# Scaling up Bayesian Inference

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# Outline

# Complex & high-dimensional data

- Interest in developing new methods for analyzing & interpreting complex, high-dimensional data

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- Arise routinely in broad fields of sciences, engineering & even arts & humanities
- Despite huge interest in big data, there are vast gaps that have fundamentally limited progress in many fields

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- Rapidly obtaining a point estimate even when sample size $n$ & overall ‘size’ of data is immense
- Bandwagons: many people work on quite similar problems, while critical open problems remain untouched

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- Accurate uncertainty quantification (UQ) is a critical issue
- Robustness of inferences also crucial
- Particular emphasis on scientific applications - limited labeled data


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- Scaling MCMC to big & complex settings challenging

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- Usually multiple likelihood and/or gradient evaluations at each iteration

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- **Designer MCMC**: define clever kernels that solve mixing problems in high dimensions
- I’ll focus on EP-MCMC & aMCMC in remainder

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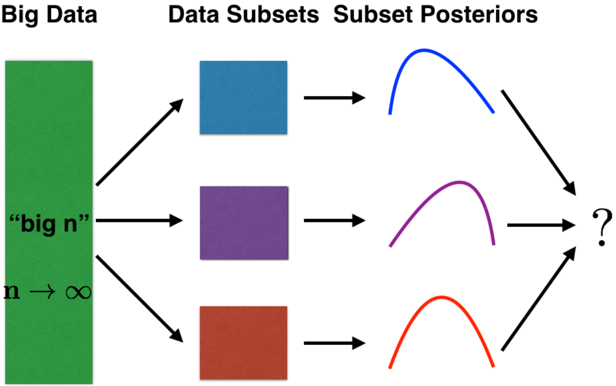
# Embarrassingly parallel MCMC

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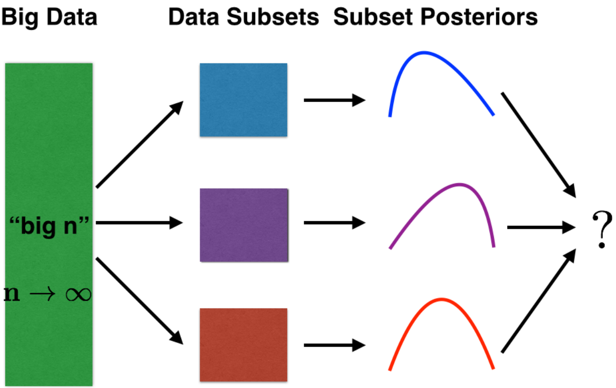


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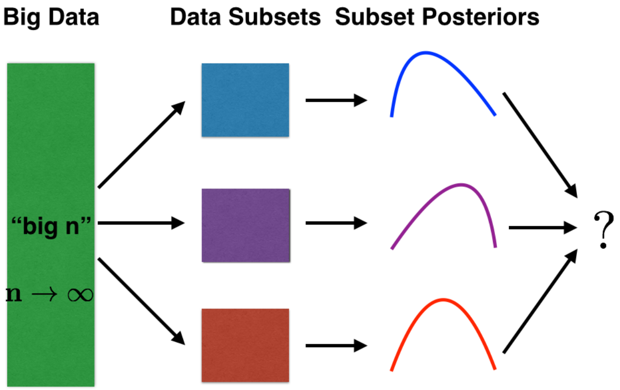
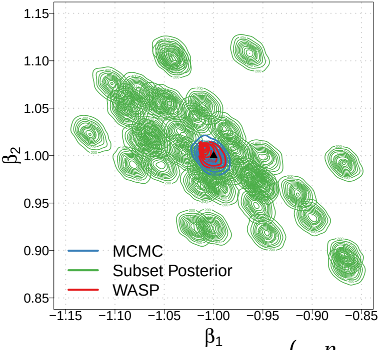



- Divide large sample size $n$ data set into many smaller data sets stored on different machines
- Draw posterior samples for each subset posterior in parallel
- 'Magically' combine the results quickly & simply

# Toy Example: Logistic Regression




$$\text{pr}(y_i = 1 | x_{i1}, \ldots, x_{ip}, \theta) = \frac{\exp\left(\sum_{j=1}^{p} x_{ij}\beta_j\right)}{1 + \exp\left(\sum_{j=1}^{p} x_{ij}\beta_j\right)}.$$

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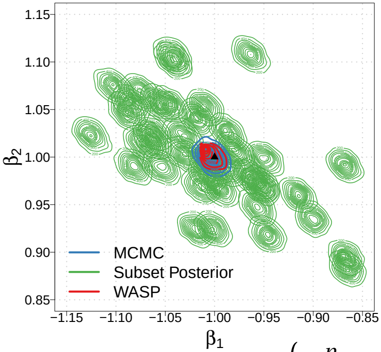


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- Subset posteriors: ‘noisy’ approximations of full data posterior.
- ‘Averaging’ of subset posteriors reduces this ‘noise’ & leads to an accurate posterior approximation.

## Stochastic Approximation

- Full data posterior density of *inid* data $Y^{(n)}$

$$\pi_n(\theta \mid Y^{(n)}) = \frac{\prod_{i=1}^n p_i(y_i \mid \theta)\pi(\theta)}{\int_\Theta \prod_{i=1}^n p_i(y_i \mid \theta)\pi(\theta) d\theta}.$$

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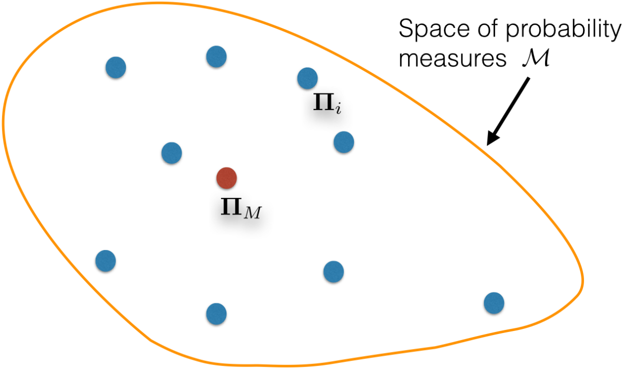
- $\gamma = O(k)$ - chosen to minimize approximation error

# Barycenter in Metric Spaces



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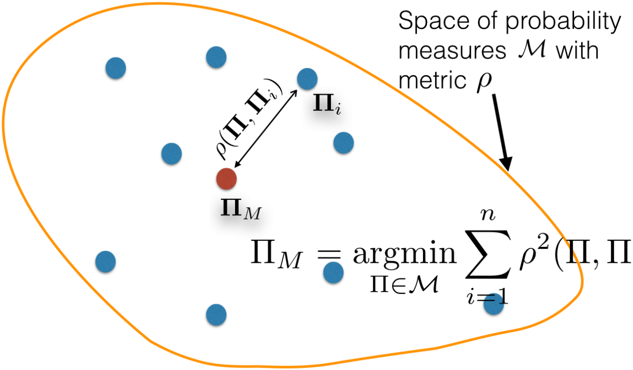


$$\Pi_M = \underset{\Pi \in \mathcal{M}}{\operatorname{argmin}} \sum_{i=1}^{n} \rho^2(\Pi, \Pi_i)$$

# WAsserstein barycenter of Subset Posteriors (WASP)

*Srivastava, Li & Dunson (2015)*

- 2-Wasserstein distance between $\mu, \nu \in \mathscr{P}_2(\Theta)$

$$W_2(\mu, \nu) = \inf\left\{\left(\mathbb{E}[d^2(X, Y)]\right)^{\frac{1}{2}} : \text{law}(X) = \mu, \text{law}(Y) = \nu\right\}.$$

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- Plugging in $\widehat{\Pi}^{\gamma}_{m}(\cdot \mid Y_{[j]})$ for $j = 1, \ldots, k$, a linear program (LP) can be used for fast estimation of an atomic approximation

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- For WASP, generalize to multimargin optimal transport problem - entropy smoothing has been used previously
- We can avoid such smoothing & use sparse LP solvers - neglible computation cost compared to sampling

# WASP: Theorems

## Theorem (Subset Posteriors)

*Under "usual" regularity conditions, there exists a constant* $C_1$ *independent of subset posteriors, such that for large* $m$*,*

$$\mathbb{E}_{P_{\theta_0}^{[j]}} W_2^2 \left\{ \Pi_m^{\gamma}(\cdot \mid Y_{[j]}), \delta_{\theta_0}(\cdot) \right\} \leq C_1 \left( \frac{\log^2 m}{m} \right)^{\frac{1}{\alpha}} \qquad j = 1, \ldots, k,$$

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## Theorem (WASP)

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- Can be implemented in *STAN*, which allows powered likelihoods

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- Conditions: standard, mild conditions on likelihood + prior finite 2nd moment & uniform integrabiity of subset posteriors

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- WASP/PIE is <u>much</u> faster than MCMC & highly accurate

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- Nonparametric models, dependence, hierarchical models, etc.
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- WASP/PIE is <u>much</u> faster than MCMC & highly accurate
- Carefully designed VB implementations often do very well

# Outline

## **aMCMC** ***Johndrow, Mattingly, Mukherjee & Dunson (2015)***

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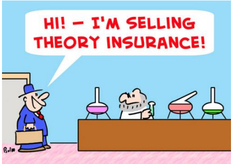


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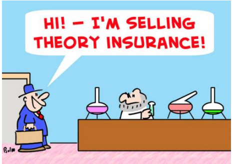


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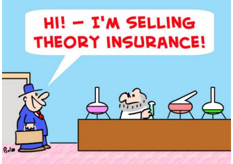


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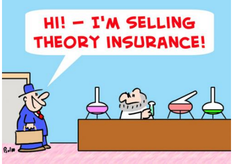


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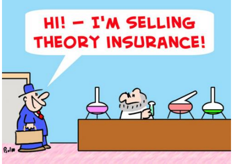


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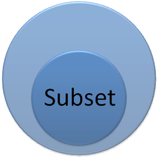
- Often larger approximation error → larger $s_\epsilon$ & rougher approximations are better when speed super important

# Ex 1: Approximations using subsets

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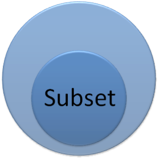


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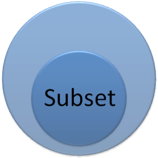


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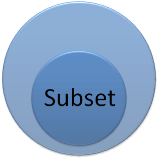


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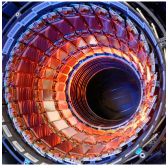

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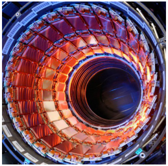

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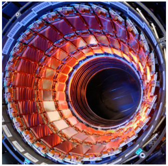

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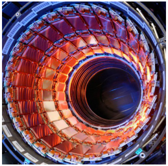

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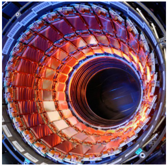

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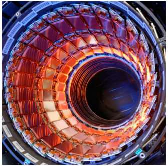

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- As budget increases & loss focused more on tails (e.g., for interval estimation), optimal $|V|$ increases

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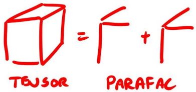


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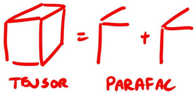


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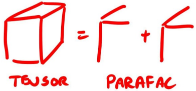


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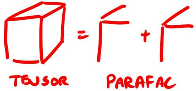


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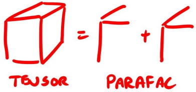


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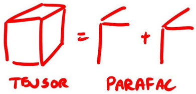


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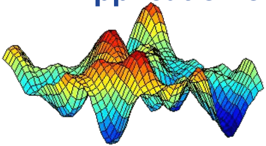

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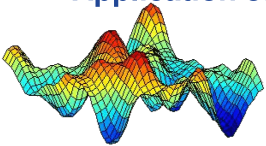

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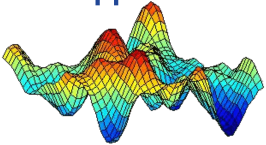

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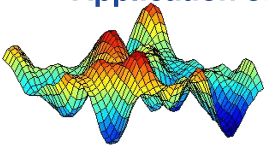

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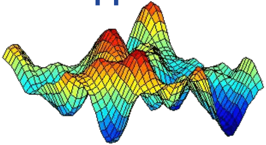

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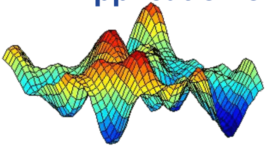

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- Also, very interested in hybrid frequentist-Bayes algorithms

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- Scalable, excellent mixing & empirical/predictive performance

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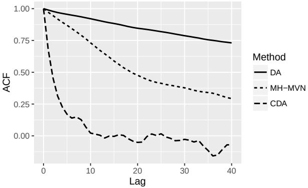


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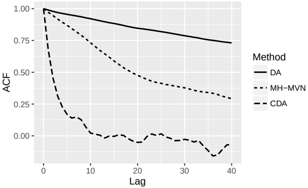


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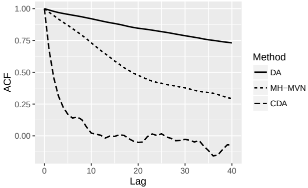


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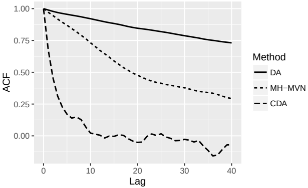
- Common data augmentation algorithms for discrete data fail badly for large imbalanced data (*Johndrow et al. 2016*)

# What about mixing?

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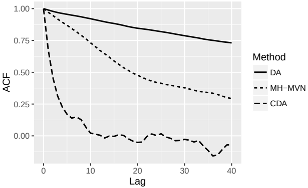


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- But such problems can be fixed via calibration (*Duan et al. 2016*)
- Interesting area for further research

# Primary References


- Duan L, Johndrow J, Dunson DB (2017) Calibrated data augmentation for scalable Markov chain Monte Carlo. *arXiv:1703.03123*.
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