History of Mathematics

Math 395 Spring 2010 — Fowler 310 MWF 10:30am - 11:25am

 — http://faculty.oxy.edu/ron/math/395/10/

Class 23: **Friday April 2**

TITLE The Calculus: Sir Isaac Newton
CURRENT READING: Katz, §16
NEXT READING: Katz, §17

Homework #8 DUE Friday April 9 (in class)
Katz, p. 539: #1, #2, #6, #11, #26. EXTRA CREDIT: page 540, #18, #19.

SUMMARY
We will discuss one of the co-inventors of Calculus, Isaac Newton, one of the greatest minds of the last millennium.

Isaac Newton (1642-1727) was born on Christmas Day, 1642 some 100 miles north of London in the town of Woolsthorpe. In 1661 he started to attend Trinity College, Cambridge and assisted Isaac Barrow in the editing of the latter's *Geometrical Lectures*.

For parts of 1665 and 1666, the college was closed due to The Plague and Newton went home and studied by himself, leading to what some observers have called "the most productive period of mathematical discover ever reported" (Boyer, *A History of Mathematics*, 430).

During this period Newton made four monumental discoveries

1. the binomial theorem
2. the calculus
3. the law of gravitation
4. the nature of colors

Newton is widely regarded as the most influential scientist of all-time even ahead of Albert Einstein, who demonstrated the flaws in Newtonian mechanics two centuries later. Newton's epic work *Philosophiæ Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy)* more often known as the *Principia* was first published in 1687 and revolutionalized science itself.

Infinite Series
One of the key tools Newton used for many of his mathematical discoveries were power series. He was able to show that he could expand the expression $(a+bx)^n$ for any n

$$(a+bx)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}bx + \binom{n}{2}a^{n-2}b^2x^2 + \binom{n}{3}a^{n-3}b^3x^3 + \ldots + \binom{n}{n}b^nx^n$$

He was able to check his intuition that this result must be true by showing that the power series for $(1+x)^{-1}$ could be used to compute a power series for log(1+x) which he then used to calculate the logarithms of several numbers close to 1 to over 50 decimal paces.

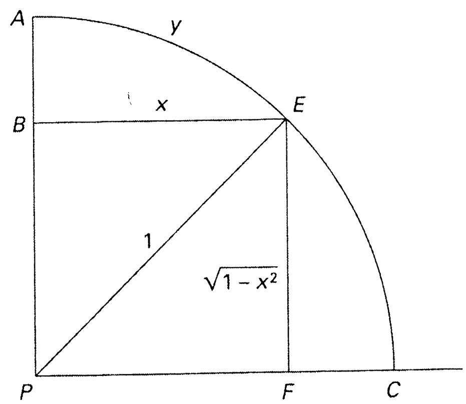
Newton's *De analysi per aequationes numero terminorum infinitas (On Analysis by Equations with Infinitely Many Terms)* was written in 1669 but was not published, although it was circulated to several of his friends and collaborators. In it, he showed that

$$y = \arcsin x = 2\int_0^x \sqrt{1-x^2}\,dx - x\sqrt{1-x^2} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$

Since

$$\sqrt{1-x^2} = (1-x^2)^{1/2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \cdots$$

As demonstrated by the figure below



Newton used these results to obtain the power series expansion for the sine function, and the cosine function, writing

$$\sin y = y - \frac{1}{6}y^3 + \frac{1}{120}y^5 - \frac{1}{5040}y^7 + \ldots$$

And using $\cos y = \sqrt{1-\sin^2 y}$ to obtain an equivalent expression for cos y.

Exercise

Obtain the series of cos y from the power series for sine from the series for sin y the way Newton did it

Newton's Calculus: Fluxions and Fluents
Newton defined a **fluxion** $\dot{x}$ as the speed of a quantity x (which he called the **fluent**) that depended on time. Newton basically thought of all derivatives as time derivatives.

He defined the **moment** of a fluent to be the amount it increases in an "infinitely small" amount of time o. Thus the moment of the fluent x is $x+\dot{x}o$.

In*Tractatus de methodis serierum et fluxionum (A Treatise on the Method of Series and Fluxions)* of 1671 Newton says:

> "[A]n equation which expresses a relationship of fluent quantities without variance at all times will express that relationship equally between $x+\dot{x}o$ and $y+\dot{y}o$ as between x and y; so $x+\dot{x}o$ and $y+\dot{y}o$ may be substituted in place of the later quantities, x and y, in the said equation."

The example Newton used was the equation $x^3 - ax^2 + axy - y^3 = 0$ which he was able to get the expression $3x^2\dot{x} - 2ax\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0$

Exercise

Considering $f(x,y)=0$ where x and y are both functions of t, obtain an expression for $\frac{df}{dt}$ in general (using the Chain Rule) and then consider $f(x,y) = x^3 - ax^2 + axy - y^3 = 0$ to replicate Newton's result.

Example

Let's use Newton's fluxion and moments to obtain the derivative of $f(x,y) = x^3 - ax^2 + axy - y^3 = 0$ with respect to time.

Finding Fluents from Fluxions

Given $\dot{y}^2 = \dot{x}\dot{y} + x^2\dot{x}^2$ Newton would re-write the expression as $\frac{\dot{y}^2}{\dot{x}^2} = x^2 + \frac{\dot{y}}{\dot{x}}$ which is a quadratic expression in $\frac{\dot{y}}{\dot{x}}$ and can be solved using the quadratic formula, obtaining

$$\frac{\dot{y}}{\dot{x}} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + x^2}$$

Which he could then use the binomial theorem on to obtain

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 + \dots \text{ and } y = -\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{7}x^7 + \dots$$

GroupWork

Let's try and replicate Newton's result by using the binomial theorem. Using modern methods, what kind of equation and solution techniques would typically be used instead?

Newton's Table of Integrals

Of course, using this method to obtain fluents from fluxions would be time consuming so Newton generated a table of expressions for fluents (on the right) given the fluxion (on the left). He knew that the expression on the right could be used to obtain the area under the function on the left.

$$y = \frac{ax^{n-1}}{(b+cx^n)^2} \qquad z = \frac{(a/nb)x^n}{b+cx^n}$$

$$y = ax^{n-1}\sqrt{b+cx^n} \qquad z = \frac{2a}{3nc}(b+cx^n)^{3/2}$$

$$y = ax^{2n-1}\sqrt{b+cx^n} \qquad z = \frac{2a}{nc}\left(-\frac{2}{15}\frac{b}{c} + \frac{1}{5}x^n\right)(b+cx^n)^{3/2}$$

$$y = \frac{ax^{2n-1}}{\sqrt{b+cx^n}} \qquad z = \frac{2a}{nc}\left(-\frac{2}{3}\frac{b}{c} + \frac{1}{3}x^n\right)\sqrt{b+cx^n}$$