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Preface

Theories of the known, which are described by different physical ideas, may be equivalent in all their predictions and hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in one's attempt to understand what is not yet understood.

R. P. Feynman [1966]

A Parable

Imagine a society in which the citizens are encouraged, indeed compelled up to a certain age, to read (and sometimes write) musical scores. All quite admirable. However, this society also has a very curious—few remember how it all started—and disturbing law: *Music must never be listened to or performed!*

Though its importance is universally acknowledged, for some reason music is not widely appreciated in this society. To be sure, professors still excitedly pore over the great works of Bach, Wagner, and the rest, and they do their utmost to communicate to their students the beautiful meaning of what they find there, but they still become tongue-tied when brashly asked the question, "What's the point of all this?!"

In this parable, it was patently unfair and irrational to have a law forbidding would-be music students from experiencing and understanding the subject directly through "sonic intuition." But in our society of mathematicians we *have* such a law. It is not a written law, and those who flout it may yet prosper, but it says, *Mathematics must not be visualized!*

More likely than not, when one opens a random modern mathematics text on a random subject, one is confronted by abstract symbolic reasoning that is divorced from one's sensory experience of the world, *despite* the fact that the very phenomena one is studying were often discovered by appealing to geometric (and perhaps physical) intuition.

This reflects the fact that steadily over the last hundred years the honour of visual reasoning in mathematics has been besmirched. Although the great mathematicians have always been oblivious to such fashions, it is only recently that the "mathematician in the street" has picked up the gauntlet on behalf of geometry.

The present book openly challenges the current dominance of purely symbolic logical reasoning by using new, visually accessible arguments to explain the truths of elementary complex analysis.

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Computers

In part, the resurgence of interest in geometry can be traced to the mass-availability of computers to draw mathematical objects, and perhaps also to the related, somewhat breathless, popular interest in chaos theory and in fractals. This book instead advocates the more sober use of computers as an aid to geometric *reasoning*.

I have tried to encourage the reader to think of the computer as a physicist would his laboratory—it may be used to check existing ideas about the construction of the world, or as a tool for discovering new phenomena which then demand new ideas for their explanation. Throughout the text I have suggested such uses of the computer, but I have deliberately avoided giving *detailed* instructions. The reason is simple: whereas a mathematical idea is a timeless thing, few things are more ephemeral than computer hardware and software.

Having said this, the program "$f(z)$" is currently the best tool for visually exploring the ideas in this book; a free demonstration version can be downloaded directly from Lascaux Graphics [http://www.primenet.com/~lascaux/]. On occasion it would also be helpful if one had access to an all-purpose mathematical engine such as *Maple*® or *Mathematica*®. However, I would like to stress that none of the above software is essential: the entire book can be fully understood without *any* use of a computer.

Finally, some readers may be interested in knowing how computers were used to produce this book. Perhaps five of the 501 diagrams were drawn using output from *Mathematica*®; the remainder I drew by hand (or rather "by mouse") using CorelDRAW™, occasionally guided by output from "$f(z)$". I typeset the book in LaTeX using the wonderful *Y&Y TeX System for Windows* [http://www.YandY.com/], the figures being included as EPS files. The text is Times, with Helvetica heads, and the mathematics is principally MathTime™, though nine other mathematical fonts make cameo appearances. All of these Adobe Type 1 fonts were obtained from Y&Y, Inc., with the exception of Adobe's *MathematicalPi-Six* font, which I used to represent quaternions. Having typeset the book, I used the DVIPSONE™ component of the *Y&Y TeX System for Windows* to generate a fully page-independent, DSC-compliant PostScript® file, which I transmitted to Oxford via the Internet (using FTP) in the form of a single ZIP file. Finally, OUP printed the book directly from this PostScript® file.

The Book's Newtonian Genesis

In the summer of 1982, having been inspired by Westfall's [1980] excellent biography, I made an intense study of Newton's [1687] masterpiece, *Philosophiae Naturalis Principia Mathematica*. While the Nobel physicist S. Chandrasekhar [1995] has sought to lay bare the remarkable nature of Newton's *results* in the *Principia*, the present book instead arose out of a fascination with Newton's *methods*.

It is fairly well known that Newton's original 1665 version of the calculus was different from the one we learn today: its essence was the manipulation of power series, which Newton likened to the manipulation of decimal expansions in arithmetic. The symbolic calculus—the one in every standard textbook, and the one now associated with the name of Leibniz—was also perfectly familiar to

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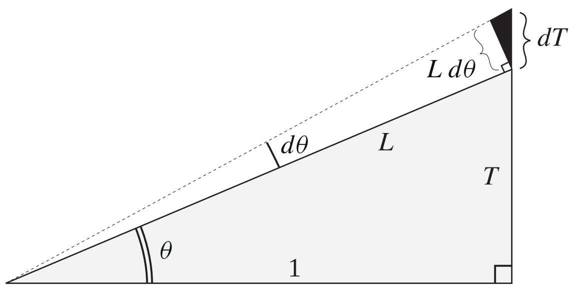
Newton, but apparently it was of only incidental interest to him. After all, armed with his power series, Newton could evaluate an integral like $\int e^{-x^2}\,dx$ just as easily as $\int \sin x\,dx$. Let Leibniz try *that!*

It is less well known that around 1680 Newton became disenchanted with both these approaches, whereupon he proceeded to develop a *third* version of calculus, based on *geometry*. This "geometric calculus" is the mathematical engine that propels the brilliant physics of Newton's *Principia*.

Having grasped Newton's method, I immediately tried my own hand at using it to simplify my teaching of introductory calculus. An example will help to explain what I mean by this. Let us show that if $T = \tan\theta$, then $\frac{dT}{d\theta} = 1 + T^2$. If we increase θ by a small amount $d\theta$ then T will increase by the amount dT in the figure below. To obtain the result, we need only observe that in the limit as $d\theta$ tends to zero, the black triangle is ultimately similar [exercise] to the shaded triangle. Thus, in this limit,



$$\frac{dT}{L\,d\theta} = \frac{L}{1} \quad\Longrightarrow\quad \frac{dT}{d\theta} = L^2 = 1 + T^2.$$

Only gradually did I come to realize how naturally this mode of thought could be applied—almost exactly 300 years later!—to the geometry of the complex plane.

Reading This Book

In the hope of making the book fun to read, I have attempted to write as though I were explaining the ideas directly to a friend. Correspondingly, I have tried to make you, the reader, into an active participant in developing the ideas. For example, as an argument progresses, I have frequently and deliberately placed a pair of logical stepping stones sufficiently far apart that you may need to pause and stretch slightly to pass from one to the next. Such places are marked "[exercise]"; they often require nothing more than a simple calculation or a moment of reflection.

This brings me to the exercises proper, which may be found at the end of each chapter. In the belief that the essential prerequisite for finding the answer to a question is the *desire* to find it, I have made every effort to provide exercises that provoke curiosity. They are considerably more wide-ranging than is common, and they often establish important facts which are then used freely in the text itself. While problems whose be all and end all is routine calculation are thereby avoided,

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I believe that readers will automatically develop considerable computational skill *in the process* of seeking solutions to these problems. On the other hand, my intention in a large number of the exercises is to illustrate how geometric thinking can often *replace* lengthy calculation.

Any part of the book marked with a star ("*") may be omitted on a first reading. If you do elect to read a starred section, you may in turn choose to omit any starred *sub*sections. Please note, however, that a part of the book that is starred is not necessarily any more difficult, nor any less interesting or important, than any other part of the book.

Teaching from this Book

The entire book can probably be covered in a year, but in a single semester course one must first decide what *kind* of course to teach, then choose a corresponding path through the book. Here I offer just three such possible paths:

• **Traditional Course.** Chapters 1 to 9, *omitting all starred material* (e.g., the whole of Chapter 6).

• **Vector Field Course.** In order to take advantage of the Pólya vector field approach to visualizing complex integrals, one could follow the "Traditional Course" above, omitting Chapter 9, and adding the unstarred parts of Chapters 10 and 11.

• **Non-Euclidean Course.** At the expense of teaching any integration, one could give a course focused on Möbius transformations and non-Euclidean geometry. These two related parts of complex analysis are probably the most important ones for contemporary mathematics and physics, and yet they are also the ones that are almost entirely neglected in undergraduate-level texts. On the other hand, graduate-level works tend to assume that you have already encountered the main ideas as an undergraduate: Catch 22!

Such a course might go as follows: All of Chapter 1; the unstarred parts of Chapter 2; all of Chapter 3, including the starred sections but (possibly) omitting the starred *sub*sections; all of Chapter 4; all of Chapter 6, including the starred sections but (possibly) omitting the starred *sub*sections.

Omissions and Apologies

If one believes in the ultimate unity of mathematics and physics, as I do, then a very strong case for the necessity of complex numbers can be built on their apparently fundamental role in the quantum mechanical laws governing *matter*. Also, the work of Sir Roger Penrose has shown (with increasing force) that complex numbers play an equally central role in the relativistic laws governing the structure of *space–time*. Indeed, if the laws of matter and of space–time are ever to be reconciled, then it seems very likely that it will be through the auspices of the complex numbers. This book cannot explore these matters; instead, we refer the interested reader to Feynman [1963, 1985], to Penrose [1989, 1994], and to Penrose and Rindler [1984].

A more serious omission is the lack of discussion of Riemann surfaces, which I had originally intended to treat in a final chapter. This plan was aborted once it be-

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came clear that a serious treatment would entail expanding the book beyond reason. By this time, however, I had already erected much of the necessary scaffolding, and this material remains in the finished book. In particular, I hope that the interested reader will find the last three chapters helpful in understanding Riemann's original physical insights, as expounded by Klein [1881]. See also Springer [1957, Chap. 1], which essentially reproduces Klein's monograph, but with additional helpful commentary.

I consider the history of mathematics to be a vital tool in understanding both the current state of mathematics, and its trajectory into the future. Sadly, however, I can do no more than touch on historical matters in the present work; instead I refer you to the remarkable book, *Mathematics and Its History*, by John Stillwell [1989]. Indeed, I strongly encourage you to think of his book as a companion to mine: not only does it trace and explain the development of complex analysis, but it also explores and illuminates the connections with other areas of mathematics.

To the expert reader I would like to apologize for having invented the word "amplitwist" [Chapter 4] as a synonym (more or less) for "derivative", as well the component terms "amplification" and "twist". I can only say that the need for *some* such terminology was forced on me in the classroom: if you try teaching the ideas in this book *without* using such language, I think you will quickly discover what I mean! Incidentally, a precedence argument in defence of "amplitwist" might be that a similar term was coined by the older German school of Klein, Bieberbach, *et al*. They spoke of "eine Drehstreckung", from "drehen" (to twist) and "strecken" (to stretch).

A significant proportion of the geometric observations and arguments contained in this book are, to the best of my knowledge, new. I have not drawn attention to this in the text itself as this would have served no useful purpose: students don't need to know, and experts will know without being told. However, in cases where an idea is clearly unusual but I am aware of it having been published by someone else, I have tried to give credit where credit is due.

In attempting to rethink so much classical mathematics, I have no doubt made mistakes; the blame for these is mine alone. Corrections will be gratefully received, and then posted, at http://www.usfca.edu/vca.

My book will no doubt be flawed in many ways of which I am not yet aware, but there is one "sin" that I have intentionally committed, and for which I shall not repent: many of the arguments are not rigorous, at least as they stand. This is a serious crime if one believes that our mathematical theories are merely elaborate mental constructs, precariously hoisted aloft. Then rigour becomes the nerve-racking balancing act that prevents the entire structure from crashing down around us. But suppose one believes, as I do, that our mathematical theories are attempting to capture aspects of a robust Platonic world that is not of our making. I would then contend that an initial lack of rigour is a small price to pay if it allows the reader to see into this world more directly and pleasurably than would otherwise be possible.

San Francisco, California T. N.
June, 1996

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