CONTENTS

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GENERALIZED FACE–PRODUCTS OF MATRICES IN MODELS OF DIGITAL ANTENNA ARRAYS WITH NONIDENTICAL CHANNELS

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The paper describes new matrix operations for compact notation of responses in radio–engineering systems, which use the technology of digital formation of radiation patterns of antenna arrays in the case of nonidentical reception channels.

An internet search made it possible to discover a number of foreign dissertations and materials of international conferences dated 1996–2001, in which the authors suggest using the Khatri–Rao matrix product for signal processing in digital antenna arrays (DAA) [1]. Independently of the authors of [2], this mathematical apparatus was first applied by the author of [3] to the problems of many–coordinate radar measurements, who, like Khatri and Rao, decided to use this matrix operation and called it the transposed face–splitting product.

Based on the principle of symmetry, in [3] the operation of face–product was also introduced, which, as distinguished from the Khatri–Rao procedure, permitted row–by–row Kronecker multiplication of matrices with equal numbers of rows, and the main properties of this procedure were investigated. This innovation was ahead of the initiative of Prof. Fortiana of the University of Barcelona in mathematics, who, independently of [3, 4], also suggested the operation of face–splitting product of matrices and called it the semi–Hadamard product [5]. In the course of email communications, Mr. Fortiana appreciated the results described in [3, 4] and confirmed their priority. The time elapsed since the publication of [3, 4] proved the validity of this approach, and the increased number of its followers among foreign specialists points to the inconsistency of a skeptical attitude to this sphere of research.

This paper develops the theory of face–products of matrices as applied to treatment of the problems of radar and communication based on DAA applications.

It is well known that when considering many–coordinate information and measurement systems with digital shaping of the radiation pattern in the case of nonidentical channels of antenna arrays, there arises a problem of compact matrix formalization of the reception channel responses. To resolve this problem, we propose a family of new versions of face–products based on the penetrating face–multiplication of matrices of different size [6], and on generalized operations of face–products and block face–products [7].

The essence of the penetrating face–multiplication [6] is that for the rows in which the many–dimensional right–hand matrix is to be "split", we consider those rows or columns of numbers, which are arranged in the dimension complementary with respect to the left–hand matrix. By **the penetrating face–product** of a $p\times g$–matrix $A = [a_{ij}]$ and an n–dimensional matrix B ($n \geq 3$), deployed into the block row or block column with $p\times g$–blocks ($B = [B_r]$), we mean a matrix (with dimension B) of the type

$$A \boxdot B = [A \circ B_r] \qquad (1)$$

where the product $A \circ B_r$ is the Hadamard–type, and for the matrix B represented as a block–row or a block–column, respectively,

$$A \boxdot B = \left[A \circ B_1 \;\vdots\; A \circ B_2 \;\vdots \cdots \vdots\; A \circ B_r \;\vdots \cdots \right],$$

$$A \boxdot B = \left[A^{\mathrm{T}} \circ B_1^{\mathrm{T}} \;\vdots\; A^{\mathrm{T}} \circ B_2^{\mathrm{T}} \;\vdots\; A^{\mathrm{T}} \circ B_r^{\mathrm{T}} \;\vdots \right]^{\mathrm{T}},$$

where "T" denotes transposition.

In the case of a p–vector C and of a two–dimensional matrix B matched to it in the number of rows, we obtain the identity $C \,\square\, B = C \boxdot B$ [6]. For the p–row C^{T} and a two–dimensional matrix B, which are matched in their number of columns, the relation $C^{\mathrm{T}} + B = C^{\mathrm{T}} \boxdot B$ is valid. Among other properties of the penetrating face–multiplication, we must mention its commutativity ($A \boxdot B = B \boxdot A$), but, for convenience, we assume in what follows that the matrix of lesser dimension is situated to the left.

A peculiar feature of the penetrating face–product is "infiltration" of the two–dimensional matrix into the three–dimensional one, so that dimension of the latter remains unchanged. The operation introduced by (1) makes it possible to formalize the process of infiltration of discrete sets through sets of a larger dimension. Mathematical simulation of this procedure is often required in analysis of radio–engineering systems. For example, for a three–coordinate Doppler radar system with a planar digital antenna array (DAA), in which the radiation patterns of its antenna elements cannot be factored, for the single–signal case expression (1) permits us to describe the noiseless analytical model of signals at the outputs of R frequency filters, where this model can take the AFR discrepancies of the reception channels into account [7]:

$$U = \dot{a} \cdot (Q \boxdot F) = \dot{a} \cdot \left[Q \circ F_1 \vdots Q \circ F_2 \vdots \cdots \vdots Q \circ F_r \vdots \cdots \right]. \quad (2)$$

Here $\dot{a}$ is the signal complex amplitude,

$$Q = \begin{bmatrix} \dot{Q}_{11}(x,y) & \dot{Q}_{12}(x,y) & \cdots & \dot{Q}_{1R}(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{Q}_{R1}(x,y) & \dot{Q}_{R2}(x,y) & \cdots & \dot{Q}_{RR}(x,y) \end{bmatrix}$$

is the matrix of complex characteristics of directivity of antenna elements of the planar DAA,

$$F = \begin{bmatrix} F_{111}(\omega) & \cdots & F_{1R1}(\omega) & F_{112}(\omega) & \cdots & F_{1R2}(\omega) & & F_{11N}(\omega) & \cdots & F_{1RN}(\omega) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \cdots \vdots & \vdots & \ddots & \vdots \\ F_{R11}(\omega) & \cdots & F_{RR1}(\omega) & F_{R12}(\omega) & \cdots & F_{RR2}(\omega) & & F_{R1N}(\omega) & \cdots & F_{RRN}(\omega) \end{bmatrix}$$

is the block matrix of complex–valued frequency responses of digital filters $\dot{F}_{rmn}(\omega)$ at the signal frequency ω for $R \times R$ reception channels of the DAA (each rmth channel is characterized by a particular AFR of the nth frequency filter $F_{rmn}(\omega)$, and the ordinal number n of the block corresponds to the numbers of the Doppler filters while the first two numbers in the indices of rm elements of every block are identical to ordinal numbers of indices of the matrix Q and to the number of the respective reception channel).

Thus, every nth block of the product $Q \boxdot F$ can be written as

$$Q \circ F_n = \begin{bmatrix} \dot{Q}_{11}(x,y) \cdot \dot{F}_{11n}(\omega) & \dot{Q}_{12}(x,y) \cdot \dot{F}_{12n}(\omega) & \cdots & \dot{Q}_{1R}(x,y) \cdot \dot{F}_{1Rn}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{Q}_{R1}(x,y) \cdot \dot{F}_{R1n}(\omega) & \dot{Q}_{R2}(x,y) \cdot \dot{F}_{R2n}(\omega) & \cdots & \dot{Q}_{RR}(x,y) \cdot \dot{F}_{RRn}(\omega) \end{bmatrix}.$$

When solving the problem of range measurement and direction finding, in a pulse radar system with the planar DAA the nonidentitites of reception channels can be considered integrally in the whole reception band. This is done by the respective description of the pulses' envelope in each channel of the array. The appropriate model of responses of a three–coordinate pulse radar system to a single source can also be written with the use of the penetrating face–product in the form [7]

$$U = \dot{a}\cdot(Q \boxdot S) = \dot{a}\cdot[Q\circ S_1 \vdots Q\circ S_2 \vdots \cdots \vdots Q\circ S_t \vdots \cdots],$$

where $Q\circ S_g = \begin{bmatrix} \dot{Q}_{11}(x,y)\cdot S_{11g}(z) & \dot{Q}_{12}(x,y)\cdot S_{12g}(z) & \cdots & \dot{Q}_{1R}(x,y)\cdot S_{1Rg}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{Q}_{R1}(x,y)\cdot S_{R1g}(z) & \dot{Q}_{R_2}(x,y)\cdot S_{R2g}(z) & \cdots & \dot{Q}_{RR}(x,y)\cdot S_{RRg}(z) \end{bmatrix}$ are the blocks of Hadamard products, and $S_{nmg}(z)$ is the response of the gth strobe of range in the nmth reception channel.

It should be noted that, depending on the variant of processing, for $S_{nmg}(z)$ we may consider not only the results of additional gating of ADC samples, but the discrete envelope of the pulse signal in the sample having the zth ordinal number. This fact must be taken into account, because, for brevity, we will indicate only the first of the possible interpretations of the above function of the parameter z.

In order to formalize the model of a 4–coordinate radar system with a planar DAA, when we simultaneously estimate the range, frequency, and angular coordinates of targets, in the case of nonidentity of channels it is convenient to use the generalized face–product (GFP) [7]. This type of multiplication is intended exclusively for block matrices whose blocks have equal dimensions, and may be defined as follows:

The **generalized face–product** of block matrices $A = [A_{ij}]$ and $B = [B_{ig}]$ with a matched partition into blocks of equal dimensions and with the same number of block–rows is the matrix $A \tilde{\square} B$, in which every ith block–row represents a totality of penetrating face–products of all blocks A_{ij} of the ith block–row of the left–hand matrix by the block–row $B_i = [B_{i1} \dots B_{i2} \dots B_{iG}]$, with the corresponding number of the right–hand matrix B:

$$A \tilde{\square} B = \left[A_{ij} \boxdot [B_{i1} \quad B_{i2} \quad \cdots \quad B_{iG}]\right] \tag{3}$$

where $\boxdot$ denotes the penetrating face–product.

Having compared the face–product with matrix operation (3), it can be easily seen that the GFP represents, in essence, its counterpart, but at a higher level of generalization. Here the role of matrix elements acting in the previous face–multiplication is now played by the matrix blocks. In addition, instead of the ordinary product, in GFP we use the Hadamard product (see the definition of the penetrating face–multiplication (1)).

The methodological significance of the new type of matrix product permits us to consider it as a promising tool for systems analysis and synthesis.

Having established the necessary terminology and the features of the matrix apparatus used, we consider the sought mathematical model of a four–coordinate radar with a DAA. In the case of a solitary signal, whose source can be called pointwise, the array of output (noiseless for simplicity) voltages of the reception channels of a digital pattern–shaping network, after additional gating of the ADC samples in range and formation of frequency filters, can be written as

$$U = \left(Q \boxdot \left\{S \tilde{\square} F\right\}\right)\cdot \dot{a} = Q \boxdot [S_1 \boxdot F \vdots S_2 \boxdot F \vdots \cdots \vdots S_T \boxdot F]\cdot \dot{a}, \tag{4}$$

where $Q = \begin{bmatrix} \dot{Q}_{11}(x,y) & \dot{Q}_{12}(x,y) & \cdots & \dot{Q}_{1R}(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{Q}_{R1}(x,y) & \dot{Q}_{R2}(x,y) & \cdots & \dot{Q}_{RR}(x,y) \end{bmatrix}$ is the matrix of complex–valued directivity characteristics of the antenna elements in the planar DAA,

$$S = [S_1\ S_2 \cdots S_G] = \begin{bmatrix} S_{111}(z) & \cdots & S_{1R1}(z) \vdots S_{112}(z) & \cdots & S_{1R2}(z) \vdots & \vdots S_{11G}(z) & \cdots & S_{1RG}(z) \\ \vdots & \cdots & \vdots \ \vdots \ \vdots & \cdots & \vdots \cdots \vdots & \vdots & \cdots & \vdots \\ S_{R11}(z) & \cdots & S_{RR1}(z) \vdots S_{R12}(z) & \cdots & S_{RR2}(z) \vdots & \vdots S_{R1G}(z) & \cdots & S_{RRG}(z) \end{bmatrix}$$

is the block matrix of responses of G range strobes from $R\times R$ reception channels, whose different AFR lead to different envelopes of the pulse signals (each rmth channel in the gth range strobe will have its own, quite peculiar, envelope of the pulse signal $S_{rmg}(z)$), and F is the block matrix of characteristics of N frequency filters, which is similar to that considered in (2).

In a more sophisticated multiposition case, when all the positions represent 4–coordinate radar stations with DAA (4), the general analytical model of the multistatic system, when resolving the range – bearing problem for a single source, can be formalized with the aid of GFP (3):

$$\tilde{U}=\left(\tilde{Q}\,\tilde{\square}\left\{\tilde{S}\,\tilde{\square}\,\tilde{F}\right\}\right)\cdot\dot{a}. \tag{5}$$

Here the block matrices $\tilde{U}$, $\tilde{Q}$, $\tilde{S}$, and $\tilde{F}$ have the same meaning as in (4), but differ from the latter in that they have an additional index in the block matrix elements — corresponding to the positional number of the multiposition system:

$$\tilde{Q}=\begin{bmatrix}\tilde{Q}_1\\ \tilde{Q}_2\\ \vdots\\ \tilde{Q}_P\end{bmatrix}=\begin{bmatrix}\dot{Q}_{111}(x,y) & \dot{Q}_{121}(x,y) & \cdots & \dot{Q}_{1R1}(x,y)\\ \vdots & \vdots & \cdots & \vdots\\ \dot{Q}_{R11}(x,y) & \dot{Q}_{R21}(x,y) & \cdots & \dot{Q}_{RR1}(x,y)\\ \cdots & \cdots & \cdots & \cdots\\ & \vdots & \vdots & \\ \cdots & \cdots & \cdots & \cdots\\ \dot{Q}_{11P}(x,y) & \dot{Q}_{12P}(x,y) & \cdots & \dot{Q}_{1RP}(x,y)\\ \vdots & \vdots & \cdots & \vdots\\ \dot{Q}_{R1P}(x,y) & \dot{Q}_{R2P}(x,y) & \cdots & \dot{Q}_{RRP}(x,y)\end{bmatrix},$$

$$\tilde{S}=\begin{bmatrix}\tilde{S}_{11} & \tilde{S}_{12} & \cdots & \tilde{S}_{1G}\\ \vdots & \vdots & \cdots & \vdots\\ \tilde{S}_{P1} & \tilde{S}_{P2} & \cdots & \tilde{S}_{PG}\end{bmatrix}=$$

$$=\begin{bmatrix}S_{1111}(z) & \cdots & S_{1R11}(z) & S_{1112}(z) & \cdots & S_{1R12}(z) & & S_{111G}(z) & \cdots & S_{1R1G}(z)\\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ S_{R111}(z) & \cdots & S_{RR11}(z) & S_{R112}(z) & \cdots & S_{RR12}(z) & & S_{R11G}(z) & \cdots & S_{RR1G}(z)\\ & & & & \cdots & & & & & \\ S_{11P1}(z) & \cdots & S_{1RP1}(z) & S_{11P2}(z) & \cdots & S_{1RP2}(z) & & S_{11PG}(z) & \cdots & S_{1RPG}(z)\\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ S_{R1P1}(z) & \cdots & S_{RRP1}(z) & S_{R1P2}(z) & \cdots & S_{RRP2}(z) & & S_{R1PG}(z) & \cdots & S_{RRPG}(z)\end{bmatrix},$$

while the matrix F turns into the matrix S if we replace the symbols S and z in it by symbols F and ω, respectively.

With the use of block–type notations, relationships (5) can be represented in a more detailed form based on definition (3):

$$U=\left(\begin{bmatrix}\tilde{Q}_1\\ \vdots\\ \tilde{Q}_P\end{bmatrix}\tilde{\square}\begin{bmatrix}\tilde{S}_{11} & \tilde{S}_{12} & \cdots & \tilde{S}_{1G}\\ \vdots & \vdots & \cdots & \vdots\\ \tilde{S}_{P1} & \tilde{S}_{P2} & \cdots & \tilde{S}_{PG}\end{bmatrix}\tilde{\square}\begin{bmatrix}\tilde{F}_{11} & \tilde{F}_{12} & \cdots & \tilde{F}_{1N}\\ \vdots & \vdots & \cdots & \vdots\\ \tilde{F}_{P1} & \tilde{F}_{P2} & \cdots & \tilde{F}_{PN}\end{bmatrix}\right)\cdot\dot{a}=$$

$$\begin{bmatrix} \tilde{Q}_1 \square \left[\tilde{S}_{11} \square [F_{11} \ \cdots \ F_{1N}]\right] & \tilde{Q}_1 \square \left[\tilde{S}_{12} \square [F_{11} \ \cdots \ F_{1N}]\right] & \cdots & \tilde{Q}_1 \square \left[\tilde{S}_{1G} \square [F_{11} \ \cdots \ F_{1N}]\right] \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{Q}_P \square \left[\tilde{S}_{P1} \square [F_{P1} \ \cdots \ F_{PN}]\right] & \tilde{Q}_P \square \left[\tilde{S}_{P2} \square [F_{P1} \ \cdots \ F_{PN}]\right] & \cdots & \tilde{Q}_P \square \left[\tilde{S}_{PG} \square [F_{P1} \ \cdots \ F_{PN}]\right] \end{bmatrix} \dot{a},$$

where each *pgn*th block of the resulting matrix U is defined by Hadamard's product of the respective blocks of matrices $\tilde{Q}$, $\tilde{S}$, $\tilde{F}$:

$$\tilde{U}_{pgn} = \left(\tilde{Q}_p \circ \tilde{S}_{pg} \circ \tilde{F}_{pn}\right) \cdot \dot{a} =$$

$$= \begin{bmatrix} \dot{a} \cdot \tilde{Q}_{11p}(x,y) \cdot S_{11pg}(z) \cdot F_{11pn}(\omega) & \dots & \dot{a} \cdot \tilde{Q}_{1Rp}(x,y) \cdot S_{1Rpg}(z) \cdot F_{1Rpn}(\omega) \\ \dots & \dots & \dots \\ \dot{a} \cdot \tilde{Q}_{R1p}(x,y) \cdot S_{R1pg}(z) \cdot F_{R1pn}(\omega) & \cdots & \dot{a} \cdot \tilde{Q}_{RRp}(x,y) \cdot S_{RRpg}(z) \cdot F_{RRpn}(\omega) \end{bmatrix}.$$

If we have to estimate simultaneously the angular coordinates and ranges of M sources, and the characteristics of reception channels of DAA are nonidentical, the solution of this measurement problem requires introduction of the concept of **block generalized face–product (BGFP)** of matrices [7]. The operation reduces to implementation of the block–by–block procedure of generalized face–product applied to the blocks of the same hierarchical level.

By the **block generalized face–product** of a $dbp \times ngs$ matrix $A = [A_{bg}]_{dn}$ and a $dbp \times nks$ matrix $B = [B_{bk}]_{dn}$, consisting of equal numbers ($d \times n$) of super–blocks with dimensions $b \times g$ and $b \times k$, respectively, formed by b block–rows each, and comprising g (the matrix A) and k (the matrix B) $p \times s$ blocks, is meant the matrix $A \tilde{\circledcirc} B$, each *dn*th super–block of which represents the generalized face–product of the respective super–blocks of the initial matrices:

$$A \tilde{\circledcirc} B = \left[A_{bg} \tilde{\square} B_{bk} \right]_{dn}.$$

Based on this definition, consider now a single–position radar system, which resolves the range–bearing problem with M signal sources. The voltages of its output pulse mixture, without noise taken into account, can be represented by the operation of the block GFP in the form

$$U = (Q \tilde{\circledcirc} S)(A \otimes 1_R \otimes 1_G), \qquad (6)$$

where $Q = \begin{bmatrix} \dot{Q}_{11}(x_1,y_1) & \cdots & \dot{Q}_{1R}(x_1,y_1) & & \dot{Q}_{11}(x_M,y_M) & \cdots & \dot{Q}_{1R}(x_M,y_M) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \dot{Q}_{R1}(x_1,y_1) & \cdots & \dot{Q}_{RR}(x_1,y_1) & & \dot{Q}_{R1}(x_M,y_M) & \cdots & \dot{Q}_{RR}(x_M,y_M) \end{bmatrix}$, $S = [S_1 \ S_2 \cdots S_M]$, and

$$S_m = \begin{bmatrix} S_{111}(z_m) & \cdots & S_{1R1}(z_m) & & S_{11G}(z_m) & \cdots & S_{1RG}(z_m) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ S_{R11}(z_m) & \cdots & S_{RR1}(z_m) & & S_{R1G}(z_m) & \cdots & S_{RRG}(z_m) \end{bmatrix}.$$

$A = [\dot{a}_1 \ \dot{a}_2 \dots \dot{a}_m]^T$ is the vector of complex amplitudes of M signals, 1_R and 1_G are identity matrices with their dimensions $R \times R$ and $G \times G$, respectively, while the block–matrix U with its entries $\dot{U}_{krt} = \sum_{m=1}^{M} \dot{a}_m \cdot S_{nrg}(z_m) \cdot Q_{kr}(x_m, y_m)$ has the form

$$U = \begin{bmatrix} \dot{U}_{111} & \cdots & \dot{U}_{1R1} & & \dot{U}_{11G} & \cdots & \dot{U}_{1RG} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \dot{U}_{R11} & \cdots & \dot{U}_{RR1} & & \dot{U}_{R1G} & \cdots & \dot{U}_{RRG} \end{bmatrix}.$$

In the process of 4–coordinate measurements, the model (6), supplemented with frequency selection, is transformed into the following matrix expression:

$$U = (Q \tilde{\circledcirc} S \tilde{\circledcirc} F)(A \otimes 1_R \otimes 1_G \otimes 1_N), \tag{7}$$

where, as distinct from (6), the block–matrix U is supplemented with new blocks, whose indices correspond to ordinal numbers of the frequency filters:

$$U = \begin{bmatrix} \dot{U}_{1111} & \cdots & \dot{U}_{1R11} & & \dot{U}_{11G1} & \cdots & \dot{U}_{1RG1} & & \dot{U}_{11GN} & \cdots & \dot{U}_{1RGN} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \dot{U}_{R111} & \cdots & \dot{U}_{RR11} & & \dot{U}_{R1G1} & \cdots & \dot{U}_{RRG1} & & \dot{U}_{R1GN} & \cdots & \dot{U}_{RRGN} \end{bmatrix},$$

and

$$\dot{U}_{krgn} = \sum_{m=1}^{M} \dot{a}_m \cdot S_{krg}(z_m) \cdot Q_{kr}(x_m, y_m) \cdot F_{krn}(\omega_m), \; F = [F_1 \; F_2 \cdots F_M],$$

$$F_m = \begin{bmatrix} F_{111}(\omega_m) & \cdots & F_{1R1}(\omega_m) & & F_{11N}(\omega_m) & \cdots & F_{1RN}(\omega_m) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ F_{R11}(\omega_m) & \cdots & F_{RR1}(\omega_m) & & F_{R1N}(\omega_m) & \cdots & F_{RRN}(\omega_m) \end{bmatrix},$$

and 1_G is the $G \times G$ identity matrix.

For the multi–position case, (6) and (7) can be generalized as follows:

$$U = (\tilde{Q} \tilde{\circledcirc} \tilde{S})(A \otimes 1_R \otimes 1_G), U = (\tilde{Q} \tilde{\circledcirc} \tilde{S} \tilde{\circledcirc} \tilde{F})(A \otimes 1_R \otimes 1_G \otimes 1_N), \tag{8}$$

where

$$\tilde{Q} = \begin{bmatrix} \dot{Q}_{111}(x_1, y_1) & \cdots & \dot{Q}_{1R1}(x_1, y_1) & & \dot{Q}_{111}(x_M, y_M) & \cdots & \dot{Q}_{1R1}(x_M, y_M) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \dot{Q}_{R11}(x_1, y_1) & \cdots & \dot{Q}_{RR1}(x_1, y_1) & & \dot{Q}_{R11}(x_M, y_M) & \cdots & \dot{Q}_{RR1}(x_M, y_M) \\ & \vdots & & \cdots & & \vdots & \\ \dot{Q}_{11P}(x_1, y_1) & \cdots & \dot{Q}_{1RP}(x_1, y_1) & & \dot{Q}_{11P}(x_M, y_M) & \cdots & \dot{Q}_{1RP}(x_M, y_M) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \dot{Q}_{R1P}(x_1, y_1) & \cdots & \dot{Q}_{RRP}(x_1, y_1) & & \dot{Q}_{R1P}(x_M, y_M) & \cdots & \dot{Q}_{RRP}(x_M, y_M) \end{bmatrix},$$

$$\tilde{S} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} & \cdots & \tilde{S}_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{S}_{P1} & \tilde{S}_{P2} & \cdots & \tilde{S}_{PM} \end{bmatrix}, \; \tilde{F} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} & \cdots & \tilde{F}_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{F}_{P1} & \tilde{F}_{P2} & \cdots & \tilde{F}_{PM} \end{bmatrix},$$

while

$$\tilde{S}_{pm} = \begin{bmatrix} S_{11p1}(z_m) & \cdots & S_{1Rp1}(z_m) & & S_{11pG}(z_m) & \cdots & S_{1RpG}(z_m) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ S_{R1p1}(z_m) & \cdots & S_{RRp1}(z_m) & & S_{R1pG}(z_m) & \cdots & S_{RRpG}(z_m) \end{bmatrix},$$

$$\tilde{F}_{pm} = \begin{bmatrix} F_{11p1}(\omega_m) & \cdots & F_{1Rp1}(\omega_m) & & F_{11pN}(\omega_m) & \cdots & F_{1RpN}(\omega_m) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ F_{R1p1}(\omega_m) & \cdots & F_{RRp1}(\omega_m) & & F_{R1pN}(\omega_m) & \cdots & F_{RRpN}(\omega_m) \end{bmatrix}.$$

It can be easily seen that the peculiar feature of the above many–signal models is representation of the amplitude cofactor as the Kronecker product of the vector of amplitudes by the identity matrices. In order to reduce the number of the latter, it is expedient to use formalization of the radar many–signal models based on the operations of transposed GFP and BGFP [7]. By analogy with the concepts considered above, we shall give the following definitions of these operations.

The transposed generalized face–product (TGFP) of block–matrices $A = [A_{ij}]$ and $B = [B_{gj}]$, with the matched decomposition into blocks of equal dimensions and with the same number of block–columns, represents a matrix $A \tilde{\blacksquare} B$, in which every jth block–column is a set of the penetrating face–products of all blocks A_{ij} of the jth block–column of the left–hand matrix by the respective (in terms of ordinal number) block–column $B_j = [B_{1j}^{\mathrm{T}} \dots B_{Gj}^{\mathrm{T}}]^{\mathrm{T}}$ of the right–hand matrix B:

$$A \tilde{\blacksquare} B = \left[A_{ij} \boxdot \begin{bmatrix} B_{1j} \\ \vdots \\ B_{Gj} \end{bmatrix} \right] \qquad (9)$$

where $\boxdot$ denotes the penetrating face–product.

By the **transposed block generalized face–product (TBGFP)** of a $dbp \times ngs$ matrix $A = [A_{bg}]_{dn}$ and a $dkp \times ngs$ matrix $B = [B_{kg}]_{dn}$, consisting of equal numbers ($d \times n$) of super–blocks with dimensions $b \times g$ and $k \times g$, respectively, formed by g block–columns each, and comprising b (the matrix A) and k (the matrix B) $p \times s$ blocks, is meant the $dbkp \times ngs$ matrix $A \tilde{\circledbullet} B$, where each of its dnth super–blocks represents the transposed generalized face–product of the respective super–blocks of the initial matrices, i.e., $A \tilde{\circledbullet} B = \left[A_{bg} \tilde{\blacksquare} B_{kg} \right]_{dn}$.

In order to preserve the continuity in the system of matrix notations, which was used earlier for description of the direction–finding characteristics of DAA of Doppler's filter AFR and responses of the procedures including the range gating, let us supplement the above theory with a description of the operation of block–rotation of matrices [7].

The **block–rotation** of a matrix A, each block A_{ij} of which represents a block–row or a block–column, is an operation consisting of non–transposed rotation of the indicated block–rows (block–columns) of the sought matrix about their first blocks in the clockwise direction (in the counter–clockwise direction, respectively), so that the block–rows turn into block–columns, and vice versa. Inside the blocks comprising the block–rows (block–columns), no changes occur, and the structure of the matrix A also remains unchanged at the level of the blocks A_{ij}. To denote this operation — by analogy with transposition, we shall use for the superscript in the block matrices, the letter R corresponding to *rotation*.

In the MatLab 5.0 package and in its subsequent versions there is a built–in variant of the rotation procedure rot(A), which differs considerably from that suggested in this paper. Unfortunately, the standard conception, if applied to block matrices, leads to a result unsuitable for applications to our problem.

Based on the set of new concepts, we come to alternative (with respect to (6)–(8)) models of many–signal radar measurers for simultaneous estimation of range, angles, and frequency of several point signal sources:

$$U = (Q \tilde{\circledbullet} S^R)(A \otimes 1_R), \; U = \left(Q \tilde{\circledbullet} (S^R \tilde{\circledbullet} F^R)\right)(A \otimes 1_R),$$
$$U = (\tilde{Q} \tilde{\circledbullet} \tilde{S}^R)(A \otimes 1_R), \; U = \left(\tilde{Q} \tilde{\circledbullet} (\tilde{S}^R \tilde{\circledbullet} \tilde{F}^R)\right)(A \otimes 1_R), \qquad (10)$$

Here the matrices Q, $\tilde{Q}$, 1_R, and the vector A have their former meaning, while $S^R, \tilde{S}^R, F^R, \tilde{F}^R$ are the block matrices $S, \tilde{S}, F, \tilde{F}$ having undergone rotation and interpreted like relations (6)–(8), and

$$S^R = \left[S_1^R \, S_2^R \cdots S_M^R\right], F^R = \left[F_1^R \, F_2^R \cdots F_M^R\right],$$

$$S_m^R = \begin{bmatrix} S_{111}(z_m) & \cdots & S_{1R1}(z_m) \\ \vdots & \cdots & \vdots \\ S_{R11}(z_m) & \cdots & S_{RR1}(z_m) \\ S_{11G}(z_m) & \vdots & S_{1RG}(z_m) \\ \vdots & \cdots & \vdots \\ S_{R1G}(z_m) & \cdots & S_{RRG}(z_m) \end{bmatrix}, F_m^R = \begin{bmatrix} F_{111}(\omega_m) & \cdots @ & F_{1R1}(\omega_m) \\ \vdots & \cdots & \vdots \\ F_{R11}(\omega_m) & \cdots & F_{RR1}(\omega_m) \\ F_{11N}(\omega_m) & \vdots & F_{1RN}(\omega_m) \\ \vdots & \cdots & \vdots \\ F_{R1N}(\omega_m) & \cdots & F_{RRN}(\omega_m) \end{bmatrix};$$

$$\tilde{S}^R = \begin{bmatrix} \tilde{S}^R_{11} & \tilde{S}^R_{12} & \cdots & \tilde{S}^R_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{S}^R_{P1} & \tilde{S}^R_{P2} & \cdots & \tilde{S}^R_{PM} \end{bmatrix}, \quad \tilde{F}^R = \begin{bmatrix} \tilde{F}^R_{11} & \tilde{F}^R_{12} & \cdots & \tilde{F}^R_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{F}^R_{P1} & \tilde{F}^R_{P2} & \cdots & \tilde{F}^R_{PM} \end{bmatrix},$$

$$\tilde{S}^R_{pm} = \begin{bmatrix} S_{1|p1}(z_m) & \cdots & S_{1Rp1}(z_m) \\ \vdots & \cdots & \vdots \\ S_{R|p1}(z_m) & \cdots & S_{RRp1}(z_m) \\ & \vdots & \\ S_{1|pG}(z_m) & \cdots & S_{1RpG}(z_m) \\ \vdots & \cdots & \vdots \\ S_{R|pG}(z_m) & \cdots & S_{RRpG}(z_m) \end{bmatrix}, \quad \tilde{F}^R_{pm} = \begin{bmatrix} F_{1|p1}(\omega_m) & \cdots & F_{1Rp1}(\omega_m) \\ \vdots & \cdots & \vdots \\ F_{R|p1}(\omega_m) & \cdots & F_{RRp1}(\omega_m) \\ & \vdots & \\ F_{1|pN}(\omega_m) & \cdots & F_{1RpN}(\omega_m) \\ \vdots & \cdots & \vdots \\ F_{R|pN}(\omega_m) & \cdots & F_{RRpN}(\omega_m) \end{bmatrix}.$$

It is noteworthy that in all relations (10) we can find the same amplitude factor $A \otimes 1_R$. The further reduction of its dimensionality consisting in its interpretation by the vector of amplitudes, is based on the so–called **block vectorization,** which, as distinct from the known vec–operators, is applicable not to the whole matrix but to individual blocks.

The **operation of block vectorization** of a blockwise $dp{\times}sc$–matrix A represents its block–by–block transformation with the aid of the vec–operator, i.e.,

$$b\mathrm{vec}_{pc} A = b\mathrm{vec}_{pc}[A_{ds}] = \left[\mathrm{vec}[a_{pc}]\right]_{ds}. \tag{11}$$

The double subscript pc at the operator $b\mathrm{vec}_{pc}$ defines the dimensions of the blocks subject to the vec–operator. Generation of these blocks is performed beginning from the left–hand upper corner of the transformed matrix, with account for the fact that the index p denotes the number of rows while the index c — the number of columns in the block.

Having applied the procedure of (11), we can derive from (10) the following analytical expressions of the pulse signal voltages in a 3– and 4–coordinate radar system with DAA:

$$U = \left[b\mathrm{vec}_{pc}\left(Q \tilde{\circledcirc} S^R\right)\right]A, \quad U = \left[b\mathrm{vec}_{pc}\left(Q \tilde{\circledcirc}\left(S^R \tilde{\circledcirc} F^R\right)\right)\right]A,$$

$$U = \left[b\mathrm{vec}_{pc}\left(\tilde{Q} \tilde{\circledcirc} \tilde{S}^R\right)\right]A, \quad U = \left[b\mathrm{vec}_{pc}\left(\tilde{Q} \tilde{\circledcirc}\left(\tilde{S}^R \tilde{\circledcirc} \tilde{F}^R\right)\right)\right]A.$$

The subsequent use of the generated models of responses of the radar systems with DAA is identical to the variants of resolving the measurement problems described in [4], and to the procedures of analysis of potential accuracy and estimation of maximum attainable resolving capacity of many–coordinate measurements as applied to identical channels of the antenna array. The concrete mechanism of application of the above models may vary substantially depending on the problem to be solved. With the availability of this powerful matrix apparatus, researchers must only make their choice and select the most convenient variant of formalization of the matrix model of DAA response from those considered above. Then, based on the unified approach, we can perform the further analysis of the potentialities of a particular radio–engineering system or synthesis of the coordinate measurement methods corresponding to the system structure, particularly, simultaneous estimation of coordinates in a many–signal situation. The development of these lines of research was hampered because of imperfection of the traditional matrix algebra. In the theory of radar, communication, and systems analysis there are other problems, whose solution may be facilitated by the use of the approach suggested.

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14 January 2002