EE364a Review Session 4

session outline:

- transformations
- dual problem
- homework hints

Transformations

- transformation of objective
- transformation of constraints

example: objective transformation

$$\begin{array}{ll} \text{minimize} & \int_{-\infty}^{c^T x} \frac{1}{\sqrt{2\pi}} e^{t^2/2} dt \\ \text{subject to} & Ax \preceq b \\ & Hx = g \end{array}$$

– is it a convex problem?
– is it a quasiconvex problem?

solution:

– nonconvex: objective is not convex

– quasiconcave: sublevel sets are convex

– $\int_{-\infty}^{c^T x} \frac{1}{\sqrt{2\pi}} e^{t^2/2} dt = \Phi(c^T x)$ where $\Phi(u)$ is monotone increasing in u, so minimizing $\Phi(c^T x)$ is the same as minimizing $c^T x$

– thus equivalent problem is an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & Hx = g \end{array}$$

example: transform the following constraint to a set of linear constraints

$$a_1^T x + b_1 + \max(a_2^T x + b_2, a_3^T x + b_3) \leq 0$$

solution 1:

– introduce a new variable t

– thus

$$\begin{aligned} a_1^T x + b_1 + t &\leq 0 \\ a_2^T x + b_2 - t &\leq 0 \\ a_3^T x + b_3 - t &\leq 0 \end{aligned}$$

solution 2:

– put $a_1^T x + b_1$ inside the $\max$ function

– then we get

$$\begin{aligned} (a_1 + a_2)^T x + (b_1 + b_2) &\leq 0 \\ (a_1 + a_3)^T x + (b_1 + b_3) &\leq 0 \end{aligned}$$

example: what about the following constraint?

$$a_1^T x + b_1 - \max(a_2^T x + b_2, a_3^T x + b_3) \leq 0$$

solution

– non-convex constraint
– cannot be transformed into a set of linear inequalities
– consider the following problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x - \bar{b} \preceq 0 \\ & Hx - g = 0 \\ & a_1^T x + b_1 - \max(a_2^T x + b_2, a_3^T x + b_3) \leq 0 \end{array}$$

This is not an LP, but can be solved easily by solving two LPs.

Consider the last inequality:

$*$ if $a_2^T x + b_2 \geq a_3^T x + b_3$, then $a_1^T x + b_1 - a_2^T x - b_2 \leq 0$

$*$ if $a_2^T x + b_2 \leq a_3^T x + b_3$, then $a_1^T x + b_1 - a_3^T x - b_3 \leq 0$

Thus, optimal solution can be found by solving two LPs:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x - \bar{b} \preceq 0 \\ & Hx - g = 0 \\ & a_2^T x + b_2 \geq a_3^T x + b_3 \\ & a_1^T x + b_1 - a_2^T x - b_2 \leq 0 \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{A}x - \bar{b} \preceq 0 \\ & Hx - g = 0 \\ & a_2^T x + b_2 \leq a_3^T x + b_3 \\ & a_1^T x + b_1 - a_3^T x - b_3 \leq 0 \end{array}$$

Then choose optimal solution with smaller objective value.

Finding dual problem

- primal problem

$$\begin{array}{ll} \text{maximize} & f_0(x) \\ \text{subject to} & f(x) \preceq 0 \\ & h(x) = 0 \end{array}$$

- Lagrangian

$$L(x,\lambda,\nu) = f_0(x) + \lambda^T f(x) + \nu^T h(x)$$

- Lagrange dual function

$$g(\lambda,\nu) = \inf_x L(x,\lambda,\nu)$$

- dual problem

$$\begin{array}{ll} \text{maximize} & g(\lambda,\nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

example: entropy maximization

$$\begin{array}{ll} \text{minimize} & f_0(x) = \sum_{i=1}^n x_i \log x_i \\ \text{subject to} & Ax \preceq b \\ & \mathbf{1}^T x = 1 \end{array}$$

solution:

– Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \lambda^T (Ax - b) + \nu(\mathbf{1}^T x - 1)$$

– find Lagrange dual function

$$\begin{aligned} g(\lambda, \nu) &= \inf_x \left(f_0(x) + \lambda^T (Ax - b) + \nu(\mathbf{1}^T x - 1) \right) \\ &= -b^T \lambda - \nu - \sup_x \left((-A^T \lambda - \nu \mathbf{1})^T x - f_0(x) \right) \\ &= -b^T \lambda - \nu - f_0^*(-A^T \lambda - \nu \mathbf{1}), \end{aligned}$$

where $f_0^*(y) = \sup_x (y^T x - f_0(x)) = \sum_{i=1}^n e^{y_i - 1}$

– we get the dual problem

$$\begin{array}{ll} \text{maximize} & -b^T\lambda - \nu - \sum_{i=1}^n e^{-a_i^T\lambda - \nu - 1} \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

– to simplify, minimize $g(\lambda, \nu)$ over ν (i.e., $\nu^\star = \log \sum_{i=1}^n e^{-a_i^T\lambda} - 1$)

$$\begin{array}{ll} \text{maximize} & -\log\left(\sum_{i=1}^n e^{-a_i^T\lambda}\right) - b^T\lambda \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

– finally we get a GP in convex form (why?)

Homework hints

- P4.29
  - how to deal with $\mathbf{prob}(c^T x \geq \alpha)$?
    see robust LP example (stochastic approach via SOCP)

$$\mathbf{prob}(c^T x \geq \alpha) = 1 - \Phi\left(\frac{\alpha - \bar{c}^T x}{\sqrt{x^T \Sigma x}}\right)$$

  - how to deal with nonconvex/nonconcave objective?
    $\Phi(u)$ is monotone increasing in u, so transform objective to get quasiconvex problem

- FIR filter design
  - how to represent the objective w_c in (b)?
    remind approximation width problem in HW3

    $$W(x) = \sup\{T|\ |x_1 f_1(t) + \cdots + x_1 f_1(t) - f_0(t)| \leq \epsilon \text{ for } 0 \leq t \leq T\}$$

  - how to represent the objective N in (c)?
    express filter of length N in terms of coefficients a_i, and then apply the hint for ω_c