# The Klein four-group

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We describe alternative ways to present the famous Klein four-group

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## 1

It is well-known that the Klein four-group, or Klein group in short, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the addition modulo 2 as the group multiplication. Please see [1]. Here we show further possibilities to present the Klein four-group. The last is new, as far we know. See [2] in the internet. We define the group $G := (\{1, -1\}, \cdot)$ with the integers 1 and $-1$, and '$\cdot$' is the ordinary multiplication. We take the sets $G \times G$ and $G \times G \times G$ and we multiply componentwise. Note that the orders of all elements which we deal with are two, except the order of the neutral element.

The Klein four-group is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ and to $(G \times G, \cdot)$.

| '+' | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|
| $(0,0)$ | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
| $(0,1)$ | $(0,1)$ | $(0,0)$ | $(1,1)$ | $(1,0)$ |
| $(1,0)$ | $(1,0)$ | $(1,1)$ | $(0,0)$ | $(0,1)$ |
| $(1,1)$ | $(1,1)$ | $(1,0)$ | $(0,1)$ | $(0,0)$ |

$\cong$

| '$\cdot$' | $(1,1)$ | $(1,-1)$ | $(-1,1)$ | $(-1,-1)$ |
|---|---|---|---|---|
| $(1,1)$ | $(1,1)$ | $(1,-1)$ | $(-1,1)$ | $(-1,-1)$ |
| $(1,-1)$ | $(1,-1)$ | $(1,1)$ | $(-1,-1)$ | $(-1,1)$ |
| $(-1,1)$ | $(-1,1)$ | $(-1,-1)$ | $(1,1)$ | $(1,-1)$ |
| $(-1,-1)$ | $(-1,-1)$ | $(-1,1)$ | $(1,-1)$ | $(1,1)$ |

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It follows the group $(G\times G\times G,\cdot)$. It consists of 8 elements, and their operations are given in the following way. We omit the multiplication with the neutral element $(1,1,1)$ due to the lack of space. There are 7 subgroups of $(G\times G\times G,\cdot)$ isomorphic to the Klein group. The Klein four-group is generated by four elements $(1,1,1),(-1,-1,1),(-1,1,-1)$ and $(1,-1,-1)$, and also by $(1,1,1)$, $(-1,-1,1)$, $(-1,1,1)$, $(1,-1,1)$ and by $(1,1,1)$, $(-1,1,-1)$, $(-1,1,1)$, $(1,1,-1)$, and also by $(1,1,1)$, $(1,-1,-1)$, $(1,-1,1)$, $(1,1,-1)$, respectively. Further the Klein group is generated by the elements $(1,1,1)$, $(-1,-1,-1)$, $(1,-1,-1)$, $(-1,1,1)$, and by $(1,1,1)$, $(-1,-1,-1)$, $(1,-1,1)$, $(-1,1,-1)$, and by $(1,1,1)$, $(-1,-1,-1)$, $(1,1,-1)$, $(-1,-1,1)$, respectively.

| '$\cdot$' | $(-1,-1,1)$ | $(-1,1,-1)$ | $(1,-1,-1)$ | $(-1,1,1)$ | $(1,-1,1)$ | $(1,1,-1)$ | $(-1,-1,-1)$ |
|---|---|---|---|---|---|---|---|
| $(-1,-1,1)$ | $(1,1,1)$ | $(1,-1,-1)$ | $(-1,1,-1)$ | $(1,-1,1)$ | $(-1,1,1)$ | $(-1,-1,-1)$ | $(1,1,-1)$ |
| $(-1,1,-1)$ | $(1,-1,-1)$ | $(1,1,1)$ | $(-1,-1,1)$ | $(1,1,-1)$ | $(-1,-1,-1)$ | $(-1,1,1)$ | $(1,-1,1)$ |
| $(1,-1,-1)$ | $(-1,1,-1)$ | $(-1,-1,1)$ | $(1,1,1)$ | $(-1,-1,-1)$ | $(1,1,-1)$ | $(1,-1,1)$ | $(-1,1,1)$ |
| $(-1,1,1)$ | $(1,-1,1)$ | $(1,1,-1)$ | $(-1,-1,-1)$ | $(1,1,1)$ | $(-1,-1,1)$ | $(-1,1,-1)$ | $(1,-1,-1)$ |
| $(1,-1,1)$ | $(-1,1,1)$ | $(-1,-1,-1)$ | $(1,1,-1)$ | $(-1,-1,1)$ | $(1,1,1)$ | $(1,-1,-1)$ | $(-1,1,-1)$ |
| $(1,1,-1)$ | $(-1,-1,-1)$ | $(-1,1,1)$ | $(1,-1,1)$ | $(-1,1,-1)$ | $(1,-1,-1)$ | $(1,1,1)$ | $(-1,-1,1)$ |
| $(-1,-1,-1)$ | $(1,1,-1)$ | $(1,-1,1)$ | $(-1,1,1)$ | $(1,-1-1)$ | $(-1,1,-1)$ | $(-1,-1,1)$ | $(1,1,1)$ |

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There are 35 subgroups of $(G\times G\times G\times G,\cdot)$ isomorphic to the Klein group, due to the following proposition. Correspondingly there are 'a lot' of subgroups of $(G^n,\cdot) := (G\times G\times G\times G\times\ldots\times G,\cdot)$ isomorphic to the Klein group, where '$\cdot$' means multiplication of components. Let us abbreviate $A(n)$ for that number. We have $A(1)=0,\ A(2)=1,\ A(3)=7,\ A(4)=35$.

**Proposition 1.1.** *Let $n$ be a natural number, $n>2$. There are at least $4\cdot\binom{n}{3}+\binom{n}{2}$ subgroups of $(G^n,\cdot)$ isomorphic to the Klein group. There are exactly $A(n)=\frac{1}{3}\cdot\binom{2^n-1}{2}=\frac{(2^n-1)\cdot(2^{n-1}-1)}{3}$ subgroups of $(G^n,\cdot)$ isomorphic to the Klein group.*

*Proof.* In an element of $(G^n,\cdot)$ are $n$ positions. We choose three or two or three positions, respectively. From this we build either $\binom{n}{3}$ or $\binom{n}{2}$ or $3\cdot\binom{n}{3}$ Klein groups, respectively, as the following examples show. We fix $n=4$. In the first example we choose position two, three and four. We fill three quadruples with two '$-1$' at these positions. In the next example we choose the positions two and four. In the third example we choose the positions one, two and three. We fill them with '$-1$'. We construct three Klein groups. Note that we omit always the neutral element $(1,1,1,1)$.
$(1,-1,1,-1)$, $(1,-1,-1,1)$, $(1,1,-1,-1)$ and $(1,-1,1,-1)$, $(1,-1,1,1)$, $(1,1,1,-1)$, and last but not least $(-1,-1,-1,1)$, $(-1,1,1,1)$, $(1,-1,-1,1)$, respectively.
We prove the exact formula. In the group $(G^n,\cdot)$ are $2^n$ elements. This means there are $2^n-1$ elements which are not the neutral element $e:=(1,1,1,\ldots,1,1)$. Two elements of the set $\{a,b\in G^n \mid a,b\neq e, a\neq b\}$ generate a Klein group by four elements $\{e,a,b,a\cdot b\}$. Two of these elements generate three times the same group. ☐

We get $A(5)\geq 4\cdot\binom{5}{3}+\binom{5}{2}=4\cdot 10+10=50,\ A(6)\geq 4\cdot\binom{6}{3}+\binom{6}{2}=4\cdot 20+15=95$. We have $A(3)=7,\ A(4)=35,\ A(5)=155,\ A(6)=651$.

**Proposition 1.2.** *Every commutative finite group where all elements have the order one or two is isomorphic to some group $(G\times G\times G\times\ldots\times G,\cdot)$.*

*Proof.* Let's take a finite abelian group $Ab$ with the above conditions. By the fundamental theorem of finite abelian groups there is a number $n$ such that $Ab$ is isomorphic to the group $((\mathbb{Z}_2)^n,+)$, since the elements of $Ab$ have orders less or equal two. Since $(G,\cdot)$ is isomorphic to $(\mathbb{Z}_2,+)$ it follows $Ab\cong(G^n,\cdot)$. ☐

## References

[1] Siegfried Bosch: *Algebra* Springer 2004

[2] https://groupprops.subwiki.org/wiki/Klein_four-group

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