Canonical Polyhedra

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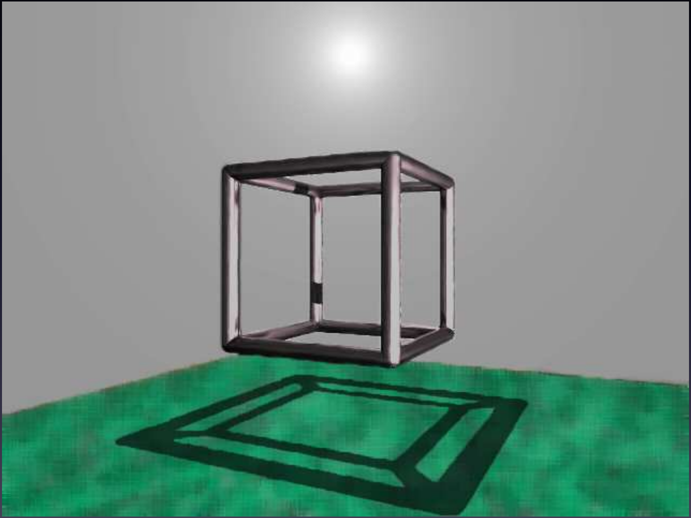
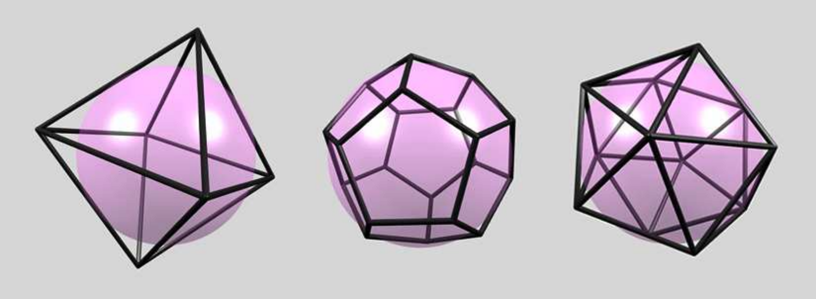
Polyhedral graphs



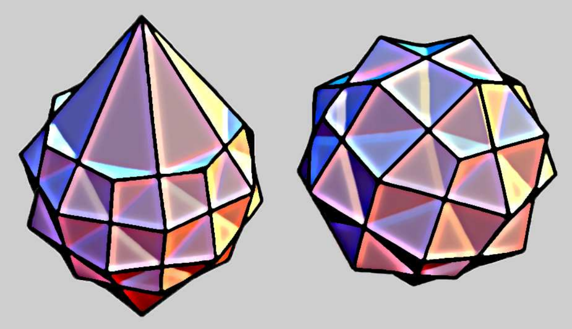
Canonical polyhedron

1. All edges are tangent to the unit sphere
2. The tangent points have a centre of mass at the centre of the sphere



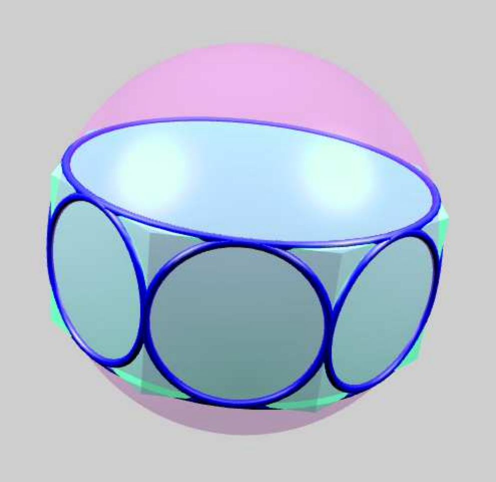
Koebe polyhedron

1. All edges are tangent to the unit sphere
2. ~~The tangent points have a centre of mass at the centre of the sphere~~



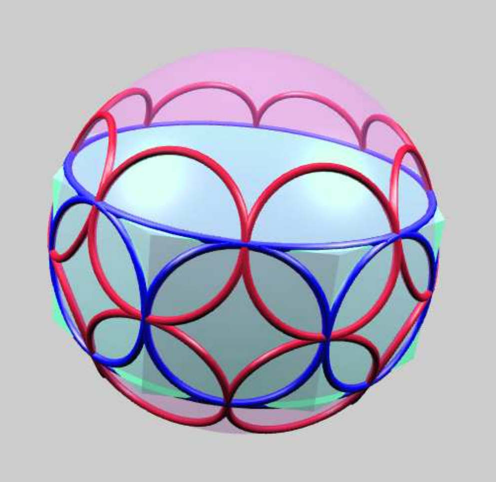
Circle packing

Every Koebe polyhedron admits a circle packing.



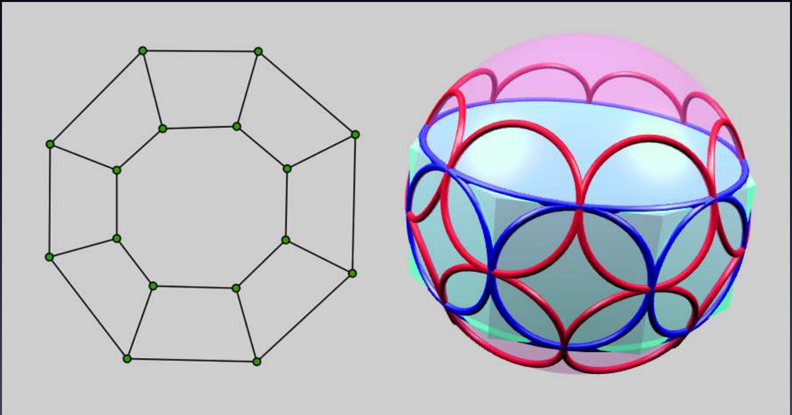
Circle pattern

Primal packing + dual packing = circle pattern



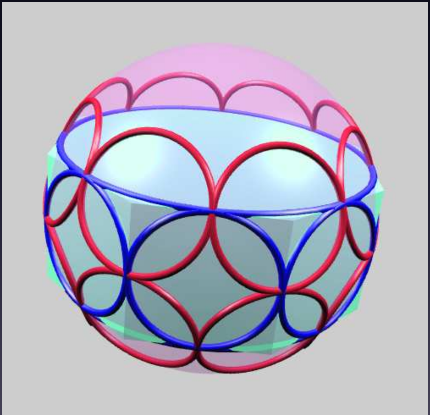
Orthogonal circle pattern

Every polyhedral graph admits an orthogonal circle pattern.



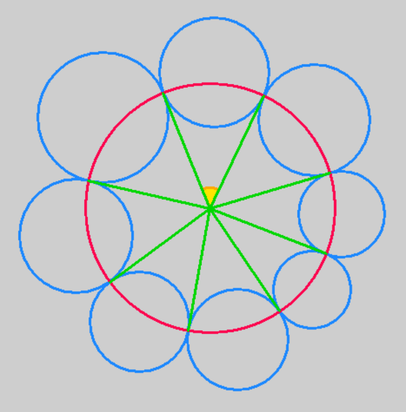
Parameters defining a circle pattern

A circle pattern is defined uniquely (up to rotation) by the intersection angles and the circle radii.

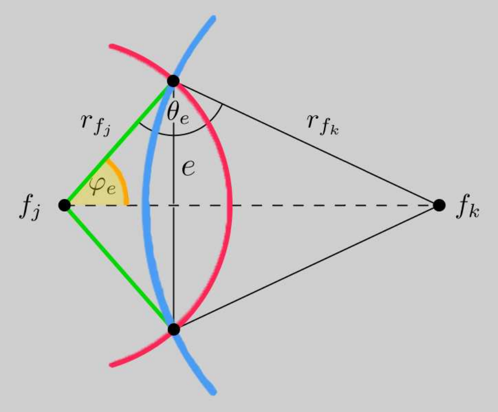


Circle pattern on the Euclidean plane

Angles around a vertex add up to π.



Circle pattern on the Euclidean plane



$$\varphi_e = \arctan\left(\frac{r_{f_k}\sin\theta_e}{r_{f_j} - r_{f_k}\cos\theta_e}\right) \Rightarrow 2\pi - 2\sum_{f_j \mid f_k} \varphi_e = 0$$

Rewrite as critical point problem

Had: $$0 = 2\pi - 2 \sum_{f_j \mid f_k} \arctan\left(\frac{r_{f_k} \sin\theta_e}{r_{f_j} - r_{f_k} \cos\theta_e}\right) \qquad \forall f_j$$

Want: $$\frac{\partial S}{\partial r_{f_j}} = 2\pi - 2 \sum_{f_j \mid f_k} \arctan\left(\frac{r_{f_k} \sin\theta_e}{r_{f_j} - r_{f_k} \cos\theta_e}\right) \qquad \forall f_j$$

So that: $$\frac{\partial S}{\partial r_{f_j}} = 0 \qquad \forall f_j$$

Rewrite as critical point problem

Critical points of S

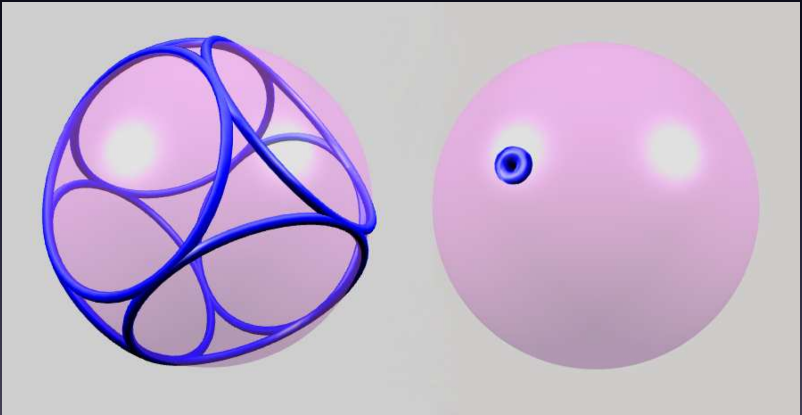
$\longrightarrow$ Solutions to the system of equations

$\longrightarrow$ Circle patterns!

Goal: Find a critical point of S

Solving numerically

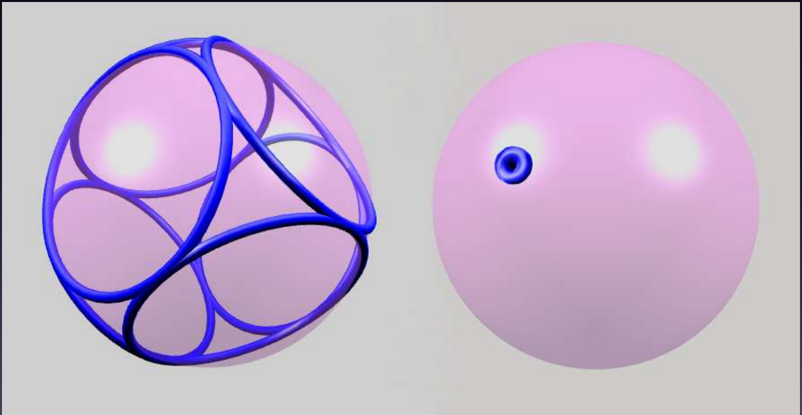
Idea: Find a critical point with Newton's method
Problem: Might get a degenerate circle pattern



Degenerate solutions

Fact: No degenerate solutions on the plane

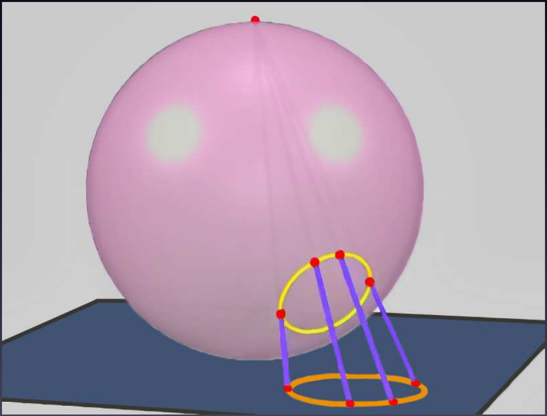
Idea: Solve on the Euclidean plane first



Map solution from plane to sphere

Fact: Stereographic projections preserves angles

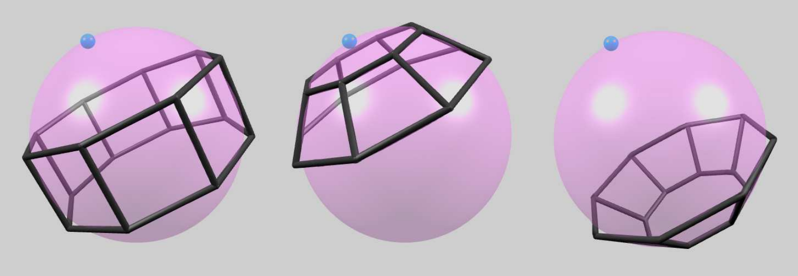
Idea: Map Euclidean solution onto the sphere



Obtaining a family of Koebe polyhedra

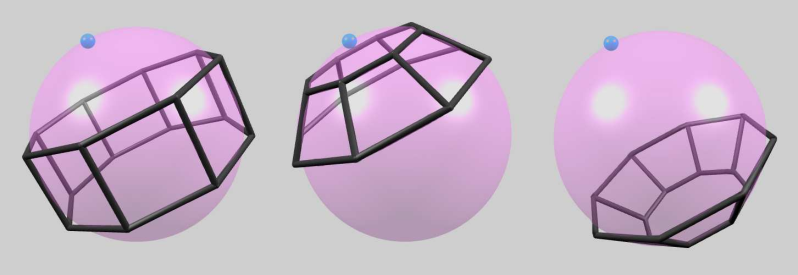
Apply a projective transformation that fixes the sphere

i.e. move towards or away from a fixed point



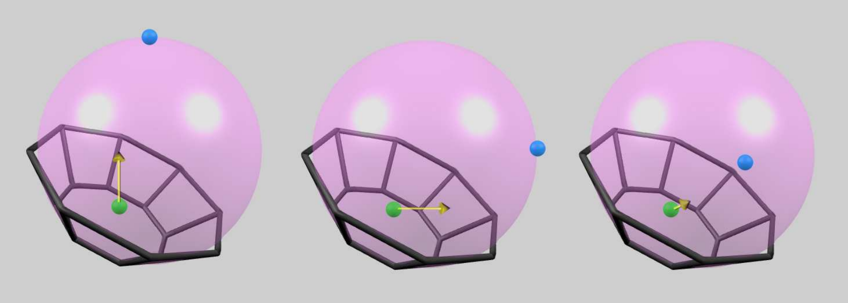
Centre of mass

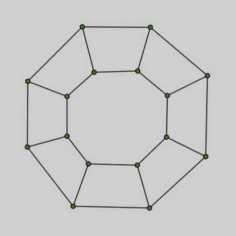
Notice that the centre of mass of the tangent points moves continuously towards or away from the fixed point.



Canonicalizing a Koebe polyhedron

Sequentially adjust the polyhedron with fixed points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ until the centre of mass lies on $(0, 0, 0)$ (centre of the unit sphere).





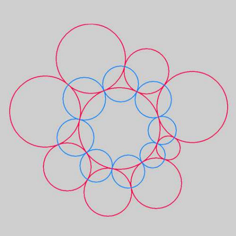
1. polyhedral graph

$$2\pi - 2\sum_{\substack{f_j \mid f_k \\ e}} \arctan\left(\frac{r_{f_k}\sin\theta_e}{r_{f_j} - r_{f_k}\cos\theta_e}\right) = 0 \quad \forall f_j$$

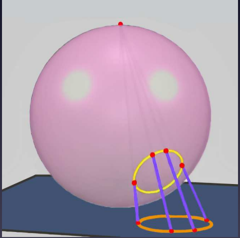
$$\Downarrow$$

$$S = \sum_{f_j \mid f_k}\left[\operatorname{Im}\operatorname{Li}_2\left(e^{\rho_{f_k}-\rho_{f_j}+i\theta}\right) + \operatorname{Im}\operatorname{Li}_2\left(e^{\rho_{f_j}-\rho_{f_k}+i\theta}\right)\right] - \sum_{f_j \mid f_k}(\pi-\theta)\left(\rho_{f_j}+\rho_{f_k}\right) + \sum_{f_j} 2\pi\rho_{f_j}$$

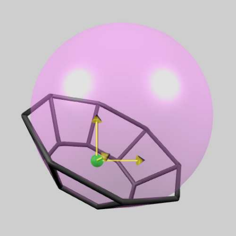
2. algebraic form



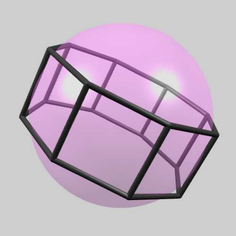
3. solve on plane



4. map to sphere

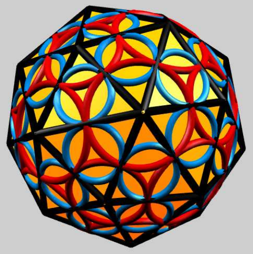
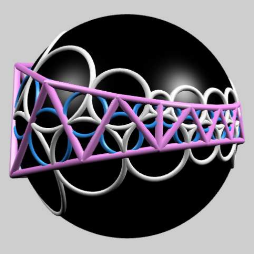
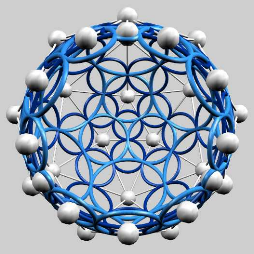


5. canonicalize



6. result

The End

References:
Boris A. Springborn, *Variational Principles for Circle Patterns*. Berlin, 2003.
Stefan Sechelmann, *Discrete Minimal Surfaces, Koebe Polyhedra, and Alexandrovs Theorem. Variational Principles, Algorithms, and Implementation*. Berlin, 2007.