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Theory of Stagnation Point Heat Transfer in Dissociated Air[†]

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SUMMARY

The boundary-layer equations are developed in general for the case of very high speed flight where the external flow is in a dissociated state. In particular the effects of diffusion and of atom recombination in the boundary layer are included. It is shown that at the stagnation point the equations can be reduced exactly to a set of nonlinear ordinary differential equations even when the chemical reactions proceed so slowly that the boundary layer is not in thermochemical equilibrium.

Two methods of numerical solution of these stagnation point equations are presented, one for the equilibrium case and the other for the nonequilibrium case. Numerical results are correlated in terms of the parameters entering the numerical formulation so as not to depend critically on the physical assumptions made.

For the nonequilibrium boundary layer, both catalytic (to atom recombination) and noncatalytic wall surfaces are considered. A solution is represented which shows the transition from the "frozen" boundary layer (very slow recombination rates) to the equilibrium boundary layer (fast recombination rates). A recombination rate parameter is introduced to interpret the nonequilibrium results, and it is shown that a scale factor is involved in relating the equilibrium state of a boundary layer on bodies of different sizes.

It is concluded that the heat transfer through the equilibrium stagnation point boundary layer can be computed accurately by a simple correlation formula [see Eq. (63)] and that the heat transfer is almost unaffected by a nonequilibrium state of the boundary layer provided the wall is catalytic and the Lewis Number near unity.

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SYMBOLS

- c_i = mass fraction of component i
- c_{pi} = specific heat per unit mass at constant pressure of component i , Eq. (19)
- c = \bar{c}_p/\bar{c}_{pw} , Eq. (A-11)
- \bar{c}_p defined by Eq. (19)
- C_1 = recombination rate parameter, Eq. (57)
- C_2 = $h_D/\bar{c}_{pw}T_s$
- C_3 defined by Eq. (A-14)
- d defined by Eqs. (A-4), and (A-7)
- D = diffusion coefficient
- D^T = thermal diffusion coefficient
- e defined by Eq. (A-13)
- e_i = internal energy per unit mass of component i
- f defined by Eq. (24)
- F defined by Eq. (A-9)
- g defined by Eq. (25)
- h = enthalpy per unit mass of mixture, Eq. (16)
- h_i = perfect gas enthalpy per unit mass of component i , Eq. (10)
- h_i^0 = heat evolved in the formation of component i at 0°K. per unit mass
- h_A^0 = dissociation energy per unit mass of atomic products, Eq. (59)
- h_D = average atomic dissociation energy times atom mass fraction in external flow, Eq. (61)
- k = thermal conductivity
- K_1 = recombination rate constant, Eq. (50)
- K_2 = dissociation rate constant, Eq. (51)
- l defined by Eqs. (33), (A-5), and (A-9)
- L_i = $D_i\rho\bar{c}_p/k$, Lewis Number
- L_i^T = $D_i^T\rho\bar{c}_p/k$, thermal Lewis Number
- L = Lewis Number for atom-molecule mixture
- M = molecular weight of molecules in atom-molecule mixture
- N = number density, moles per unit volume
- Nu = Nusselt Number, Eq. (43)
- p = pressure
- q = heat flux
- \vec{q} = vector mass velocity
- \vec{q}_i = vector diffusion velocity, Eq. (2)
- r = cylindrical radius of body
- R = body nose radius
- Re = Reynolds Numbers, Eq. (44)

R_i	= gas constant of component i
R_m	= gas constant of mixture
R	= universal gas constant
s	defined by Eq. (27)
T	= absolute temperature
T_V	= vibrational temperature for molecules, taken as 800° K.
u	= x component of velocity
v	= y component of velocity
w_i	= mass rate of formation of component i per unit volume and time
x	= distance along meridian profile
y	= distance normal to the surface
α_1, α_2	defined by Eq. (A-5)
β_1, β_2	defined by Eq. (A-7)
γ_1, γ_2	defined by Eq. (A-6)
η	defined by Eq. (22)
θ	defined by Eq. (26)
μ	= absolute viscosity
ν	= kinematic viscosity
ξ	defined by Eq. (23)
ρ	= mass density
σ	= Prandtl Number $\bar{c}_p\mu/k$, taken as 0.71 in numerical calculations
Φ	= dissipation function

Subscripts

A	= atom
i	= i th component of mixture
e	= external flow conditions
w	= wall
s	= stagnation point in external flow
M	= molecule
m	= mixture
E	= equilibrium

(1) INTRODUCTION

THE PROCESS of aerodynamic heat transfer at hypersonic velocities is complicated by two features not normally present at low velocities. The first is the possible dissociation and ionization of air due to high static temperatures encountered where the air is decelerated by shock waves, by viscous forces in the boundary layer, or at a stagnation point. Because dissociation and ionization (and their reverse processes, recombination) proceed at finite rates, thermochemical equilibrium is not necessarily achieved throughout the flow field, and such rates are therefore an essential ingredient of the flow process. Secondly, diffusion of atoms and ions, which subsequently recombine with a high specific energy release, may appreciably add to the heat transferred by normal molecular conduction. While there may be other physical phenomena also present, such as radiative effects, it is the purpose of the present analysis to include only the two effects previously noted in an otherwise classical viscous flow problem.

Some aspects of this problem have already received attention. Moore¹ considered a dissociated laminar boundary layer on a flat plate in air with a local composition determined by the thermochemical equilibrium—i.e., a recombination rate constant sufficiently great to maintain local equilibrium. Hansen² noted that Moore had miscalculated the Prandtl Number for dissociated air and an analysis similar to Moore's but

following Hansen's suggestion was made by Romig and Dore.³ Beckwith⁴ considered the heat transfer to the stagnation region of a blunt nosed body, using integral methods.* Crown⁵ also considered the stagnation point problem, using a modified Crocco method for solving the boundary-layer equations. Finally, Mark⁶ treated the stagnation point equilibrium boundary layer with variable fluid properties.†

In all these analyses no detailed distinction is made between the roles of atomic diffusion and molecular conduction in transporting energy to the wall.‡ If one considers the energy transport through a motionless dissociated gas with temperature and concentration gradients, the energy flux is approximately

$$q = k \text{ grad } T + h_A^0 D \rho \text{ grad } c_A$$

where k is the ordinary thermal conductivity, T is the temperature, h_A^0 is the dissociation energy per unit mass of atomic products, D is the atomic diffusion coefficient, ρ the density, and c_A the atomic mass fraction. The first term is the usual transport of kinetic, vibrational and rotational energy and the second is the transport of potential (recombination) energy. Even in the "equilibrium" boundary layer, the latter term should be taken into account in the energy equation, and will constitute a significant contribution where the atom concentration and diffusional velocities are noticeable.

These distinctions were first pointed out by Fay.¹³ Subsequently, Lees⁷ considered in detail the laminar heat-transfer problem in dissociated air, including the effects of atomic diffusion, and suggested several approximations to facilitate the solution of the boundary-layer equations. He considered the limiting extremes of the recombination rate constant which (as discussed below) lead to simpler solutions than the general case, and suggested expressions for the heat transfer for both cases. Similar arguments, but in less detail, are advanced by Kuo.¹⁴

Apart from the physical mechanism of heat transfer, there are several relevant aerodynamic considerations, the foremost of which is the question of shock-wave boundary-layer interaction at a sharp leading edge. From a mechanical point of view, it does not appear possible to maintain sharp leading edges with their attendant high heat-transfer rates, so that a finite radius of curvature appears mandatory. Under these conditions a distinct layer will exist independent of the detached bow shock wave if the boundary-layer thickness is much less than the shock detachment distance.

* It is noted that the variation of viscosity through the boundary layer, which materially affects the heat-transfer rate, cannot be accounted for by the integral method.

† For a more detailed discussion of numerical results, see Fay, Riddell, and Kemp.¹²

‡ Crown⁵ for example, assumes that the potential energy of dissociation (which is actually carried by diffusion of atoms) is transported in the same manner as the internal energy of the molecules, which implies equal diffusivities of molecules and atoms.

Since the former varies as the inverse square root of the Reynolds Number and the latter is independent of Reynolds Number, there is a minimum Reynolds Number below which the shock wave and boundary layer merge. However, since low Reynolds Numbers at hypersonic velocities can only be attained by reducing the density (and hence increasing the mean free path), this limit may not be reached before free molecule flow ensues. It thus appears that stagnation point boundary layers are quite relevant to hypersonic continuum flows.

Boundary layers at locations other than the stagnation point are also of interest, but will not be considered herein in detail. There are certain difficulties inherent in such solutions when the recombination rate is finite which will be discussed more fully below.

For any boundary-layer calculation, the behavior of the free stream outside the boundary layer must be known. If the boundary-layer thickness is small compared with the nose (and also detached shock wave) radius of curvature, then only fluid which has passed through the normal shock wave close to the axis (or plane) of symmetry will enter the boundary layer, and this constitutes the "free stream." If it is assumed that this stream has attained thermochemical equilibrium by the time it reaches the stagnation point, then the free-stream conditions for the stagnation point boundary layer are those of an equilibrium gas. This would appear to be the case most likely to be encountered.*

Whether the free stream in regions other than the stagnation point remains at equilibrium again depends upon the recombination rate since, in flowing to regions of lower pressure, the gas is expanded and cooled. Again, there will be no general solution which will include all possible situations.

The two extremes of recombination rate give rise to simpler solutions than the general case. For sufficiently small recombination rate, the concentration of atoms (or ion pairs) is determined by the diffusive flow from the free stream to the wall where recombination would occur, and would bear no relation to the thermochemical equilibrium concentration corresponding to the local temperature. In such a "frozen" boundary layer, the temperature and concentration distributions are practically independent of one another. On the other hand, for a sufficiently large recombination rate, constant thermochemical equilibrium would prevail throughout and either the temperature or concentration distribution is a sufficient description of the thermodynamic state of the boundary layer. It is, therefore, to be expected that the distribution of atoms in the

* The time for the gas to come to equilibrium behind the normal shock wave depends upon the kinetics of the dissociation process. It is merely noted here that immediately behind the shock wave, before dissociation begins, the translational temperature is extremely high and thereby promotes high dissociation rates. Even at high temperatures equilibrium may not be attained soon enough if the density is sufficiently low, but this would probably occur only near the free molecule flow regime.

"frozen" and "equilibrium" boundary layers will be quite different.

Despite the importance of the processes of dissociation and recombination in determining the thermodynamic state of the air throughout the flow field, their effect on heat transfer is secondary. This is most easily seen by again considering the heat flux through a stagnant gas as given above, and replacing $\text{grad } T$ by $(1/c_p) \text{grad } h$, where h is the perfect gas enthalpy. If we also make the approximation that $D\rho c_p/k$ (Lewis number) is unity, the heat flux becomes

$$q = (k/c_p) \text{grad } (h + c_A h_A^0)$$

that is, the heat flux is determined by the chemical enthalpy (perfect gas enthalpy plus enthalpy of formation) difference between free stream and wall. Whether atoms recombine in the boundary layer or on the wall makes no great difference since the energy is conducted about as readily by normal conduction as by diffusion when the Lewis Number is approximately one. On the other hand, if the heat transfer is to be known more exactly, then such relevant effects as variation of transport coefficients with temperature, variation in heat capacities, actual Lewis Number, etc., must be accounted for properly. It will be seen that it is the influence of these effects which constitutes the principal departure from an extrapolation of the classical theory.

Since a considerable fraction of the heat may be transported by atomic diffusion toward the wall followed by recombination on the surface, it would be possible to eliminate this fraction of the heat transfer by using a noncatalytic surface. However, such a scheme is useful only if the atoms do not first recombine in the gas before reaching the wall. The flight conditions under which the gas phase recombination is slow enough to permit atoms to reach the wall, and the resulting heat transfer with both catalytic and noncatalytic surface, have been determined for the stagnation point flow.

For a discussion of experimental techniques and results of shock tube measurements of stagnation point heat transfer in dissociated air, see reference 15.

(2) LAMINAR BOUNDARY-LAYER EQUATIONS IN A DISSOCIATED GAS

The general equation of continuity for any species i is

$$\text{div}\{\rho(\vec{q} + \vec{q}_i)c_i\} = w_i \quad (1)$$

where ρ is the mixture density, c_i is the mass fraction of species i , w_i is the mass rate of formation of species i per unit volume, \vec{q} is the mass averaged velocity and \vec{q}_i the diffusional velocity of species i measured with respect to \vec{q} . \vec{q}_i may be given by

$$\vec{q}_i = -(D_i/c_i) \text{grad } c_i - (D_i^T/T) \text{grad } T \quad (2)$$

where D_i and D_i^T are, respectively, the molecular and thermal diffusion coefficients of species i , and T is the temperature. The first term on the right of Eq. (2) is

due to concentration diffusion and the second is due to thermal diffusion (pressure diffusion is neglected).

The use of the gradient of mass fraction instead of mole fraction is particularly useful since, in the case of a bimolecular mixture, D_i becomes the bimolecular diffusion coefficient (D_{12}) which is practically independent of composition. For multicomponent mixtures, the diffusional velocity of any one component depends, in general, upon the concentration gradients of all the components, and a method for determining the diffusional velocities has been suggested by Hirschfelder, Curtiss, and Bird.⁸ In such a case Eq. (2) is not exact (except for equal diffusivities and molecular weights of all components), but is a useful approximation. A dissociated gas (such as air) in which all the molecules (or atoms) have nearly the same molecular weight and probably similar collision diameters may be considered primarily a two-component mixture with atoms and molecules as species, for which Eq. (2) is adequate.

In order to conserve mass in any chemical change it is necessary that

$$\sum w_i = 0 \quad (3)$$

and from the definition of the mass averaged velocity it follows that

$$\sum \vec{q}_i c_i = 0 \quad (4)$$

Hence, Eq. (1) summed over all species gives the usual form for the continuity equation,

$$\text{div}(\rho \vec{q}) = 0 \quad (5)$$

The energy equation may be written for a moving element of fluid

$$\rho \vec{q} \cdot \text{grad}(\sum c_i e_i) = \text{div}(k \text{ grad } T) - \text{div}(\sum \rho \vec{q}_i c_i h_i) + \sum w_i h_i^0 + \dot{p} \text{ div } \vec{q} + \Phi \quad (6)$$

where e_i , h_i , and h_i^0 are, respectively, the specific internal energy, enthalpy, and heat of formation of species i , k is the thermal conductivity for transport of kinetic, rotational, and vibrational energy, \dot{p} is the pressure, and Φ is the dissipation function.

The first term on the right of Eq. (6) is the internal energy increase due to normal heat conduction, the second that due to fluid diffusing across the boundary of the element, the third that due to chemical reaction,* and the fourth and fifth that due to the work of the pressure and the viscous forces, respectively.

It will be assumed that the gas is a mixture of perfect gases, so that for each component

$$p_i = \rho_i R_i T \quad (7)$$

where R_i is the gas constant for species i , and for the mixture

$$p = \rho R_m T \quad (8)$$

where the gas constant for the mixture (R_m) is given by

* The heat of formation h_i^0 may be taken as zero for the molecules and negative for the atoms, in which case the enthalpy h [see Eq. (16)] is always positive.

$$R_m = \sum c_i R_i \quad (9)$$

It then follows that the enthalpy and internal energy are related by

$$h_i = e_i + R_i T \quad (10)$$

By combining Eqs. (5) and (7) through Eq. (10) with Eq. (6), the steady-state energy equation reduces to

$$\rho \vec{q} \cdot \text{grad}(\sum c_i h_i) = \text{div}(k \text{ grad } T - \sum \rho \vec{q}_i c_i h_i) + \sum w_i h_i^0 + \vec{q} \cdot \text{grad } p + \Phi \quad (11)$$

A further simplification occurs by combining Eqs. (1) and (11) to eliminate the term $\sum w_i h_i^0$:

$$\rho \vec{q} \cdot \text{grad}\{\sum c_i (h_i - h_i^0)\} = \text{div}\{k \text{ grad } T - \sum \rho \vec{q}_i c_i (h_i - h_i^0)\} + \vec{q} \cdot \text{grad } p + \Phi \quad (12)$$

By making the usual boundary-layer assumptions, Eqs. (1), (5), and (12) reduce to the following form for a body of revolution, if the boundary-layer thickness is small compared with the radius of curvature and centrifugal forces are neglected:†

$$(\rho r u)_x + (\rho r v)_y = 0 \quad (13)$$

$$\rho u c_{ix} + \rho v c_{iy} = \{D_i \rho c_{iy} + D_i^T \rho c_i T_y / T\}_y + w_i \quad (14)$$

$$\rho u h_x + \rho v h_y = (k T_y)_y + u \dot{p}_x + \mu (u_y)^2 + \{\sum D_i \rho (h_i - h_i^0) c_{iy} + \sum D_i^T \rho c_i (h_i - h_i^0) T_y / T\}_y \quad (15)$$

where r is the radial distance of the body surface from the axis of revolution, u and v are the velocity components in the x and y directions (tangential and normal to the surface, respectively), μ is the absolute viscosity, and the enthalpy for the mixture is

$$h = \sum c_i (h_i - h_i^0) \quad (16)$$

The corresponding equation of motion is

$$\rho u u_x + \rho v u_y = -\dot{p}_x + (\mu u_y)_y \quad (17)$$

Since the transport coefficients are, in general, functions of temperature and composition, it may be desirable to use T rather than h as the dependent variable in the energy equation. Noting that h_i is a function of temperature alone, we have

$$\text{grad } h = \{\sum c_i (dh_i/dT)\} \text{ grad } T + \sum (h_i - h_i^0) \text{ grad } c_i \quad (18)$$

Letting

$$\bar{c}_p \equiv \sum c_i (dh_i/dT) = \sum c_i c_{pi} \quad (19)$$

where c_{pi} is the constant pressure specific heat for translation, rotation, and vibration, the energy equation (15), in combination with Eq. (14), becomes

$$\bar{c}_p \{\rho u T_x + \rho v T_y\} = (k T_y)_y + u \dot{p}_x + \mu (u_y)^2 + \sum w_i (h_i^0 - h_i) + \sum c_{pi} (D_i \rho c_{iy} + D_i^T \rho c_i T_y / T) T_y \quad (20)$$

For a gas at equilibrium, on the other hand, it is found

† Subscripts x and y (or ξ and η below) are used to denote partial differentiation.

more convenient to use the enthalpy as a variable rather than temperature, in which case by combining Eqs. (17) and (18) with Eq. (15), we obtain

$$\rho u (h + u^2/2)_x + \rho v (h + u^2/2)_y = \left\{ (k/\bar{c}_p) \times (h + u^2/2)_y \right\}_y + \left\{ (1/2) [\mu - (k/\bar{c}_p)] (u^2)_y \right\}_y + \left\{ \Sigma [D_i \rho - (k/\bar{c}_p)] (h_i - h_i^0) c_{iy} + \Sigma (D_i^T \rho c_i / T) (h_i - h_i^0) T_y \right\}_y \quad (21)$$

Eqs. (13), (14), (17), and (20) or (21) thus constitute the system whose solution is required.

(3) SIMILAR SOLUTIONS

As is usual in boundary-layer problems, one first seeks solutions of restricted form which permit reducing exactly the partial differential equations to ordinary differential form. An easily recognizable case is that of the stagnation point flow, where, because of symmetry, all the dependent variables are chosen to be functions of y alone, except u which must be taken proportional to x times a function of y . This also appears to be the *only* case for which exact ordinary differential equations may be obtained regardless of the recombination rate. For the flat plate and cone, exact solutions exist only for the extreme values of the recombination rate constant, that is, when the boundary layer is either "frozen" or in the thermodynamic equilibrium.

For all other cases, certain degrees of approximation are required. Following standard procedures, one first tries solutions for which the velocity and enthalpy profiles remain similar to themselves, at least for an appreciable distance along the body. Such "locally similar" approximate solutions may be obtained for the "frozen" or "equilibrium" boundary layer, and also for an arbitrary recombination rate in certain restricted cases.

The existence of a finite recombination rate is not the only hindrance to obtaining exact solutions in regions other than the stagnation point. In general, the variation of the velocity, thermodynamic variables and transport coefficients in the free stream and along the wall of the body of arbitrary shape preclude exact solutions (except for the cone and flat plate). An important aspect of the locally similar solutions is a proper accounting for these variations such that the heat transfer may be determined for any point of the body of arbitrary (but regular) shape. Although this problem will not be treated in this paper, we will start with a transformation suitable for locally similar solutions.

With this in mind, we choose the following transformation of the independent variables x and y , which includes the usual Howarth and Mangler transformations, as proposed by Lees.⁷

$$\eta \equiv (ru_e/\sqrt{2\xi}) \int_0^y \rho dy \quad (22)^*$$

$$\xi \equiv \int_0^x \rho_w \mu_w u_e r^2 dx \quad (23)^*$$

where u_e is the velocity at the outer edge of the boundary layer. In addition, the following dimensionless dependent variables are chosen:

$$\partial f / \partial \eta \equiv u/u_e; \quad f = \int_0^\eta (\partial f / \partial \eta) d\eta \quad (24)$$

$$g = (h + u^2/2)/h_s \quad (25)$$

$$\theta = T/T_e \quad (26)$$

$$s_i = c_i/c_{ie} \quad (27)$$

where the subscript e refers to values in the local free stream, and s to values in the free stream at the stagnation point. Substituting in Eqs. (13), (14), (17), (20), and (21), there results

$$\rho v = -r^{-1} [(\sqrt{2\xi} f_\xi + f/\sqrt{2\xi}) \xi_x + \sqrt{2\xi} f_y \eta_x] \quad (28)$$

$$[(l/\sigma)(L_i s_{i\eta} + L_i^T s_i \theta_\eta / \theta)]_\eta + f s_{i\eta} + [2\xi w_i / \rho u_e c_{ie} (d\xi/dx)] = 2\xi (f_\eta s_{i\xi} - f_\xi s_{i\eta}) + 2f_\eta s_i [d(\ln c_{ie})/d(\ln \xi)] \quad (29)$$

$$(lf_\eta)_\eta + ff_\eta + 2[d(\ln u_e)/d(\ln \xi)] [(\rho_e/\rho) - f_\eta^2] = 2\xi (f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}) \quad (30)$$

$$[(\bar{c}_p/\bar{c}_{pw}) (l/\sigma) \theta_\eta]_\eta + (\bar{c}_p/\bar{c}_{pw}) f \theta_\eta + \Sigma [2\xi w_i / \rho u_e (d\xi/dx)] [(h_i^0 - h_i)/\bar{c}_{pw} T_e] + (u_e^2/\bar{c}_{pw} T_e) lf_\eta^2 + \Sigma (c_{pi}/\bar{c}_{pw}) (c_{ie}/\sigma) (L_i s_{i\eta} + L_i^T s_i \theta_\eta / \theta) = f_\eta \{ 2(\bar{c}_p/\bar{c}_{pw}) \theta [d(\ln T_e)/d(\ln \xi)] + (2u_e^2/\bar{c}_{pw} T_e) (\rho_e/\rho) [d(\ln u_e)/d(\ln \xi)] \} + 2(\bar{c}_p/\bar{c}_{pw}) \xi (f_\eta \theta_\xi - f_\xi \theta_\eta) \quad (31)$$

$$[(l/\sigma) g_\eta]_\eta + f g_\eta + (u_e^2/h_s) \{ (1 - \sigma^{-1}) lf_\eta f_{\eta\eta} \}_\eta + \{ (l/\sigma) \Sigma (c_{ie}/h_s) (h_i - h_i^0) [(L_i - 1) s_{i\eta} + L_i^T s_i \theta_\eta / \theta] \}_\eta = 2\xi (f_\eta g_\xi - f_\xi g_\eta) \quad (32)$$

where

$$l \equiv \rho \mu / \rho_w \mu_w$$

and where σ and L_i are the Prandtl Number ($\bar{c}_p \mu / k$) and Lewis Number ($D_i \rho \bar{c}_p / k$), respectively. The subscript w refers to values of the variables at the wall.

For the dependent variables f , g , θ , and s_i to be functions of η alone, it is first necessary that the thermodynamic state variables be unchanging in the free stream and along the wall as ξ increases, a condition satisfied at a stagnation point, or along a cone or flat plate. It is further necessary that the source term—i.e., the term

involving w_i —in Eqs. (29) and (31) depend upon η alone. Since the chemistry of the recombination process is believed to depend only upon the local thermodynamic variables—that is, w_i is a function of ρ , T , and s_i —this second condition may be satisfied in three different ways: (a) $w_i = 0$ (frozen boundary layer), (b) $u_e d(\ln \xi)/dx = \text{constant}$, which is satisfied at a stagnation point, or (c) the recombination rate is sufficiently

* For two-dimensional flows let $r = \text{constant}$.

large to maintain thermodynamic equilibrium,* in which case w_i is determined through Eq. (29) from the

$$[(l/\sigma)(L_i s_{i\eta} + L_i^T s_i T_\eta/T)]_\eta + f s_{i\eta} + \{2(du_e/dx)_s\}^{-1}(w_i/\rho c_{is}) = 0 \quad (34)$$

$$(lf_{\eta\eta})_\eta + ff_{\eta\eta} + (1/2)\{(\rho_s/\rho) - f_\eta^2\} = 0 \quad (35)$$

$$[(\bar{c}_p/\bar{c}_{pw})(l/\sigma)]_\eta + (\bar{c}_p/\bar{c}_{pw})f\theta_\eta + \{2(du_e/dx)_s\}^{-1}\Sigma(w_i/\rho)[(h_i^0 - h_i)/\bar{c}_{pw}T_s] + \Sigma(c_{pi}/\bar{c}_{pw})(c_{is}l/\sigma)(L_i s_{i\eta} + L_i^T s_i \theta_\eta/\theta)_\eta = 0 \quad (36)$$

$$[(l/\sigma)g_\eta]_\eta + fg_\eta + ((l/\sigma)\Sigma[c_{is}(h_i - h_i^0)/h_s])\{(L_i - 1)s_{i\eta} + L_i^T s_i \theta_\eta/\theta\}_\eta = 0 \quad (37)$$

where it has been assumed that $u^2 \ll h_s$.

For a "locally similar" solution away from the stagnation point it is required that the terms on the right-hand side of Eqs. (29) through (32) all be negligible compared with those on the left, and thus the equations may be integrated with respect to η . The dependence upon ξ of f , g , θ , and s_i is thereby implicitly determined by the variation of ρ , T , s_i , u , and the transport coefficients in the free stream and along the wall, this variation being determined by the aerodynamics of the external flow.

For Newtonian flow the distribution around the body of the relevant quantities is easily found, all quantities except du_e/dx being determined by the local body slope—i.e., local pressure. Since du_e/dx depends upon the pressure gradient and hence body curvature in the meridian plane, there is introduced an extra parameter in the locally similar solutions. Lees⁷ points out that in the equation of motion (30) the pressure gradient affects only a single term, that this term is generally small, and that neglecting this term in the equation of motion will only slightly affect the solutions for g , θ , and s_i . This appears to be a greatly desirable simplification in determining the heat transfer at other than the stagnation point.

The use of locally similar solutions can only be justified a posteriori in each particular case by determining from such a solution the magnitude of the terms in Eqs. (29) through (32) which were neglected. Presumably an iterative scheme could be devised to improve on such solutions. It does not appear possible to formulate a general criterion for determining the limit of applicability of the locally similar solutions except a loose physical argument that conditions in the free stream and along the body must change only slightly in a distance of many boundary-layer thicknesses.

(4) HEAT TRANSFER

The local heat-transfer rate to the body q is determined by the sum of the conductive and diffusive transports, the latter being included only when the atoms recombine on the wall. Thus,

$$q = [k(\partial T/\partial y)]_{y=0} + [\Sigma\rho(h_i - h_i^0) \times \{D_i(\partial c_i/\partial y) + (D_i^T c_i/T)(\partial T/\partial y)\}]_{y=0} \quad (38)$$

* Strictly speaking, the fluid cannot be exactly in thermodynamic equilibrium, for if this were so, there could be no net rate of change of composition following a particle path. For large enough rate constants, the fluid will be very close to equilibrium.

solution of Eqs. (28), (30), and (32).

For the stagnation point, then, Eqs. (29) to (32) reduce to

or

$$q = [(k/\bar{c}_p)(\partial h/\partial y)]_{y=0} + [\Sigma(k/\bar{c}_p)(h_i - h_i^0) \times \{(L_i - 1)(\partial c_i/\partial y) + (L_i^T c_i/T)(\partial T/\partial y)\}]_{y=0} \quad (39)$$

In terms of the dimensionless temperature and enthalpy distributions, this becomes

$$q = (rk_w \rho_w u_e T_e / \sqrt{2\xi}) \{ \theta_\eta + \Sigma c_{ie} \times [(h_i - h_i^0)/\bar{c}_p T_e] (L_i s_{i\eta} + L_i^T s_i \theta_\eta/\theta) \}_{\eta=0} \quad (40)$$

or

$$q = (rk_w \rho_w u_e h_s / \sqrt{2\xi} \bar{c}_{pw}) \times \{ g_\eta + \Sigma c_{ie} [(h_i - h_i^0)/h_s] \times [(L_i - 1)s_{i\eta} + L_i^T s_i \theta_\eta/\theta] \}_{\eta=0} \quad (41)$$

For stagnation point heat transfer we note that

$$r \rho_w u_e / \sqrt{2\xi} = \{(2/\nu_w)(du_e/dx)_s\}^{1/2} \quad (42)$$

where ν_w is the kinematic viscosity at the wall.

It is also possible to define local Nusselt and Reynolds Numbers based on the local coordinate x . Thus, defining†

$$Nu = qx\bar{c}_{pw}/k_w(h_s - h_w) \quad (43)$$

$$Re = u_e x / \nu_w \quad (44)$$

the heat transfer at the stagnation point may be written as

$$q = (Nu/\sqrt{Re})\sqrt{\rho_w \mu_w (du_e/dx)_s} [(h_s - h_w)/\sigma] \quad (45)$$

where

$$Nu/\sqrt{Re} = [\sqrt{2} \bar{c}_{pw} T_s / (h_s - h_w)] \times \{ \theta_\eta + \Sigma c_{is} [(h_i - h_i^0)/\bar{c}_p T_s] \times (L_i s_{i\eta} + L_i^T s_i \theta_\eta/\theta) \}_{\eta=0} \quad (46a) \ddagger$$

$$Nu/\sqrt{Re} = [\sqrt{2}/(1 - g_w)] \times \{ g_\eta + \Sigma c_{is} [(h_i - h_i^0)/h_s] \times [(L_i - 1)s_{i\eta} + L_i^T s_i \theta_\eta/\theta] \}_{\eta=0} \quad (46b) \ddagger$$

It should be emphasized that the heat transfer is independent of the particular choice of reference density and viscosity ($\rho_w \mu_w$) which appears in the definition of Eq. (23). The use of wall values of ρ and μ for this pur-

† It is clear from the expressions (40) and (41) that the enthalpy difference between wall and free stream, not the temperature difference, is the proper "driving force" for heat transfer.

‡ For conical or flat plate flows, the factor $\sqrt{2}$ in Eq. (46) should be replaced by $\sqrt{3/2}$ or $1/\sqrt{2}$, respectively, when the reference dimensions for Nu and Re are the distances from the apex or leading edge, respectively, measured along the surface.

pose is merely a convenience, but a logical choice in that it ensures that $l = 1$ [Eq. (33)] at the wall ($\eta = 0$). However, another choice, such as free-stream ρ and μ , or even $\rho\mu$ equal to an arbitrary function of x , could also be used. In each such case, however, the solutions of the boundary-layer equations give different $g(\eta)$, $\theta(\eta)$, etc. It is clear from an examination of these equations that each such solution may be transformed into another by a simple change of scale of η and f defined by

$$\sqrt{\rho_w\mu_w} \eta = \sqrt{\rho_0\mu_0} \eta_0 \quad (47)$$

$$\sqrt{\rho_w\mu_w} f = \sqrt{\rho_0\mu_0} f_0 \quad (48)$$

and

$$g(\eta) = g_0(\sqrt{\rho_0\mu_0/\rho_w\mu_w} \eta_0), \quad \text{etc.} \quad (49)$$

and where $\rho_0\mu_0$ is the new reference value used in defining η_0 . As a consequence the heat transfer [see Eqs. (38) and (39)] can be seen to be independent of the choice of $\rho_0\mu_0$.

On the other hand, if one attempts to estimate $g_\eta(0)$ for the case of $\rho\mu$ not constant from a solution for $\rho\mu$ constant,* which is the problem considered by Lees,⁷ the reference state ($\rho_0\mu_0$) is quite relevant since by its proper (and arbitrary) selection the wall gradient of g may be made to equal numerically that for the case of $\rho\mu$ constant. A discussion of this point has been given by Probstein,⁹ and by Fay, Riddell, and Kemp.¹²

(5) TRANSPORT PROPERTIES

To obtain a numerical solution of the boundary-layer equations it is necessary to know the viscosity, Prandtl Number, Lewis Number, and thermal diffusivity as functions of the local temperature, density, and composition. For the high temperatures under consideration, these properties have been estimated,[†] the principal uncertainty resulting from lack of knowledge of atom-molecule interaction potentials. The particular results of this study which are pertinent to the boundary-layer calculation may be summarized as follows.

(a) The viscosity of equilibrium air, determined by assuming that all particles possess the same interaction potentials, does not vary more than 10 per cent from Sutherland's formula below 9,000°K.

(b) Both Prandtl and Lewis Numbers do not change appreciably with temperature (below 9,000°K.), the value of the latter being uncertain, but estimated as about 1.4.

In light of the uncertainties in viscosity and Lewis Number, it is clear that the calculated heat transfer using these estimates is also uncertain. However, the numerical solutions of the boundary-layer equations may be obtained for a range of possible Lewis Number and viscosity variation, such solutions being generally valid when expressed explicitly in terms of the property

* As, for example, the solutions of Cohen and Reshotko.¹⁰

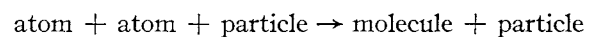
† These estimates were made by Dr. S. Penner of California Institute of Technology and M. M. Litvak of Cornell University.

variations. Thus for the numerical solutions, the Prandtl Number was held fixed at 0.71, the Lewis Number was assumed constant through the boundary layer at values ranging from one to two, and the viscosity variation was determined by Sutherland's law for the equilibrium air and as calculated by Penner and Litvak for nonequilibrium composition.[‡]

It is easily shown that thermal diffusion is unimportant for the equilibrium boundary layer at stagnation temperatures less than 10,000°K. For the frozen boundary layer this is no longer necessarily the case; however, thermal diffusion was neglected in all the numerical solutions reported in this paper.

(6) RECOMBINATION RATE

As suggested by Davidson,** the recombination rate of atoms is determined by a three-body collision:



For this process the rate of disappearance of atoms may be written as

$$dN_A/dt = -K_1 N_A^2 N T^{-1.5} \quad (50)$$

where the temperature dependence is as suggested by Davidson, who also estimated $K_1 T^{-1.5}$ to be 5×10^{14} cc.² mole⁻² sec.⁻¹ for oxygen recombination when $T = 300^\circ\text{K}$.^{††} N_A and N are the number of moles per cm.³ of atoms and particles, respectively.

Atoms will be produced by the reverse of the above reaction, so that the net rate of production may be written as

$$dN_A/dt = -K_1 N_A^2 N T^{-1.5} + K_2(T) N_M N \quad (51)$$

Since under equilibrium conditions, there is no net production of atoms, $K_2(T)$ may be solved for in Eq. (51) in terms of the equilibrium atom and molecule concentrations which would exist at the local temperature T and pressure. Substituting this value of K_2 in Eq. (51), there results

$$dN_A/dt = -K_1 N_A^2 N T^{-1.5} \times \{1 - (N_M/N_{ME})(N_{AE}/N_A)^2\} \quad (52)$$

where the subscript E refers to the concentrations which would exist at thermodynamic equilibrium at the local pressure and temperature, and N_M is the number of moles of molecules per cm.³.

Considering a gas mixture of diatomic molecules of

‡ Results of the experiments of reference 15 indicate good agreement with the present estimates of transport properties with Prandtl Number 0.71 and Lewis Number 1.4.

** Dr. Norman Davidson, California Institute of Technology (private communication).

†† Recent experiments reported by J. Camm and J. Keck (see Bulletin of the American Physical Society, Series II, Vol. 2, No. 4, p. 216, 1957) indicate that the recombination rate at 6,000°K. may be higher than that obtained by extrapolating Davidson's estimate to such a temperature. There is nothing known concerning this rate at the low wall temperatures pertinent to the boundary-layer problem.

molecular weight M and atoms of weight $M/2$, the atom mass fraction and total density are, respectively,

$$c_A = N_A/(N_A + 2N_M) \quad (53)$$

and
$$\rho = M(N_M + N_A/2) \quad (54)$$

The net mass rate of formation of atoms (w_A) may now be determined in terms of these quantities to give

$$\left. \begin{aligned} w_A &= (M/2)(dN_A/dt) \\ &= -(2K_1\rho^3T^{-1.5}/M^2) \times \\ &\quad [(1 + c_A)(c_A^2 - c_{AE}^2)/(1 - c_{AE}^2)] \end{aligned} \right\} \quad (55)$$

Neglecting the term $(1 - c_{AE}^2)$, which does not vary by 25 per cent in the cases considered, the relevant source term in the diffusion equation (34) for atoms reduces to

$$\left\{ 2(du_e/dx)_s \right\}^{-1}(w_i/\rho) = -[K_1\rho_s^2T_s^{-3.5}/\mathcal{R}^2(du_e/dx)_s] \times \left\{ \theta^{-3.5}(c_A^2 - c_{AE}^2)/(1 + c_A) \right\} \quad (56)$$

where \mathcal{R} is the universal gas constant. The coefficient of the term in braces on the right of Eq. (56) contains all the dimensional effects of the finite recombination rate problem, and will be termed the *recombination rate parameter* (C_1)—i.e.,

$$C_1 \equiv K_1\rho_s^2T_s^{-3.5}\mathcal{R}^{-2}(du_e/dx)_s^{-1} \quad (57)$$

The term $(du_e/dx)_s^{-1}$ is approximately the time for a particle in the free stream to move a distance equal to the nose radius, and thus also the time for a particle to diffuse through the boundary layer at the stagnation point. The factor which multiplies it in Eq. (57) is the reciprocal of the lifetime of an atom, so that the recombination rate parameter is the ratio of the diffusion time to the lifetime of an atom. Because the diffusion time contains the body nose radius while the lifetime does not, a scale effect is introduced by the chemical change which is not accounted for in the Reynolds Number. Thus, similar flows require equal Reynolds Numbers and recombination rate parameters.

A term similar to the left side of Eq. (56) appears in the energy equation (36) except that it is multiplied by a dimensionless dissociation energy. A part of this term, $\Sigma w_i h_i$, is negligible since $\Sigma w_i = 0$ and h_i per unit mass for vibrationally excited molecules and atoms are nearly equal, being in the ratio 9/10.

(7) NUMERICAL SOLUTIONS

The boundary-layer equations derived above [Eqs. (34) through (37)] can be put in a form suitable for numerical integration. The details of how this was done are given in the Appendix. Numerical solutions were obtained by use of an IBM 650 digital computer.

Many combinations of the parameters σ , L , and C_1 are possible, as well as extreme ranges in the free-stream (stagnation point) and wall boundary values. It was therefore decided to restrict the solutions to the following values:

- (a) $\sigma = 0.71$
- (b) $L = 1.0, 1.4, 2.0; L^T = 0$

(c) $0 \leq C_1 \leq \infty$

(d) Stagnation point conditions corresponding to thermodynamic equilibrium at velocities between 5,800 ft. per sec. and 22,800 ft. per sec. and at altitudes of 25,000 ft. to 120,000 ft. (Equilibrium air properties were determined from the tables prepared by the National Bureau of Standards¹¹.)

(e) Wall temperatures from 300° K. to 3,000° K.

Wherever possible, only one parameter or boundary condition was varied in a series of calculations in order to determine its individual effect. It was not believed necessary to establish such effects for all possible combinations of the remaining parameters. Tables giving the values of the parameters and the boundary conditions for each computation may be obtained directly from the authors.

The heat-transfer parameter (Nu/\sqrt{Re}) determined from each integration depends not only upon the parameters and boundary values of the dependent variables involved, but also upon the assumed variation of viscosity with temperature and composition. In addition, further simplifications as explained below for the individual cases were also made. In order to make the results less dependent upon the specific assumptions made, the heat-transfer parameter was numerically correlated with the parameters and boundary values in what seemed to be a suitable manner. It is believed that not too widely different assumptions would give results which would fall within the same correlation.

(8) THE EQUILIBRIUM BOUNDARY LAYER

A first series of equilibrium boundary layers was computed by solving Eqs. (35) and (37) simultaneously (hereafter denoted as Method 1). The solution requires specifying ρ_s/ρ and $l = \rho\mu/\rho_w\mu_w$ as functions of g , typical variations of these quantities being shown in Fig. 1. The calculated values at various altitudes of interest are compared with the fitted curves used in the computing program, from which it was concluded that there was a negligible effect of altitude variation on these functions. Specific details of the method of calculation are given in the Appendix. It was found that moderate changes in the distribution for identical end values resulted in negligible changes in the heat-transfer parameter.

For $L_i = 1$, the equations are similar in form to those solved by Cohen and Reshotko,¹⁰ and become identical at low enough stagnation temperatures when l is approximately constant. Solutions were obtained for the range of velocities and wall temperatures given above, and the heat-transfer parameter was found to depend only upon the total variation in $\rho\mu$ across the boundary layer, in accordance with the relation

$$Nu/\sqrt{Re} = 0.67(\rho_s\mu_s/\rho_w\mu_w)^{0.4} \quad (58)$$

The numerical correlation leading to Eq. (58) is shown in Fig. 2. The solution of Cohen and Reshotko¹⁰ for $l = 1$ is also plotted, after correcting for Prandtl Number by multiplying their result by $(0.71)^{0.4}$.

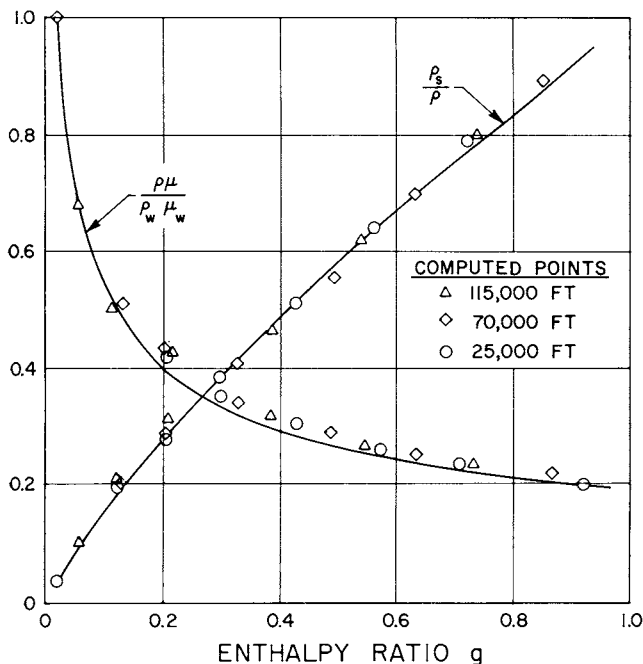


FIG. 1. Variation of $\rho\mu/\rho_w\mu_w$ and ρ_s/ρ through a stagnation point boundary layer, shown as a function of enthalpy ratio g . Stagnation conditions are for a flight velocity of 19,700 ft. per sec. at various altitudes. $T_w = 300^\circ\text{K}$.

An alternative procedure for the equilibrium boundary layer is to solve Eqs. (34), (35) and (36) simultaneously (hereafter denoted as Method 2). An appreciable simplification results if we consider air to be composed of only "air" molecules and "air" atoms having an average heat of formation given by

$$h_A^0 = \frac{\sum_{\text{atoms}} c_{is}(-h_i^0)}{\sum_{\text{atoms}} c_{is}} \quad (59)$$

where the summation extends over atomic oxygen and nitrogen only. Thus only one diffusion equation (34) is needed for the diffusion of air atoms. With this simplification the equilibrium boundary layer may be treated by eliminating the term involving w_i between Eq. (34) and (36). A solution is then possible when \bar{c}_p and l are specified as functions of s_A and θ , and s_A is specified as a function of θ through the known equilibrium atom fraction as a function of temperature. (Details of the approximations made are given in the Appendix.)

This alternative solution was found to give very closely the same results as the Method 1 for a Lewis Number of unity, and the results are compared with Eq. (58) in Fig. 2. For a Lewis Number of unity, Method 2 is believed to be less accurate than Method 1, since it involves more approximations to the real gas properties.

For other values of the Lewis Number, the effect of Lewis Number on the heat-transfer parameter was found by Method 2 to be best given by

$$\frac{(Nu/\sqrt{Re})}{(Nu/\sqrt{Re})_{L=1}} = 1 + \frac{(L^{0.52} - 1)(h_D/h_s)}{L} \quad (60)$$

where the "dissociation enthalpy" h_D is defined as

$$h_D = \sum_{\text{atoms}} c_{is}(-h_i^0) = h_A^0 \sum_{\text{atoms}} c_{is} \quad (61)$$

i.e., h_D is the dissociation enthalpy per unit mass of air in the external flow. The numerical results are plotted in Fig. 3 for comparison with Eq. (60).

It was also possible to determine the effect of Lewis Number from the Method 1 computations by evaluating the additional term in Eq. (37) involving $(L - 1)$ from the equilibrium properties of air. (The approximation for this evaluation is discussed in the Appendix.) Two such cases were computed, and the results are plotted in Fig. 3 for comparison with Eq. (60). While there is some disagreement with the results of Method 2, it is not too unreasonable considering the many different approximations involved in fitting curves to the functions l , \bar{c}_p , etc. It is the authors' opinion that the Method 2 solutions give a better indication of the Lewis Number effect for the equilibrium boundary layer, as embodied in Eq. (60). However, for Lewis Number unity the effect of $\rho\mu$ variation is believed to be better given by the results of Method 1 [Eq. (58)], so that the total effect may be obtained by combination of Eqs. (58) and (60) in the form

$$Nu/\sqrt{Re} = 0.67(\rho_s\mu_s/\rho_w\mu_w)^{0.4} \times \{1 + (L^{0.52} - 1)(h_D/h_s)\} \quad (62)$$

The stagnation point heat-transfer rate for $\sigma = 0.71$ thus becomes, by virtue of Eq. (45),*

$$q = 0.94(\rho_w\mu_w)^{0.1}(\rho_s\mu_s)^{0.4} \times \{1 + (L^{0.52} - 1)(h_D/h_s)\}(h_s - h_w)\sqrt{(du_e/dx)_s} \quad (63)$$

It is interesting to note that the external flow properties are much more important than the wall values in determining the heat-transfer rate, so that the uncertainty in the heat transfer is about 40 per cent of the un-

* For σ not equal to 0.71, it is recommended that the factor 0.94 be replaced by $(0.76 \sigma^{-0.6})$.

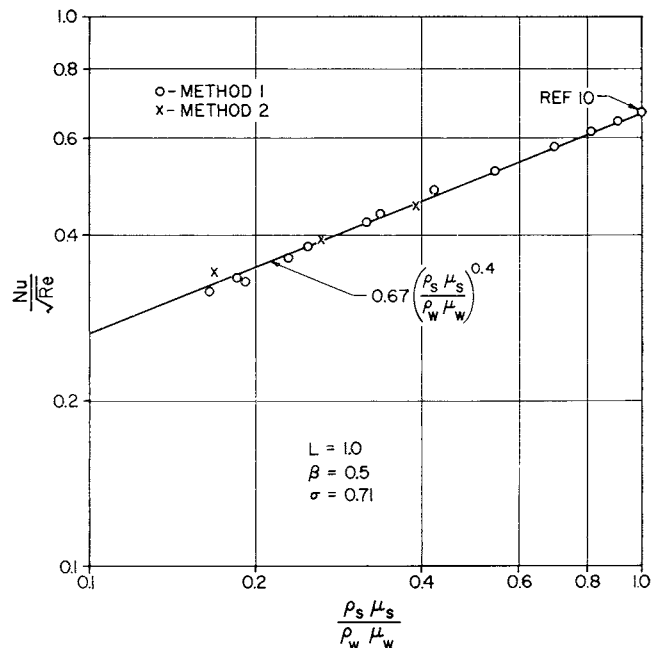


FIG. 2. Correlation of the heat-transfer parameter Nu/\sqrt{Re} as a function of the $\rho\mu$ ratio across the boundary layer, $\rho_s\mu_s/\rho_w\mu_w$ for the equilibrium stagnation point boundary layer with $L = 1$.

certainty in the external viscosity. The physical reason for the importance of the external viscosity is that the growth of the boundary layer, and hence the heat transfer to the wall, depends mostly upon the external properties. The analogy with turbulent boundary layers is easily seen.

For a modified Newtonian flow, the stagnation point velocity gradient is

$$(du_e/dx)_s = (1/R)\sqrt{2(p_s - p_\infty)/\rho_s} \quad (64)$$

where R is the nose radius and p_∞ is the ambient pressure.

(9) THE "FROZEN" BOUNDARY LAYER

When atomic gas phase recombination is negligible ($C_1 = 0$), atoms diffusing from the free stream will reach the wall. If the wall is *noncatalytic* to surface recombination, the atom fraction at the wall will build up to the free-stream value. On the other hand, if the wall is *catalytic*, the atom concentration will be reduced to its equilibrium value at the wall temperature. Intermediate cases of wall catalyticity are of course possible, but only these extremes were computed.

Eqs. (34), (35), and (36) were solved with $C_1 = 0$ for various stagnation conditions and Lewis Numbers as discussed in the Appendix (Method 2).

For Lewis Number unity, the effect of the $\rho\mu$ variation was very close to that found for the equilibrium boundary layer by Method 2, and could suitably be expressed by Eq. (58). For other values of L , the dependence could best be given by*

$$(Nu/\sqrt{Re})/(Nu/\sqrt{Re})_{L=1} = 1 + \frac{1}{(L^{0.63} - 1)(h_D/h_s)} \quad (65)$$

The calculated values are compared with Eq. (65) in Fig. 4.

The difference in the exponent of L for the frozen as compared with the equilibrium boundary layer [Eqs. (60) and (65)] is quite certain since exactly the same property variations were used in both cases, and also seems reasonable in view of the greater importance of diffusion throughout the whole of the frozen boundary layer. It can be seen, however, that for a Lewis Number not too far from unity there is little difference in heat transfer for the frozen as opposed to the equilibrium boundary layer.

A few cases for noncatalytic wall were also computed. The resultant heat-transfer parameter could be given approximately by Eq. (62) with $L = 0$ —i.e., the heat transfer becomes proportional to $h_s - h_D$.

A comparison of the distributions of enthalpy, temperature, and atom concentration for an equilibrium and a frozen boundary layer with identical catalytic wall and free-stream conditions is shown in Figs. 5 and 6. Both cases give very nearly the same heat transfer; however, the enthalpy distributions are slightly different (since $L = 1.4$) and the temperature and concentration

distributions are markedly different, as is the mechanism of heat transfer.

(10) FINITE RECOMBINATION RATE

This most general case was solved using Method 2 with values of the recombination rate parameter (C_1) varying from zero (frozen) to infinity (equilibrium). As for the frozen boundary layer, the wall may be either catalytic or noncatalytic, and both of these alternatives were calculated. It is to be expected, of course, that for large values of C_1 (near equilibrium) there should be little effect of wall catalysis, since few atoms reach the wall.

The heat-transfer parameter for one flight condition and wall temperature is plotted in Fig. 7 for the complete range of C_1 . The solid lines are the total heat transfer for both catalytic (upper curve) and noncatalytic (lower curve) surfaces. For the catalytic wall, the fraction of heat transfer by conduction alone is shown by the dotted curve, so that the freezing of the boundary layer as recombination slows down (C_1 decreasing) is easily evident.

For either wall condition, C_1 must change by a factor of 10^4 in order for the boundary layer to change from substantially frozen to equilibrium throughout. Within this region of variation of C_1 , the boundary layer will be partly frozen (near the outer edge) and partly in equilibrium (near the wall). Since the recombination term [Eq. (56)] varies as $T^{-3.5}$, and the temperature changes by a factor of twenty between wall and external flow for the case considered, large variations in the recombination rate are possible across the boundary layer, thus permitting it to be partly frozen and partly in equilibrium.

For the noncatalytic wall, the distributions of atom mass fraction for several recombination rate parameters are shown in Fig. 8. For C_1 very large, no atoms reach the wall, all having recombined in the gas. For lower values of C_1 , some atoms reach the wall and, because none recombine on the wall, a finite atom concentration builds up. For C_1 approaching zero, there is no recombination and hence no concentration gradients exist.

It can be seen in Fig. 7 that a much lower value of C_1 is necessary to "freeze" the boundary layer when a noncatalytic wall is used than would be the case otherwise. This is caused by the "damming up" of the atoms at the noncatalytic wall, resulting in greater recombination because of high local concentrations.

From Eqs. (57) and (63) it can be seen that, for a given flight velocity (hence T_s), C_1 varies as the square of the stagnation point density (and thus for strong shock waves, as the square of the ambient density), and also as the nose radius. Thus the boundary layer would become frozen at a high enough altitude, this altitude being less for small nose radii than for large. In order to change from a frozen to an equilibrium boundary layer, C_1 must change by 10^4 , and thus the density by 10^2 , which is an altitude change of about 100,000 ft.

* Note that Lees [see reference 7, Eq. (19)] suggested that, for the frozen boundary layer, the exponent of L in Eq. (65) be $2/3$.

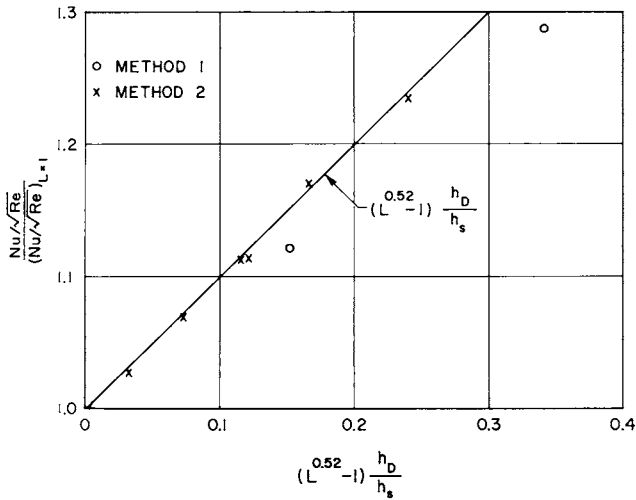


FIG. 3. Correlation of Lewis Number effects on the heat-transfer parameter Nu/\sqrt{Re} for the equilibrium stagnation point boundary layer with $L = \text{constant} \neq 1$.

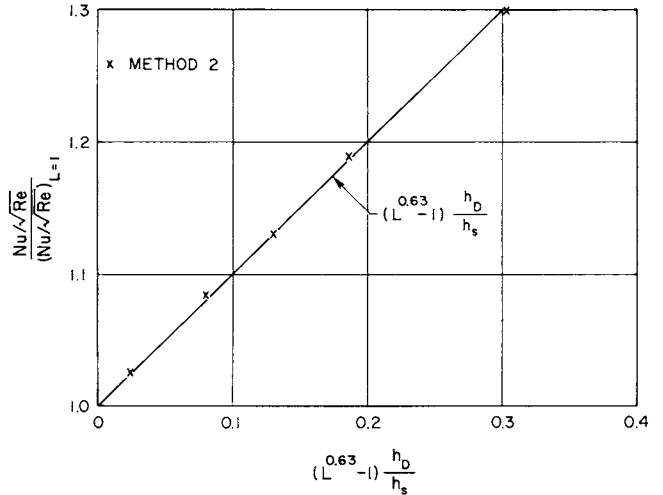


FIG. 4. Correlation of Lewis Number effects on the heat-transfer parameter Nu/\sqrt{Re} for the frozen stagnation point boundary layer, $L = \text{constant} \neq 1$.

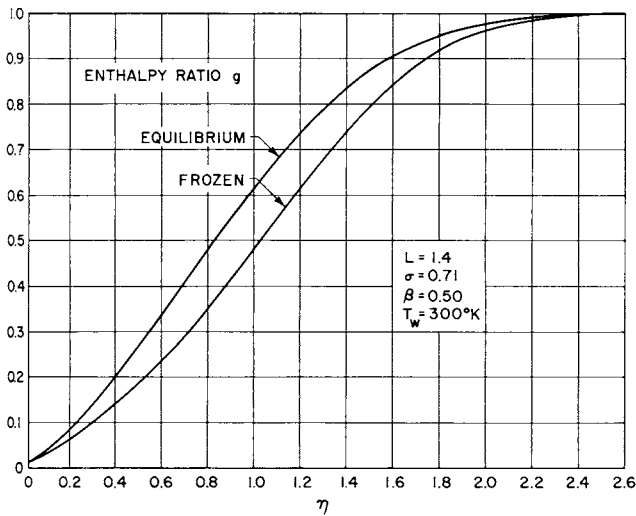


FIG. 5. Comparison of enthalpy distributions for an equilibrium and a frozen stagnation point boundary layer with the same external flow and wall conditions, $g_w = 0.0123$, $c_{AS} = 0.499$.

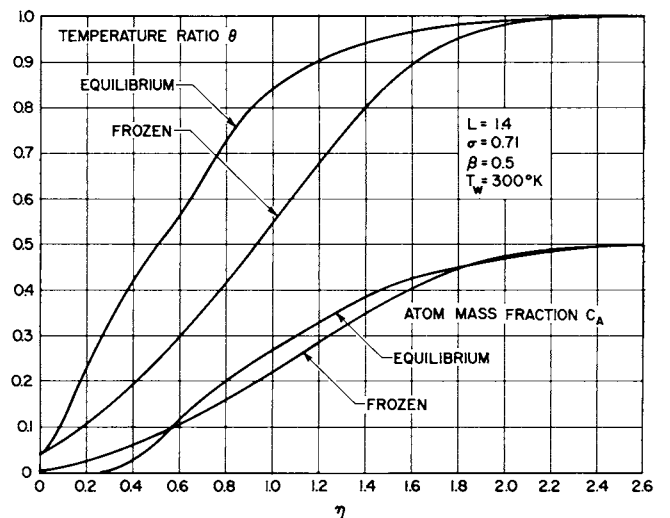


FIG. 6. Comparison of temperature and atom concentration distributions for an equilibrium and a frozen boundary layer with the same external flow and wall conditions, $g_w = 0.0123$, $c_{AS} = 0.499$.

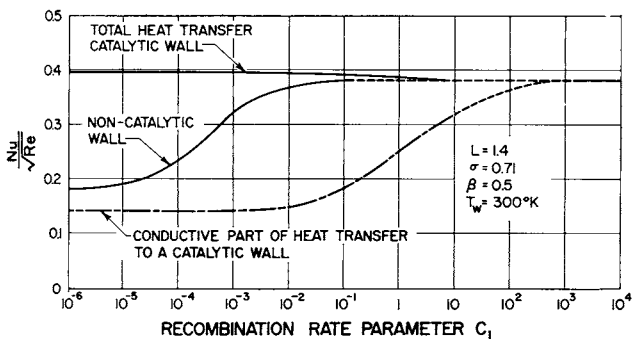


FIG. 7. Heat-transfer parameter Nu/\sqrt{Re} for a boundary layer with finite recombination rates—i.e., various values of the recombination rate parameter C_1 , $g_w = 0.0123$, $c_{AS} = 0.536$. Note the reduction of heat transfer to a noncatalytic wall for low values of C_1 .

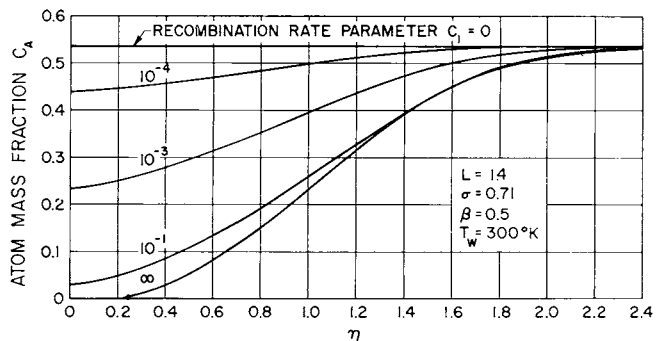


FIG. 8. Distributions of atom mass fraction in a stagnation point boundary layer on a noncatalytic wall for several values of the recombination rate parameter C_1 , $g_w = 0.0123$, $c_{AS} = 0.536$.

(11) CONCLUSIONS

The laminar stagnation point heat transfer in dissociated air can be given by Eqs. (63) and (65) for the equilibrium and frozen boundary layers, respectively. These results were computed for a Prandtl Number of 0.71 and for a Lewis Number which was constant throughout the boundary layer.

The major deviation in the heat-transfer parameter from the low temperature, perfect gas value is due to the variation of $\rho\mu$ across the boundary layer. The heat transfer [Eq. (63)] is mainly dependent upon the value of $\rho\mu$ at the outer edge of the boundary layer.

If the wall catalyzes atomic recombination, the total heat transfer is not much affected by a nonequilibrium state of the boundary layer if the Lewis Number is near unity.

If the wall is noncatalytic, the heat transfer may be appreciably reduced when the boundary layer is frozen throughout—i.e., when the recombination reaction time becomes much longer than the time for a particle to diffuse through the boundary layer. Since the ratio of these times depends upon altitude and nose radius, there is a scale effect which determines the chemical state of the boundary layer.

APPENDIX—DETAILS OF THE NUMERICAL SOLUTIONS

Method 1

The momentum equation for the stagnation point boundary layer is given by Eq. (35)—namely,

$$(lf_{\eta\eta})_{\eta} + ff_{\eta\eta} + (1/2)[(\rho_s/\rho) - f_{\eta}^2] = 0 \quad (\text{A-1})$$

If thermal diffusion is neglected (i.e., $L_i^T = 0$) then the energy equation in terms of the enthalpy becomes, from Eq. (37),

$$[(l/\sigma)g_{\eta}]_{\eta} + fg_{\eta} + \left\{ (l/\sigma)\Sigma[c_{is}(h_i - h_i^0)/h_s](L_i - 1)s_{i\eta} \right\}_{\eta} = 0 \quad (\text{A-2})$$

or

$$[(l/\sigma)(1 + d)g_{\eta}]_{\eta} + fg_{\eta} = 0 \quad (\text{A-3})$$

where

$$d \equiv \left. \begin{aligned} &\Sigma[c_{is}(h_i - h_i^0)/h_s](L_i - 1)(\partial s_i/\partial g)_p \\ &= (L - 1)\Sigma(h_i - h_i^0)(\partial c_i/\partial h)_p \end{aligned} \right\} \quad (\text{A-4})$$

where the subscript p denotes that the differentiation is at constant pressure and it is assumed $L_i = L = \text{constant}$ for all species.

In Method 1, which is suitable only for the equilibrium boundary layer, Eqs. (A-1) and (A-3) were solved simultaneously with the boundary conditions

$$\begin{aligned} f(0) &= 0, \quad f_{\eta}(0) = 0, \quad f_{\eta}(\infty) = 1 \\ g(0) &= g_w, \quad g(\infty) = 1 \end{aligned}$$

The functions l , ρ_s/ρ and d were evaluated from the calculated equilibrium properties of air¹¹ and by taking the viscosity to vary according to Sutherland's formula [see Section (5) above]. For given external (stagnation

point) flow conditions these quantities were plotted as functions of g . For numerical computation it was convenient to use analytic expressions of the following form:

$$l \equiv \rho\mu/\rho_w\mu_w = (\alpha_1/\sqrt{g}) - (\alpha_2/g) \quad (\text{A-5})$$

$$\rho_s/\rho = 1 - \gamma_1(1 - g) - \gamma_2(1 - g)^4 \quad (\text{A-6})$$

$$d = (L - 1)\Sigma(h_i - h_i^0)(\partial c_i/\partial h)_p = \beta_1 e^{-\beta_2/g} \quad (\text{A-7})$$

The constants α , γ , β in each expression were determined by fitting these expressions to the equilibrium air calculations (see Fig. 1).

Numerical solutions for this problem were obtained on an IBM 650 computer. The method of solution was to pick values of $f_{\eta\eta}(0)$ and $g_{\eta}(0)$ and integrate the equations directly, recording the resultant asymptotic values of f_{η} and g for large values of η . After three such integrations an interpolation will produce better values of $f_{\eta\eta}(0)$ and $g_{\eta}(0)$. The interpolation procedure was repeated until the required conditions at "infinity" were met—i.e., $f_{\eta} \rightarrow 1$ and $g \rightarrow 1$. This interpolation was made an integral part of the numerical program so by starting with three initial guesses for $f_{\eta\eta}(0)$ and $g_{\eta}(0)$ the program would run automatically to completion.

It should be noted that Eqs. (A-1) and (A-3) are formally identical with the stagnation point equations solved by Cohen and Reshotko¹⁰ except for the function d [Eq. (A-4)]. If $L = 1$, however, $d \equiv 0$; thus by specifying $L = 1$, $\sigma = 1$, $l = 1$ and $\rho_s/\rho = g$, the stagnation point solutions given by Cohen and Reshotko could be duplicated. (A table giving the specific values of the parameters for which solutions were obtained by the method described above may be obtained directly from the authors.)

Method 2

This method is a more general formulation in that it allows computation of the nonequilibrium boundary layer. As may be expected, however, it involves more approximations than the rather straightforward procedure of Method 1.

In the nonequilibrium case, the concentration of the various species is not determined by the enthalpy and the (known) pressure. It is necessary, therefore, to add a continuity equation for each species. Furthermore, it is convenient to express the thermodynamic properties in terms of the temperature and the concentrations of the species. The energy equation should, therefore, be written in terms of the temperature. To make this problem tractable it was assumed that air is a diatomic gas composed of "air" molecules and "air" atoms with properties properly averaged between oxygen and nitrogen. The dissociation energy of an air atom was taken to be the average dissociation energy in the external flow [see Eq. 59]. With this assumption the problem is reduced to the simultaneous solution of three equations (momentum, energy and atom concentration) and the thermodynamic properties are to be expressed in terms of the temperature and atom concentration.

The momentum equation is still Eq. (A-3), with

$$\rho_s/\rho = [(1 + c_A)/(1 + c_{As})]\theta \quad (\text{A-8})$$

$$l \equiv \rho\mu/\rho_w\mu_w = [1/(1 + c_A)]^{3/2}(\theta_w/\theta)F(\theta) \quad (\text{A-9})$$

where

$$F(\theta) \equiv (T_s\theta/300)^{3/2}[413/(T_s\theta + 113)] + 3.7(T_s\theta/10,000)^2 - 2.35(T_s\theta/10,000)^4$$

and the stagnation temperature T_s is given in degrees Kelvin. The function $F(\theta)$ is a fitted curve giving the temperature dependence of the viscosity under the assumption that the atoms and molecules have the same collision cross-sections.

The energy equation in terms of the temperature is, from Eq. (36) with $L_i^T = 0$,

$$[(cl/\sigma)\theta_\eta]_\eta + cf\theta_\eta + (l/\sigma)\theta_\eta \times \frac{\Sigma(c_{pA}/\bar{c}_{pw})L_i c_{is} s_{i\eta} + \{2(du_e/dx)_s\}^{-1} \times \Sigma(w_i/\rho)[(h_i^0 - h_i)/\bar{c}_{pw}T_s]}{\Sigma(w_i/\rho)[(h_i^0 - h_i)/\bar{c}_{pw}T_s]} = 0 \quad (\text{A-10})$$

where

$$c \equiv \bar{c}_p/\bar{c}_{pw}$$

With the assumption of a simple diatomic gas and taking $L_i = L = \text{constant}$, the third term may be rewritten as

$$(Ll/\sigma)\theta_\eta c_{A\eta} [(c_{pA} - c_{pM})/\bar{c}_{pw}]$$

and the fourth term, using Eqs. (56), (57), and (61), and taking $h_A = h_M$, becomes

$$C_1(h_D/\bar{c}_{pw}T_s)[(c_A^2 - c_{AE}^2)/\theta^{3.5}(1 + c_{As})]$$

$$\text{Now } c_{pM} \approx (R/M)\{(7/2) + e^{-(T_V/T)^2}\}$$

where the exponential is the vibrational heat capacity and $T_V \approx 800^\circ\text{K}$. for air; also

$$c_{pA} = (5/2)(R/(M/2))$$

Hence,

$$c \equiv \bar{c}_p/\bar{c}_{pw} = (10/7)c_A + \{1 + (2/7)e^{-(\theta_V/\theta)^2}\}(1 - c_A) \quad (\text{A-11})$$

$$(c_{pA} - c_{pM})/\bar{c}_{pw} = (3/7) - (2/7)e^{-(\theta_V/\theta)^2} \quad (\text{A-12})$$

For computation then the energy equation becomes

$$[(cl/\sigma)\theta_\eta]_\eta + cf\theta_\eta + e(Ll/\sigma)\theta_\eta c_{A\eta} + C_1 C_2 m = 0 \quad (\text{A-13})$$

where

$$l \equiv \rho\mu/\rho_w\mu_w = [1/(1 + c_A)]^{3/2}(\theta_w/\theta)F(\theta), \quad \text{see Eq. (A-9)}$$

$$e \equiv (c_{pA} - c_{pM})/\bar{c}_{pw} = (3/7) - (2/7)e^{-(\theta_V/\theta)^2}$$

$$c \equiv \bar{c}_p/\bar{c}_{pw} = (10/7)c_A + \{1 + (2/7)e^{-(\theta_V/\theta)^2}\}(1 - c_A)$$

$C_1 = \text{parameter, see Eq. (57)}$

$C_2 \equiv h_D/\bar{c}_{pw}T_s = \text{parameter, see Eq. (61)}$

$m \equiv (c_A^2 - c_{AE}^2)/\theta^{3.5}(1 + c_A)$

The equilibrium atom mass fraction c_{AE} can be determined from reference 11. For computation c_{AE} was approximated by

$$c_{AE} = c_{As}e^{C_3(1-1/\theta)} \quad (\text{A-14})$$

where C_3 is a constant.

The continuity equation for atoms was written in terms of the atom mass fraction c_A instead of the normalized atom mass fraction s . Thus Eq. (34) becomes

$$[(Ll/\sigma)c_{A\eta}]_\eta + fc_{A\eta} - \{2(du_e/dx)_s\}^{-1}\Sigma(w_i/\rho) = 0 \quad (\text{A-15})$$

or

$$[(Ll/\sigma)c_{A\eta}]_\eta + fc_{A\eta} - C_1 m = 0 \quad (\text{A-16})$$

Method 2 for the nonequilibrium boundary layer is the simultaneous solution of Eqs. (A-1), (A-13) and (A-16) with the boundary conditions

$$f(0) = 0, \quad f_\eta(0) = 0, \quad f_\eta(\infty) = 1 \\ \theta(0) = \theta_w, \quad \theta(\infty) = 1$$

$$c_A(0) = 0 \quad \text{for catalytic wall} \\ c_{A\eta}(0) = 0 \quad \text{for noncatalytic wall} \left. \vphantom{\begin{matrix} c_A(0) = 0 \\ c_{A\eta}(0) = 0 \end{matrix}} \right\} c_A(\infty) = c_{As}$$

Solutions were obtained on a digital computer using an iterative procedure similar to that used in Method 1.

The limiting case of the equilibrium boundary layer was obtained by Method 2 by eliminating the term $C_1 m$ between Eqs. (A-13) and (A-16) and solving the resulting equation simultaneously with Eq. (A-1), taking the equilibrium atom concentration as a known quantity in the form of Eq. (A-14). The limiting case of the frozen boundary was obtained by putting $C_1 \equiv 0$.*

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* Numerical results of these calculations may be obtained directly from the authors.

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