A CUBICAL TYPE THEORY FOR HIGHER INDUCTIVE TYPES

SIMON HUBER

1. Introduction

This note describes a variation of cubical type theory [1] better suited for the extension with higher inductive types. The basic idea is to decompose the composition operation into a generalized version of transport and a homogeneous composition, i.e., a composition in a constant type. A similar approach was already taken in earlier versions of [1] which where then dropped due to problems with a regularity assumption on composition present in the earlier versions.

2. New Primitives

2.1. **Transport.** The generalization of the transport operation from [1] where one can also specify where the given type is known to be constant; on this part the output is equal to the input.

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0) \qquad \Gamma \vdash u_0 : A(i/0)}{\Gamma \vdash \mathsf{transp}^i\, A\, \varphi\, u_0 : A(i/1)[\varphi \mapsto u_0]}$$

Note that since $\Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0)$ also $\Gamma, \varphi \vdash A(i/0) = A(i/1)$ (and hence this equation also holds in context $\Gamma, i : \mathbb{I}, \varphi$).

We can also derive a corresponding "filling" operation which connects the input to transp to its output by:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0) \qquad \Gamma \vdash u_0 : A(i/0)}{\Gamma, i : \mathbb{I} \vdash \mathsf{transpFill}^i\, A\, \varphi\, u_0 = \mathsf{transp}^j\, A(i/i \wedge j)\, (\varphi \vee (i = 0))\, u_0 : A}$$

Note that $\Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0)$ entails

$$\Gamma, i : \mathbb{I}, j : \mathbb{I}, \varphi \vee (i = 0) \vdash A(i/i \wedge j) = A(i/i \wedge j)(j/0).$$

This operation satisfies

$$\Gamma \vdash (\mathsf{transpFill}^i\, A\, \varphi\, u_0)(i/0) = u_0 : A(i/0), \text{ and}$$
$$\Gamma \vdash (\mathsf{transpFill}^i\, A\, \varphi\, u_0)(i/1) = \mathsf{transp}^i\, A\, \varphi\, u_0 : A(i/1),$$

and the induced path is constant u_0 on φ.

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This old write-up is based on work figuring out higher inductive types in cubical type theory together with Thierry Coquand, Anders Mörtberg, and Cyril Cohen, which later also lead to [2]. Section 3.6 was slightly modified in February 2022, after a possible problem was pointed out by András Kovács (see `https://github.com/agda/agda/issues/5755`).

Using the involution on $\mathbb{I}$ we can also derive the corresponding operation going from $(i/1)$ to $(i/0)$ by:

$$\mathsf{transp}^{-i}\, A\, \varphi\, u = (\mathsf{transp}^{i}\, A(i/1-i)\, \varphi\, u)(i/1-i) : A(i/0)$$

where now $u : A(i/1)$. Similarly one can define $\mathsf{transpFill}^{-i}\, A\, \varphi\, u$.

Another derived operation is $\mathsf{forward}$:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash r : \mathbb{I} \qquad \Gamma \vdash u : A(i/r)}{\Gamma \vdash \mathsf{forward}^{i}\, A\, r\, u = \mathsf{transp}^{i}\, A(i/i \vee r)\, (r = 1)\, u : A(i/1)}$$

satisfying $\mathsf{forward}^{i}\, A\, 1\, u = u$.

2.2. **Homogeneous Composition.** Homogeneous composition is like composition from [1] but in a constant type:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash u_0 : A[\varphi \mapsto u(i/0)]}{\Gamma \vdash \mathsf{hcomp}^{i}\, A\, [\varphi \mapsto u]\, u_0 : A[\varphi \mapsto u(i/1)]}$$

We have a derived analogous homogeneous filling operation given by

$$\Gamma, i : \mathbb{I} \vdash \mathsf{hfill}^{i}\, A\, [\varphi \mapsto u]\, u_0 = \mathsf{hcomp}^{j}\, A\, [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto u_0]\, u_0 : A.$$

2.3. **Composition.** The general composition operation from [1] can be defined in terms of transp and hcomp as follows.

$$\frac{\begin{array}{c}\Gamma, i : \mathbb{I} \vdash A \\ \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash u_0 : A(i/0)[\varphi \mapsto u(i/0)]\end{array}}{\begin{array}{c}\Gamma \vdash \mathsf{comp}^{i}\, A\, [\varphi \mapsto u]\, u_0 = \\ \mathsf{hcomp}^{i}\, A(i/1)\, [\varphi \mapsto \mathsf{forward}^{j}\, A(i/j)\, i\, u]\, (\mathsf{forward}^{i}\, A\, 0\, u_0) : A(i/1)\end{array}}$$

Note $\mathsf{forward}^{j}\, A(i/j)\, i\, u$ binds j only in A we can also simply write this as $\mathsf{forward}^{i}\, A\, i\, u$. The required judgmental equality for comp follows from the one of hcomp and $\mathsf{forward}^{i}\, A\, 1\, u = u$.

It might be illustrative to the reader to see that such a generalized transport operation $\mathsf{transp}^{i}\, A\, \varphi\, u_0$ can be defined in terms of composition by $\mathsf{comp}^{i}\, A\, [\varphi \mapsto u_0]\, u_0$.

3. Recursive Definition of Transport

We now explain $\mathsf{transp}^{i}\, A\, \varphi\, u_0$ by induction on the type A.

3.1. **Natural Numbers.**

$$\mathsf{transp}^{i}\, \mathsf{N}\, \varphi\, 0 = 0$$

$$\mathsf{transp}^{i}\, \mathsf{N}\, \varphi\, (\mathsf{S}\, u_0) = \mathsf{S}(\mathsf{transp}^{i}\, \mathsf{N}\, \varphi\, u_0)$$

We could also directly take $\mathsf{transp}^{i}\, \mathsf{N}\, \varphi\, u_0 = u_0$.

3.2. **Dependent Paths.** Let $\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash A$, $\Gamma, i : \mathbb{I} \vdash u : A(j/0)$, and $\Gamma, i : \mathbb{I} \vdash v : A(j/1)$.

$$\begin{aligned}&\mathsf{transp}^{i}\, (\mathsf{Path}^{j}\, A\, v\, w)\, \varphi\, u_0 = \\ &\qquad\qquad \langle j \rangle\, \mathsf{comp}^{i}\, A\, [\varphi \mapsto u_0\, j, (j = 0) \mapsto v, (j = 1) \mapsto w]\, (u_0\, j)\end{aligned}$$

Note that we can in general not take an hcomp here as A might depend on i.

3.3. **Dependent Pairs.** Let $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I}, x : A \vdash B$ with $\Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0)$ and $\Gamma, i : \mathbb{I}, \varphi, x : A \vdash B = B(i/0)$.

$$\mathsf{transp}^i\,((x:A) \times B)\,\varphi\,u_0 = (\mathsf{transp}^i\,A\,\varphi\,(u_0.1), \mathsf{transp}^i\,B(x/v)\,\varphi\,(u_0.2))$$

where $v = \mathsf{transpFill}^i\,A\,\varphi\,u_0.1$.

3.4. **Dependent Functions.** Let $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I}, x : A \vdash B$ with $\Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0)$ and $\Gamma, i : \mathbb{I}, \varphi, x : A \vdash B = B(i/0)$.

$$\mathsf{transp}^i\,((x:A) \to B)\,\varphi\,u_0\,v = \mathsf{transp}^i\,B(x/w)\,\varphi\,(u_0\,w(i/0))$$

where $v : A(i/1)$ and $w = \mathsf{transpFill}^{-i}\,A\,\varphi\,v$.

3.5. **Universe.**

$$\mathsf{transp}^i\,\mathsf{U}\,\varphi\,A = A$$

3.6. **Glue.** Let

$$\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I} \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash T \qquad \Gamma, i : \mathbb{I}, \varphi \vdash w : \mathsf{Equiv}\,T\,A$$

and write B for $\mathsf{Glue}\,[\varphi \mapsto (T, w)]\,A$. We will indicate the (usually omitted) arguments of unglue by a subscript of the involved partial element, so, e.g., unglue_φ denotes the unglue of B, and $\mathsf{unglue}_{\varphi(i/0)}$ the unglue of $B(i/0)$.

Let further $\Gamma \vdash \psi : \mathbb{F}$ and $\Gamma, i : \mathbb{I}, \psi \vdash B = B(i/0)$ and $\Gamma \vdash u_0 : B(i/0)$. We are going to define

$$\Gamma \vdash \mathsf{transp}^i\,B\,\psi\,u_0 : B(i/1)$$

satisfying[1]

(i) $\Gamma, \psi \vdash \mathsf{transp}^i\,B\,\psi\,u_0 = u_0 : B(i/1)$, and
(ii) $\Gamma, \forall i \varphi \vdash \mathsf{transp}^i\,B\,\psi\,u_0 = \mathsf{transp}^i\,T\,\psi\,u_0 : T(i/1)$.

Note that since B is constant when restricted to ψ, so is T:

$$\Gamma, i : \mathbb{I}, \varphi, \psi \vdash T = B = B(i/0) = T(i/0),$$

and so the right-hand side in (ii) is well typed.

First, we set

$$\Gamma, \forall i \varphi, i : \mathbb{I} \vdash \tilde{t} = \mathsf{transpFill}^i\,T\,\psi\,u_0 : T$$

and $\Gamma, \forall i \varphi \vdash t_1 = \tilde{t}(i/1) : T(i/1)$.

Next, define

$$\Gamma \vdash a_1 = \mathsf{comp}^i\,A\,[\psi \mapsto \mathsf{unglue}_\varphi\,u_0, \forall i \varphi \mapsto w.1\,\tilde{t}]\,(\mathsf{unglue}_{\varphi(i/0)}\,u_0) : A(i/1).$$

Note that we have

$$\Gamma, \psi \wedge \forall i \varphi, i : \mathbb{I} \vdash w.1\,\tilde{t} = w.1\,u_0 = \mathsf{unglue}_\varphi\,u_0,$$

and $\Gamma, \forall i \varphi \vdash (w.1\,\tilde{t})(i/0) = w(i/0).1\,u_0 = \mathsf{unglue}_{\varphi(i/0)}\,u_0$, so the previous composition is well formed.

We get a partial element

$$\begin{aligned} \Gamma, \varphi(i/1), \psi \vee \forall i \varphi \vdash [&\psi \mapsto (u_0, \langle\_\rangle a_1), \\ &\forall i \varphi \mapsto (t_1, \langle\_\rangle a_1)] : \mathsf{fib}\,w(i/1).1\,a_1 \end{aligned} \tag{1}$$

[1] Note that these are of course rules of the system. What this really shows is that these rules are admissible in this case, and should also suggest how to define a constructive semantics based on cubical sets similar to [1]. Similar remarks apply later for our calculations.

which we can extend to an element

$$\Gamma, \varphi(i/1) \vdash (t_1', \alpha) : \mathsf{fib}\, w(i/1).1\, a_1$$

using that $w(i/1).1$ is an equivalence [1, Lemma 5].

Now set

$$\Gamma \vdash a_1' = \mathsf{hcomp}^j\, A(i/1)\, [\varphi(i/1) \mapsto \alpha\, j, \psi \mapsto a_1]\, a_1 : A(i/1).$$

Note that $\Gamma, j : \mathbb{I}, \varphi(i/1) \wedge \psi \vdash \alpha j = a_1$ since (t_1', α) extends (1), and trivially $\Gamma, \varphi(i/1) \vdash \alpha\, 0 = a_1$ as α is in the fiber of a_1.

Finally, we can set

$$\Gamma \vdash \mathsf{transp}^i(\mathsf{Glue}\, [\varphi \mapsto (T, w)]\, A)\, \psi\, u_0 = \mathsf{glue}\, [\varphi(i/1) \mapsto t_1']\, a_1' : B(i/1)$$

which is well defined as $\Gamma, \varphi(i/1) \vdash a_1' = \alpha\, 1 = w(i/1).1\ t_1'$.

Let us now check (i) and (ii). For (i) we have $\Gamma, \psi, \varphi(i/1) \vdash t_1' = u_0 : T(i/1)$ as (t_1', α) extends (1).

Concerning (ii) we have

$$\Gamma, \forall i \varphi \vdash \mathsf{transp}^i(\mathsf{Glue}\, [\varphi \mapsto (T, w)]\, A)\, \psi\, u_0 = t_1' = t_1$$

using $\forall i \varphi \leqslant \varphi(i/1)$ and (1).

4. Recursive Definition of Homogeneous Composition

We explain hcomp by induction on the type.

4.1. Natural Numbers.

$$\mathsf{hcomp}^i\, \mathsf{N}\, [\varphi \mapsto 0]\, 0 = 0$$
$$\mathsf{hcomp}^i\, \mathsf{N}\, [\varphi \mapsto \mathsf{S}\, u]\, (\mathsf{S}\, u_0) = \mathsf{S}(\mathsf{hcomp}^i\, \mathsf{N}\, [\varphi \mapsto u]\, u_0)$$

4.2. Dependent Paths.

$$\mathsf{hcomp}^i(\mathsf{Path}^j\, A\, v\, w)\, [\varphi \mapsto u]\, u_0 = \langle j \rangle\, \mathsf{hcomp}^i\, A\, [\varphi \mapsto u\, j, (j = 0) \mapsto v, (j = 1) \mapsto w]\, (u_0\, j)$$

4.3. Dependent Pairs.

$$\mathsf{hcomp}^i\, ((x : A) \times B)\, [\varphi \mapsto u]\, u_0 = (v(i/1), \mathsf{comp}^i\, B(x/v)\, [\varphi \mapsto u.2]\, u_0.2)$$

where $v = \mathsf{hfill}^i\, A\, [\varphi \mapsto u.1]\, u_0.1$. As v depends on i we cannot use hcomp in the second component on the right-hand side.

4.4. Dependent Functions.

$$\mathsf{hcomp}^i\, ((x : A) \to B)\, [\varphi \mapsto u]\, u_0\, v = \mathsf{hcomp}^i\, B(x/v)\, [\varphi \mapsto u\, v]\, (u_0\, v)$$

4.5. Universe.

$$\mathsf{hcomp}^i\, \mathsf{U}\, [\varphi \mapsto E]\, A = \mathsf{Glue}\, [\varphi \mapsto (E(i/1), \mathsf{equiv}^i\, E(i/1 - i))]\, A$$

4.6. **Glue.** Given $\Gamma \vdash A$, $\Gamma \vdash \varphi : \mathbb{F}$, $\Gamma, \varphi \vdash T$, and $\Gamma, \varphi \vdash w : \mathsf{Equiv}\, T\, A$. Let us write B for $\mathsf{Glue}\,[\varphi \mapsto (T, w)]\, A$. Moreover, we are given

$$\Gamma \vdash \psi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \psi \vdash u : B \qquad \Gamma \vdash u_0 : B[\psi \mapsto u(i/0)]$$

and we want to define

$$\Gamma \vdash \mathsf{hcomp}^i\, B\, [\psi \mapsto u]\, u_0 : B[\psi \mapsto u(i/1)]$$

such that

$$\Gamma, \varphi \vdash \mathsf{hcomp}^i\, B\, [\psi \mapsto u]\, u_0 = \mathsf{hcomp}^i\, T\, [\psi \mapsto u]\, u_0 : T. \tag{2}$$

First, we set

$$\Gamma, i : \mathbb{I}, \varphi \vdash \tilde{t} = \mathsf{hfill}^i\, T\, [\psi \mapsto u]\, u_0 : T$$

and write $t_1 = \tilde{t}(i/1)$.

Now define $\Gamma \vdash a_1 = \mathsf{hcomp}^i\, A\, [\psi \mapsto \mathsf{unglue}\, u, \varphi \mapsto w.1\, \tilde{t}]\, (\mathsf{unglue}\, u_0) : A$. This composition is well formed since

$$\Gamma, \varphi \vdash w.1\, \tilde{t}(i/0) = w.1\, u_0 = \mathsf{unglue}\, u_0 : A$$

and

$$\Gamma, i : \mathbb{I}, \varphi \wedge \psi \vdash \mathsf{unglue}\, u = w.1\, u = w.1\, \tilde{t} : A.$$

We can now set $\Gamma \vdash \mathsf{hcomp}^i\, B\, [\psi \mapsto u]\, u_0 = \mathsf{glue}\, [\varphi \mapsto t_1]\, a_1 : B$. This is well defined as $\Gamma\varphi \vdash a_1 = w.1\, \tilde{t}(i/1) = w.1\, t_1 : A$. Note that we also have $\Gamma, \psi \vdash \mathsf{hcomp}^i\, B\, [\psi \mapsto u]\, u_0 = u(i/1) : B$ since $\Gamma, \psi, \varphi \vdash t_1 = u(i/1) : T$ and $\Gamma, \psi \vdash a_1 = \mathsf{unglue}\, u(i/1) : A$, so

$$\begin{aligned}\Gamma, \psi \vdash \mathsf{hcomp}^i\, B\, [\psi \mapsto u]\, u_0 &= \mathsf{glue}\, [\varphi \mapsto t_1]\, a_1 \\ &= \mathsf{glue}\, [\varphi \mapsto u(i/1)]\, (\mathsf{unglue}\, u(i/1)) = u(i/1) : B\end{aligned}$$

Also (2) trivially follows from $\Gamma, \varphi \vdash \mathsf{glue}\, [\varphi \mapsto t_1]\, a_1 = t_1 : T$.

Observe that we didn't use the fact that $w.1$ is an equivalence.

References

1. C. Cohen, T. Coquand, S. Huber, and A. Mörtberg, *Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom*, 21st International Conference on Types for Proofs and Programs (TYPES 2015) (T. Uustalu, ed.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 69, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018.
2. T. Coquand, S. Huber, and A. Mörtberg, *On higher inductive types in cubical type theory*, Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science (New York, NY, USA), LICS '18, ACM, 2018, pp. 255–264.