

**Session 4:
loglinear
regression
part 1**

Levi Waldron

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outline

Brief review
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Session 4: loglinear regression part 1

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CUNY SPH Biostatistics 2

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Learning objectives and outline

Learning objectives

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- 1 Define log-linear models in GLM framework
- 2 Identify situations that motivate use of log-linear models
- 3 Define the Poisson distribution and the log-linear Poisson GLM
- 4 Identify applications and properties of the Poisson distribution
- 5 Define multicollinearity and identify resulting issues

Outline

- 1 Brief review of GLMs
- 2 Motivating example for log-linear models
- 3 Poisson log-linear GLM
- 4 Notes on Multicollinearity

Reading: Vittinghoff textbook chapter 8.1-8.3

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- **Random component** specifies the conditional distribution for the response variable - it doesn't have to be normal but can be any distribution that belongs to the "exponential" family of distributions
- **Systematic component** specifies linear function of predictors (linear predictor)
- **Link** [denoted by $g(\cdot)$] specifies the relationship between the expected value of the random component and the systematic component, can be linear or nonlinear

Linear Regression as GLM

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- **The model:**

$$y_i = E[y|x] + \epsilon_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi} + \epsilon_i$$

- **Random component** of y_i is normally distributed:

$$\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$$

- **Systematic component** (linear predictor):

$$\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$$

- **Link function** here is the *identity link*:

$g(E(y|x)) = E(y|x)$. We are modeling the mean directly,
no transformation.

Logistic Regression as GLM

- **The model:**

$$\text{Logit}(P(x)) = \log \left(\frac{P(x)}{1 - P(x)} \right) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$$

- **Random component:** y_i follows a Binomial distribution (outcome is a binary variable)
- **Systematic component:** linear predictor

$$\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$$

- **Link function:** *logit* (Converts Prob \rightarrow log-odds)

$$g(P(x)) = \text{logit}(P(x)) = \log \left(\frac{P(x)}{1 - P(x)} \right)$$

$$P(x) = g^{-1}(\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi})$$

Additive vs. Multiplicative models

- Linear regression is an *additive* model
 - e.g. for two binary variables $\beta_1 = 1.5$, $\beta_2 = 1.5$.
 - If $x_1 = 1$ and $x_2 = 1$, this adds 3.0 to $E(y|x)$
- Logistic regression is a *multiplicative* model
 - If $x_1 = 1$ and $x_2 = 1$, this adds 3.0 to $\log(\frac{P}{1-P})$
 - Odds-ratio $\frac{P}{1-P}$ increases 20-fold: $\exp(1.5 + 1.5)$ or $\exp(1.5) * \exp(1.5)$

Motivating example for log-linear models

Effectiveness of a depression case-management program

- Research question: can a new treatment reduce the number of needed visits to the emergency room, compared to standard care?
- *outcome*: # of emergency room visits for each patient in the year following initial treatment
- *predictors*:
 - *race* (white or nonwhite)
 - *treatment* (treated or control)
 - *amount of alcohol consumption* (numerical measure)
 - *drug use* (numerical measure)

Statistical issues

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- 1 about 1/3 of observations are exactly 0 (did not return to the emergency room within the year)
- 2 highly nonnormal and cannot be transformed to be approximately normal
- 3 even $\log(y_i + 1)$ transformation will have a “lump” at zero + over 1/2 the transformed data would have values of 0 or $\log(2)$
- 4 a linear regression model would give negative predictions for some covariate combinations
- 5 some subjects die or cannot be followed up on for a whole year

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Towards a reasonable model

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- A *multiplicative* model will allow us to make inference on *ratios* of mean emergency room usage
- Modeling *log* of the *mean* emergency usage ensures positive means, and does not suffer from $\log(0)$ problem
- Random component of GLM, or residuals (was $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$ for linear regression) may still not be normal, but we can choose from other distributions

Proposed model without time

$$\log(E[Y_i]) = \beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i$$

Or equivalently:

$$E[Y_i] = \exp(\beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i)$$

where $E[Y_i]$ is the expected number of emergency room visits for patient i .

- Important note: Modeling $\log(E[Y_i])$ is *not* equivalent to modeling $E(\log(Y_i))$

Accounting for follow-up time

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Instead, model mean count per unit time:

$$\log(E[Y_i]/t_i) = \beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i$$

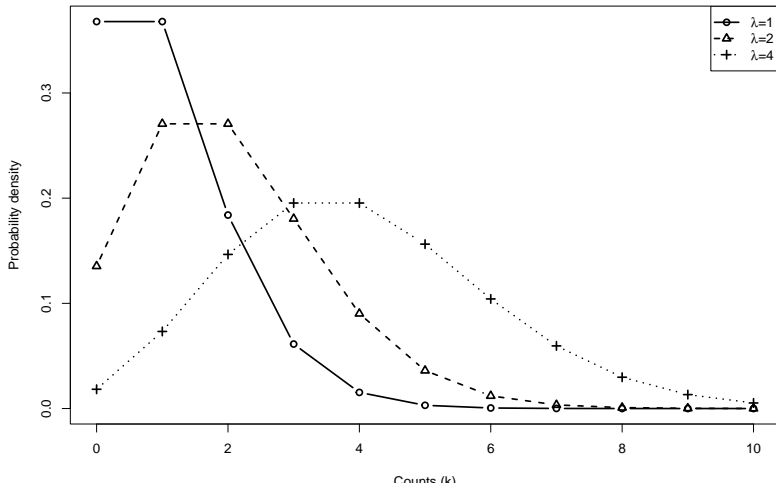
Or equivalently:

$$\log(E[Y_i]) = \beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i + \log(t_i)$$

- $\log(t_i)$ is not a covariate, it is called an *offset*

The Poisson distribution

- Count data are often modeled as Poisson distributed:
 - mean λ is greater than 0
 - variance is also λ
 - Probability density $P(k, \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$



When the Poisson distribution works

- Individual events are low-probability (small p), but many opportunities (large n)
 - e.g. # 911 calls per day
 - e.g. # emergency room visits
- Approximates the binomial distribution when n is large and p is small
 - e.g. $n > 20$, $np < 5$ or $n(1 - p) < 5$
- When mean of residuals is approx. equal to variance

GLM with log-linear link and Poisson error model

- Model the number of counts per unit time as Poisson-distributed + so the expected number of counts per time is λ_i

$$E[Y_i]/t_i = \lambda_i$$

$$\log(E[Y_i]/t_i) = \log(\lambda_i)$$

$$\log(E[Y_i]) = \log(\lambda_i) + \log(t_i)$$

Recalling the log-linear model systematic component:

$$\log(E[Y_i]) = \beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i + \log(t_i)$$

GLM with log-linear link and Poisson error model (cont'd)

Then the systematic part of the GLM is:

$$\log(\lambda_i) = \beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i$$

Or alternatively:

$$\lambda_i = \exp(\beta_0 + \beta_1 \text{RACE}_i + \beta_2 \text{TRT}_i + \beta_3 \text{ALCH}_i + \beta_4 \text{DRUG}_i)$$

Interpretation of coefficients

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- Suppose that $\hat{\beta}_1 = -0.5$ in the fitted model, where $RACE_i = 0$ for white and $RACE_i = 1$ for non-white.
- The mean rate of emergency room visits per unit time for white relative to non-white, all else held equal, is estimated to be:

$$\begin{aligned} & \frac{\exp(\beta_0 + 0 + \beta_2 TRT_i + \beta_3 ALCH_i + \beta_4 DRUG_i)}{\exp(\beta_0 - 0.5 + \beta_2 TRT_i + \beta_3 ALCH_i + \beta_4 DRUG_i)} \\ &= \frac{e^{\beta_0} e^0 e^{\beta_2 TRT_i} e^{\beta_3 ALCH_i} e^{\beta_4 DRUG_i}}{e^{\beta_0} e^{-0.5} e^{\beta_2 TRT_i} e^{\beta_3 ALCH_i} e^{\beta_4 DRUG_i}} \\ &= \frac{e^0}{e^{-0.5}} \\ &= e^{0.5} \approx 1.65 \end{aligned}$$

Interpretation of coefficients (cont'd)

- If $\hat{\beta}_1 = -0.5$ with whites as the reference group:
 - after adjustment for treatment group, alcohol and drug usage, whites tend to use the emergency room at a rate 1.65 times higher than non-whites.
 - equivalently, the average rate of usage for whites is 65% higher than that for non-whites
- Multiplicative rules apply for other coefficients as well, because they are exponentiated to estimate the mean rate.

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Multi-collinearity

What is Multicollinearity?

- 1 *Multicollinearity* exists when two or more of the independent variables in regression are moderately or highly correlated.
- 2 High correlation among continuous predictors or high concordance among categorical predictors
- 3 Impacts the ability to estimate regression coefficients
 - larger standard errors for regression coefficients
 - ie, coefficients are unstable over repeated sampling
 - exact collinearity produces infinite standard errors on coefficients
- 4 Can also result in unstable (high variance) prediction models

Identifying multicollinearity

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- 1 Pairwise correlations of data or of model matrix (latter works with categorical variables)
- 2 Heat maps
- 3 Variance Inflation Factor (VIF) of regression coefficients

Example: US Judge Ratings dataset

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See ?USJudgeRatings for dataset, ?pairs for plot code:

```
## Warning in par(usr): argument 1 does not name a gr
```

```
## Warning in par(usr): argument 1 does not name a gr
```

```
## Warning in par(usr): argument 1 does not name a gr
```

```
## Warning in par(usr): argument 1 does not name a gr
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## Warning in par(usr): argument 1 does not name a gr
```

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## Warning in par(usr): argument 1 does not name a gr
```

```
## Warning in par(usr): argument 1 does not name a gr
```

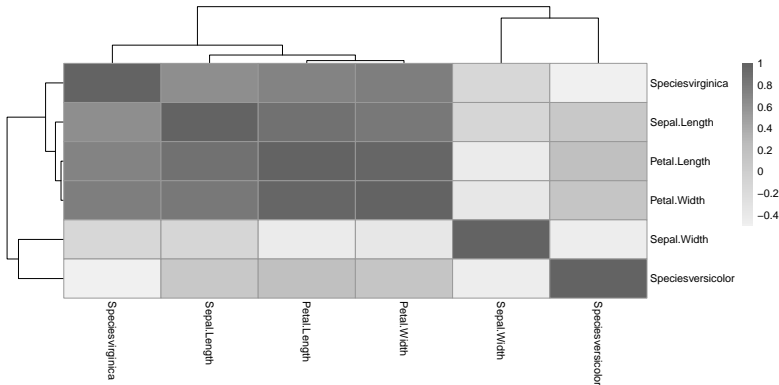
```
## Warning in par(usr): argument 1 does not name a gr
```

Example: iris dataset

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One categorical variable, so use model matrix. Make a simple heatmap.

```
mm <- model.matrix( ~ ., data = iris)
pheatmap::pheatmap(cor(mm[, -1]), #-1 gets rid of intercept column
  color = colorRampPalette(c("#f0f0f0", "#bdbdbd", "#636363"))(100))
```



Note: multicollinearity exists between multiple predictors, not between predictor and outcome

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Confirm what in iris dataset using Variance Inflation Factor of a linear regression model:

```
fit <- lm(Sepal.Width ~ ., data = iris)
car::vif(fit)
```

```
##                GVIF Df GVIF^(1/(2*Df))
## Sepal.Length  6.124653  1      2.474804
## Petal.Length 45.132550  1      6.718076
## Petal.Width  18.373804  1      4.286468
## Species      32.701564  2      2.391344
```

Approaches for dealing with multicollinearity

Options:

- 1 Select a representative variable
- 2 Average variables
- 3 Principal Component Analysis or other dimension reduction
- 4 For prediction modeling, special methods like penalized regression, Support Vector Machines, . . .

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- 1 Log-linear models are appropriate for non-negative, skewed count data
 - probability of each event is low
- 2 The coefficients of log-linear models are *multiplicative*
- 3 An *offset* term can account for varying follow-up time or otherwise varying opportunity to be counted
- 4 Poisson distribution is limit of binomial distribution with high number of trials, low probability
- 5 Inference from log-linear models is sensitive to the choice of error model (assumption on the distribution of residuals)
- 6 We will cover other options next week for when the Poisson error model doesn't fit:
 - Variance proportional to mean, instead of equal
 - Negative Binomial
 - Zero Inflation