u}{c} \right) E_0 \hat{\imath} = \frac{1 - \frac{u}{c}}{\sqrt{1 - \left(\frac{u}{c} \right)^2}} E_0 \hat{\imath} \\ &= \sqrt{\frac{c-u}{c+u}} \, E_0 \hat{\imath} \end{aligned}$$

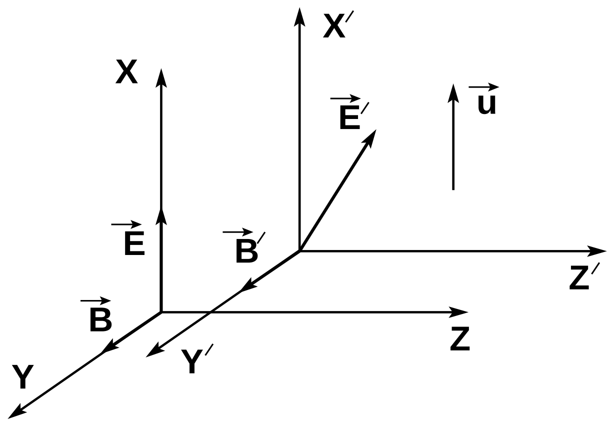
and

$$\begin{aligned} c\vec{B}' &= \gamma \left(1 - \frac{u}{c} \right) cB_0 \hat{\jmath} \\ &= \sqrt{\frac{c-u}{c+u}} \, cB_0 \hat{\jmath} \end{aligned}$$

Thus, the ratio $|\vec{E}'|/|\vec{B}'| = |E_0|/|B_0|$ is constant and independent of u.

Example 2

Suppose that the frame S' moves in the direction of the electric field, i.e. $\vec{u} = u\hat{i}$.



In this case

$$\begin{aligned} \vec{E}' &= E_0\hat{i} + \gamma\beta c B_0\hat{k} \\ &= \left(\hat{i} + \gamma\frac{u}{c}\hat{k}\right) E_0 \end{aligned}$$

and

$$c\vec{B}' = \gamma c B_0 \hat{j}$$

Thus, the magnetic field remains unchanged but the electric field turns towards the direction of propagation of the wave.

18.5 Doppler Effect

Consider a plane wave propagating in vacuum

$$\vec{E}(t) = \vec{E}_0 \mathrm{e}^{i(\omega t - \vec{k}\cdot\vec{r})}$$

In the moving frame S' this wave will have a different frequency ω' and the wave vector $\vec{k}'$, but the phase of the wave will remain unchanged as in invariant under the transformation

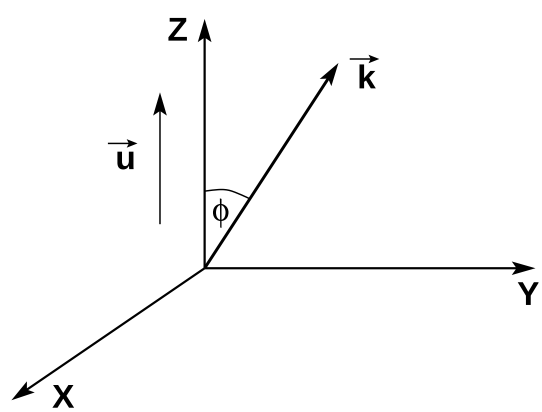
$$\phi = \omega t - \vec{k} \cdot \vec{r} = \omega' t' - \vec{k}' \cdot \vec{r}'$$

Using the Lorentz transformations

$$\begin{aligned} \vec{r} &= \bar{\alpha} * \vec{r}' + \gamma \vec{\beta} c t' \\ ct &= \gamma \left(ct' + \vec{\beta} \cdot \vec{r}' \right) \end{aligned}$$

Hence

$$\begin{aligned} &\omega' t' - \vec{k}' \cdot \vec{r}' = \omega t - \vec{k} \cdot \vec{r} \\ &= \frac{\omega\gamma}{c} \left(ct' + \vec{\beta} \cdot \vec{r}' \right) - \bar{\alpha} * \vec{k} \cdot \vec{r}' - \gamma \vec{k} \cdot \vec{\beta} c t' \\ &= \omega\gamma t' - \gamma t' \vec{k} \cdot \vec{u} + \left(\frac{\omega\gamma}{c} \vec{\beta} - \bar{\alpha} * \vec{k} \right) \cdot \vec{r}' \\ &= \gamma \left(\omega - \vec{k} \cdot \vec{u} \right) t' - \vec{k}' \cdot \vec{r}' \end{aligned}$$



Thus

$$\begin{aligned} \omega' &= \gamma \left(\omega - \vec{k} \cdot \vec{u} \right) \\ \vec{k}' &= \bar{\alpha} * \vec{k} - \frac{\omega\gamma}{c^2} \vec{u} \end{aligned}$$

If the wave propagates in vacuum, $\vec{k} = \frac{\omega}{c}\hat{\vec{k}}$, and then

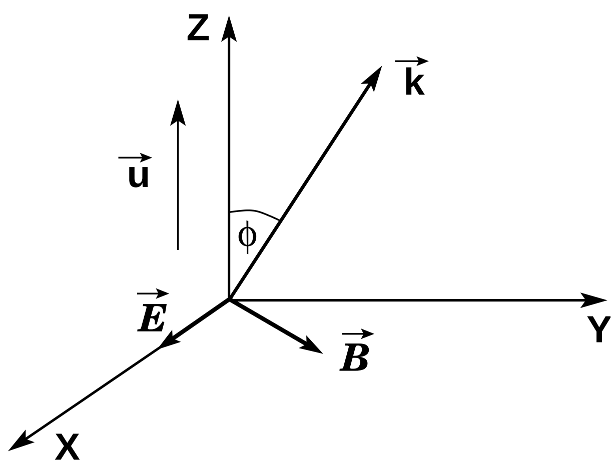
$$\omega' = \gamma\omega\left(1 - \frac{u}{c}\cos\phi\right)$$

where ϕ is the angle between the direction of propagation of the wave and $\vec{u}$. For $\phi = 0$

$$\omega' = \omega\sqrt{\frac{c-u}{c+u}}$$

18.6 Transformation of Energy of a Plane EM Wave

Consider an EM wave propagating in the $\vec{k}$ direction and an observer moving in the z direction, as shown in the Figure 18.6.



Let $\vec{E} = E\ \hat{\imath}$. Then, in the frame S':

$$\begin{aligned}\vec{E}' &= \gamma\vec{E} + \gamma\vec{\beta} \times c\vec{B}_{\perp} \\ &= \gamma\vec{E} + \gamma\cos\phi\ \vec{\beta} \times c\vec{B}\end{aligned}$$

Since $\vec{B} = B\hat{j} + B_{\parallel}\,\hat{k}$ (the wave propagates in the plane yz), and $cB = E$, we obtain

$$\begin{aligned}\vec{E}' &= \gamma\vec{E} - \gamma\cos\phi\;\beta cB\;\hat{i} \\ &= \gamma\,(1-\beta\cos\phi)\,\vec{E}\end{aligned}$$

Consider now energy of an electric field of a plane EM wave confined in a volume ΔV. In the S frame

$$W_e = \frac{1}{4}\varepsilon_0 E_0^2 \Delta V$$

In the S' frame

$$W_e' = \frac{1}{4}\varepsilon_0 (E_0')^2 \Delta V'$$

However

$$\Delta V = \Delta x \Delta y \Delta z$$

Assume that S' frame moves in the direction of the z axis, and the wave propagates in the direction $\vec{k}\cdot\vec{u} = u\cos\phi$. Then

$$\Delta V' = \Delta x \Delta y \Delta z'$$

where

$$\Delta z' = \frac{\Delta z}{\gamma\left(1 - \frac{u}{c}\cos\phi\right)}$$

Hence

$$\begin{aligned}W_e' &= \frac{1}{4}\varepsilon_0 E_0^2 \gamma^2 \left(1 - \frac{u}{c}\cos\phi\right)^2 \frac{\Delta V}{\gamma\left(1 - \frac{u}{c}\cos\phi\right)} \\ &= W_e \gamma \left(1 - \frac{u}{c}\cos\phi\right)\end{aligned}$$

It is interesting to compare the transformation of energy with the transformation of frequency

$$\begin{aligned}W_e' &= W_e \gamma \left(1 - \frac{u}{c}\cos\phi\right) \\ \omega' &= \omega\gamma \left(1 - \frac{u}{c}\cos\phi\right)\end{aligned}$$

We see that the energy and frequency transform in the similar way, indicating that $W_e \sim \omega$. This proportionality was predicted in quantum physics as $W_e = \hbar\omega$ and forms backgrounds of the so called **quantum electrodynamics**.

Questions:

(1) Find the condition under which the continuity equation for ρ and $\vec{J}$ is invariant under the Lorentz transformation.

Appendix A

PHYS3050 Facts and Formulae

$$\text{Gauss}' \text{ Divergence Theorem}: \quad \oint_S \vec{F} \cdot \hat{n}\, dS = \int_V \nabla \cdot \vec{F}\, dV$$

$$\text{Stokes}'\text{s Theorem}: \quad \oint_\ell \vec{F} \cdot d\vec{\ell} = \int_S \nabla \times \vec{F} \cdot \hat{n}\, dS$$

Numerical values in SI units:
$\varepsilon_0 = 8.85 \times 10^{-12}$
$\mu_0 = 4\pi \times 10^{-7}$
$c = 3 \times 10^8$ [ms^{-1}]
For the electron: $e = 1.6 \times 10^{-19}$ [C], $m = 9.11 \times 10^{-31}$ [kg]

$$\text{The Lorentz force law}: \quad \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\text{Coulomb}'\text{s Law}: \quad \vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$$

$$\text{Biot} - \text{Savart Law}: \quad d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{\ell} \times \hat{r}}{r^2}$$

$$\text{Gauss}' \text{ Law}: \quad \oint \vec{E} \cdot \hat{n}\, dS = \frac{Q}{\varepsilon_0}$$

$$\text{Ampère}'\text{s Circuital Law}: \quad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

Maxwell's Equations in vacuum:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations in material bodies:

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Poynting vector:

$$\vec{N} = \varepsilon_0 c^2 \vec{E} \times \vec{B}$$

Poynting's Theorem:

$$\oint_S \varepsilon_0 c^2 (\vec{E} \times \vec{B}) \cdot \hat{n}\, dS = -\int_V \vec{E} \cdot \vec{J}\, dV - \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \frac{B^2}{\mu_0} \right) dV$$

Or in polarizable materials where it is convenient to define $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$:

$$\oint_S \vec{E} \times \vec{H} \cdot \hat{n}\, dS = -\int_V \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\, dV - \int_V \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\, dV - \int_V \vec{E} \cdot \vec{J}\, dV$$

A theorem on the calculation of the mean Poynting vector from complex fields:

$$\bar{\vec{N}} = \frac{1}{2} \operatorname{Re} \left[\vec{E}_c \times \vec{H}_c^* \right]$$

The rate of doing work in magnetization

$$\frac{dW}{dt} = H \frac{dB}{dt}$$

and $B = \mu_0 (H + M)$.

An arbitrary vector $\vec{F}$ can be written as:

$$\begin{aligned} \vec{F} &= -\frac{1}{4\pi} \nabla \int_V \frac{\nabla \cdot \vec{F}}{r} dV + \frac{1}{4\pi} \nabla \times \int_V \frac{\nabla \times \vec{F}}{r} dV \\ &= \vec{F}_l + \vec{F}_t \end{aligned}$$

Fields and potentials:

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \qquad\qquad \vec{B} = \nabla \times \vec{A}$$

Differential equation for the vector potential:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)$$

In the Lorentz gauge:

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

the differential equations for the electromagnetic potentials are:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

and these have solutions of the form:

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r}\, dV$$

$$\vec{A} = \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{\vec{J}}{r}\, dV$$

The field of a Hertzian dipole:

$$E_r = \frac{I_0 \Delta\ell \cos\theta}{4\pi\varepsilon_0 c}\left[\frac{2}{ikr^3} + \frac{2}{r^2}\right] \mathrm{e}^{i(\omega t - kr)}$$

$$E_\theta = \frac{I_0 \Delta\ell \sin\theta}{4\pi\varepsilon_0 c}\left[\frac{1}{ikr^3} + \frac{1}{r^2} + \frac{ik}{r}\right] \mathrm{e}^{i(\omega t - kr)}$$

$$B_\phi = \frac{I_0 \Delta\ell \sin\theta}{4\pi\varepsilon_0 c^2}\left[\frac{ik}{r} + \frac{1}{r^2}\right] \mathrm{e}^{i(\omega t - kr)}$$

The mean energy flux from the Hertzian dipole:

$$\bar{N} = \frac{I_0^2}{8\varepsilon_0 c}\left[\frac{\Delta\ell}{\lambda}\right]^2 \frac{\sin^2\theta}{r^2}$$

A series solution in 2 dimensions to Laplace's equation in Cartesian coordinates:

$$\Phi(x,z) = \sum_k [A_k \sin(\alpha x) + B_k \cos(\alpha x)][C_k \sinh(\alpha z) + D_k \cosh(\alpha z)]$$

A series solution in 3 dimensions to Laplace's equation in spherical polar coordinates:

$$\Phi(r,\theta,\phi) = \sum_\ell \left\{\left(C_{1\ell}\, r^\ell + C_{2\ell} r^{-(\ell+1)}\right)\left[\sum_m [a_{lm} \cos(m\phi) + b_{lm} \sin(m\phi)]\, P_\ell^m(\cos\theta)\right]\right\}$$

Useful properties of trigonometrical functions:

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin^2\alpha &= \frac{1}{2}(1 - \cos 2\alpha) \\
\cos^2\alpha &= \frac{1}{2}(1 + \cos 2\alpha) \\
\int_0^\pi \sin^3\theta\, d\theta &= \frac{4}{3} \\
\int_0^{2\pi} \sin(m\phi)\sin(n\phi)\ d\phi &= \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases} \\
\int_0^{2\pi} \cos(m\phi)\cos(n\phi)\ d\phi &= \begin{cases} 0 & \text{for } m \neq n \\ \pi & \text{for } m = n \end{cases} \\
\int_0^{2\pi} \sin(m\phi)\cos(n\phi)\ d\phi &= 0 \quad \text{for all} \quad m \text{ and } n
\end{aligned}$$

Properties of Legendry polynomials:

$$\int_{-1}^{1} P_l^m(\cos\theta)\, P_k^n(\cos\theta)\, d(\cos\theta) \;=\; 0 \qquad \text{unless} \quad m=n \quad \text{and} \quad l=k$$

$$\int_{-1}^{1} [P_l^m(\cos\theta)]^2\, d(\cos\theta) \;=\; \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$$

$$P_0 = 1 \qquad P_1^0 = \cos\theta \qquad P_1^1 = \sin\theta \qquad P_2^0 = \frac{1}{4}(3\cos(2\theta)+1)$$

$$P_2^1 = \frac{3}{2}\sin(2\theta) \qquad P_2^2 = \frac{3}{2}(1-\cos(2\theta))$$

$$P_l(1) = 1 \qquad \text{for all} \quad l$$

A theorem on the electrostatic potential due to a distribution of electric dipoles of moment per unit volume $\vec{P}$:

$$\Phi = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\vec{P}\cdot\hat{n}}{r}\, dS + \frac{1}{4\pi\varepsilon_0}\int_V -\frac{\nabla\cdot\vec{P}}{r}\, dV$$

A theorem on the vector potential due to a distribution of magnetic dipoles of moment per unit volume $\vec{M}$:

$$\vec{A} = \frac{1}{4\pi\varepsilon_0 c^2}\int_V \frac{\nabla\times\vec{M}}{r}\, dV + \frac{1}{4\pi\varepsilon_0 c^2}\oint_S \frac{\vec{M}\times\hat{n}}{r}\, dS$$

A dispersion equation:

$$k = \omega\left[\left\{\varepsilon^{'}\mu^{'} - \mu^{''}\left[\varepsilon^{''} + \frac{\sigma}{\omega}\right]\right\} - i\left\{\mu^{'}\left[\varepsilon^{''} + \frac{\sigma}{\omega}\right] + \varepsilon^{'}\mu^{''}\right\}\right]^{\frac{1}{2}}$$

The skin depth in a good conductor:

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

General boundary conditions:

- The normal component of $\vec{B}$ is continuous across a boundary.
- The normal component of $\vec{D}$ is continuous across a boundary.
- The tangential component of $\vec{E}$ is continuous across a boundary.
- The tangential component of $\vec{H}$ is continuous across a boundary.

Special boundary conditions at the surface between a dielectric and a perfect conductor:

- The tangential component of $\vec{E} = 0$.
- The normal component of $\vec{B}$ or $\vec{H} = 0$.

The characteristic impedance of free space:

$$Z_0 = \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377 \simeq 120\pi \quad \text{ohms}$$

The Fresnel equations:
Case 1: $\vec{E}$ normal to the plane of incidence.
Reflection:

$$E_1 = \frac{k_1\mu_2\, \cos\theta_i - \mu_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}{k_1\mu_2\, \cos\theta_i + \mu_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}\, E_0$$

Transmission:

$$E_2 = \frac{2k_1\mu_2\, \cos\theta_i}{k_1\mu_2\, \cos\theta_i + \mu_1\sqrt{k_2^2 - k_1^2\sin^2\theta_i}}\, E_0$$

Case 2: $\vec{E}$ in the plane of incidence.
Reflection :

$$H_1 = \frac{\mu_1 k_2^2 \cos\theta_i - \mu_2 k_1 \sqrt{k_2^2 - k_1^2 \sin^2\theta_i}}{\mu_1 k_2^2 \cos\theta_i + \mu_2 k_1 \sqrt{k_2^2 - k_1^2 \sin^2\theta_i}}$$

Transmission:

$$H_2 = \frac{2\mu_1 k_2^2 \cos\theta_i}{\mu_1 k_2^2 \cos\theta_i + \mu_2 k_1 \sqrt{k_2^2 - k_1^2 \sin^2\theta_i}}$$

In dielectric media the Fresnel equations become:
$\vec{E}$ normal to the plane of incidence:

$$\frac{E_1}{E_0} = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$
$$\frac{E_2}{E_0} = \frac{2\cos\theta_i\ \sin\theta_t}{\sin(\theta_i + \theta_t)}$$

$\vec{E}$ in the plane of incidence:

$$\frac{E_1}{E_0} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$
$$\frac{E_2}{E_0} = \frac{2\cos\theta_i\ \cos\theta_t}{\sin(\theta_i + \theta_t)\ \cos(\theta_i - \theta_t)}$$

Rectangular waveguides:

If the fields vary as $\mathrm{e}^{-\gamma z}$:
For TE modes the longitudinal component of $\vec{H}$ satisfies:

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k^2 H_z = 0$$

where

$$k^2 = \gamma^2 - i\omega\mu(\sigma + i\omega\varepsilon)$$

Then satisfying the boundary conditions (assuming the walls are perfect conductors) requires:

$$\gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + i\mu\omega(\sigma + i\varepsilon\omega)$$

For the lossless waveguide:
Cut-off frequency for the mn mode:

$$f_{mn} = v_m \sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}$$

Phase velocity:

$$v_p = \frac{v_m}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Group velocity:

$$v_g = v_m \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

VECTOR FORMULAS

$$\begin{aligned}
\nabla(\Phi + \Psi) &= \nabla\Phi + \nabla\Psi \\
\nabla \cdot (\vec{A} + \vec{B}) &= \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \\
\nabla \times (\vec{A} + \vec{B}) &= \nabla \times \vec{A} + \nabla \times \vec{B} \\
\nabla(\Phi\Psi) &= \Phi\nabla\Psi + \Psi\nabla\Phi \\
\nabla \cdot (\Phi\vec{A}) &= \vec{A} \cdot \nabla\Phi + \Phi\nabla \cdot \vec{A} \\
\nabla \cdot (\vec{A} \times \vec{B}) &= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \\
\nabla \times (\Phi\vec{A}) &= \nabla\Phi \times \vec{A} + \Phi\nabla \times \vec{A} \\
\nabla \times (\vec{A} \times \vec{B}) &= \vec{A}\nabla \cdot \vec{B} - \vec{B}\nabla \cdot \vec{A} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} \\
\nabla \cdot \nabla\Phi &= \nabla^2\Phi \\
\nabla \cdot (\nabla \times \vec{A}) &= 0 \\
\nabla \times \nabla\Phi &= 0 \\
\nabla \times (\nabla \times \vec{A}) &= \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} \\
\vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \\
\vec{A} \times (\vec{B} \times \vec{C}) &= \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})
\end{aligned}$$

FORMS OF VECTOR OPERATIONS IN CYLINDRICAL COORDINATES

$$\nabla\Phi = \hat{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{z}\frac{\partial\Phi}{\partial z}$$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\vec{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\frac{1}{\rho}\left(\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

VECTOR AND DIFFERENTIAL OPERATIONS IN SPHERICAL COORDINATES

$$\begin{aligned}
\vec{A} &= (A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta)\,\hat{r} \\
&+ (A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta)\,\hat{\theta} \\
&+ (-A_x\sin\phi + A_y\cos\phi)\,\hat{\phi} \\
&= A_r\ \hat{r} + A_\theta\ \hat{\theta} + A_\phi\ \hat{\phi}
\end{aligned}$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\phi}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\begin{aligned}\nabla\times\vec{A} &= \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial\left(\sin\theta A_\phi\right)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial\left(rA_\phi\right)}{\partial r}\right] \\ &+ \frac{\hat{\phi}}{r}\left[\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\end{aligned}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$