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# QUARTERLY OF APPLIED MATHEMATICS

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Vol. XI

October, 1953

No. 3

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## SOME MATHEMATICAL METHODS AND TECHNIQUES IN ECONOMICS<sup>1</sup>

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That mathematics plays a certain role in various phases of economic theory is, of course, quite well known. The number of mathematicians and economists unaware of the discipline known as "Mathematical Economics" is surely small. Unfortunately, the number of mathematicians and economists who are aware of the aims, the methods and the successes (the failures are, as is always the case, only too well known) of this rapidly expanding field, is likewise small.

The majority of us are exposed in the very early stages of our training to simple applications of mathematics in economics. To cite an instance, the classical high-school course in algebra which does not use the price-demand relation to illustrate the ideas of an inverse variation, is probably non-existent. A good many mathematicians think that this is almost the sum total of pure mathematical depth and sophistication that the economist encounters.

On the other hand, since statistics is, in so many obvious ways, ideally suited for analyzing certain types of economic phenomena, one is often fooled into believing that this represents all of the applied mathematical ideas that can play a central position in an economic investigation.

The appearance, in recent years, of the book "The Theory of Games and Economic Behavior" by von Neumann and Morgenstern [40] has been instrumental in dispelling from the minds of many mathematicians and economists some of the false ideas about what mathematics entered, and possibly more important, what mathematics does not enter, in the problems arising in everyday mathematical economics.

The purpose of this paper is, in part, to give a presentation of some phases of pure mathematics that are in current use in the economic world. We do not claim or pretend that this paper is either exhaustive or definitive—we merely propose to touch on a few boundary points with the hope that some of the readers will feel an urge to dig deeper into the interior. This paper is being written by a mathematician primarily for a math-

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<sup>1</sup>Received April 17, 1953. The author is gratefully indebted to the members of the Cowles Commission research staff for the fruitful discussions, suggestions and criticisms which they have contributed to the making of this paper. This paper is a result of the work being done at the Cowles Commission for Research in Economics under subcontract to the RAND Corporation; it is to be reprinted as Cowles Commission Paper, New Series, No. 78.

ematical reader. The topics we discuss are thus selected, not so much for their economic depth or significance as for their mathematical interest.

Historically, the calculus entered very quickly into the study of price-demand relations. The calculus was, in fact, essentially the only tool used, until fairly recently in all of mathematical economics. (For a history and description of both the methodology and problems considered up to the beginning of the 20th Century see the authoritative article by V. Pareto [43]).

To say that the calculus, as a fundamental weapon, has by now played out its role would be both exaggerated and misleading. However, its initial foothold has been weakened considerably, and in certain places, arguments previously employing calculus techniques have been reworked using more powerful and more modern methods. There are several fundamental reasons for this trend to get away from using the calculus. Firstly, in a desire for generality, the conditions of differentiability placed on functions, especially when not needed, are aesthetically unsatisfactory. Secondly, the existence of derivatives for functions arising from a study of complex and oftentimes highly discrete situations is by no means a natural assumption. Thirdly the appropriate calculus conditions imposed on the functions considered at times obscure the essential nature of the problem. Added to this are the huge successes achieved in other fields (physics, for one) by the introduction of more of the full mathematical power available today.

This accounts for the tendency, with certain economists, to introduce some "natural" tools in their domain. The surprising effect (to some people) has been a great simplification and reinterpretation of old results and a satisfying surge forward in new research.

Although it is clear that mathematical statistics and game theory play a vital part in economic theory, we shall not consider their applications here. How and why they are used can be found very easily, and in many places [3, 33, 38, 40, 41, 50].

As we have pointed out previously, the calculus has performed (and is still performing) a striking function in the discipline. We begin the paper proper with an example that arises in economics and is handled via the calculus. This example has its origin in the theory of consumer's behavior; the derivation we present is of the so-called Slutsky Equation (Slutsky [47]; for a recent treatment of it see Samuelson [45, p. 97-103]). The problem itself is of classical stature in the economics of consumption. Its solution is a striking example of how a serious economic result emerges from an elementary mathematical analysis.<sup>2</sup>

We suppose there are  $n$  commodities which are labeled 1, 2,  $\dots$ ,  $n$  and a given consumer. A commodity bundle is a vector  $x$  whose  $i$ th coordinate represents an amount of the  $i$ th commodity. We suppose that we are also given prices for each commodity. Let  $p$  be the price vector. Given a commodity bundle  $x$  it is assumed that the consumer derives a certain satisfaction  $s(x)$  where  $s$  is a real-valued, twice differentiable strictly convex function. (Economists usually consider a more general class of function, but for the sake of simplicity we restrict ourselves to the convex case). For economic purposes  $s$  could be equally well replaced by a monotonic increasing transformation of itself. In order for our results to be economically meaningful, our description of the consumer's behavior should be independent of the particular transformation applied to  $s$ . Finally, we suppose the consumer has a given amount of money,  $\mu$ . Subject to the budgetary

<sup>2</sup>The form of the proof given here is due to G. Debreu.

constraint

$$p'x = \mu \quad (' \text{ denotes transposes}) \quad (1)$$

the consumer tries to maximize his satisfaction  $s(x)$ . Out of this desire, there results the necessary condition

$$\frac{ds}{dx} = \sigma p, \quad \sigma \text{ a scalar and} \quad (2)$$

where  $\sigma$ , of course, depends on the particular form of  $s$ . ( $\sigma$  has traditionally been called the *marginal utility of money*).

On differentiating (2) we obtain

$$S dx = \sigma dp + p d\sigma \quad (3)$$

where  $S$  is the Hessian matrix of  $s$ .  $S$  is thus a symmetric matrix.

Differentiation of (1) yields

$$p' dx + (dp')x = d\mu. \quad (4)$$

Let

$$\Sigma = \begin{pmatrix} \frac{1}{\sigma} S & p \\ p' & 0 \end{pmatrix}.$$

From the symmetry of  $S$ , we have that  $\Sigma$  is also symmetric. The system (3) and (4) now becomes

$$\Sigma \begin{pmatrix} dx \\ -\frac{d\sigma}{\sigma} \end{pmatrix} = \begin{pmatrix} dp \\ d\mu - x' dp \end{pmatrix}. \quad (5)$$

Since  $s$  is strictly convex,  $\Sigma^{-1}$  exists. Since  $\Sigma$  is symmetric,

$$\Sigma^{-1} = \begin{pmatrix} X & \gamma \\ \gamma' & c \end{pmatrix},$$

where  $X$  is again symmetric and where  $\gamma$  is a column vector and  $c$  is a number. Equation (5) then gives rise to

$$\begin{pmatrix} dx \\ -\frac{d\sigma}{\sigma} \end{pmatrix} = \begin{pmatrix} X & \gamma \\ \gamma' & c \end{pmatrix} \begin{pmatrix} dp \\ d\mu - x' dp \end{pmatrix}. \quad (6)$$

Expanding the right-hand side of (6), we arrive at  $dx = X dp + \gamma (d\mu - x' dp)$ . Thus

$$\frac{\partial x}{\partial p} = X - \gamma x', \quad \frac{\partial x}{\partial \mu} = \gamma.$$

Combining these we have

$$\frac{\partial x}{\partial p} = X - \frac{\partial x}{\partial \mu} x', \quad (7)$$

and so

$$X = \frac{\partial x}{\partial p} + \frac{\partial x}{\partial \mu} x'. \quad (8)$$

Since  $X$  is symmetric, picking out the  $(i, j)$  and  $(j, i)$  element we have

$$\frac{\partial x_i}{\partial p_j} + x_i \frac{\partial x_i}{\partial \mu} = \frac{\partial x_j}{\partial p_i} + x_j \frac{\partial x_j}{\partial \mu}. \quad (9)$$

This is the Slutsky Equation. Notice that the result is independent of the particular transformation applied to  $s$ . The respective sides of (9) are termed the *substitution coefficients*. We now seek an economic interpretation for these. We ask: what changes in amounts of commodities and what changes in income would leave the consumer's satisfaction unchanged. From (2), since  $s$  is being kept constant,  $p' dx = 0$ . So (4) becomes  $d\mu = x' dp$ . Thus  $dx = X dp$ , whence the matrix elements of  $X$ , that is, the substitution coefficients describe the consumer behavior when satisfaction is assumed to be held constant.

This simple illustration of the use of the calculus, and of a calculus of the most rudimentary form, accounts, to a small degree, for the security the economist felt in the continued use of the calculus as his primary tool. As a consequence, he felt no particular urgency to broaden the mathematical repertoire to be applied to his problems. I should not like to imply that the preceding example is typical, in its mathematical depth, of the category of problems attacked. Mathematical analysis of a far more penetrating character was employed by some economic writers. Nontrivial existence theorems in the theory of differential equations, results from the theory of integral equations, and a great wealth of related mathematics was put to very effective use. (For an excellent treatment, along such lines, of a wide variety of topics, see Samuelson's stimulating book [45].)

However, in the late 1930's, the variety of mathematical topics finding application became more and more diversified and modern mathematical developments began reflecting on the nature of the work undertaken by the economist. In the remainder of this paper we shall touch on some of these. The order of their presentation is not meant to be chronological. These often involve convex sets or the calculus; this is quite natural, since such an important part of economics can be viewed as a maximizing activity.

We now consider a situation arising in the phase of economics known as "Welfare Economics." Before discussing the problem *per se* we need a brief, crude description of the general scope of the Welfare Economics. (For a much more thorough description see Samuelson [45], Arrow [1]).

Suppose that there are  $n$ -individuals in a community, designated by  $1, 2, \dots, n$ . We suppose that each of these has a preference ordering by which he ranks the possible prospects or social states that can confront him; that is, to the  $i$ th individual there belongs a binary relation  $R_i$ , defined on the set of possible social states  $X, Y, Z$  with the following properties:

- 1)  $XR_i Y$  or  $YR_i X$ ,
- 2)  $XR_i Y, YR_i Z$  implies  $XR_i Z$ .

Here,  $R_i$  is a complete (or as it is sometimes called, a weak simple) ordering;  $XR, Y$  can be thought of as “ $X$  is at least as good as  $Y$  as far as the  $i$ th individual is concerned.” Let  $XP, Y$  denote that  $XR, Y$  but not  $YR, X$ . A social state  $X$  will be said to be optimal if the following is true: if for any  $i$  there exists a  $Y$  with  $YP, X$ , then for some other individual  $j$ ,  $XP, Y$ . That is,  $X$  is optimal if whenever a social state  $Y$  is preferred to  $X$  by some individual, then  $Y$  is not “at least as good as  $X$ ” according to all other individuals. Another way of looking at this type of optimality can be formulated as: let  $XRY$  mean  $XR, Y$  for all  $i = 1, 2, \dots, n$ ;  $R$  then defines a partial order in the set of social states; a state  $X$  is now optimal if it is a maximal element in this partial order. The problem of the Welfare Economics is to give a description of (and a prescription as to how to attain) such an optimal social state.

Here is a situation which has been handled within the last ten years by two totally different approaches—one in the framework of the orthodox calculus and the other in that of the theory of convex sets. In order to exhibit the differences (and the metamorphosis) in these two polar attacks on the problem we present two solutions of the same problem arising in the Welfare Economics. The first treatment we give is the calculus discussion, and is due to Lange [35].

We assume each individual's preference ordering  $R_i$  can be represented by a real valued function  $u^{(i)}$ , the  $i$ th person's utility function. The purpose is to maximize  $u^{(i)}$ , for each  $i$ , subject to the conditions that  $u^{(i)}$  is kept constant for  $j \neq i$ .

We suppose the utility function of each individual to be a function of the amounts of each commodity that he gets; that is, if  $x_j^{(i)}$  is the amount of the  $j$ th commodity held by the  $i$ th individual,  $j = 1, 2, \dots, m$ , then  $u^{(i)} = u^{(i)}(x_1^{(i)}, x_2^{(i)}, \dots, x_m^{(i)})$ . We further assume that each  $u^{(i)}$  is appropriately differentiable. Let  $X_r = \sum_{i=1}^n x_r^{(i)}$  be the total amount of the  $r$ th commodity which is obtained by the whole community. The  $X_r$  are interrelated by a “technological function”  $F(X_1, X_2, \dots, X_m) = 0$ . The problem then becomes: maximize  $u^{(i)}(x_1^{(i)}, \dots, x_m^{(i)})$   $i = 1, 2, \dots, n$  subject to the constraints

- 1)  $u^{(i)}(x_1^{(i)}, \dots, x_m^{(i)}) = \text{constant for } j \neq i,$
- 2)  $X_r = \sum_{i=1}^n x_r^{(i)}$  for each  $r = 1, 2, \dots, m,$
- 3)  $F(X_1, X_2, \dots, X_m) = 0.$

From the theory of Lagrange multipliers, this is equivalent to “extremizing” the unconstrained expression

$$\sum_{i=1}^n \lambda_i u^{(i)} + \sum_{r=1}^m \mu_r \left( \sum_{i=1}^n x_r^{(i)} - X_r \right) + \mu F(X_1, \dots, X_m),$$

where the  $\lambda$ 's and  $\mu$ 's are Lagrange multipliers. Differentiation and elimination yields

$$\frac{\partial u^{(i)}}{\partial x_r^{(i)}} \bigg/ \frac{\partial u^{(i)}}{\partial x_s^{(i)}} = \frac{\partial F}{\partial X_r} \bigg/ \frac{\partial F}{\partial X_s} \quad \text{for all } i, r \text{ and } s.$$

Equivalently,

$$\frac{\partial u^{(i)}}{\partial x_r^{(i)}} \bigg/ \frac{\partial u^{(i)}}{\partial x_r^{(j)}} = \frac{\partial u^{(i)}}{\partial x_s^{(i)}} \bigg/ \frac{\partial u^{(i)}}{\partial x_s^{(j)}} \quad \text{for all } i, j, r \text{ and } s$$

Economically,  $\partial u^{(i)}/\partial x_r^{(i)}$  is the "marginal utility of the  $r$ th commodity for the  $i$ th individual," and the above gives, as a necessary condition, that certain ratios of marginal utilities be equal. The result is invariant with regard to monotone increasing transformations of the utility functions. The Lagrange multipliers could be interpreted as prices, and the solution of the problem could be stated in terms of the existence of prices with certain properties. This will be done in the second treatment we give of this same problem using the theory of convex sets.

Before turning to the second version of this problem, we digress into several pathways suggested by factors which have arisen in the examples which we have already discussed.

In the situations which we have considered, the following kind of assumptions have been made: a complete ordering, occurring naturally as a preference ranking, confronts the economist; he, in turn, wishes to discuss the optimal behavior under this ordering. In order to do so he resorts to a real-valued function, having desirable differentiability properties. However, the final description of the optimal behavior is given in a form independent, to a major degree, of this function. Two questions immediately present themselves. Firstly, what conditions on the ordering insure the existence of such order-preserving functions? Secondly, since the final form of the solutions does not depend on these functions, can all these problems be handled without the artifice of a real-valued representation of the ordering?

The first of these problems, important as it seems to be, has received very little attention from the economist. In fact, some economists have even gone as far as to tacitly assume that every complete ordering can be so represented. (The lexicographic ordering of the unit square offers an easy counter-example to this). In case the completely ordered set is, in addition, endowed with a "probability-mixing" operation, this problem has been thoroughly thrashed out by von Neumann-Morgenstern, Marschak [37], and Herstein-Milnor [25]. For the case in which no such mixture operation exists, H. Wold [54] did give conditions on the ordering which guaranteed the existence of an order-preserving function; however, his conditions were somewhat restrictive. In the mathematical literature, in a paper by Eilenberg [15] there is given a fairly general set of conditions for the existence of a continuous order-preserving function. This has recently been rederived by Debreu [10]. The result can be stated as follows:

Let  $S$  be a completely ordered topological space such that

I)  $S$  is separable and connected

II) for every  $a_0 \in S$  the sets  $\{a \in S \mid a \leq a_0\}$ ,  $\{a \in S \mid a_0 \leq a\}$  are closed. Then there exists on  $S$  a continuous, real, order-preserving function.

Since the economist usually works in a finite-dimensional Euclidian space, or at worst, in a separable Hilbert space, Debreu's condition that his space be separable and connected is really not very restrictive. Incidentally, Debreu has recently shown that if (I) is replaced by: (I)'  $S$  is perfectly separable, then the final result still holds.

As for the second problem, namely that of entirely avoiding the real-valued representation of the ordering, the alternative approach, which we are about to present, to the problem in Welfare Economics furnishes a typical method which has been successfully employed. Although this was hardly the main goal of the paper to be cited, it was an interesting by-product of the approach. The discussion which follows stems from a paper by Debreu [11].

Let us again suppose that there are  $m$  commodities labeled  $1, 2, \dots, m$  and  $n$  consumption units, labeled  $1, 2, \dots, n$ . The activity of a consumption unit is characterized by a vector  $x$ , the consumption vector, in the  $m$ -dimensional Euclidian space; the  $i$ th component of this vector being the amount of  $i$ th commodity used by the consumption unit. We suppose that each consumer has his individual complete order  $R_i$ , defined on the set of all consumption vectors. It may happen that  $xR_i y$  and  $yR_i x$  without  $y = x$ ; but if we define  $xI_i y$  if this does occur, this yields a proper equivalence relation, and the equivalence classes,  $S_i(x)$ , now form a linearly ordered set  $T_i$ . The satisfaction space  $\mathfrak{S}$  is introduced as the set of all vectors  $S = (S_1, S_2, \dots, S_n)$  where  $S_i \in T_i$ . In  $\mathfrak{S}$  a partial order is defined by

$$S^{(2)} = (S_1^{(2)}, S_2^{(2)}, \dots, S_n^{(2)}) \geq S^{(1)} = (S_1^{(1)}, S_2^{(1)}, \dots, S_n^{(1)})$$

if and only if  $S_i^{(2)}R_i S_i^{(1)}$  for every  $i = 1, 2, \dots, n$ .

The production activity of the system is represented by an input vector  $y = (y_1, y_2, \dots, y_m)$ , where  $y_i$  is the input (positive or negative) of the  $i$ th commodity. Technological conditions restrain  $y$  to belong to a set  $\eta$ . Let  $\eta^{\text{min}}$  be the set of efficient production vectors. A family of sets  $\eta_j, j = 1, 2, \dots, r$  is a decomposition of  $\eta$  if  $\eta = \sum \eta_j$ , the sum being in the sense of vector sums of sets.  $\eta_j$  characterizes activities of the  $j$ th production unit. If  $X_i = X_i(s_i^0)$  denote the  $\{x_i \mid s_i(x_i) \geq s_i^0\}$  the  $i$ th consumption unit,  $X = \sum X_i$  is the set of all total consumption vectors.  $z = x + y, x \in X, y \in \eta$  is the total net consumption of the whole economy. Let  $z^0$  be the vector whose components are the available amounts of each commodity. The economic system is constrained by

$$y \in \eta, \quad z \leq z^0 \quad \text{in the vector sense.}$$

The goal of the economic system is to find an  $S \in \mathfrak{S}$ , maximal in the partial order defined on  $\mathfrak{S}$ , which is consistent with the above constraints. We now assume all the sets  $X_i, \eta_j$  are *convex* and *closed*.

$$Z = \sum X_i + \sum \eta_j \quad \text{is then convex.}$$

Using the separation theorem for convex sets, namely, if two closed convex sets with interior points have only one point in common then there is at least one plane through that point separating the convex sets, one readily obtains the following result:

A necessary and sufficient condition for  $S^0$  to be maximal (or for  $z_0 = \sum x_i^0 + \sum y_j^0$  to be minimal) is the existence of a price vector  $p > 0$  and a set of numbers  $a_i, i = 1, 2, \dots, n$  so that

$\alpha)$   $x$ , being constrained by  $p'x \geq a_i, s_i(x_i)$  reaches its maximum at  $x_i^0$  for every  $i$

$\beta)$   $y_j$  being constrained by  $y_j \in \eta_j, p'y_j$  reaches its minimum at  $y_j^0$  for every  $j$ .

Here  $p$  is a positive, normal vector to the separating plane. The theorem restates the following rules of behavior for consumption and production units: each consumption unit, subject to a budgetary constraint maximizes its satisfaction and each production unit, subject to the technological constraints, maximizes its profit.

We leave these problems concerning convexity, maximization under constraints and related topics and turn to a phase of economics which employs completely different mathematical techniques.

Matrix theory, to some degree, enters into many portions of many diverse fields.

Matrices are used in these as a pure notational device, as a compact and transparent representation of systems of linear equations, and in many other subsidiary, convenient roles, without use being made of essential matrix theory. In economics, also, matrices make their appearance in this costume. But deep, significant, intrinsic results of the matrix theory do play an important part. In the recent literature one finds a number of research papers in economics which employ certain corners of matrix theory in the fullest. (To cite a few examples, we refer the reader to Chipman [9], Metzler [39], Goodwin [23], Simon and Hawkins [24] and Solow [48].)

The mathematical origin of a good many of these results is a sequence of papers by the German mathematician Frobenius [19, 20]. These results have been rederived and somewhat extended in a simple, modern fashion by Wielandt [53] and Debreu and Herstein [13]. In these above-mentioned papers, among other results, are obtained theorems concerning the nature of the characteristic roots of a non-negative matrix  $A$  and the properties of  $(sI - A)^{-1}$  for  $s$  a sufficiently large real number.

We now consider an economic situation in which these theorems function effectively. We follow here the treatment of a problem in the theory of international trade as given by Solow [48].

Suppose we consider  $n$  countries carrying the labels  $1, 2, \dots, n$ . Let  $a_{ij}$  denote the marginal propensity (that is, the increase in imports from country  $i$  by country  $j$  per unit increase in income of country  $j$ ) of the  $j$ th country to import from the  $i$ th country, and  $a_{i.}$  the marginal propensity to consume domestic goods. Furthermore, let  $x_i$  represent the national income of the  $i$ th country,  $c_i$  the autonomous expenditure in country  $i$ . Then we have, assuming linearity (which can be viewed as a first approximation for more general cases)

$$\text{i) } x_i = \sum_{j=1}^n a_{ij}x_j + c_i .$$

In matrix form this becomes

$$\text{ii) } (I - a)x = c,$$

where  $a = (a_{ij})$ ;  $x, c$  are column vectors. Economic meaningfulness demands that the quantities  $x_i, c_i$  be all nonnegative. We assume that all  $a_{ij} \geq 0$ .

Thus we are immediately forced to consider conditions under which the systems  $x = (I - a)^{-1}c$  is solvable in nonnegative terms.

The linear equation system (ii) is easily seen to be the static solution of the linear difference equation system

$$\text{iii) } Ix(t+1) - ax(t) = c.$$

A question of importance for this dynamic system will naturally concern the stability properties of its solutions. These can be shown to be equivalent to the condition that all the characteristic roots of  $a$  are less than 1 in absolute value. For this system Metzler [39] has proved: if  $a_{i.} \geq 0$  it is necessary and sufficient for the stability of (iii), that  $I - a$  have all its principal minors positive. (As a pure matrix theorem this is an easy consequence of the results of Frobenius, when we use that stability is equivalent to the condition that characteristic roots of  $a$  be less than 1 in absolute value.) These types of questions are treated in Section 3 of the paper by Debreu and Herstein [13].



Let us consider  $a = (a_{ij})$ , a nonnegative,  $n \times n$  matrix. A set,  $S$ , of indices will be said to be closed if  $a_{pq} = 0$  for  $q \in S$ ,  $p \notin S$ . That is, a closed set is associated with a collection of countries each of which spends in no country which is not in the collection. If no such proper closed set exists the matrix  $a$  is said to be indecomposable. This definition coincides with the purely mathematically motivated definition given by Frobenius [20], namely: the matrix  $a$  is said to be indecomposable if, for no permutation matrix  $\pi$ , the product  $\pi a \pi^{-1}$  can be represented in the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where the  $A_{i,i}$  are square submatrices. A matrix which is not indecomposable is called decomposable. If  $b$  is a decomposable matrix then there is a permutation matrix  $\pi$  so that

$$\pi b \pi^{-1} = \begin{pmatrix} B_1 & & & * \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_r \end{pmatrix}$$

where the  $B_i$  are square indecomposable submatrices on the diagonal.

In terms of the economic model, if the import matrix  $a$  is indecomposable, then a dollar spent in any one country will eventually induce spending in every other country.

Let us now assume that the import matrix  $a$  is indecomposable. From the results of Wielandt [53] or Solow [48] we can find a permutation  $\pi$  which puts  $a$  into the form

$$\pi a \pi^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & G_w \\ G_1 & 0 & \cdots & 0 & 0 \\ 0 & G_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & G_{w-1} & 0 \end{pmatrix},$$

where the 0's on the diagonal are square matrices. Economically this can be interpreted as: the countries in  $G_i$  spend only in those in  $G_{i+1}$ .

From the Frobenius theorems the following main result can be extracted: "if  $a$  is a nonnegative, indecomposable matrix then  $a$  has a positive characteristic root  $r$  so that

- 1) If  $\alpha$  is any other characteristic root of  $a$  then  $|\alpha| \leq r$ ;
- 2) to  $r$  can be associated a positive characteristic vector;
- 3)  $r$  is a simple root;
- 4) an increase of any element of  $a$  yields an increase in  $r$ .

Moreover if there should be exactly  $k$  roots,  $\alpha_\lambda$ , with  $|\alpha_\lambda| = r$  then  $\alpha_\lambda = r \exp(2\pi\lambda/k)$ ,

$\lambda = 1, 2, \dots, k$  and a permutation  $\pi$  exists so that

$$\pi a \pi^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & A_k \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{k-1} & 0 \end{pmatrix}.$$

Solow's  $w$ , which had a purely economic origin now can be interpreted as  $k$ , the number of characteristic roots of largest absolute value. After  $w$  time units have elapsed, the original expenditures from  $G_i$  in  $G_{i+1}$  has an influence in  $G_i$ , giving rise to a cyclicity of spending.

The above discussion encroaches on a tiny part of this territory, which has already been staked out by the economist. However, it is typical in the sense that the theorems of Frobenius play such a central role.

We now concern ourselves with several questions which have a combinatorial flavor to them. While these are not all phrased purely in economic terms, they have, or should have, many applications in economics.

The first of these is the so-called "personnel assignment" problem. Suppose that there are  $n$  individuals available to fill  $n$  positions. Suppose further that the  $i$ th individual obtains a rating of  $a_{ij}$  in his ability to handle the  $j$ th job. The question then is: how shall individuals be assigned to jobs in order to have the "overall efficiency" a maximum?

The problem can be put into purely mathematical garb in following vein: given a matrix  $A$ , for what permutation matrix  $P$  is the trace of  $AP$  a maximum? Of course this would merely require testing  $n!$  possible permutations. Even for  $n$  relatively small this straightforward procedure would be out of the realm of practical feasibility.

The problem then becomes one of reducing the number of necessary computational steps. This can be achieved by (at least) two continuizations of the problem. We shall describe one of these, due to von Neumann [41] in very little detail, later in the paper. But before doing so, we should like to describe several related combinatorial questions.

A situation very similar to that in the personnel assignment problem is the "desk-cabinet" problem. This runs as follows: Suppose that there are  $n$  desks and  $n$  filing cabinets. Let  $d_{ij}$  be the distance from the  $i$ th desk to the  $j$ th cabinet. On the assumption that the individuals assigned to each desk make the same number of trips per day to their respective filing cabinets, what assignment of desks to cabinets should be made to insure that the total distance walked is a minimum. Formulated in mathematical terms, the problem is that of finding a permutation matrix  $P_0$  which minimizes the trace of  $DP$  where  $D$  is the matrix of distances.

A purely economic variant of this can be phrased as follows: minimize the cost of production of  $n$  plants at  $n$  locations if  $d_{ij}$  is the cost of production for the  $i$ th plant at the  $j$ th location.

A generalization of the two above-mentioned situations, arising fairly naturally in economics, involves finding a permutation  $P_0$  which minimizes or maximizes the trace of  $DP$  where  $P$  is no longer left free to roam over the whole symmetric group but is

restricted to range only over a subset thereof. In an actual problem this might be realized, for instance, by the condition that in any reshuffling of locations, railway depots would be restricted to occupy space only along railroad lines.

A solution to this problem would immediately yield a solution to another one, the "travelling salesman" problem. The travelling salesman problem, verbally described is the following: suppose a travelling salesman on his route must visit  $n$  cities and return to his home base which is one of these cities. If he must visit each of these cities once, and if he knows the distances between all of them, how should he plan his trip so that the distance he travels is minimal. This can be shown to be equivalent to minimizing the trace of  $CP$  where  $P$  is restricted to be a permutation which in its cycle decomposition is representable as an  $n$  cycle.

This relative of the Hamiltonian game can also be exhibited as a special case of a purely economic consideration due to Beckmann. Let there be  $n$  plants, and  $n$  fixed locations where these plants could be situated. Suppose  $a_{ij}$  represents the flow from the  $i$ th plant to the  $j$ th plant. Suppose also that  $k_{ij}$  is the cost of transportation from location  $i$  to location  $j$ . Then how should the plants be located in order that the total transportation costs among plants are a minimum. If  $A = (a_{ij})$  and  $K = (k_{ij})$  and if  $A'$  denotes the transpose, this simply becomes a question of finding a permutation  $P_0$  which minimizes the trace of  $A'P_0KP$ .

We return to the personnel assignment problem, and present a sketch of von Neumann's game theory approach to it [41].

Consider the following two person game: we have an  $n \times n$  checker board, with each square having two indices, its row index and its column index. The first player picks a square; the second player then guesses either of the indices of the square which the first player has picked. He must state which index he is guessing. If he guesses correctly he receives an amount  $\alpha_{ij}$ , where  $i, j$  are the indices of the square involved, from the first player. Otherwise he receives 0.

This game is related to the solution of the personnel assignment problem via the following theorem proved by von Neumann, (where the strategies refer to those of the first player).

"The extreme optimal strategies of the above game are precisely the following ones:

Consider those permutations  $P_0$  which maximize the trace of  $AP$ , where  $A = (1/\alpha_{ij})$ . For each  $P_0$  assign the probability  $x_{ij}$  to the square  $i, j$  where  $x_{ij} = (a/\alpha_{ij})\delta_{P_0(i), j}$  where  $\delta$  is the Kronecker delta; and where  $a$  is the value of the game for the second player.

Using techniques of Brown and von Neumann [7] to get approximate solutions of games, the number of steps in solving the problem is reduced from  $n!$  to a power of  $n$ .

Let  $z = (z_{ij})$  be a vector in the  $n^2$  Euclidian space. We define:

$$R = \{z = (z_{ij}) \mid z_{ij} \geq 0, \quad \sum_j z_{ij} = 1, \quad \sum_i z_{ij} = 1\},$$

$$S = \{z = (z_{ij}) \mid z_{ij} \geq 0, \quad \sum_j z_{ij} \leq 1, \quad \sum_i z_{ij} \leq 1\},$$

$$P = \{z = (z_{ij}) \mid z_{ki} = \delta_{P(i), j}, \quad j \text{ for some permutation } P\}.$$

Von Neumann's proof then hinges on the following two lemmas,

$$1) \quad \{S = \{z \mid z \leq w \text{ for some } w \in R\},$$

where  $z \leq w$  means the inequality is true in each component;

2)  $R =$  convex hull of  $P$ .

The reader will notice that certain very important current research in economics have scarcely been mentioned. For instance nothing has been said about the input-output models of Leontief or the applications of linear programming to economics. We felt that to do justice to these two would require much more space than would be appropriate for an article such as this. The reader is referred to Leontief's book, *The Structure of the American Economy*, [36] for the first topic, the book, *Application of Linear Programming to the Theory of the Firm*, by Dorfman, [14] for the second topic, and the book, *Activity Analysis of Production and Allocation* [29] for both.

With this discussion of the combinatorial problems we conclude the phase of the paper of "detailed" discussion of the applications of mathematics to economics. However, before concluding we wish to point out briefly some sources where one can find fine applications of other mathematical techniques.

The theories of differential equations and of difference equations play fundamental parts, in many connections, in economic theories. Chapter X of Samuelson's book [45] gives splendid illustrations of the use of techniques from these regions of mathematics.

Beckmann [5] has employed the classical theory of the calculus of variations in his study of continuous models of transportation. Earlier applications of the calculus of variation occur in papers by Hotelling [27] and Roos [44] and Evans [17].

Stone has used the theory of graphs in his economic studies. An example of this is [49]. The theory of graphs has also entered into the considerations of Koopmans and Reiter [31] of their transportation model. Charnes [8] has utilized the graph theory to extend certain computational techniques.

Following the trend towards axiomatization in mathematics there have been some purely axiomatic studies of economic questions; a pioneer effort in this direction is a postulational study of utility by Frisch [18]. Another example is the study of the existence of a social welfare function by Arrow [1] and Hildreth [26].

Many other applications of the theory of convex sets can be found than those already described in this paper. The theory of convex polyhedral cones was first developed by Weyl [52], and detailed mathematical investigations of these cones were carried out by Gerstenhaber [22]. These results, and many others, have a variety of applications, as the reader can see by looking at Koopmans [30], Georgescu-Roegen [21], Samuelson [46], and Arrow [2] in the monograph *Activity Analysis of Production and Allocation*. [29].

It is well known how the mathematician's interest in physical and astronomical problems have led to advances in mathematics itself. We cite here an example of a similar advance in mathematics which had as its origin a pure economic motivation. Not surprisingly, these new mathematical advances brought about have themselves stimulated the economic problem from which they sprang.

The Menger seminar in Vienna on mathematical economics, amongst many other topics, concerned itself with that of the existence of equilibrium. This led to the paper by A. Wald [51]. Soon after von Neumann [42] in proving the existence of an equilibrium point for an economic system, found it necessary to extend the Brouwer fixed point theorem. Kakutani [28] then simplified von Neumann's proof and cast the theorem in a somewhat different light. This led to an even more powerful topological fixed point theorem by Eilenberg and Montgomery [16]. Begle [6] took up from there and gave a

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very general fixed point theorem which subsumed that of Eilenberg and Montgomery. To complete the cycle, Arrow and Debreu [4, 12] have applied those theorems to obtain rigorous proofs of the existence of equilibrium points for fairly general economic systems.

In conclusion, the author should like to offer an apology to those economists whose contributions to the field may have been overlooked in this paper. The limitations of space, as well as the author's incomplete knowledge of the field, make such omissions unavoidable. An expository paper of this type cannot aim at complete coverage; rather, the author tried to convey an idea of the range and variety of mathematical results which have found applications in economics.

#### REFERENCES

- [1] Arrow, Kenneth J., *Social Choice and Individual Values*, Cowles Commission for Research in Economics, Monograph 12. Wiley, 1951.
- [2] ———, "Alternative Proof of the Substitution Theorem for Leontief Models in the General Case" in [29], pp. 155-165.
- [3] ———, "Mathematical Models in the Social Sciences" in "the Policy Sciences," edited by Daniel Lerner and Harold Lasswell, Stanford University Press, pp. 129-155.
- [4] Arrow, Kenneth J. and Debreu, Gerard, "Existence of an Equilibrium for a Competitive Economy." Forthcoming.
- [5] Beckmann, Martin, "A Continuous Model of Transportation," *Econometrica* 20 (1952) 643-661.
- [6] Begle, E. G., "A Fixed Point Theorem," *Annals of Mathematics* 51, (1950) 544-550.
- [7] Brown, G. W. and von Neumann, J., "Solutions of Games by Differential Equations," in [33], pp. 73-79.
- [8] Charnes, A., "Optimality and Degeneracy in Linear Programming," *Econometrica* 20 (1952) 160-171.
- [9] Chipman, John, "The Multi-Sector Multiplier," *Econometrica* 18 (1950) 387-395.
- [10] Debreu, Gerard, "Real Representation of a Preference Ordering," (Forthcoming).
- [11] ———, "The Coefficient of Resource Utilization," *Econometrica* 19 (1951) 273-291.
- [12] ———, "A Social Equilibrium Existence Theorem," *Proc. Nat. Ac.* 38 (1952) 886-893.
- [13] Debreu, Gerard and Herstein, I. N., "Nonnegative Square Matrices," Cowles Commission Discussion Paper, Mathematics 417, and forthcoming in *Econometrica*.
- [14] Dorfman, Robert, *Application of Linear Programming to the Theory of the Firm*, University of California Press, 1951.
- [15] Eilenberg, Samuel, "Ordered Topological Spaces," *Amer. Journal Math.* 43 (1941) 39-46.
- [16] Eilenberg, Samuel, and Montgomery, Deane, "Fixed Point Theorems for Multi-valued Transformations," *Amer. Journal Math.* 68 (1946) 214-222.
- [17] Evans, G. C., "The Dynamics of Monopoly," *Amer. Math. Monthly* 31 (1924), pp. 77-83.
- [18] Frisch, Ragnar, "Sur un Problème d'Economie Pure," *Norsk Mathematisk Forenings Skrifter* 1 (1926) 1-40.
- [19] Frobenius, G., "Über Matrizen aus Positiven Elementen," *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, (1908), pp. 471-476.
- [20] ———, "Über Matrizen aus nicht-negativen Elementen," *Sitzungsberichte Preuss. Ak. Wiss.* (1912) 456-477.
- [21] Georgescu-Roegen, Nicholas, "Some Properties of a Generalized Leontief Model," in [29], pp. 165-173.
- [22] Gerstenhaber, Murray, "Theory of Convex Polyhedral Cones," in [29], pp. 298-317.
- [23] Goodwin, Richard M., "The Multiplier as a Matrix," *Economic Journal* 59 (1949) 537-555.
- [24] Hawkins, D. and Simon, H. A., "Some Conditions of Macroeconomic Stability," *Econometrica* 17 (1949) 245-248.
- [25] Herstein, I. N. and Milnor, John, "An Axiomatic Approach to Measurable Utility," forthcoming in *Econometrica*.
- [26] Hildreth, Clifford, "Alternative Conditions for Social Ordering," *Econometrica* 21 (1953) 81-94.
- [27] Hotelling, H. "The Economics of Exhaustible Research," *Journal Political Economy* 39 (1931) 137-176.

- [28] Kakutani, S., "A. Generalization of Brouwer's Fixed Point Theorem," *Duke Math. Journal* **8** (1941) 457-459.
- [29] Koopmans, Tjalling C., *Activity Analysis of Production and Allocation*, Cowles Commission for Research in Economics, Monograph 13, Wiley, 1951.
- [30] ———, "Analysis of Production" in [29], pp. 33-97.
- [31] ———, "Alternative Proof of the Substitution Theorem for Leontief Models in the Case of Three Industries," [29], pp. 147-155.
- [32] Koopmans, Tjalling C., and Reiter, Stanley, "A Model of Transportation," in [29], pp. 222-260.
- [33] Kuhn, H. W. and Tucker, A. W., *Contributions to the Theory of Games*, Annals of Mathematics Study, No. 24, Princeton, 1950.
- [34] ———, "Nonlinear Programming" in *Proceedings of the Second Berkeley Symposium of Mathematical Statistics and Probability*, Berkeley, 1951.
- [35] Lange, Oscar, "The Foundations of Welfare Economics" *Econometrica*, **10** (1942) 215-228.
- [36] Leontief, Wassily, *The Structure of American Economy, 1919-1939*; Second Edition, N. Y., Oxford University Press, 1951.
- [37] Marschak, Jacob, "Rational Behavior, Uncertain Prospects and Measurable Utility," *Econometrica* **18** (1905) 111-141.
- [38] McKinsey, J. C. C., *Introduction to the Theory of Games, RAND Series*, McGraw-Hill, 1952.
- [39] Metzler, Lloyd, "A Multiple Region Theory of Income and Trade," *Econometrica* **18** (1950) 329-354.
- [40] Morgenstern, Oskar and Von Neumann, John, *Theory of Games and Economic Behavior*.
- [41] Von Neumann, John, "The Personnel Assignment Problem and a Certain Two Person Game," to appear in *Contributions to the Theory of Games, Annals of Mathematics Studies*.
- [42] ———, "Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eines Mathematischen Kolloquiums* **1** (1937) 73-83.
- [43] Pareto, V., "Anwendung der Mathematik auf Nationalökonomie," *Encyklopadie Math. Wiss.* (Arithmetik und Algebra), Zweiter Teil, Leipzig, 1900-1904, pp. 1094-1120.
- [44] Roos, C. F., "A Mathematical Theory of Competition," *Amer. Journal Math.* **67** (1925) 163-175.
- [45] Samuelson, Paul A., *Foundations of Economic Analysis*, Harvard, 1947.
- [46] ———, "Abstract of a Theorem Concerning Substitutability in Open Leontief Models," in [29], pp. 142-147.
- [47] Slutsky, E., "Sulla Teoria de Bilancio del Consumatore," *Giornale Economisti* **60** (1915) 19-23.
- [48] Solow, Robert, "On the Structure of Linear Models," *Econometrica* **20** (1952) 29-47
- [49] Stone, Richard, "Comptabilité Social, Agrégation, Et Invariance," *Economie Appliquee* **59** (1949) 26-54.
- [50] Wald, Abraham, *Statistical Decision Function*, Wiley, 1950.
- [51] ———, "Über einige Gleichungssysteme der mathematischen Ökonomie," *Zeitsch. Nationalökonomie* **7** (1936) 637-670. Translated in *Econometrica* **29** (1951) 368-403.
- [52] Weyl, Herman, "Elementary Theory of Convex Polyhedra," in [33], pp. 3-19, [translated by H. W. Kuhn from *Commentarii Math. Helvet.* **8** (1935) 290-306.]
- [53] Weilandt, Helmut, "Unzerlegbare, nicht-negative Matrizen," *Math. Zeitschr.* **52** (1950) 642-648.
- [54] Wolf, Herman, "A Synthesis of Pure Demand Analysis," Part II, *Skand. Aktuarietidskrift* **26** (1943) 220-263.